Algorithmics

Spring 2020 - Tutorial #2

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Exercise 2.1

Task

Consider the following LP:

$$\begin{aligned} \textbf{maximize} \quad & x_1 + x_2 \\ \textbf{subject to} \quad & 4x_1 - x_2 \leq 8 \\ & 2x_1 + x_2 \leq 10 \\ & 5x_1 - 2x_2 \geq -2 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

a) Find all extremal points of this LP.
b) Transform the given LP into normal form and find the admissible basic points.
c) Solve the LP.

Exercise 2.1 (a) - Solution

We use the same technique as in *Exercise 1.6* (exhaustive search) to obtain the following extremal points (x_1, x_2) and respective values $c(x_1, x_2)$:

(x_1, x_2)	$c(x_1, x_2)$
$(0, 0)$	0
$(2, 0)$	2
$(0, 1)$	1
$(2, 6)$	8
$(3, 4)$	7

Exercise 2.1 (b) - Solution

$$\begin{array}{rrcrcr} \textbf{maximize} & 1x_1 & + & 1x_2 & & \\ \textbf{subject to} & 4x_1 & - & 1x_2 & \leq & 8 \\ & 2x_1 & + & 1x_2 & \leq & 10 \\ & -5x_1 & + & 2x_2 & \leq & 2 \\ & x_1 & , & x_2 & \geq & 0 \end{array}$$

The extremal points are the same as in (a). The respective slacking extensions yield the admissible basic points.

I. Extremal Point: (0, 0)
   $\Rightarrow$ Corresponding ABP: (0, 0, 8, 10, 2)

II Extremal Point: (2, 0)
   $\Rightarrow$ Corresponding ABP: (2, 0, 0, 6, 12)

III Extremal Point: (0, 1)
   $\Rightarrow$ Corresponding ABP: (0, 1, 9, 9, 0)

IV Extremal Point: (2, 6)
   $\Rightarrow$ Corresponding ABP: (2, 6, 6, 0, 0)

V Extremal Point: (3, 4)
   $\Rightarrow$ Corresponding ABP: (3, 4, 0, 0, 9)

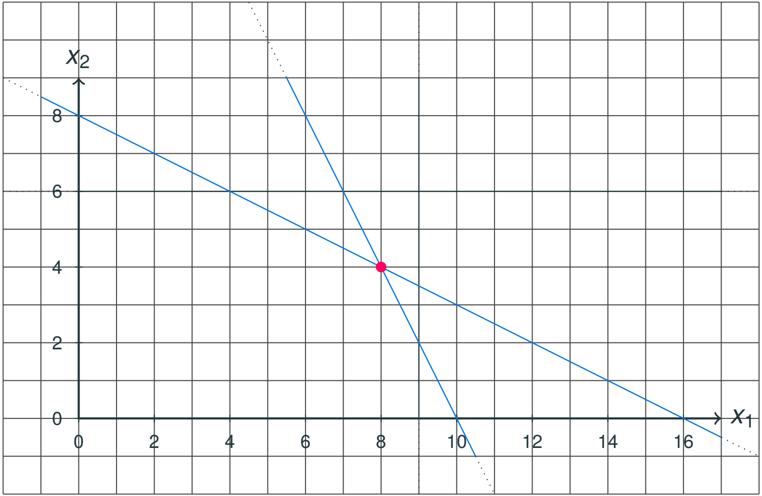
Solution

c) Based on (a), we see that the extremal point $(2, 6)$ maximizes the LP with $c\,(2, 6) = 8$.

Exercise 2.2

Compare this exercise to exercise 1.6!

The Simplex Algorithm



$$
\begin{array}{rrcrcrl}
c_0: & x_1 & , & x_2 & \geq & 0 \\
c_1: & x_1 & & & \leq & 9 \\
c_2: & & & x_2 & \leq & 6 \\
c_3: & x_1 & + & 2x_2 & \leq & 16 \\
c_4: & 2x_1 & + & x_2 & \leq & 20
\end{array}
$$

EP: (8,4)

Exercise 2.3

Task

Consider the following LP:

$$\begin{aligned} \textbf{maximize} \quad & 3x_1 + x_2 + 2x_3 \\ \textbf{subject to} \quad & x_1 + x_2 + 3x_3 \leq 30 \\ & 2x_1 + 2x_2 + 5x_3 \leq 24 \\ & 4x_1 + x_2 + 2x_3 \leq 36 \\ & x_1,\ x_2,\ x_3 \geq 0 \end{aligned}$$

a) Determine the initial Simplex tableau and highlight all pivot positions.

b) Solve the LP using the Simplex method.

Exercise 2.3

$$\begin{array}{lrrrrr} \textbf{maximize} & t = & 0 & +3x_1 & +1x_2 & +2x_3 \\ \textbf{subject to} & s_1 = & 30 & -1x_1 & -1x_2 & -3x_3 \\ & s_2 = & 24 & -2x_1 & -2x_2 & -5x_3 \\ & s_3 = & 36 & -4x_1 & -1x_2 & -2x_3 \end{array}$$

- x_1, x_2, x_3 are the non-basic variables and thus equal to zero.
- s_1, s_2, s_3 are the basic variables.

Which non-basic variable increases the target value t the most?

The Simplex Algorithm

$$
\begin{array}{lrrrrr|l}
\textbf{maximize} & t = & 0 & +3x_1 & +1x_2 & +2x_3 & \\
\textbf{subject to} & s_1 = & 30 & -1x_1 & -1x_2 & -3x_3 & \frac{30}{1} = 30 \\
 & s_2 = & 24 & -2x_1 & -2x_2 & -5x_3 & \frac{24}{2} = 12 \\
 & s_3 = & 36 & -4x_1 & -1x_2 & -2x_3 & \frac{36}{4} = 9
\end{array}
$$

- All variables are required to be nonnegative.
- x_1 may be increased by at most 9.
- This will set s_3 to zero and thus leave the basis.
- x_1 will enter the basis.

$$
\begin{array}{rrrrr}
s_3 = & 36 & -4x_1 & -1x_2 & -2x_3 \\
x_1 = & 9 & -\frac{1}{4}s_3 & -\frac{1}{4}x_2 & -\frac{1}{2}x_3
\end{array}
$$

Now we can plug x_1 into the remaining terms.

$$x_1 = \quad 9 \quad -\tfrac{1}{4}s_3 \quad -\tfrac{1}{4}x_2 \quad -\tfrac{1}{2}x_3$$

Modifying the s_1 term:

$$s_1 = 30 - 1x_1 - 1x_2 - 3x_3$$
$$s_1 = 30 - 1 \cdot (9 - \frac{1}{4}s_3 - \frac{1}{4}x_2 - \frac{1}{2}x_3) - 1x_2 - 3x_3$$
$$s_1 = 21 + \frac{1}{4}s_3 - \frac{3}{4}x_2 - \frac{5}{2}x_3$$

The Simplex Algorithm

$$x_1 = 9 - \tfrac{1}{4}s_3 - \tfrac{1}{4}x_2 - \tfrac{1}{2}x_3$$

Modifying the s_2 term:

$$s_2 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$s_2 = 24 - 2 \cdot (9 - \frac{1}{4}s_3 - \frac{1}{4}x_2 - \frac{1}{2}x_3) - 2x_2 - 5x_3$$

$$s_2 = 6 + \frac{1}{2}s_3 - \frac{3}{2}x_2 - 4x_3$$

The Simplex Algorithm

$$x_1 = \quad 9 \quad -\tfrac{1}{4}s_3 \quad -\tfrac{1}{4}x_2 \quad -\tfrac{1}{2}x_3$$

Modifying the t term:

$$t = 0 + 3x_1 + 1x_2 + 2x_3$$
$$t = 0 + 3 \cdot (9 - \frac{1}{4}s_3 - \frac{1}{4}x_2 - \frac{1}{2}x_3) + 1x_2 + 2x_3$$
$$t = 27 - \frac{3}{4}s_3 + \frac{1}{4}x_2 + \frac{1}{2}x_3$$

The Simplex Algorithm

$$
\begin{array}{llrrrr}
\textbf{maximize} & t = & 27 & -\frac{3}{4}s_3 & +\frac{1}{4}x_2 & +\frac{1}{2}x_3 \\
\textbf{subject to} & s_1 = & 21 & +\frac{1}{4}s_3 & -\frac{3}{4}x_2 & -\frac{5}{2}x_3 \\
 & s_2 = & 6 & +\frac{1}{2}s_3 & -\frac{3}{2}x_2 & -4x_3 \\
 & x_1 = & 9 & -\frac{1}{4}s_3 & -\frac{1}{4}x_2 & -\frac{1}{2}x_3
\end{array}
$$

- s_3, x_2, x_3 are the non-basic variables and thus equal to zero.
- s_1, s_2, x_1 are the basic variables.

Which non-basic variable increases the target value t the most?

Solution (a)

Consider the following LP:

$$\begin{aligned} \textbf{maximize} \quad & 3x_1 + x_2 + 2x_3 \\ \textbf{subject to} \quad & x_1 + x_2 + 3x_3 \leq 30 \\ & 2x_1 + 2x_2 + 5x_3 \leq 24 \\ & 4x_1 + x_2 + 2x_3 \leq 36 \\ & x_1,\ x_2,\ x_3 \geq 0 \end{aligned}$$

a) Determine the initial Simplex tableau and highlight all pivot positions.

		x_1	x_2	x_3
	0	3	1	2
s_1	30	1	1	3
s_2	24	2	(2)	(5)
s_3	36	(4)	1	2

Solution (b)

Consider the following LP:

$$\begin{aligned} \textbf{maximize} \quad & 3x_1 + x_2 + 2x_3 \\ \textbf{subject to} \quad & x_1 + x_2 + 3x_3 \leq 30 \\ & 2x_1 + 2x_2 + 5x_3 \leq 24 \\ & 4x_1 + x_2 + 2x_3 \leq 36 \\ & x_1,\ x_2,\ x_3 \geq 0 \end{aligned}$$

b) Solve the LP using the Simplex method.

Initial Tableau:

		x_1	x_2	x_3
	0	3	1	2
s_1	30	1	1	3
s_2	24	2	2	5
s_3	36	4	1	2

The Simplex Algorithm

		x_1	x_2	x_3
	0	3	1	2
s_1	30	1	1	3
s_2	24	2	2	5
s_3	36	4	1	2

		s_3	x_2	x_3
		-3/4		
s_1		-1/4		
s_2		-1/2		
x_1	9	1/4	1/4	1/2

1. $\tilde{t}_{3,1} = 1/4$
2. $\tilde{t}_{3,0} = 36/4 = 9$

   $\tilde{t}_{3,2} = 1/4$

   $\tilde{t}_{3,3} = 2/4 = 1/2$
3. $\tilde{t}_{0,1} = -3/4$

   $\tilde{t}_{1,1} = -1/4$

   $\tilde{t}_{2,1} = -2/4 = -1/2$

The Simplex Algorithm

		x_1	x_2	x_3
	0	3	1	2
s_1	30	1	1	3
s_2	24	2	2	5
s_3	36	4	1	2

		s_3	x_2	x_3
	-27	-3/4	1/4	1/2
s_1	21	-1/4	3/4	5/2
s_2	6	-1/2	3/2	4
x_1	9	1/4	1/4	1/2

4. $\tilde{t}_{0,0} = 0 - \frac{3}{4} \cdot 36 = -27$

$\tilde{t}_{0,2} = 1 - \frac{3}{4} \cdot 1 = \frac{1}{4}$

$\tilde{t}_{0,3} = 2 - \frac{3}{4} \cdot 2 = \frac{1}{2}$

$\tilde{t}_{1,0} = 30 - \frac{1}{4} \cdot 36 = 21$

$\tilde{t}_{1,2} = 1 - \frac{1}{4} \cdot 1 = \frac{3}{4}$

$\tilde{t}_{1,3} = 3 - \frac{1}{4} \cdot 2 = \frac{5}{2}$

$\tilde{t}_{2,0} = 24 - \frac{2}{4} \cdot 36 = 6$

$\tilde{t}_{2,2} = 2 - \frac{2}{4} \cdot 1 = \frac{3}{2}$

$\tilde{t}_{2,3} = 5 - \frac{2}{4} \cdot 2 = 4$

Solution (b)

Initial Tableau:

		x_1	x_2	x_3
	0	3	1	2
s_1	30	1	1	3
s_2	24	2	2	5
s_3	36	4	1	2

Iteration 1:

		s_3	x_2	x_3
	-27	-3/4	1/4	1/2
s_1	21	-1/4	3/4	5/2
s_2	6	-1/2	3/2	4
x_1	9	1/4	1/4	1/2

Solution (b)

Iteration 1:

		s_3	x_2	x_3
	-27	-3/4	1/4	1/2
s_1	21	-1/4	3/4	5/2
s_2	6	-1/2	3/2	4
x_1	9	1/4	1/4	1/2

Iteration 2:

		s_3	x_2	s_2
	-111/4	-11/16	1/16	-1/8
s_1	69/4	1/16	-3/16	-5/8
x_3	3/2	-1/8	3/8	1/4
x_1	33/4	5/16	1/16	-1/8

Solution (b)

Iteration 2:

		s_3	x_2	s_2
	-111/4	-11/16	1/16	-1/8
s_1	69/4	1/16	-3/16	-5/8
x_3	3/2	-1/8	3/8	1/4
x_1	33/4	5/16	1/16	-1/8

Final Tableau:

		s_3	x_3	s_2
	-28	-2/3	-1/6	-1/6
s_1	18	0	1/2	-1/2
x_2	4	-1/3	8/3	2/3
x_1	8	1/3	-1/6	-1/6

Optimal Solution: $(\overline{x}_1, \overline{x}_2, \overline{x}_3) = (8, 4, 0)$; $c(8, 4, 0) = 28$.

Exercise 2.4

Task (a)

A farmer owns 200 acres of land, on which he plans to grow barley, wheat and potatoes. The respective details are provided in the following table:

	Potatoes	Barley	Wheat	Capacity
Cultivation costs (€/acre)	1,000	2,000	3,000	150,000 €
Days of work (d/acre)	1	3	4	160 days
Net profit (€/acre)	2,000	2,000	3,000	

Maximize the farmer's net profit.

Solution (a)

$$\begin{array}{ll} \textbf{maximize} & 2x_1 + 2x_2 + 3x_3 \\ \textbf{subject to} & 1 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 \leq 200 \\ & 1 \cdot x_1 + 2 \cdot x_2 + 3 \cdot x_3 \leq 150 \\ & 1 \cdot x_1 + 3 \cdot x_2 + 4 \cdot x_3 \leq 160 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Initial Tableau:

		x_1	x_2	x_3
	0	2	2	3
s_1	200	1	1	1
s_2	150	1	2	3
s_3	160	1	3	4

Solution (a)

Initial Tableau:

		x_1	x_2	x_3
	0	2	2	3
s_1	200	1	1	1
s_2	150	1	2	3
s_3	160	1	3	4

Iteration 1:

		x_1	x_2	s_3
	-120	5/4	-1/4	-3/4
s_1	160	3/4	1/4	-1/4
s_2	30	1/4	-1/4	-3/4
x_3	40	1/4	3/4	1/4

Solution (a)

Iteration 1:

		x_1	x_2	s_3
	-120	5/4	-1/4	-3/4
s_1	160	3/4	1/4	-1/4
s_2	30	1/4	-1/4	-3/4
x_3	40	1/4	3/4	1/4

Iteration 2:

		s_2	x_2	s_3
	-270	-5	1	3
s_1	70	-3	1	2
x_1	120	4	-1	-3
x_3	10	-1	1	1

Solution (a)

Iteration 2:

		s_2	x_2	s_3
	-270	-5	1	3
s_1	70	-3	1	2
x_1	120	4	-1	-3
x_3	10	-1	1	1

Final Tableau:

		s_2	x_2	x_3
	-300	-2	-2	-3
s_1	50	-1	-1	-2
x_1	150	1	2	3
s_3	10	-1	1	1

Optimal Sol.: $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (150, 0, 0)$; $c(150, 0, 0) = 300000$.

Task (b)

The owner of a chicken farm uses two sorts of grain (called *A* and *B*) to feed the animals. Each kind contains protein, fat, carbohydrates and additional indigestible components. On the basis of *A* and *B* the farmer now wants to create a blend which contains at least 1 kg of protein, 800 g of fat and 1.8 kg of carbs. The composition and prices per kg of each sort of grain are given in the table below:

	Grain *A*	Grain *B*
Protein (per kg)	100 g	200 g
Fat (per kg)	200 g	100 g
Carbs (per kg)	100 g	600 g
Price (per kg)	8 €	12 €

Minimize the price of the blend using the Simplex method based on the normal form of the original LP instance and a *helper tableau* T^{aux}.

Solution (b)

Linear Program:

$$\begin{aligned} \textbf{minimize} \quad & 8x_1 + 12x_2 \\ \textbf{s.t.} \quad & 100x_1 + 200x_2 \geq 1000 \\ & 200x_1 + 100x_2 \geq 800 \\ & 100x_1 + 600x_2 \geq 1800 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Normal Form:

$$\begin{aligned} \textbf{maximize} \quad & -8x_1 - 12x_2 \\ \textbf{s.t.} \quad & -100x_1 - 200x_2 \leq -1000 \\ & -200x_1 - 100x_2 \leq -800 \\ & -100x_1 - 600x_2 \leq -1800 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The Auxiliary Problem

$$\begin{array}{rrrrrrrrr}
\textbf{maximize} & & & & & - & 1x_0 & & \\
\textbf{maximize} & - & 8x_1 & - & 12x_2 & & & & \\
\textbf{subject to} & - & 100x_1 & - & 200x_2 & - & 1x_0 & \leq & -1000 \\
 & - & 200x_1 & - & 100x_2 & - & 1x_0 & \leq & -800 \\
 & - & 100x_1 & - & 600x_2 & - & 1x_0 & \leq & -1800 \\
 & & x_1 & , & x_2 & , & x_0 & \geq & 0
\end{array}$$

The Auxiliary Problem

$$
\begin{array}{rrrrrrrrrr}
\textbf{maximize} & & & & & - & 1x_0 & & & & & \\
\textbf{maximize} & - & 8x_1 & - & 12x_2 & & & & & & & \\
\textbf{subject to} & - & 100x_1 & - & 200x_2 & - & 1x_0 & +s_1 & & & = & -1000 \\
 & - & 200x_1 & - & 100x_2 & - & 1x_0 & & +s_2 & & = & -800 \\
 & - & 100x_1 & - & 600x_2 & - & 1x_0 & & & +s_3 & = & -1800
\end{array}
$$

The Auxiliary Problem

$$
\begin{array}{rcrcrcrcr}
t_{\textbf{aux}} & = & & & & & & - & 1x_0 \\
t_{\textbf{orig}} & = & 0 & - & 8x_1 & - & 12x_2 & & \\
s_1 & = & -1000 & + & 100x_1 & + & 200x_2 & + & 1x_0 \\
s_2 & = & -800 & + & 200x_1 & + & 100x_2 & + & 1x_0 \\
s_3 & = & -1800 & + & 100x_1 & + & 600x_2 & + & 1x_0
\end{array}
$$

What's next?

$$
\text{If } \begin{array}{l} x_0 = 1800 \\ x_1 = 0 \\ x_2 = 0 \end{array} \quad \text{we have } \begin{array}{l} s_1 = 800 \\ s_2 = 1000 \\ s_3 = 0 \end{array}.
$$

Solve s_3 for x_0 to obtain $x_0 = 1800 - 100x_1 - 600x_2 + s_3$.

The Auxiliary Problem

$$
\begin{aligned}
x_0 &= 1800 - 100x_1 - 600x_2 + s_3 \\
s_1 &= -1000 + 100x_1 + 200x_2 + (1800 - 100x_1 - 600x_2 + s_3) \\
&= 800 - 400x_2 + s_3 \\
s_2 &= -800 + 200x_1 + 100x_2 + (1800 - 100x_1 - 600x_2 + s_3) \\
&= 1000 + 100x_1 - 500x_2 + s_3 \\
t_{\textbf{orig}} &= 0 - 8x_1 - 12x_2 \\
t_{\textbf{aux}} &= -1800 + 100x_1 + 600x_2 - s_3
\end{aligned}
$$

Solution (b)

Normal Form:

maximize $- 8x_1 - 12x_2$

s.t.

$$-100x_1 - 200x_2 \leq -1000$$
$$-200x_1 - 100x_2 \leq -800$$
$$-100x_1 - 600x_2 \leq -1800$$
$$x_1, x_2 \geq 0$$

[T_{aux}] Initial Tableau:

		x_0	x_1	x_2
	0	-1	0	0
	0	0	-8	-12
s_1	-1000	-1	-100	-200
s_2	-800	-1	-200	-100
s_3	-1800	-1	-100	-600

Solution (b)

[T_{aux}] Initial Tableau:

		x_0	x_1	x_2
	0	-1	0	0
	0	0	-8	-12
s_1	-1000	-1	-100	-200
s_2	-800	-1	-200	-100
s_3	-1800	-1	-100	-600

[T_{aux}] Iteration 1:

		s_3	x_1	x_2
	1800	-1	100	600
	0	0	-8	-12
s_1	800	-1	0	400
s_2	1000	-1	-100	500
x_0	1800	-1	100	600

Solution (b)

[T_{aux}] Iteration 1:

		s_3	x_1	x_2
	1800	-1	100	600
	0	0	-8	-12
s_1	800	-1	0	400
s_2	1000	-1	-100	500
x_0	1800	-1	100	600

[T_{aux}] Iteration 2:

		s_3	x_1	s_1
	600	1/2	100	-3/2
	24	-3/100	-8	3/100
x_2	2	-1/400	0	1/400
s_2	0	1/4	-100	-5/4
x_0	600	1/2	100	-3/2

Solution (b)

[T_{aux}] Iteration 2:

		s_3	x_1	s_1
	600	1/2	100	-3/2
	24	-3/100	-8	3/100
x_2	2	-1/400	0	1/400
s_2	0	1/4	-100	-5/4
x_0	600	1/2	100	-3/2

[T_{aux}] Final Tableau:

		s_3	x_0	s_1
	0	0	-1	0
	72	1/100	2/25	-9/100
x_2	2	-1/400	0	1/400
s_2	600	3/4	1	-11/4
x_1	6	1/200	1/100	-3/200

Solution (b)

[T_{aux}] Final Tableau:

		s_3	x_0	s_1
	0	0	-1	0
	72	1/100	2/25	-9/100
x_2	2	-1/400	0	1/400
s_2	600	3/4	1	-11/4
x_1	6	1/200	1/100	-3/200

Initial Tableau:

		s_3	s_1
	72	1/100	-9/100
x_2	2	-1/400	1/400
s_2	600	3/4	-11/4
x_1	6	1/200	-3/200

Solution (b)

Initial Tableau:

		s_3	s_1
	72	1/100	-9/100
x_2	2	-1/400	1/400
s_2	600	3/4	-11/4
x_1	6	1/200	-3/200

Final Tableau:

		s_2	s_1
	64	-1/75	-4/75
x_2	4	1/300	-1/150
s_3	800	4/3	-11/3
x_1	2	-1/150	1/300

Optimal solution of the initial minimization problem: $(\bar{x}_1, \bar{x}_2) = (2, 4)$; Costs: 64 €.

Exercise 2.5

Task

Consider the following LP instance I:

$$\begin{aligned} \textbf{maximize} \quad & -2x_1 - 3x_2 \\ \textbf{subject to} \quad & x_1 + x_2 \leq 7 \\ & x_1 - x_2 \leq -16 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

Determine $opt(I_{aux})$. Does I have any valid solutions?

Solution

Consider the following LP instance I:

$$\begin{aligned} \textbf{maximize} \quad & -2x_1 - 3x_2 \\ \textbf{subject to} \quad & x_1 + x_2 \leq 7 \\ & x_1 - x_2 \leq -16 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

Determine $opt(I_{aux})$. Does I have any valid solutions?

[T^{aux}] Initial Tableau:

		x_0	x_1	x_2
	0	-1	0	0
	0	0	-2	-3
s_1	7	-1	1	1
s_2	-16	-1	1	-1

Solution

[T^{aux}] Initial Tableau:

		x_0	x_1	x_2
	0	-1	0	0
	0	0	-2	-3
s_1	7	-1	1	1
s_2	-16	-1	1	-1

[T^{aux}] Iteration 1:

		s_2	x_1	x_2
	16	-1	-1	1
	0	0	-2	-3
s_1	23	-1	0	2
x_0	16	-1	-1	1

Solution

[T^{aux}] Iteration 1:

		s_2	x_1	x_2
	16	-1	-1	1
	0	0	-2	-3
s_1	23	-1	0	2
x_0	16	-1	-1	1

[T^{aux}] Iteration 2:

		s_2	x_1	s_1
	9/2	-1/2	-1	-1/2
	69/2	-3/2	-2	3/2
x_2	23/2	-1/2	0	1/2
x_0	9/2	-1/2	-1	-1/2

$$\Rightarrow opt(I_{aux}) = -9/2 \neq 0 \Rightarrow X(I) = \emptyset$$