Algorithmics

Spring 2020 - Tutorial #1

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Exercise 1.1

What is the difference between O and o, resp. Ω and ω?

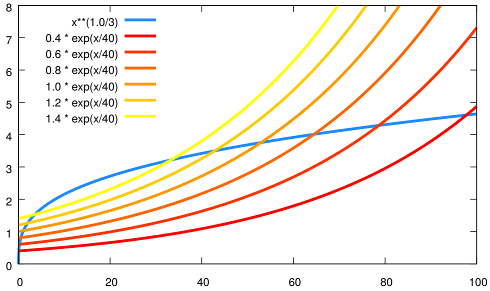
$f \in O(g)$: For some $C \in \mathbb{R}^{>0}$ there exists a $n_0 \in \mathbb{N}$, such that for all $n \in \mathbb{N}$ with $n > n_0$ it holds: $f(n) \leq C \cdot g(n)$.

$f \in o(g)$: For all $C \in \mathbb{R}^{>0}$ there exists a $n_0 \in \mathbb{N}$, such that for all $n \in \mathbb{N}$ with $n > n_0$ it holds: $f(n) < C \cdot g(n)$.

From $f \in o(g)$ it follows directly: $f \in O(g)$.

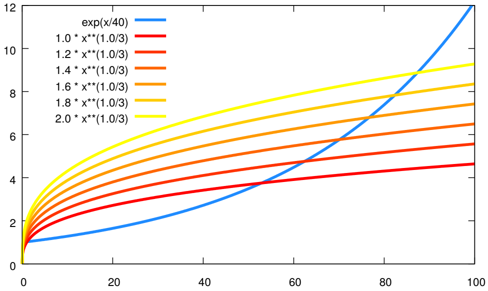
Analogously for Ω and ω.
(Replace $\leq$ by $\geq$ and $<$ by $>$)

$f \in o(g)$



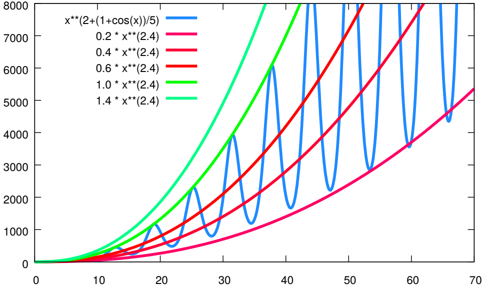
$f \in o(g)$ informal: No matter how I choose $C \in \mathbb{R}^{>0}$, from a certain point on $f(n)$ is always smaller than $C \cdot g(n)$.

$f \in \omega(g)$



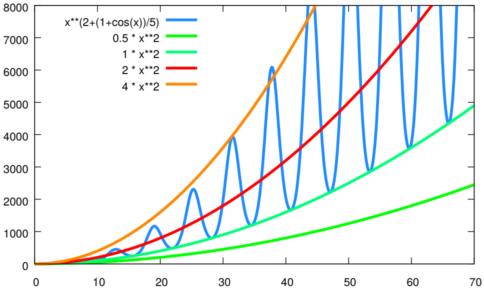
$f \in \omega(g)$ informal: No matter how I choose $C \in \mathbb{R}^{>0}$, from a certain point on $f(n)$ is always greater than $C \cdot g(n)$.

$f \in O(g)$ (in the example: $f \notin (o(g))$)



$f \in O(g)$ informal: I can choose a $C \in \mathbb{R}^{>0}$, such that from a certain point on $f(n)$ is never greater than $C \cdot g(n)$.

$f \in \Omega(g)$ (in the example: $f \notin (\omega(g))$)



$f \in \Omega(g)$ informal: I can choose a $C \in \mathbb{R}^{>0}$, such that from a certain point on $f(n)$ is never smaller than $C \cdot g(n)$.

Transitivity and Reflexivity

Transitivity

- $f(n) \in \Theta(g(n))$ and $g(n) \in \Theta(h(n))$ imply $f(n) \in \Theta(h(n))$.
- $f(n) \in O(g(n))$ and $g(n) \in O(h(n))$ imply $f(n) \in O(h(n))$.
- $f(n) \in \Omega(g(n))$ and $g(n) \in \Omega(h(n))$ imply $f(n) \in \Omega(h(n))$.
- $f(n) \in o(g(n))$ and $g(n) \in o(h(n))$ imply $f(n) \in o(h(n))$.
- $f(n) \in \omega(g(n))$ and $g(n) \in \omega(h(n))$ imply $f(n) \in \omega(h(n))$.

Reflexivity

- $f(n) \in \Theta(f(n))$.
- $f(n) \in O(f(n))$.
- $f(n) \in \Omega(f(n))$.

Symmetry

Symmetry

- $f(n) \in \Theta(g(n))$ if and only if $g(n) \in \Theta(f(n))$.

Transpose Symmetry

- $f(n) \in O(g(n))$ if and only if $g(n) \in \Omega(f(n))$.
- $f(n) \in o(g(n))$ if and only if $g(n) \in \omega(f(n))$.

Exercise 1.1

Complete the following table with the symbols $O, o, \Omega, \omega, \Theta$.

	$\log n$	$2^{n/2}$	$\sqrt{n}$	5	2^n	$1/n$	n	e^n	n^2
$\log n$							o		
$2^{n/2}$									
$\sqrt{n}$									
5									
2^n									
$1/n$									
n									
e^n									
n^2									

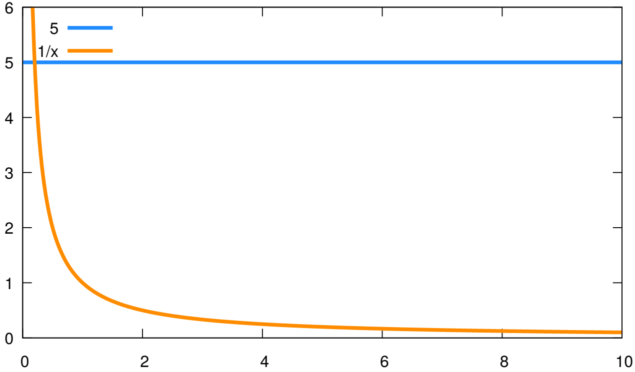
Exercise 1.1

Complete the following table with the symbols $O, o, \Omega, \omega, \Theta$.

	$\log n$	$2^{n/2}$	$\sqrt{n}$	5	2^n	$1/n$	n	e^n	n^2
$\log n$	Θ						o		
$2^{n/2}$		Θ							
$\sqrt{n}$			Θ						
5				Θ					
2^n					Θ				
$1/n$						Θ			
n							Θ		
e^n								Θ	
n^2									Θ

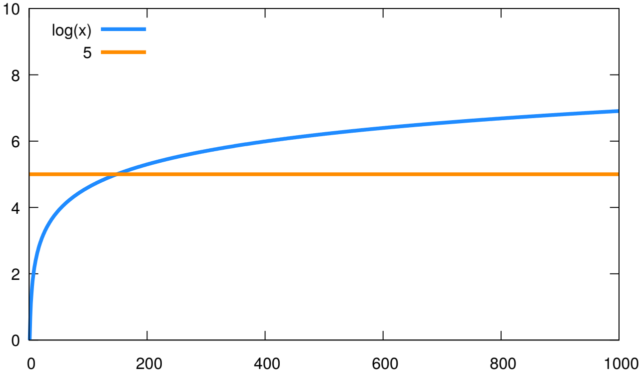
Using reflexivity.

5 vs. $\frac{1}{n}$



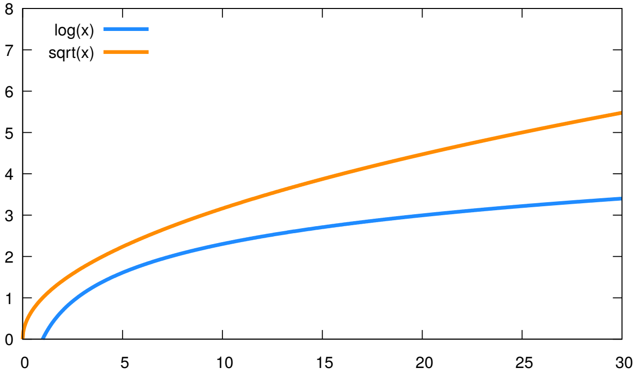
$$5 \in \omega(\tfrac{1}{n}),\ \tfrac{1}{n} \in o(5)$$

$\log n$ vs. 5



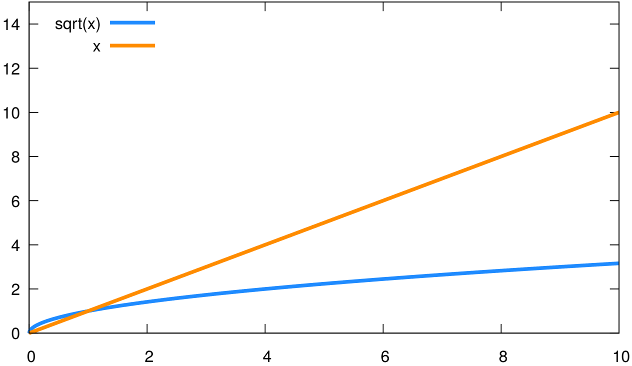
$\log n \in \omega(5)$, $5 \in o(\log n)$

$\log n$ vs. $\sqrt{n}$



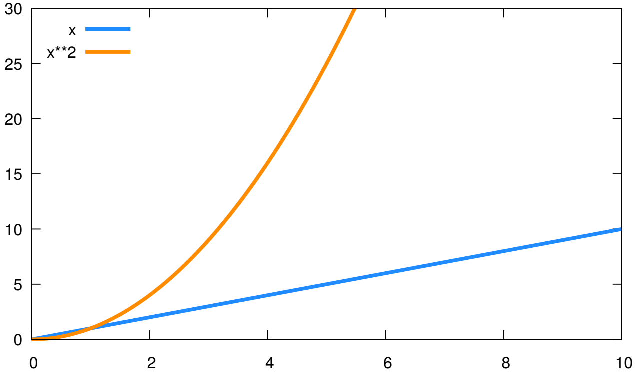
$$\log n \in o(\sqrt{n}), \ \sqrt{n} \in \omega(\log n)$$

$\sqrt{n}$ vs. n



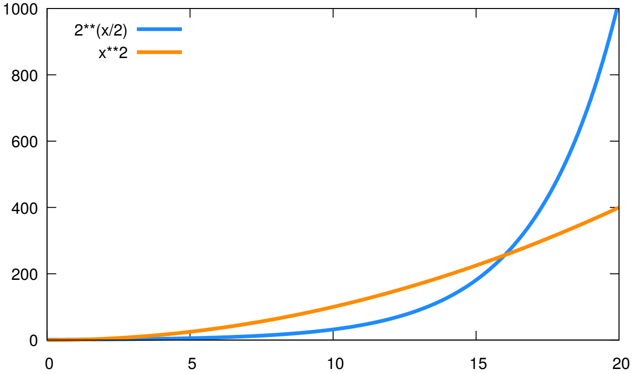
$$\sqrt{n} \in o(n),\ n \in \omega(\sqrt{n})$$

n vs. n^2



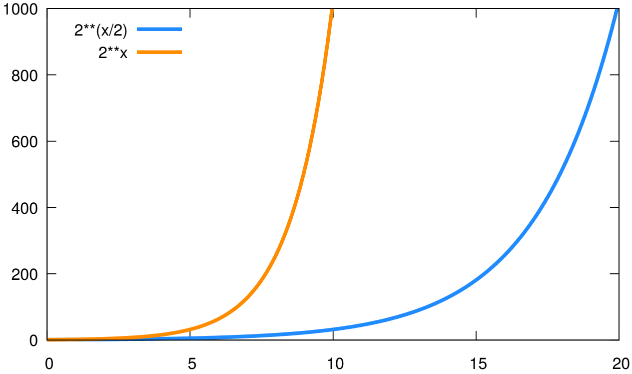
$n \in o(n^2)$, $n^2 \in \omega(n)$

$2^{\frac{n}{2}}$ vs. n^2



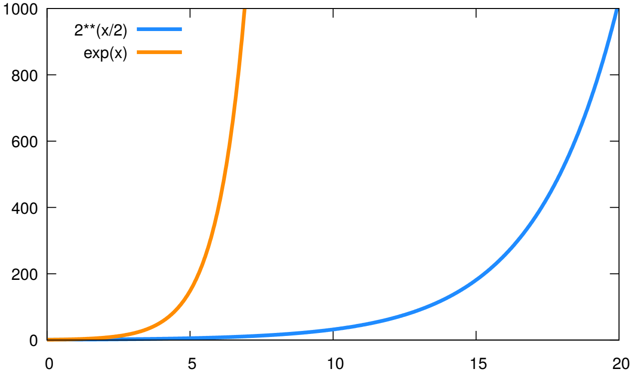
$$2^{\frac{n}{2}} \in \omega(n^2),\ n^2 \in o(2^{\frac{n}{2}})$$

$2^{\frac{n}{2}}$ vs. 2^n



$2^{\frac{n}{2}} \in o(2^n)$, $2^n \in \omega(2^{\frac{n}{2}})$

$2^{\frac{n}{2}}$ vs. e^n



$$2^{\frac{n}{2}} \in o(e^n),\ e^n \in \omega(2^{\frac{n}{2}})$$

Exercise 1.1

Summary of our observations:

$$\begin{aligned} 1/n &\in o(5) \\ 5 &\in o(\log n) \\ \log n &\in o(\sqrt{n}) \\ \sqrt{n} &\in o(n) \\ n &\in o(n^2) \\ n^2 &\in o(2^{n/2}) \\ 2^{n/2} &\in o(2^n) \\ 2^n &\in o(e^n) \end{aligned}$$

Using **transitivity** and **transpose symmetry**, that's all we need.

Exercise 1.1

Complete the following table with the symbols $O, o, \Omega, \omega, \Theta$.

	$\log n$	$2^{n/2}$	$\sqrt{n}$	5	2^n	$1/n$	n	e^n	n^2
$\log n$	Θ	o	o	ω	o	ω	o	o	o
$2^{n/2}$	ω	Θ	ω	ω	o	ω	ω	o	ω
$\sqrt{n}$	ω	o	Θ	ω	o	ω	o	o	o
5	o	o	o	Θ	o	ω	o	o	o
2^n	ω	ω	ω	ω	Θ	ω	ω	o	ω
$1/n$	o	o	o	o	o	Θ	o	o	o
n	ω	o	ω	ω	o	ω	Θ	o	o
e^n	ω	ω	ω	ω	ω	ω	ω	Θ	ω
n^2	ω	o	ω	ω	o	ω	ω	o	Θ

Exercise 1.2

Exercise 1.2

For each of the following functions f_i, find a function g_i with as few terms as possible satisfying $f_i \in \Theta(g_i)$.

$$f_1(n) = n^2 2^n + 4^n + 3^2$$

$$f_2(n) = n(n-1)/2$$

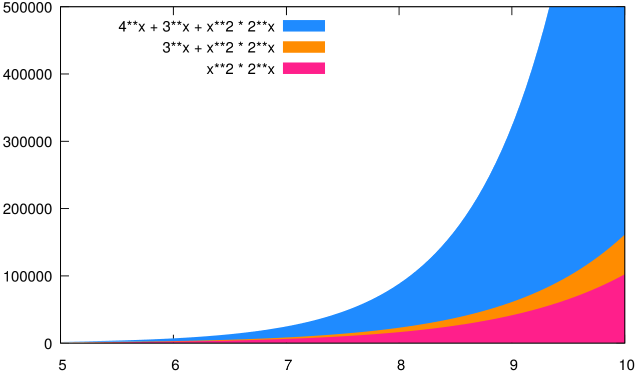
$$f_3(n) = \log n^{70}$$

$$f_4(n) = 9n \log n + 30n(\log n)^2 + n$$

$$f_5(n) = \sum_{i=1}^{n} 2^i$$

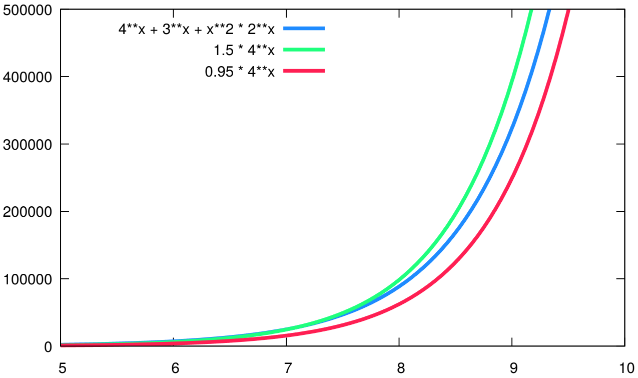
$$f_6(n) = \frac{n}{\log n} + \frac{n^2}{\log^2(n)}$$

$f_1(n) = n^2 2^n + 4^n + 3^n$



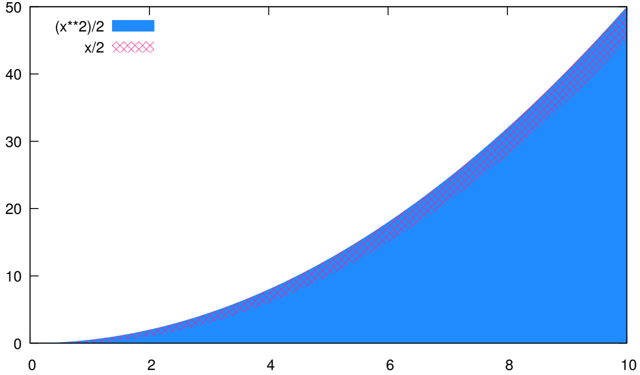
$$f_1 \in \Theta(4^n)$$

$f_1(n) = n^2 2^n + 4^n + 3^2$, $1.5 \cdot 4^n$, $0.95 \cdot 4^n$



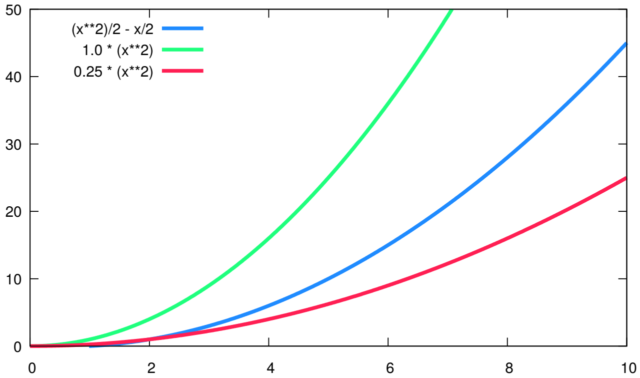
$f_1 \in \Theta(4^n)$

$f_2(n) = n(n-1)/2 = \frac{n^2}{2} - \frac{n}{2}$



$$f_2 \in \Theta(n^2)$$

$f_2(n) = n(n-1)/2$, $1.0 \cdot n^2$, $0.25 \cdot n^2$



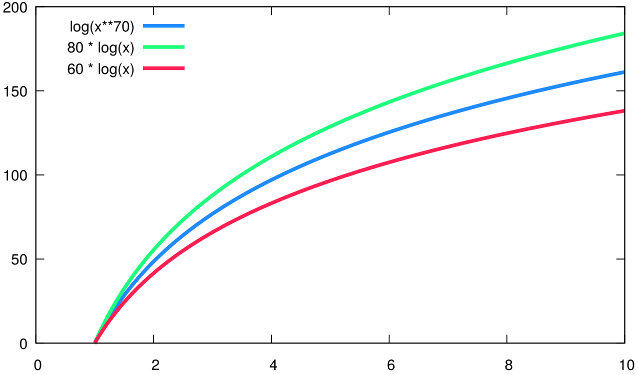
$$f_2 \in \Theta(n^2)$$

$f_3(n) = \log n^{70}$

$$f_3(n) = \log n^{70} = 70 \cdot \log n$$

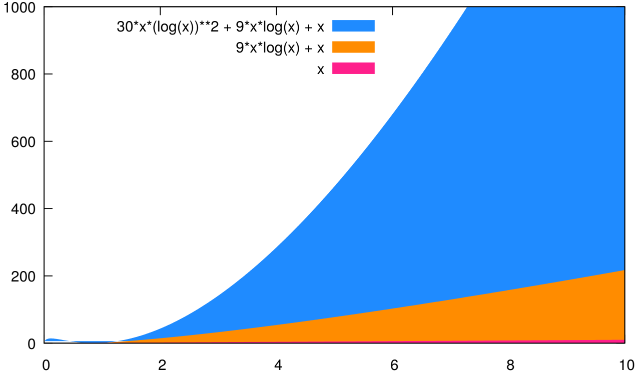
$$f_3 \in \Theta(\log n)$$

$f_3(n) = \log n^{70}$, $80 \cdot \log n$, $60 \cdot \log n$



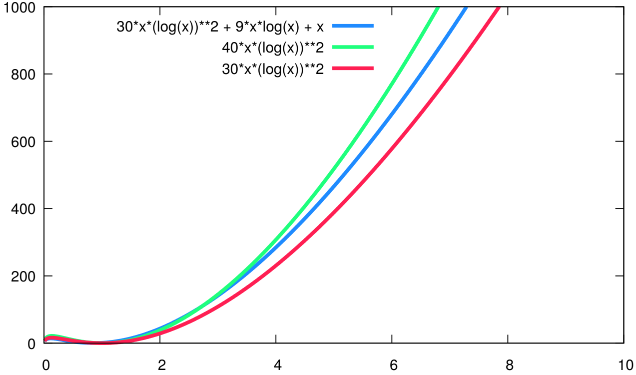
$$f_3 \in \Theta(\log n)$$

$f_4(n) = 9n \log n + 30n(\log n)^2 + n$



$$f_4 \in \Theta(n(\log n)^2)$$

$f_4(n) = 9n \log n + 30n(\log n)^2 + n$, $40 \cdot n(\log n)^2$, $30 \cdot n(\log n)^2$



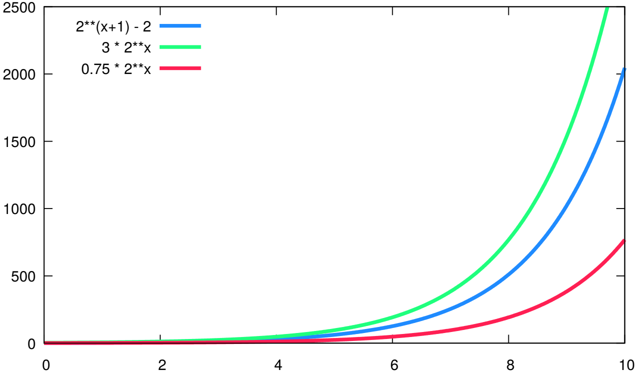
$$f_4 \in \Theta(n(\log n)^2)$$

$f_5(n) = \sum_{i=1}^{n} 2^i$

$$f_5(n) = \sum_{i=1}^{n} 2^i = 2^{n+1} - 2$$

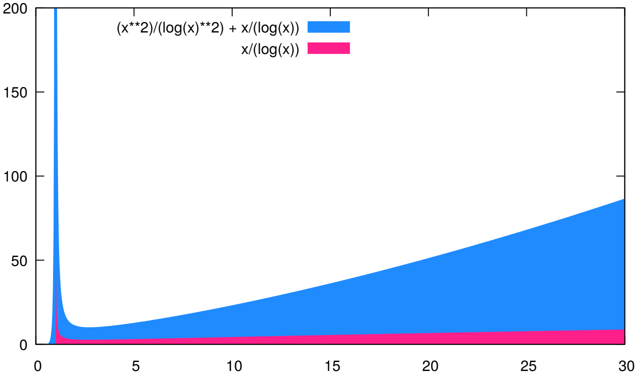
$$f_5 \in \Theta(2^n)$$

$f_5(n) = \sum_{i=1}^{n} 2^i$, $3 \cdot 2^n$, $0.75 \cdot 2^n$



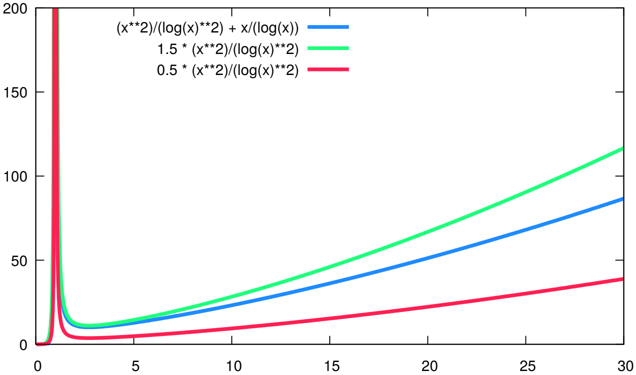
$$f_5 \in \Theta(2^n)$$

$f_6(n) = \frac{n}{\log n} + \frac{n^2}{\log^2 n}$



$$f_6 \in \Theta\left(\frac{n^2}{\log^2 n}\right)$$

$$f_6(n) = \frac{n}{\log n} + \frac{n^2}{\log^2 n}, 1.5 \cdot \frac{n^2}{\log^2 n}, 0.5 \cdot \frac{n^2}{\log^2 n}$$

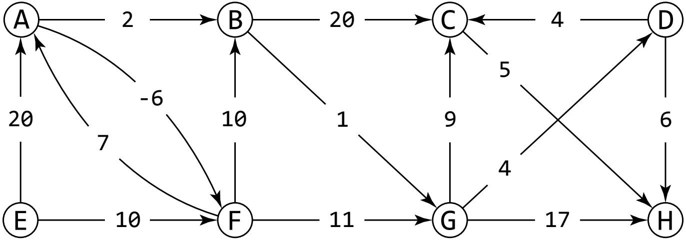


$$f_6 \in \Theta\left(\frac{n^2}{\log^2 n}\right)$$

Exercise 1.3

Exercise 1.3

We will now consider the following graph:



Find the shortest path from vertex *E* to vertex *H* using the Bellman-Ford algorithm based on the lexicographical edge order:

Exercise 1.3

	I		R_1^*		R_2^*		R_3^*		R_4^*	
	$v.d$	$v.\pi$	$v.d$	$v.\pi$	$v.d$	$v.\pi$	$v.d$	$v.\pi$	$v.d$	$v.\pi$
A	∞	-	17	F						
B	∞	-	20	F	19	A				
C	∞	-	30	G	29	D	28	D		
D	∞	-	25	G	24	G				
E	0	-								
F	∞	-	10	E						
G	∞	-	21	F	20	B				
H	∞	-	38	G	31	D	30	D		

Shortest path from E to H:

$E \rightarrow F \rightarrow A \rightarrow B \rightarrow G \rightarrow D \rightarrow H$

Distance from E to H:

30

Exercise 1.4

Exercise 1.4

Blackboard only.

Exercise 1.5

Exercise 1.5

a) Let $M_1 \subseteq \mathbb{R}^n$ be defined as

$$M_1 = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} x_i \leq 1;\, x_i \geq 0 \;\forall i = 1, \dots, n \right\}.$$

Determine all extremal points of M_1 and explain why your solution is correct.

b) Let $M_2 \subseteq \mathbb{R}^3$ be defined as

$$\begin{aligned} M_2 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid\ & x_1 + x_2 + x_3 \leq 1, \\ & x_1 + 2x_2 + 4x_3 \leq 2, \\ & x_1 + 3x_2 + 9x_3 \leq 3, \\ & x_1 + 4x_2 + 16x_3 \leq 4\}. \end{aligned}$$

How many extremal points does M_2 have? Prove your claim.

Exercise 1.5

a) Let $M_1 \subseteq \mathbb{R}^n$ be defined as

$$M_1 = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} x_i \leq 1;\, x_i \geq 0 \;\forall i = 1, \ldots, n \right\}.$$

Determine all extremal points of M_1 and explain why your solution is correct.

- There are n variables and $n+1$ restrictions.
- n restrictions have to be fulfilled with equality.
- There is a total of $n+1$ extremal points:
  - the all zero vector (0, ..., 0),
  - n unit vectors with $x_i = 1$.

Exercise 1.5

b) Let $M_2 \subseteq \mathbb{R}^3$ be defined as

$$M_2 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 \leq 1, \\ x_1 + 2x_2 + 4x_3 \leq 2, \\ x_1 + 3x_2 + 9x_3 \leq 3, \\ x_1 + 4x_2 + 16x_3 \leq 4\}.$$

How many extremal points does M_2 have? Prove your claim.

- Three restrictions have to be fulfilled with equality.
- Each subset of three restrictions is linearly independent.
- The coefficients for x_2 are equal to the last column.
- The solution is $(0, 1, 0)$.
- Due to linear independence, no other solutions exist.

Exercise 1.6

Linear Optimization

$c_0 : x_1, x_2 \geq 0$

$c_1 : x_1 \leq 9$

$c_2 : x_2 \leq 6$

$c_3 : x_1 + 2x_2 \leq 16$

$c_4 : 2x_1 + x_2 \leq 20$

$\Rightarrow$ TF(x_1, x_2)=$x_1 + x_2$

Extremal Points: TF(0,0)=0, TF(0,6)=6, TF(4,6)=10, TF(8,4)=12, TF(9,2)=11, TF(9,0)=9