

# Algorithmics

Spring 2020 - Tutorial #2

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## Exercise 2.1

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## Task

Consider the following LP:

$$\begin{array}{ll}\text{maximize} & x_1 + x_2 \\ \text{subject to} & 4x_1 - x_2 \leq 8 \\ & 2x_1 + x_2 \leq 10 \\ & 5x_1 - 2x_2 \geq -2 \\ & x_1, x_2 \geq 0\end{array}$$

- Find all extremal points of this LP.
- Transform the given LP into normal form and find the admissible basic points.
- Solve the LP.

## Exercise 2.1 (a) - Solution

We use the same technique as in *Exercise 1.6* (exhaustive search) to obtain the following extremal points  $(x_1, x_2)$  and respective values  $c(x_1, x_2)$ :

$(x_1, x_2)$	$c(x_1, x_2)$
(0, 0)	0
(2, 0)	2
(0, 1)	1
(2, 6)	8
(3, 4)	7

## Exercise 2.1 (b) - Solution

$$\begin{array}{llllll} \text{maximize} & 1x_1 & + & 1x_2 & & \\ \text{subject to} & 4x_1 & - & 1x_2 & \leq & 8 \\ & 2x_1 & + & 1x_2 & \leq & 10 \\ & - & 5x_1 & + & 2x_2 & \leq & 2 \\ & x_1 & , & x_2 & \geq & 0 \end{array}$$

The extremal points are the same as in (a).  
The respective slacking extensions yield the  
admissible basic points.

- I. Extremal Point: (0, 0)  
⇒ Corresponding ABP: (0, 0, 8, 10, 2)
- II Extremal Point: (2, 0)  
⇒ Corresponding ABP: (2, 0, 0, 6, 12)
- III Extremal Point: (0, 1)  
⇒ Corresponding ABP: (0, 1, 9, 9, 0)
- IV Extremal Point: (2, 6)  
⇒ Corresponding ABP: (2, 6, 6, 0, 0)
- V Extremal Point: (3, 4)  
⇒ Corresponding ABP: (3, 4, 0, 0, 9)

## Solution

c) Based on (a), we see that the extremal point  $(2, 6)$  maximizes the LP with  $c(2, 6) = 8$ .

## Exercise 2.2

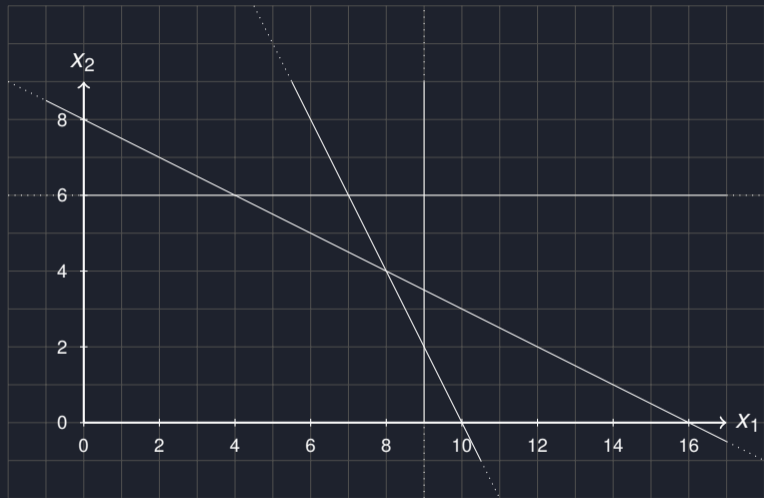
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# The Simplex Algorithm

Compare this exercise to exercise 1.6!



# The Simplex Algorithm



$$C_0 : X_1, X_2 \geq 0$$

$$C_1 : X_1 \leq 9$$

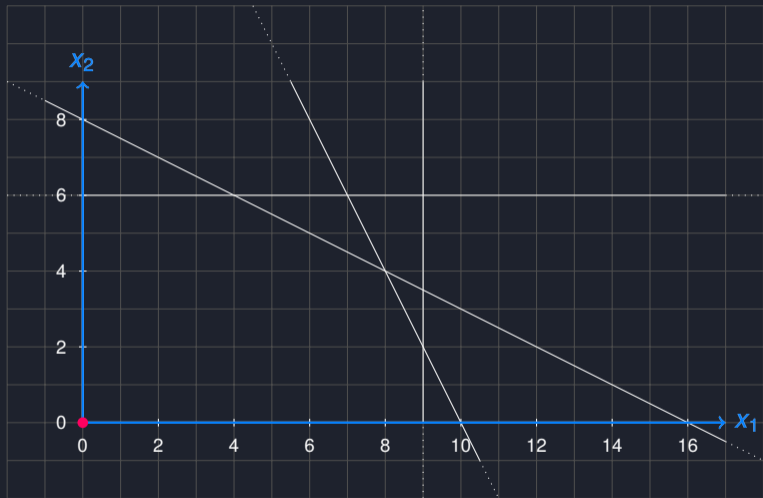
$$C_2 : X_2 \leq 6$$

$$C_3 : X_1 + 2X_2 \leq 16$$

$$C_4 : 2X_1 + X_2 \leq 20$$

	0	$X_1$	$X_2$
$C_1$	9	1	0
$C_2$	6	0	1
$C_3$	16	1	2
$C_4$	20	2	1

# The Simplex Algorithm



$$C_0 : x_1, x_2 \geq 0$$

$$C_1 : x_1 \leq 9$$

$$C_2 : x_2 \leq 6$$

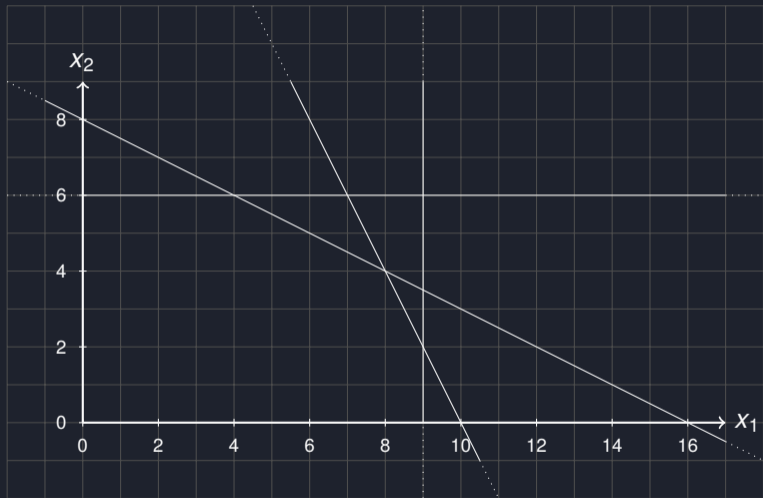
$$C_3 : x_1 + 2x_2 \leq 16$$

$$C_4 : 2x_1 + x_2 \leq 20$$

**EP: (0,0)**

		$x_1$	$x_2$
	0	1	1
$C_1$	9	1	0
$C_2$	6	0	1
$C_3$	16	1	2
$C_4$	20	2	1

# The Simplex Algorithm



$$C_0 : X_1, X_2 \geq 0$$

$$C_1 : X_1 \leq 9$$

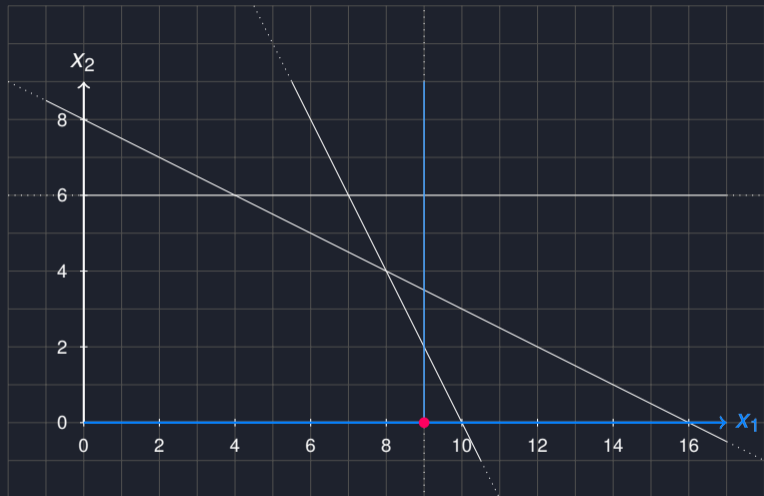
$$C_2 : X_2 \leq 6$$

$$C_3 : X_1 + 2X_2 \leq 16$$

$$C_4 : 2X_1 + X_2 \leq 20$$

		$C_1$	$X_2$
	-9	-1	1
$X_1$	9	1	0
$C_2$	6	0	1
$C_3$	7	-1	2
$C_4$	2	-2	1

# The Simplex Algorithm



$$C_0 : X_1, X_2 \geq 0$$

$$C_1 : X_1 \leq 9$$

$$C_2 : X_2 \leq 6$$

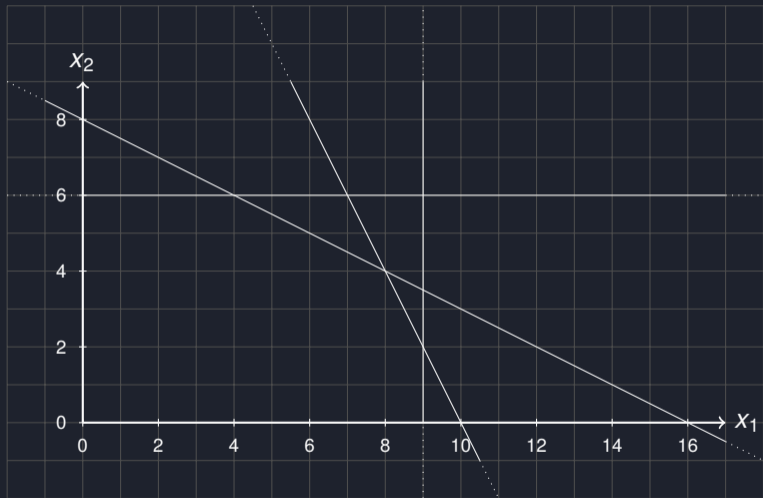
$$C_3 : X_1 + 2X_2 \leq 16$$

$$C_4 : 2X_1 + X_2 \leq 20$$

**EP: (9,0)**

		$C_1$	$X_2$
	-9	-1	1
$X_1$	9	1	0
$C_2$	6	0	1
$C_3$	7	-1	2
$C_4$	2	-2	1

# The Simplex Algorithm



$$C_0 : X_1 , X_2 \geq 0$$

$$C_1 : X_1 \leq 9$$

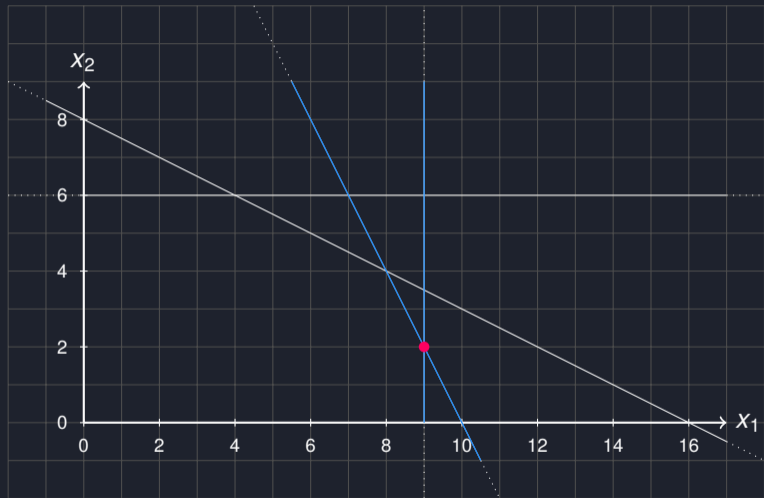
$$C_2 : X_2 \leq 6$$

$$C_3 : X_1 + 2X_2 \leq 16$$

$$C_4 : 2X_1 + X_2 \leq 20$$

	-11	$C_1$	$C_4$
$X_1$	9	1	0
$C_2$	4	2	-1
$C_3$	3	3	-2
$X_2$	2	-2	1

# The Simplex Algorithm



$$C_0 : X_1, X_2 \geq 0$$

$$C_1 : X_1 \leq 9$$

$$C_2 : X_2 \leq 6$$

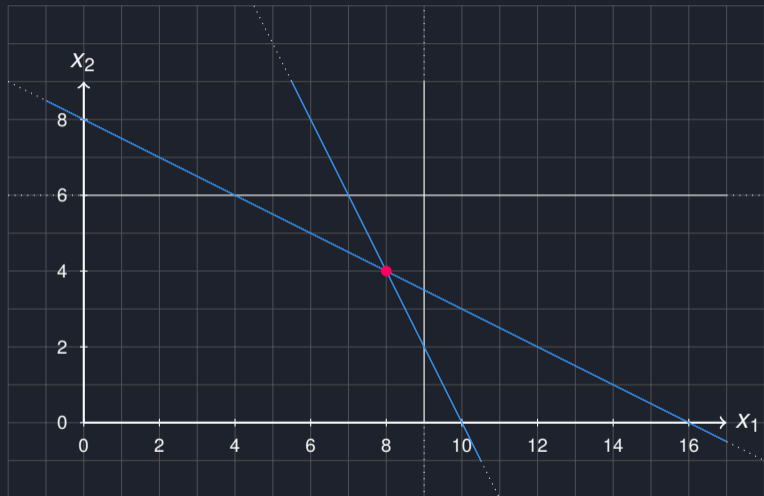
$$C_3 : X_1 + 2X_2 \leq 16$$

$$C_4 : 2X_1 + X_2 \leq 20$$

**EP: (9,2)**

	-11	$C_1$	$C_4$
$X_1$	9	1	0
$C_2$	4	2	-1
$C_3$	3	<b>3</b>	-2
$X_2$	2	-2	1

# The Simplex Algorithm



$$C_0 : X_1, X_2 \geq 0$$

$$C_1 : X_1 \leq 9$$

$$C_2 : X_2 \leq 6$$

$$C_3 : X_1 + 2X_2 \leq 16$$

$$C_4 : 2X_1 + X_2 \leq 20$$

**EP: (8,4)**

	-12	$C_3$	$C_4$
$X_1$	8	$-\frac{1}{3}$	$-\frac{1}{3}$
$C_2$	2	$-\frac{1}{3}$	$\frac{2}{3}$
$C_1$	1	$\frac{1}{3}$	$-\frac{1}{3}$
$X_2$	4	$\frac{2}{3}$	$-\frac{1}{3}$

## Exercise 2.3

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## Task

Consider the following LP:

$$\begin{array}{ll}\text{maximize} & 3x_1 + x_2 + 2x_3 \\ \text{subject to} & x_1 + x_2 + 3x_3 \leq 30 \\ & 2x_1 + 2x_2 + 5x_3 \leq 24 \\ & 4x_1 + x_2 + 2x_3 \leq 36 \\ & x_1, x_2, x_3 \geq 0\end{array}$$

- Determine the initial Simplex tableau and highlight all pivot positions.
- Solve the LP using the Simplex method.

# The Simplex Algorithm

$$\begin{array}{ll}\text{maximize} & 3x_1 + x_2 + 2x_3 \\ \text{subject to} & x_1 + x_2 + 3x_3 \leq 30 \\ & 2x_1 + 2x_2 + 5x_3 \leq 24 \\ & 4x_1 + x_2 + 2x_3 \leq 36 \\ & x_1, x_2, x_3 \geq 0\end{array}$$

# The Simplex Algorithm

$$\begin{array}{llllllll} \text{maximize} & 3x_1 & +1x_2 & +2x_3 & & & & \\ \text{subject to} & 1x_1 & +1x_2 & +3x_3 & +s_1 & & & = 30 \\ & 2x_1 & +2x_2 & +5x_3 & & +s_2 & & = 24 \\ & 4x_1 & +1x_2 & +2x_3 & & & +s_3 & = 36 \\ & x_1 & & x_2 & & x_3 & & s_1 & s_2 & s_3 & \geq 0 \end{array}$$

# The Simplex Algorithm

$$\begin{array}{llllllll} \text{maximize} & 3x_1 & +1x_2 & +2x_3 & & & & \\ \text{subject to} & 1x_1 & +1x_2 & +3x_3 & +s_1 & & & = 30 \\ & 2x_1 & +2x_2 & +5x_3 & & +s_2 & & = 24 \\ & 4x_1 & +1x_2 & +2x_3 & & & +s_3 & = 36 \end{array}$$

## Exercise 2.3

$$\begin{array}{lllll} \text{maximize} & t = & 0 & +3x_1 & +1x_2 & +2x_3 \\ \text{subject to} & s_1 = & 30 & -1x_1 & -1x_2 & -3x_3 \\ & s_2 = & 24 & -2x_1 & -2x_2 & -5x_3 \\ & s_3 = & 36 & -4x_1 & -1x_2 & -2x_3 \end{array}$$

- $x_1, x_2, x_3$  are the non-basic variables and thus equal to zero.
- $s_1, s_2, s_3$  are the basic variables.

Which non-basic variable increases the target value  $t$  the most?

# The Simplex Algorithm

$$\begin{array}{lllll} \text{maximize} & t = & 0 & +3x_1 & +1x_2 & +2x_3 \\ \text{subject to} & s_1 = & 30 & -1x_1 & -1x_2 & -3x_3 \\ & s_2 = & 24 & -2x_1 & -2x_2 & -5x_3 \\ & s_3 = & 36 & -4x_1 & -1x_2 & -2x_3 \end{array} \quad \Bigg|$$

- All variables are required to be nonnegative.

# The Simplex Algorithm

$$\begin{array}{lllll|l} \text{maximize} & t = & 0 & +3x_1 & +1x_2 & +2x_3 & \\ \text{subject to} & s_1 = & 30 & -1x_1 & -1x_2 & -3x_3 & \frac{30}{1} = 30 \\ & s_2 = & 24 & -2x_1 & -2x_2 & -5x_3 & \\ & s_3 = & 36 & -4x_1 & -1x_2 & -2x_3 & \end{array}$$

- All variables are required to be nonnegative.

# The Simplex Algorithm

$$\begin{array}{lllll|l} \text{maximize} & t = & 0 & +3x_1 & +1x_2 & +2x_3 & \\ \text{subject to} & s_1 = & 30 & -1x_1 & -1x_2 & -3x_3 & \frac{30}{1} = 30 \\ & s_2 = & 24 & -2x_1 & -2x_2 & -5x_3 & \frac{24}{2} = 12 \\ & s_3 = & 36 & -4x_1 & -1x_2 & -2x_3 & \end{array}$$

- All variables are required to be nonnegative.



# The Simplex Algorithm

$$\begin{array}{lllll|l} \text{maximize} & t = & 0 & +3x_1 & +1x_2 & +2x_3 & \\ \text{subject to} & s_1 = & 30 & -1x_1 & -1x_2 & -3x_3 & \frac{30}{1} = 30 \\ & s_2 = & 24 & -2x_1 & -2x_2 & -5x_3 & \frac{24}{2} = 12 \\ & s_3 = & 36 & -4x_1 & -1x_2 & -2x_3 & \frac{36}{4} = 9 \end{array}$$

- All variables are required to be nonnegative.

# The Simplex Algorithm

$$\begin{array}{lllll|l} \text{maximize} & t = & 0 & +3x_1 & +1x_2 & +2x_3 & \\ \text{subject to} & s_1 = & 30 & -1x_1 & -1x_2 & -3x_3 & \frac{30}{1} = 30 \\ & s_2 = & 24 & -2x_1 & -2x_2 & -5x_3 & \frac{24}{2} = 12 \\ & s_3 = & 36 & -4x_1 & -1x_2 & -2x_3 & \frac{36}{4} = 9 \end{array}$$

- All variables are required to be nonnegative.
- $x_1$  may be increased by at most 9.

# The Simplex Algorithm

$$\begin{array}{llllll} \text{maximize} & t = & 0 & +3x_1 & +1x_2 & +2x_3 \\ \text{subject to} & s_1 = & 30 & -1x_1 & -1x_2 & -3x_3 \\ & s_2 = & 24 & -2x_1 & -2x_2 & -5x_3 \\ & s_3 = & 36 & -4x_1 & -1x_2 & -2x_3 \end{array} \left| \begin{array}{l} \frac{30}{1} = 30 \\ \frac{24}{2} = 12 \\ \frac{36}{4} = 9 \end{array} \right.$$

- All variables are required to be nonnegative.
- $x_1$  may be increased by at most 9.
- This will set  $s_3$  to zero and thus leave the basis.

# The Simplex Algorithm

$$\begin{array}{llllll} \text{maximize} & t = & 0 & +3x_1 & +1x_2 & +2x_3 \\ \text{subject to} & s_1 = & 30 & -1x_1 & -1x_2 & -3x_3 \\ & s_2 = & 24 & -2x_1 & -2x_2 & -5x_3 \\ & s_3 = & 36 & -4x_1 & -1x_2 & -2x_3 \end{array} \left| \begin{array}{l} \frac{30}{1} = 30 \\ \frac{24}{2} = 12 \\ \frac{36}{4} = 9 \end{array} \right.$$

- All variables are required to be nonnegative.
- $x_1$  may be increased by at most 9.
- This will set  $s_3$  to zero and thus leave the basis.
- $x_1$  will enter the basis.

# The Simplex Algorithm

$$\begin{array}{l} \text{maximize} \\ \text{subject to} \end{array} \quad \begin{array}{r} t = \\ s_1 = \\ s_2 = \\ s_3 = \end{array} \quad \begin{array}{r} 0 \\ 30 \\ 24 \\ 36 \end{array} \quad \begin{array}{r} +3x_1 \\ -1x_1 \\ -2x_1 \\ -4x_1 \end{array} \quad \begin{array}{r} +1x_2 \\ -1x_2 \\ -2x_2 \\ -1x_2 \end{array} \quad \begin{array}{r} +2x_3 \\ -3x_3 \\ -5x_3 \\ -2x_3 \end{array} \quad \left| \quad \begin{array}{r} \frac{30}{1} = 30 \\ \frac{24}{2} = 12 \\ \frac{36}{4} = 9 \end{array} \right.$$

- All variables are required to be nonnegative.
- $x_1$  may be increased by at most 9.
- This will set  $s_3$  to zero and thus leave the basis.
- $x_1$  will enter the basis.

$$\begin{array}{r} s_3 = \\ x_1 = \end{array} \quad \begin{array}{r} 36 \\ 9 \end{array} \quad \begin{array}{r} -4x_1 \\ -\frac{1}{4}s_3 \end{array} \quad \begin{array}{r} -1x_2 \\ -\frac{1}{4}x_2 \end{array} \quad \begin{array}{r} -2x_3 \\ -\frac{1}{2}x_3 \end{array}$$

# The Simplex Algorithm

$$\begin{array}{lllll|l} \text{maximize} & t = & 0 & +3x_1 & +1x_2 & +2x_3 & \\ \text{subject to} & s_1 = & 30 & -1x_1 & -1x_2 & -3x_3 & \frac{30}{1} = 30 \\ & s_2 = & 24 & -2x_1 & -2x_2 & -5x_3 & \frac{24}{2} = 12 \\ & s_3 = & 36 & -4x_1 & -1x_2 & -2x_3 & \frac{36}{4} = 9 \end{array}$$

- All variables are required to be nonnegative.
- $x_1$  may be increased by at most 9.
- This will set  $s_3$  to zero and thus leave the basis.
- $x_1$  will enter the basis.

$$\begin{array}{lllll} s_3 = & 36 & -4x_1 & -1x_2 & -2x_3 \\ x_1 = & 9 & -\frac{1}{4}s_3 & -\frac{1}{4}x_2 & -\frac{1}{2}x_3 \end{array}$$

Now we can plug  $x_1$  into the remaining terms.

# The Simplex Algorithm

$$x_1 = 9 - \frac{1}{4}s_3 - \frac{1}{4}x_2 - \frac{1}{2}x_3$$

Modifying the  $s_1$  term:

$$s_1 = 30 - 1x_1 - 1x_2 - 3x_3$$

$$s_1 = 30 - 1 \cdot \left(9 - \frac{1}{4}s_3 - \frac{1}{4}x_2 - \frac{1}{2}x_3\right) - 1x_2 - 3x_3$$

$$s_1 = 21 + \frac{1}{4}s_3 - \frac{3}{4}x_2 - \frac{5}{2}x_3$$

# The Simplex Algorithm

$$x_1 = 9 - \frac{1}{4}s_3 - \frac{1}{4}x_2 - \frac{1}{2}x_3$$

Modifying the  $s_2$  term:

$$s_2 = 24 - 2x_1 - 2x_2 - 5x_3$$

$$s_2 = 24 - 2 \cdot \left(9 - \frac{1}{4}s_3 - \frac{1}{4}x_2 - \frac{1}{2}x_3\right) - 2x_2 - 5x_3$$

$$s_2 = 6 + \frac{1}{2}s_3 - \frac{3}{2}x_2 - 4x_3$$



# The Simplex Algorithm

$$x_1 = 9 - \frac{1}{4}s_3 - \frac{1}{4}x_2 - \frac{1}{2}x_3$$

Modifying the  $t$  term:

$$t = 0 + 3x_1 + 1x_2 + 2x_3$$

$$t = 0 + 3 \cdot \left(9 - \frac{1}{4}s_3 - \frac{1}{4}x_2 - \frac{1}{2}x_3\right) + 1x_2 + 2x_3$$

$$t = 27 - \frac{3}{4}s_3 + \frac{1}{4}x_2 + \frac{1}{2}x_3$$

# The Simplex Algorithm

$$\begin{array}{llllll} \text{maximize} & t = & 27 & -\frac{3}{4}s_3 & +\frac{1}{4}x_2 & +\frac{1}{2}x_3 \\ \text{subject to} & s_1 = & 21 & +\frac{1}{4}s_3 & -\frac{3}{4}x_2 & -\frac{5}{2}x_3 \\ & s_2 = & 6 & +\frac{1}{2}s_3 & -\frac{3}{2}x_2 & -4x_3 \\ & x_1 = & 9 & -\frac{1}{4}s_3 & -\frac{1}{4}x_2 & -\frac{1}{2}x_3 \end{array}$$

- $s_3, x_2, x_3$  are the non-basic variables and thus equal to zero.
- $s_1, s_2, x_1$  are the basic variables.

Which non-basic variable increases the target value  $t$  the most?

# The Simplex Algorithm

...

## Solution (a)

Consider the following LP:

$$\begin{aligned} \text{maximize} \quad & 3x_1 + x_2 + 2x_3 \\ \text{subject to} \quad & x_1 + x_2 + 3x_3 \leq 30 \\ & 2x_1 + 2x_2 + 5x_3 \leq 24 \\ & 4x_1 + x_2 + 2x_3 \leq 36 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

		$x_1$	$x_2$	$x_3$
	0	3	1	2
$s_1$	30	1	1	3
$s_2$	24	2	2	5
$s_3$	36	4	1	2

- a) Determine the initial Simplex tableau and highlight all pivot positions.

## Solution (b)

Consider the following LP:

$$\begin{aligned} \text{maximize} \quad & 3x_1 + x_2 + 2x_3 \\ \text{subject to} \quad & x_1 + x_2 + 3x_3 \leq 30 \\ & 2x_1 + 2x_2 + 5x_3 \leq 24 \\ & 4x_1 + x_2 + 2x_3 \leq 36 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

**Initial Tableau:**

		$x_1$	$x_2$	$x_3$
	0	3	1	2
$s_1$	30	1	1	3
$s_2$	24	2	2	5
$s_3$	36	4	1	2

**b)** Solve the LP using the Simplex method.

# The Simplex Algorithm

		$x_1$	$x_2$	$x_3$
	0	3	1	2
$s_1$	30	1	1	3
$s_2$	24	2	2	5
$s_3$	36	4	1	2

		$s_3$	$x_2$	$x_3$
$s_1$				
$s_2$				
$x_1$				

# The Simplex Algorithm

		$x_1$	$x_2$	$x_3$
	0	3	1	2
$s_1$	30	1	1	3
$s_2$	24	2	2	5
$s_3$	36	4	1	2

1.  $\tilde{t}_{3,1} = 1/4$

		$s_3$	$x_2$	$x_3$
$s_1$				
$s_2$				
$x_1$		1/4		

# The Simplex Algorithm

		$x_1$	$x_2$	$x_3$
	0	3	1	2
$s_1$	30	1	1	3
$s_2$	24	2	2	5
$s_3$	36	4	1	2

1.  $\tilde{t}_{3,1} = 1/4$

2.  $\tilde{t}_{3,0} = 36/4 = 9$

		$s_3$	$x_2$	$x_3$
$s_1$				
$s_2$				
$x_1$	9	1/4		



# The Simplex Algorithm

		$x_1$	$x_2$	$x_3$
	0	3	1	2
$s_1$	30	1	1	3
$s_2$	24	2	2	5
$s_3$	36	4	1	2

1.  $\tilde{t}_{3,1} = 1/4$

2.  $\tilde{t}_{3,0} = 36/4 = 9$

$\tilde{t}_{3,2} = 1/4$

		$s_3$	$x_2$	$x_3$
$s_1$				
$s_2$				
$x_1$	9	1/4	1/4	

# The Simplex Algorithm

		$x_1$	$x_2$	$x_3$
	0	3	1	2
$s_1$	30	1	1	3
$s_2$	24	2	2	5
$s_3$	36	4	1	2

		$s_3$	$x_2$	$x_3$
$s_1$				
$s_2$				
$x_1$	9	1/4	1/4	1/2

1.  $\tilde{t}_{3,1} = 1/4$

2.  $\tilde{t}_{3,0} = 36/4 = 9$

$\tilde{t}_{3,2} = 1/4$

$\tilde{t}_{3,3} = 2/4 = 1/2$

# The Simplex Algorithm

		$x_1$	$x_2$	$x_3$
	0	3	1	2
$s_1$	30	1	1	3
$s_2$	24	2	2	5
$s_3$	36	4	1	2

		$s_3$	$x_2$	$x_3$
		-3/4		
$s_1$				
$s_2$				
$x_1$	9	1/4	1/4	1/2

1.  $\tilde{t}_{3,1} = 1/4$

2.  $\tilde{t}_{3,0} = 36/4 = 9$

$\tilde{t}_{3,2} = 1/4$

$\tilde{t}_{3,3} = 2/4 = 1/2$

3.  $\tilde{t}_{0,1} = -3/4$

# The Simplex Algorithm

		$x_1$	$x_2$	$x_3$
	0	3	1	2
$s_1$	30	1	1	3
$s_2$	24	2	2	5
$s_3$	36	4	1	2

		$s_3$	$x_2$	$x_3$
		-3/4		
$s_1$		-1/4		
$s_2$				
$x_1$	9	1/4	1/4	1/2

1.  $\tilde{t}_{3,1} = 1/4$

2.  $\tilde{t}_{3,0} = 36/4 = 9$

$$\tilde{t}_{3,2} = 1/4$$

$$\tilde{t}_{3,3} = 2/4 = 1/2$$

3.  $\tilde{t}_{0,1} = -3/4$

$$\tilde{t}_{1,1} = -1/4$$

# The Simplex Algorithm

		$x_1$	$x_2$	$x_3$
	0	3	1	2
$s_1$	30	1	1	3
$s_2$	24	2	2	5
$s_3$	36	4	1	2

		$s_3$	$x_2$	$x_3$
		-3/4		
$s_1$		-1/4		
$s_2$		-1/2		
$x_1$	9	1/4	1/4	1/2

1.  $\tilde{t}_{3,1} = 1/4$

2.  $\tilde{t}_{3,0} = 36/4 = 9$

$$\tilde{t}_{3,2} = 1/4$$

$$\tilde{t}_{3,3} = 2/4 = 1/2$$

3.  $\tilde{t}_{0,1} = -3/4$

$$\tilde{t}_{1,1} = -1/4$$

$$\tilde{t}_{2,1} = -2/4 = -1/2$$

# The Simplex Algorithm

		$x_1$	$x_2$	$x_3$
	0	3	1	2
$s_1$	30	1	1	3
$s_2$	24	2	2	5
$s_3$	36	4	1	2

		$s_3$	$x_2$	$x_3$
		-3/4		
$s_1$		-1/4		
$s_2$		-1/2		
$x_1$	9	1/4	1/4	1/2

# The Simplex Algorithm

		$x_1$	$x_2$	$x_3$
	0	3	1	2
$s_1$	30	1	1	3
$s_2$	24	2	2	5
$s_3$	36	4	1	2

$$4. \tilde{t}_{0,0} = 0 - \frac{3}{4} \cdot 36 = -27$$

		$s_3$	$x_2$	$x_3$
	-27	-3/4		
$s_1$		-1/4		
$s_2$		-1/2		
$x_1$	9	1/4	1/4	1/2

# The Simplex Algorithm

		$x_1$	$x_2$	$x_3$
	0	3	1	2
$s_1$	30	1	1	3
$s_2$	24	2	2	5
$s_3$	36	4	1	2

$$4. \tilde{t}_{0,0} = 0 - \frac{3}{4} \cdot 36 = -27$$

$$\tilde{t}_{0,2} = 1 - \frac{3}{4} \cdot 1 = \frac{1}{4}$$

		$s_3$	$x_2$	$x_3$
	-27	-3/4	1/4	
$s_1$		-1/4		
$s_2$		-1/2		
$x_1$	9	1/4	1/4	1/2



# The Simplex Algorithm

		$x_1$	$x_2$	$x_3$
	0	3	1	2
$s_1$	30	1	1	3
$s_2$	24	2	2	5
$s_3$	36	4	1	2

$$4. \tilde{t}_{0,0} = 0 - \frac{3}{4} \cdot 36 = -27$$

$$\tilde{t}_{0,2} = 1 - \frac{3}{4} \cdot 1 = \frac{1}{4}$$

$$\tilde{t}_{0,3} = 2 - \frac{3}{4} \cdot 2 = \frac{1}{2}$$

		$s_3$	$x_2$	$x_3$
	-27	-3/4	1/4	1/2
$s_1$		-1/4		
$s_2$		-1/2		
$x_1$	9	1/4	1/4	1/2

# The Simplex Algorithm

		$x_1$	$x_2$	$x_3$
	0	3	1	2
$s_1$	30	1	1	3
$s_2$	24	2	2	5
$s_3$	36	4	1	2

		$s_3$	$x_2$	$x_3$
	-27	-3/4	1/4	1/2
$s_1$	21	-1/4		
$s_2$		-1/2		
$x_1$	9	1/4	1/4	1/2

$$4. \tilde{t}_{0,0} = 0 - \frac{3}{4} \cdot 36 = -27$$

$$\tilde{t}_{0,2} = 1 - \frac{3}{4} \cdot 1 = \frac{1}{4}$$

$$\tilde{t}_{0,3} = 2 - \frac{3}{4} \cdot 2 = \frac{1}{2}$$

$$\tilde{t}_{1,0} = 30 - \frac{1}{4} \cdot 36 = 21$$

# The Simplex Algorithm

		$x_1$	$x_2$	$x_3$
	0	3	1	2
$s_1$	30	1	1	3
$s_2$	24	2	2	5
$s_3$	36	4	1	2

		$s_3$	$x_2$	$x_3$
	-27	-3/4	1/4	1/2
$s_1$	21	-1/4	3/4	
$s_2$		-1/2		
$x_1$	9	1/4	1/4	1/2

$$4. \tilde{t}_{0,0} = 0 - \frac{3}{4} \cdot 36 = -27$$

$$\tilde{t}_{0,2} = 1 - \frac{3}{4} \cdot 1 = \frac{1}{4}$$

$$\tilde{t}_{0,3} = 2 - \frac{3}{4} \cdot 2 = \frac{1}{2}$$

$$\tilde{t}_{1,0} = 30 - \frac{1}{4} \cdot 36 = 21$$

$$\tilde{t}_{1,2} = 1 - \frac{1}{4} \cdot 1 = \frac{3}{4}$$

# The Simplex Algorithm

		$x_1$	$x_2$	$x_3$
	0	3	1	2
$s_1$	30	1	1	3
$s_2$	24	2	2	5
$s_3$	36	4	1	2

		$s_3$	$x_2$	$x_3$
	-27	-3/4	1/4	1/2
$s_1$	21	-1/4	3/4	5/2
$s_2$		-1/2		
$x_1$	9	1/4	1/4	1/2

$$4. \quad \tilde{t}_{0,0} = 0 - \frac{3}{4} \cdot 36 = -27$$

$$\tilde{t}_{0,2} = 1 - \frac{3}{4} \cdot 1 = \frac{1}{4}$$

$$\tilde{t}_{0,3} = 2 - \frac{3}{4} \cdot 2 = \frac{1}{2}$$

$$\tilde{t}_{1,0} = 30 - \frac{1}{4} \cdot 36 = 21$$

$$\tilde{t}_{1,2} = 1 - \frac{1}{4} \cdot 1 = \frac{3}{4}$$

$$\tilde{t}_{1,3} = 3 - \frac{1}{4} \cdot 2 = \frac{5}{2}$$

# The Simplex Algorithm

		$x_1$	$x_2$	$x_3$
	0	3	1	2
$s_1$	30	1	1	3
$s_2$	24	2	2	5
$s_3$	36	4	1	2

		$s_3$	$x_2$	$x_3$
	-27	-3/4	1/4	1/2
$s_1$	21	-1/4	3/4	5/2
$s_2$	6	-1/2		
$x_1$	9	1/4	1/4	1/2

$$4. \quad \tilde{t}_{0,0} = 0 - \frac{3}{4} \cdot 36 = -27$$

$$\tilde{t}_{0,2} = 1 - \frac{3}{4} \cdot 1 = \frac{1}{4}$$

$$\tilde{t}_{0,3} = 2 - \frac{3}{4} \cdot 2 = \frac{1}{2}$$

$$\tilde{t}_{1,0} = 30 - \frac{1}{4} \cdot 36 = 21$$

$$\tilde{t}_{1,2} = 1 - \frac{1}{4} \cdot 1 = \frac{3}{4}$$

$$\tilde{t}_{1,3} = 3 - \frac{1}{4} \cdot 2 = \frac{5}{2}$$

$$\tilde{t}_{2,0} = 24 - \frac{2}{4} \cdot 36 = 6$$

# The Simplex Algorithm

		$x_1$	$x_2$	$x_3$
	0	3	1	2
$s_1$	30	1	1	3
$s_2$	24	2	2	5
$s_3$	36	4	1	2

		$s_3$	$x_2$	$x_3$
	-27	-3/4	1/4	1/2
$s_1$	21	-1/4	3/4	5/2
$s_2$	6	-1/2	3/2	
$x_1$	9	1/4	1/4	1/2

$$4. \quad \tilde{t}_{0,0} = 0 - \frac{3}{4} \cdot 36 = -27$$

$$\tilde{t}_{0,2} = 1 - \frac{3}{4} \cdot 1 = \frac{1}{4}$$

$$\tilde{t}_{0,3} = 2 - \frac{3}{4} \cdot 2 = \frac{1}{2}$$

$$\tilde{t}_{1,0} = 30 - \frac{1}{4} \cdot 36 = 21$$

$$\tilde{t}_{1,2} = 1 - \frac{1}{4} \cdot 1 = \frac{3}{4}$$

$$\tilde{t}_{1,3} = 3 - \frac{1}{4} \cdot 2 = \frac{5}{2}$$

$$\tilde{t}_{2,0} = 24 - \frac{2}{4} \cdot 36 = 6$$

$$\tilde{t}_{2,2} = 2 - \frac{2}{4} \cdot 1 = \frac{3}{2}$$

# The Simplex Algorithm

		$x_1$	$x_2$	$x_3$
	0	3	1	2
$s_1$	30	1	1	3
$s_2$	24	2	2	5
$s_3$	36	4	1	2

		$s_3$	$x_2$	$x_3$
	-27	-3/4	1/4	1/2
$s_1$	21	-1/4	3/4	5/2
$s_2$	6	-1/2	3/2	4
$x_1$	9	1/4	1/4	1/2

$$4. \quad \tilde{t}_{0,0} = 0 - \frac{3}{4} \cdot 36 = -27$$

$$\tilde{t}_{0,2} = 1 - \frac{3}{4} \cdot 1 = \frac{1}{4}$$

$$\tilde{t}_{0,3} = 2 - \frac{3}{4} \cdot 2 = \frac{1}{2}$$

$$\tilde{t}_{1,0} = 30 - \frac{1}{4} \cdot 36 = 21$$

$$\tilde{t}_{1,2} = 1 - \frac{1}{4} \cdot 1 = \frac{3}{4}$$

$$\tilde{t}_{1,3} = 3 - \frac{1}{4} \cdot 2 = \frac{5}{2}$$

$$\tilde{t}_{2,0} = 24 - \frac{2}{4} \cdot 36 = 6$$

$$\tilde{t}_{2,2} = 2 - \frac{2}{4} \cdot 1 = \frac{3}{2}$$

$$\tilde{t}_{2,3} = 5 - \frac{2}{4} \cdot 2 = 4$$

## Solution (b)

Initial Tableau:

		$x_1$	$x_2$	$x_3$
	0	3	1	2
$s_1$	30	1	1	3
$s_2$	24	2	2	5
$s_3$	36	4	1	2

Iteration 1:

		$s_3$	$x_2$	$x_3$
	-27	-3/4	1/4	1/2
$s_1$	21	-1/4	3/4	5/2
$s_2$	6	-1/2	3/2	4
$x_1$	9	1/4	1/4	1/2



## Solution (b)

Iteration 1:

		$s_3$	$x_2$	$x_3$
	-27	-3/4	1/4	1/2
$s_1$	21	-1/4	3/4	5/2
$s_2$	6	-1/2	3/2	4
$x_1$	9	1/4	1/4	1/2

Iteration 2:

		$s_3$	$x_2$	$s_2$
	-111/4	-11/16	1/16	-1/8
$s_1$	69/4	1/16	-3/16	-5/8
$x_3$	3/2	-1/8	3/8	1/4
$x_1$	33/4	5/16	1/16	-1/8

## Solution (b)

Iteration 2:

		$s_3$	$x_2$	$s_2$
	-111/4	-11/16	1/16	-1/8
$s_1$	69/4	1/16	-3/16	-5/8
$x_3$	3/2	-1/8	3/8	1/4
$x_1$	33/4	5/16	1/16	-1/8

Final Tableau:

		$s_3$	$x_3$	$s_2$
	-28	-2/3	-1/6	-1/6
$s_1$	18	0	1/2	-1/2
$x_2$	4	-1/3	8/3	2/3
$x_1$	8	1/3	-1/6	-1/6

**Optimal Solution:**  $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (8, 4, 0)$ ;  $c(8, 4, 0) = 28$ .

## Exercise 2.4

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## Task (a)

A farmer owns 200 acres of land, on which he plans to grow barley, wheat and potatoes. The respective details are provided in the following table:

	Potatoes	Barley	Wheat	Capacity
Cultivation costs (€/acre)	1,000	2,000	3,000	150,000 €
Days of work (d/acre)	1	3	4	160 days
Net profit (€/acre)	2,000	2,000	3,000	

Maximize the farmer's net profit.

## Solution (a)

$$\begin{array}{ll} \text{maximize} & 2x_1 + 2x_2 + 3x_3 \\ \text{subject to} & 1 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 \leq 200 \\ & 1 \cdot x_1 + 2 \cdot x_2 + 3 \cdot x_3 \leq 150 \\ & 1 \cdot x_1 + 3 \cdot x_2 + 4 \cdot x_3 \leq 160 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Initial Tableau:

		$x_1$	$x_2$	$x_3$
	0	2	2	3
$s_1$	200	1	1	1
$s_2$	150	1	2	3
$s_3$	160	1	3	4

## Solution (a)

Initial Tableau:

		$x_1$	$x_2$	$x_3$
	0	2	2	3
$s_1$	200	1	1	1
$s_2$	150	1	2	3
$s_3$	160	1	3	4

Iteration 1:

		$x_1$	$x_2$	$s_3$
	-120	$5/4$	$-1/4$	$-3/4$
$s_1$	160	$3/4$	$1/4$	$-1/4$
$s_2$	30	$1/4$	$-1/4$	$-3/4$
$x_3$	40	$1/4$	$3/4$	$1/4$

## Solution (a)

Iteration 1:

		$x_1$	$x_2$	$s_3$
	-120	5/4	-1/4	-3/4
$s_1$	160	3/4	1/4	-1/4
$s_2$	30	1/4	-1/4	-3/4
$x_3$	40	1/4	3/4	1/4

Iteration 2:

		$s_2$	$x_2$	$s_3$
	-270	-5	1	3
$s_1$	70	-3	1	2
$x_1$	120	4	-1	-3
$x_3$	10	-1	1	1

## Solution (a)

**Iteration 2:**

		$s_2$	$x_2$	$s_3$
	-270	-5	1	3
$s_1$	70	-3	1	2
$x_1$	120	4	-1	-3
$x_3$	10	-1	1	1

**Final Tableau:**

		$s_2$	$x_2$	$x_3$
	-300	-2	-2	-3
$s_1$	50	-1	-1	-2
$x_1$	150	1	2	3
$s_3$	10	-1	1	1

**Optimal Sol.:**  $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (150, 0, 0)$ ;  $c(150, 0, 0) = 300000$ .



## Task (b)

The owner of a chicken farm uses two sorts of grain (called  $A$  and  $B$ ) to feed the animals. Each kind contains protein, fat, carbohydrates and additional indigestible components. On the basis of  $A$  and  $B$  the farmer now wants to create a blend which contains at least 1 kg of protein, 800 g of fat and 1.8 kg of carbs. The composition and prices per kg of each sort of grain are given in the table below:

	Grain $A$	Grain $B$
Protein (per kg)	100 g	200 g
Fat (per kg)	200 g	100 g
Carbs (per kg)	100 g	600 g
Price (per kg)	8 €	12 €

Minimize the price of the blend using the Simplex method based on the normal form of the original LP instance and a *helper tableau*  $T^{aux}$ .

## Solution (b)

Linear Program:

$$\begin{array}{ll} \text{minimize} & 8x_1 + 12x_2 \\ \text{s.t.} & 100x_1 + 200x_2 \geq 1000 \\ & 200x_1 + 100x_2 \geq 800 \\ & 100x_1 + 600x_2 \geq 1800 \\ & x_1, x_2 \geq 0 \end{array}$$

Normal Form:

$$\begin{array}{ll} \text{maximize} & -8x_1 - 12x_2 \\ \text{s.t.} & -100x_1 - 200x_2 \leq -1000 \\ & -200x_1 - 100x_2 \leq -800 \\ & -100x_1 - 600x_2 \leq -1800 \\ & x_1, x_2 \geq 0 \end{array}$$



## The Auxiliary Problem

$$\begin{array}{llllllll} \text{maximize} & & & & - & 1x_0 & & & \\ \text{maximize} & - & 8x_1 & - & 12x_2 & & & & \\ \text{subject to} & - & 100x_1 & - & 200x_2 & - & 1x_0 & \leq & -1000 \\ & - & 200x_1 & - & 100x_2 & - & 1x_0 & \leq & -800 \\ & - & 100x_1 & - & 600x_2 & - & 1x_0 & \leq & -1800 \\ & & x_1 & , & x_2 & , & x_0 & \geq & 0 \end{array}$$



## The Auxiliary Problem

$$\begin{aligned}t_{\text{aux}} &= && && && && - & 1x_0 \\t_{\text{orig}} &= & 0 & - & 8x_1 & - & 12x_2 & & & & \\s_1 &= & -1000 & + & 100x_1 & + & 200x_2 & + & 1x_0 & & \\s_2 &= & -800 & + & 200x_1 & + & 100x_2 & + & 1x_0 & & \\s_3 &= & -1800 & + & 100x_1 & + & 600x_2 & + & 1x_0 & & \end{aligned}$$



## The Auxiliary Problem

$$\begin{array}{rcllclclcl} t_{\text{aux}} & = & & & & & - & 1x_0 & \\ t_{\text{orig}} & = & 0 & - & 8x_1 & - & 12x_2 & & \\ s_1 & = & -1000 & + & 100x_1 & + & 200x_2 & + & 1x_0 \\ s_2 & = & -800 & + & 200x_1 & + & 100x_2 & + & 1x_0 \\ s_3 & = & -1800 & + & 100x_1 & + & 600x_2 & + & 1x_0 \end{array}$$

What's next?

$$\begin{array}{l} x_0 = 1800 \\ \text{If } x_1 = 0 \\ x_2 = 0 \end{array} \quad \text{we have} \quad \begin{array}{l} s_1 = 800 \\ s_2 = 1000 \\ s_3 = 0 \end{array} .$$



## The Auxiliary Problem

$$\begin{array}{rcllclclcl} t_{\text{aux}} & = & & & & & & - & 1x_0 \\ t_{\text{orig}} & = & 0 & - & 8x_1 & - & 12x_2 & & \\ s_1 & = & -1000 & + & 100x_1 & + & 200x_2 & + & 1x_0 \\ s_2 & = & -800 & + & 200x_1 & + & 100x_2 & + & 1x_0 \\ s_3 & = & -1800 & + & 100x_1 & + & 600x_2 & + & 1x_0 \end{array}$$

What's next?

$$\begin{array}{l} x_0 = 1800 \\ \text{If } x_1 = 0 \\ x_2 = 0 \end{array} \quad \text{we have} \quad \begin{array}{l} s_1 = 800 \\ s_2 = 1000 \\ s_3 = 0 \end{array} .$$

Solve  $s_3$  for  $x_0$  to obtain  $x_0 = 1800 - 100x_1 - 600x_2 + s_3$ .

## The Auxiliary Problem

$$x_0 = 1800 - 100x_1 - 600x_2 + s_3$$

$$s_1 = -1000 + 100x_1 + 200x_2 + (1800 - 100x_1 - 600x_2 + s_3)$$

$$= 800 - 400x_2 + s_3$$

$$s_2 = -800 + 200x_1 + 100x_2 + (1800 - 100x_1 - 600x_2 + s_3)$$

$$= 1000 + 100x_1 - 500x_2 + s_3$$

$$t_{\text{orig}} = 0 - 8x_1 - 12x_2$$

$$t_{\text{aux}} = -1800 + 100x_1 + 600x_2 - s_3$$

## Solution (b)

Normal Form:

**maximize**

$$-8x_1 - 12x_2$$

**s.t.**

$$-100x_1 - 200x_2 \leq -1000$$

$$-200x_1 - 100x_2 \leq -800$$

$$-100x_1 - 600x_2 \leq -1800$$

$$x_1, x_2 \geq 0$$

**[ $T_{aux}$ ] Initial Tableau:**

		$x_0$	$x_1$	$x_2$
	0	-1	0	0
	0	0	-8	-12
$s_1$	-1000	-1	-100	-200
$s_2$	-800	-1	-200	-100
$s_3$	-1800	-1	-100	-600

## Solution (b)

$[T_{aux}]$  Initial Tableau:

		$x_0$	$x_1$	$x_2$
	0	-1	0	0
	0	0	-8	-12
$s_1$	-1000	-1	-100	-200
$s_2$	-800	-1	-200	-100
$s_3$	-1800	-1	-100	-600

$[T_{aux}]$  Iteration 1:

		$s_3$	$x_1$	$x_2$
	1800	-1	100	600
	0	0	-8	-12
$s_1$	800	-1	0	400
$s_2$	1000	-1	-100	500
$x_0$	1800	-1	100	600

## Solution (b)

$[T_{aux}]$  Iteration 1:

		$s_3$	$x_1$	$x_2$
	1800	-1	100	600
	0	0	-8	-12
$s_1$	800	-1	0	400
$s_2$	1000	-1	-100	500
$x_0$	1800	-1	100	600

$[T_{aux}]$  Iteration 2:

		$s_3$	$x_1$	$s_1$
	600	1/2	100	-3/2
	24	-3/100	-8	3/100
$x_2$	2	-1/400	0	1/400
$s_2$	0	1/4	-100	-5/4
$x_0$	600	1/2	100	-3/2

## Solution (b)

$[T_{aux}]$  Iteration 2:

		$s_3$	$x_1$	$s_1$
	600	1/2	100	-3/2
	24	-3/100	-8	3/100
$x_2$	2	-1/400	0	1/400
$s_2$	0	1/4	-100	-5/4
$x_0$	600	1/2	100	-3/2

$[T_{aux}]$  Final Tableau:

		$s_3$	$x_0$	$s_1$
	0	0	-1	0
	72	1/100	2/25	-9/100
$x_2$	2	-1/400	0	1/400
$s_2$	600	3/4	1	-11/4
$x_1$	6	1/200	1/100	-3/200

## Solution (b)

$[T_{aux}]$  Final Tableau:

		$s_3$	$x_0$	$s_1$
	0	0	-1	0
	72	1/100	2/25	-9/100
$x_2$	2	-1/400	0	1/400
$s_2$	600	3/4	1	-11/4
$x_1$	6	1/200	1/100	-3/200

Initial Tableau:

		$s_3$	$s_1$
	72	1/100	-9/100
$x_2$	2	-1/400	1/400
$s_2$	600	3/4	-11/4
$x_1$	6	1/200	-3/200

## Solution (b)

**Initial Tableau:**

		$s_3$	$s_1$
	72	1/100	-9/100
$x_2$	2	-1/400	1/400
$s_2$	600	3/4	-11/4
$x_1$	6	1/200	-3/200

**Final Tableau:**

		$s_2$	$s_1$
	64	-1/75	-4/75
$x_2$	4	1/300	-1/150
$s_3$	800	4/3	-11/3
$x_1$	2	-1/150	1/300

**Optimal solution of the initial minimization problem:  $(\bar{x}_1, \bar{x}_2) = (2, 4)$ ; Costs: 64 €.**



## Exercise 2.5

---

# Task

Consider the following LP instance  $I$ :

$$\begin{array}{ll}\text{maximize} & -2x_1 - 3x_2 \\ \text{subject to} & x_1 + x_2 \leq 7 \\ & x_1 - x_2 \leq -16 \\ & x_1, x_2 \geq 0\end{array}$$

Determine  $\text{opt}(I_{aux})$ . Does  $I$  have any valid solutions?

# Solution

Consider the following LP instance  $I$ :

$$\begin{array}{ll} \text{maximize} & -2x_1 - 3x_2 \\ \text{subject to} & x_1 + x_2 \leq 7 \\ & x_1 - x_2 \leq -16 \\ & x_1, x_2 \geq 0 \end{array}$$

Determine  $\text{opt}(I_{aux})$ . Does  $I$  have any valid solutions?

$[T^{aux}]$  Initial Tableau:

		$x_0$	$x_1$	$x_2$
	0	-1	0	0
	0	0	-2	-3
$s_1$	7	-1	1	1
$s_2$	-16	-1	1	-1

# Solution

$[T^{aux}]$  Initial Tableau:

		$x_0$	$x_1$	$x_2$
	0	-1	0	0
	0	0	-2	-3
$s_1$	7	-1	1	1
$s_2$	-16	-1	1	-1

$[T^{aux}]$  Iteration 1:

		$s_2$	$x_1$	$x_2$
	16	-1	-1	1
	0	0	-2	-3
$s_1$	23	-1	0	2
$x_0$	16	-1	-1	1

# Solution

$[T^{aux}]$  Iteration 1:

		$s_2$	$x_1$	$x_2$
	16	-1	-1	1
	0	0	-2	-3
$s_1$	23	-1	0	2
$x_0$	16	-1	-1	1

$[T^{aux}]$  Iteration 2:

		$s_2$	$x_1$	$s_1$
	9/2	-1/2	-1	-1/2
	69/2	-3/2	-2	3/2
$x_2$	23/2	-1/2	0	1/2
$x_0$	9/2	-1/2	-1	-1/2

$$\Rightarrow \text{opt}(I_{aux}) = -9/2 \neq 0 \Rightarrow X(I) = \emptyset$$



*That's all Folks!*