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Algorithmics, Spring 2020
Exercise sheet 1

Exercise 1.1:

Complete the following table with the symbols $O, o, \Omega, \omega, \Theta$. Use O and Ω only in those cases where no other symbol can be used.

	$\log n$	$2^{n/2}$	$\sqrt{n}$	5	2^n	$1/n$	n	e^n	n^2
$\log n$							o		
$2^{n/2}$									
$\sqrt{n}$									
5									
2^n									
$1/n$									
n									
e^n									
n^2									

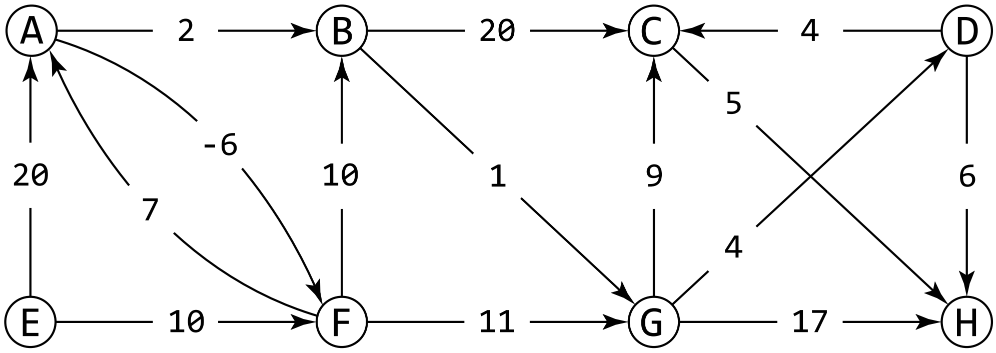
Exercise 1.2:

For each of the following functions $f_i : \mathbb{N}^{>1} \longrightarrow \mathbb{R}^{\geq 0}$, provide a function $g_i : \mathbb{N}^{>1} \longrightarrow \mathbb{R}^{\geq 0}$ having as few terms as possible and satisfying $f_i \in \Theta(g_i)$.

- $f_1(n) = n^2 2^n + 4^n + 3^n$
- $f_2(n) = n(n-1)/2$
- $f_3(n) = \log(n^{70})$
- $f_4(n) = 9n \log n + 30n(\log n)^2 + n$
- $f_5(n) = \sum_{i=1}^{n} 2^i$
- $f_6(n) = \frac{n}{\log n} + \frac{n^2}{\log^2 n}$

Exercise 1.3:

We will now consider the following graph:



Find the **shortest path** from vertex E to vertex H using the Bellman-Ford algorithm **based on the following lexicographical "edge order":**

$$\underset{2}{(A,B)},\ \underset{-6}{(A,F)},\ \underset{20}{(B,C)},\ \underset{1}{(B,G)},\ \underset{5}{(C,H)},\ \underset{4}{(D,C)},\ \underset{6}{(D,H)},\ \underset{20}{(E,A)},$$
$$\underset{10}{(E,F)},\ \underset{7}{(F,A)},\ \underset{10}{(F,B)},\ \underset{11}{(F,G)},\ \underset{9}{(G,C)},\ \underset{4}{(G,D)},\ \underset{17}{(G,H)}.$$

To do so, complete the following table. (Be advised that not all columns available in the table below may actually be needed.)

	I		R_1^*		R_2^*		R_3^*		R_4^*		R_5^*		R_6^*		R_7^*	
	$v.d$	$v.\pi$	$v.d$	$v.\pi$	$v.d$	$v.\pi$	$v.d$	$v.\pi$	$v.d$	$v.\pi$	$v.d$	$v.\pi$	$v.d$	$v.\pi$	$v.d$	$v.\pi$
A																
B																
C																
D																
E																
F																
G																
H																

* Only Relax operations are considered which actually change $(v.d, v.\pi)$, $v \in V$. If $(v.d, v.\pi)$ is changed multiple times within a single iteration, give all values (e.g., use columns R_i^* and R_{i+1}^* in the case of two changes during one iteration) respecting the aforementioned order of edges.

Exercise 1.4:

In this exercise we consider the graphs $G_1 = (V, E_1, w_1)$ and $G_2 = (V, E_2, w_2)$ with

$$V = \{v_1, ..., v_n\},$$
$$E_1 = \{(v_1, v_2), (v_2, v_3), ..., (v_{n-1}, v_n)\},$$
$$E_2 = \{(v_2, v_1), (v_3, v_2), ..., (v_n, v_{n-1})\}.$$

The edge weight functions w_1 and w_2 assign a weight of 1 to each edge in E_1 and E_2.

We say that an edge (u, v) is *relaxed* if the values $v.\pi$ and $v.d$ change as a result of the application of the `Relax`(u, v, w) operation. Further, we assume the edges are traversed in lexicographical order by the algorithm in each iteration, i.e. (v_i, v_j) is considered before (v_k, v_ℓ) if $i < k$, or $i = k$ and $j < \ell$.

Determine for each of the $|V| - 1$ iterations which edges are relaxed during the application of

1. `BellmanFord`(G_1, w_1, v_1) and
2. `BellmanFord`(G_2, w_2, v_n).

Also determine the updated distance $v.d$ and the updated predecessor $v.\pi$ for the vertex v in the respective iteration.

Exercise 1.5:

(a) Let $M_1 \subseteq \mathbb{R}^n$ be defined as

$$M_1 = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} x_i \leq 1;\; x_i \geq 0 \;\forall i = 1, \dots, n \right\}.$$

Determine all extremal points of M_1 and explain why your solution is correct.

(b)) Let $M_2 \subseteq \mathbb{R}^3$ be defined as

$$\begin{aligned} M_2 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 — \; & x_1 + x_2 + x_3 \leq 1, \\ & x_1 + 2x_2 + 4x_3 \leq 2, \\ & x_1 + 3x_2 + 9x_3 \leq 3, \\ & x_1 + 4x_2 + 16x_3 \leq 4\}. \end{aligned}$$

How many extremal points does M_2 have? Prove your claim.

Hint: You can assume that all subsets of three restrictions are linearly independent.

Exercise 1.6:

Solve the following linear program using the exhaustive search algorithm from the lecture.

$$\begin{array}{rrcrl} \textbf{maximize} & 1x_1 & + & 1x_2 & \\ \textbf{subject to} & 1x_1 & & & \leq 9 \\ & & & 1x_2 & \leq 6 \\ & 1x_1 & + & 2x_2 & \leq 16 \\ & 2x_1 & + & 1x_2 & \leq 20 \\ & x_1 & , & x_2 & \geq 0 \end{array}$$