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On the isospectral problem of Fermi curves of two-dimensional doubly periodic Schrödinger operators

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Abstract

The aim of this thesis is to parameterize the isospectral set $Iso(u_0)$ for smooth Fermi curves of two-dimensional Schrödinger operators with doubly periodic real-valued L^2-potential $u_0 : \mathbb{R}^2 \to \mathbb{R}$. This isospectral set is the set of all real-valued doubly periodic L^2-potentials u whose Fermi curve $F(u)$ equals the given Fermi curve $F(u_0)$. Our thesis essentially consists of two parts. The first part solves the isospectral problem asymptotically by investigating those part of the Fermi curve outside a sufficiently large compact set in $\mathbb{C}^2$. In this asymptotic setting, the so-called perturbed Fourier coefficients will serve as suitable coordinates for the potentials. We parameterize the asymptotic isospectral set by constructing a homeomorphism mapping it onto a topological space $\widetilde{Iso}_\delta(u_0)$, where $\widetilde{Iso}_\delta(u_0)$ can be explicitly determined. The second part of the thesis connects the asymptotic part with the so far neglected compact part of the Fermi curve. Under an additional boundedness assumption on $Iso(u_0)$, we show that $Iso(u_0)$ is homeomorphic to a Cartesian product $Iso(u_1) \times \widetilde{Iso}_\delta(u_0)$, where u_1 is a potential of *finite type*. For unbounded isospectral sets, we will show an analogous but weaker result. In the entire thesis, we use the so-called moduli $m(u)$ in order to describe the isospectral sets. These moduli are l^1-sequences. We finally show that each Fermi curve $F(u)$ is uniquely determined by its moduli $m(u)$. In particular, the moduli are invariants of the isospectral set.

Zusammenfassung

Das Ziel dieser Arbeit ist die Parametrisierung der Isospektralmenge $Iso(u_0)$ für glatte Fermikurven zweidimensionaler Schrödinger-Operatoren mit doppeltperiodischem reellwertigem L^2-Potential $u_0 : \mathbb{R}^2 \to \mathbb{R}$. Diese Isospektralmenge ist die Menge aller reellwertigen doppeltperiodischen L^2-Potentiale u, deren Fermikurve $F(u)$ gleich der gegebenen Kurve $F(u_0)$ ist. Unsere Arbeit besteht im Wesentlichen aus zwei Teilen. Der erste Teil löst das isospektrale Problem asymptotisch, indem man jenen Teil der Fermikurve außerhalb eines hinreichend großen Kompaktums in $\mathbb{C}^2$ untersucht. In diesem asymptotischen Szenario werden die so genannten gestörten Fourierkoeffizienten als geeignete Koordinaten für die Potentiale dienen. Wir parametrisieren die asymptotische Isospektralmenge, indem wir einen Homöomorphismus von ihr auf einen topologischen Raum $\widetilde{Iso}_\delta(u_0)$ konstruieren, wobei $\widetilde{Iso}_\delta(u_0)$ explizit bestimmt werden kann. Der zweite Teil der Arbeit verknüpft den asymptotischen Teil mit dem bisher vernachlässigten kompakten Teil der Fermikurve. Unter einer zusätzlichen Beschränktheitsvoraussetzung an $Iso(u_0)$ zeigen wir, dass $Iso(u_0)$ homöomorph zu einem kartesischen Produkt $Iso(u_1) \times \widetilde{Iso}_\delta(u_0)$ ist, wobei u_1 ein *finite type* Potential ist. Für unbeschränkte Isospektralmengen werden wir ein analoges, jedoch schwächeres Resultat zeigen. In der gesamten Arbeit benutzen wir die so genannten Moduli $m(u)$, um die Isospektralmengen zu beschreiben. Diese Moduli sind l^1-Folgen. Wir zeigen schließlich, dass jede Fermikurve $F(u)$ eindeutig durch ihre Moduli $m(u)$ bestimmt ist. Insbesondere sind die Moduli Invarianten der Isospektralmenge.

I have not failed. I've just found 10,000 ways that won't work.

Thomas A. Edison

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Contents

Chapter 1

Introduction

1.1 The inverse problem

We consider the time-independent two-dimensional Schrödinger equation

$$-\Delta\psi + u \cdot \psi = \lambda\psi \tag{1.1}$$

with doubly periodic potential $u : \mathbb{R}^2 \to \mathbb{C}$, eigenfunction ψ and eigenvalue $\lambda \in \mathbb{C}$. Hereby,

$$\Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

denotes the *Laplace operator* in two dimensions with respect to the variable $x = (x_1, x_2) \in \mathbb{R}^2$. Let $\Gamma \subset \mathbb{R}^2$ be the two-dimensional lattice of periods of u, i.e.

$$u(x + \gamma) = u(x) \quad \text{for all}\, \gamma \in \Gamma. \tag{1.2}$$

Since u is periodic, any solution ψ of (1.1) must be quasi-periodic (cf. [13], p. 2), that is, there is a $k \in \mathbb{C}^2$ such that

$$\psi(x + \gamma) = e^{2\pi i \langle k,\gamma\rangle}\psi(x) \quad \text{for all}\, x \in \mathbb{R}^2,\ \gamma \in \Gamma. \tag{1.3}$$

Here, $\langle\cdot,\cdot\rangle$ denotes the complex extension of the canonical euclidean bilinear form on $\mathbb{R}^2$, that is: $\langle v, w\rangle := v_1w_1 + v_2w_2$ for $v, w \in \mathbb{C}^2$. Besides, we write $|v| := \sqrt{< v, \overline{v} >}$ for the Euclidean norm of a vector $v \in \mathbb{C}^2$.
The so-called *boundary conditions* $k \in \mathbb{C}^2$ together with the eigenvalues $\lambda \in \mathbb{C}$ constitute the *Bloch variety* $B(u)$, defined by

$$\begin{aligned} B(u) :=\{(k,\lambda) \in \mathbb{C}^2 \times \mathbb{C} :\ & \text{there is a non-trivial solution } \psi \\ & \text{of the Schrödinger equation } (-\Delta + u)\psi = \lambda\psi \text{ with} \\ & \psi(x+\gamma) = e^{2\pi i\langle k,\gamma\rangle}\psi(x) \quad \text{for all}\, x \in \mathbb{R}^2,\ \gamma \in \Gamma\}. \end{aligned} \tag{1.4}$$

In this work, we are only interested in the so-called *Fermi curves* which are obtained from $B(u)$ by setting $\lambda = 0$, that is,

$$F(u) := \{k \in \mathbb{C}^2 : (k, 0) \in B(u)\}.$$

There appear two problems: The *direct* problem and the *inverse* problem. In the direct problem, one wants to parameterize for a given potential u the corresponding Fermi curve $F(u)$. It turns out that Fermi curves are subvarieties of $\mathbb{C}^2$ in the sense that at least locally, Fermi curves can be described as the zero sets of holomorphic functions $f : U \subset \mathbb{C}^2 \to \mathbb{C}$ (see for example Theorem 2.2.6 in Section 2.2 or [13, Theorem 4.1.3]).
In the inverse problem, one considers the following two subproblems:

- **The moduli problem:** Parameterize the set of all possible Fermi curves.
- **The isospectral problem:** Given a fixed potential u_0, parameterize the set of all potentials u such that $F(u) = F(u_0)$.

The complete solution of the inverse problem, which has firstly been posed by Novikov and Veselov, cf. [22], turns out to be very extensive. As the title of this thesis already suggests, we will deal with the isospectral problem in this work. We will anticipate more precisely in Section 1.3 what will be done.

1.2 The doubly periodic Schrödinger equation

As already mentioned, we deal with doubly periodic potentials. We want to recap some properties induced by this periodicity. First of all, we may restrict, due to the periodicity with respect to the lattice Γ, the domain of definition of the potential u, namely $\mathbb{R}^2$, to the *torus*

$$F := \mathbb{R}^2/\Gamma$$

which can be identified with a *fundamental domain* in $\mathbb{R}^2$. Sometimes, we speak, by abuse of notation, of the fundamental domain F (although F is defined as a torus). If we consider functions defined on F (for instance $u \in L^2(F)$), this shall mean (even though we won't always mention it explicitly) that these functions are periodic with respect to Γ (otherwise, a definition on the torus F wouldn't make any sense). The periodicity of u in the x-coordinates will have effects on the Fermi curve $F(u)$: Fermi curves turn out to be periodic (in k-coordinates, $k \in \mathbb{C}^2$) with respect to a (real) lattice $\Gamma^* \subset \mathbb{R}^2$ (cf. [13, Lemma 4.3.1]). The connection between Γ and Γ^* is that they are dual to each other. More precisely, Γ^* is defined as

$$\Gamma^* := \{x \in \mathbb{R}^2 : \langle x, \gamma\rangle \in \mathbb{Z} \text{ for all } \gamma \in \Gamma\}.$$

The Schrödinger equation (1.1) is an eigenvalue equation of the form

$$A\psi = \lambda\psi,$$

with differential operator $A := -\Delta + u$. If one wants to find eigenvalues $\lambda \in \mathbb{C}$ for which there exists a nontrivial eigenfunction ψ, one typically examines the singularities of the *resolvent*

$$\lambda \mapsto (\lambda \cdot id - A)^{-1}, \tag{1.5}$$

that is, one seeks those $\lambda \in \mathbb{C}$ for which the resolvent is not invertible, in other words $\ker(\lambda \cdot id - A) \neq \{0\}$. If there exists a (local) representation of $\lambda \cdot id - A$ as endomorphism between finite-dimensional vector spaces, the latter criterion can also be expressed as

$$\det(\lambda \cdot id - A) = 0, \tag{1.6}$$

as is well-known from linear algebra. All $\lambda \in \mathbb{C}$ for which $\lambda \cdot id - A$ is not invertible (i.e in the case above, for which (1.6) is satisfied) constitute the *point spectrum* of A. If we consider Fermi curves, we are only interested in the eigenvalue $\lambda = 0$. Now, we want to declare the function spaces the potential u and the solution ψ shall reside in. To this, we have to make clear at first what kind of solutions we are looking for. An obvious possibility would be to consider Sobolev spaces $H^{1,2}(F)$ and $H^{2,2}(F)$, respectively, if we searched for solutions in the weak or strong sense, respectively. The most general kind of solutions are those in the sense of distributions. Since we are interested in as large spaces as possible, we consider, until further notice, solutions in the sense of distributions. To this, let $S(F) := C^\infty(F)$ be the *Schwartz function space* of infinitely differentiable functions on F (these serve as test functions) and let $S^*(F)$ be the dual space to $S(F)$, i.e. the space of continuous linear functionals on $S(F)$ - the distributions. As moreover, we consider doubly periodic potentials, we will often use a Fourier representation of the potential. An apt possibilty is therefore to consider so-called *Fourier spaces*, which are defined by (cf. [13, Def. 2.5.1])

$$\mathcal{F}E := \{f \in S^*(F) : \mathcal{F}f \in E\},$$

where E is some Banach space (in our context, E will be some space of sequences, such as $l^2(\Gamma^*)$) and $\mathcal{F}f$ denotes the Fourier transform of the distribution f. Here, one should keep in mind the definition of the Fourier transform of a distribution (cf. [13, Def. 2.1.12]), namely

$$(\mathcal{F}f)(\kappa) := f(\psi_{-\kappa}), \quad \kappa \in \Gamma^*, \tag{1.7}$$

where ψ_κ denotes the κ-th Fourier mode, i.e

$$\psi_\kappa(x) := e^{2\pi i\langle\kappa,x\rangle} \quad \text{for } \kappa \in \Gamma^*. \tag{1.8}$$

In this context, let's recap the "ordinary" Fourier transform of some integrable function f (compare [13, p. 13]):

$$(\mathcal{F}f)(\kappa) := \int_F \psi_{-\kappa}(x) f(x) dx, \quad \kappa \in \Gamma^*, \tag{1.9}$$

where we also write $\hat{f}$ for $\mathcal{F}f$. Due to [13, Proposition 2.1.14], it is in some cases of *regular* distributions, i.e. distributions $f \in S^*(F)$ for which there exists a smooth function g such that $f(\phi) = \int_F g(x)\phi(x)dx$ for all test functions $\phi \in S(F)$, allowed to use the "ordinary" Fourier transform instead of the just defined "abstract" Fourier transform in (1.7) because there holds $\hat{g} = \mathcal{F}f$ due to this proposition. As we will see in a moment, we can make the quasi-periodic solution ψ periodic by some transformation. Thus, it makes sense to define for example $\mathcal{F}l^1(\Gamma^*)$ as the space for the eigenfunctions. In [13, Proposition 3.3.15], it was proved that the resolvent maps $\mathcal{F}l^{\infty,1}(\Gamma^*)$ (which is quite a large space, for the precise definition see Definition 2.2.1) boundedly into $\mathcal{F}l^1(\Gamma^*)$ (with suitably chosen λ in (1.5) such that the resolvent is a well-defined operator). This motivates why we can choose $\mathcal{F}l^1(\Gamma^*)$ as the space for the eigenfunctions. For the space of potentials $u : F \to \mathbb{C}$, we choose the Hilbert space $L^2(F)$. Note that $\hat{f} \in l^2(\Gamma^*)$ for $f \in L^2(F)$. Thus, from now on, we require for the spaces of eigenfunctions ψ and potentials u, respectively,

$$\psi \in \mathcal{F}l^1(\Gamma^*), \qquad u \in L^2(F) \quad (\Rightarrow \hat{u} \in l^2(\Gamma^*)).$$

The equations (1.1), (1.2) and (1.3) make up the doubly periodic Schrödinger equation with quasi-periodic boundary condition. In the following, we will often use an equivalent representation of these equations by using a common formulation (which has also been used in [13, p. 58], for instance), where the boundary condition $k \in \mathbb{C}^2$ is already included in the Laplace operator Δ. More precisely, we define for $k \in \mathbb{C}^2$

$$\Delta_k := (\nabla + 2\pi i k)^2 = \Delta + 4\pi i \langle k, \nabla \rangle - 4\pi^2 k^2. \tag{1.10}$$

Due to (1.3), a solution ψ of (1.1) is only quasi-periodic, but, in general, not periodic. By setting with suitable boundary condition fulfilling (1.3)

$$\psi_p(x) := e^{-2\pi i \langle k, x \rangle} \psi(x),$$

a simple calculation (see [13, Lemma 3.1.9]) shows that ψ_p is *periodic* with respect to Γ. The reformulation of (1.1), (1.2) and (1.3) is now as follows (cf. [13, Theorem 3.1.10]): For $u \in L^2(F)$, i.e. u fulfills (1.2), the quantities k, ψ and λ fulfill the equations (1.1) and (1.3) if and only if

$$(-\Delta_k + u)\psi_p = \lambda \psi_p. \tag{1.11}$$

Thus, (1.1) and (1.3) can be expressed in one single equation (1.11). Let's emphasize one more time that another advantage of this formulation is that the eigenfunctions are *periodic*. In the sequel, we will, for simplicity, write $\psi = \psi_p$ since from now on, we will continue our considerations in the setting of (1.11) so that there shouldn't be any confusion.

1.3 What is done in this work

The main goal of this work is to solve the isospectral problem as introduced in Section 1.1 for real-valued potentials with smooth Fermi curve, i.e. to determine for given real-valued $u_0 \in L^2(F)$ the isospectral set

$$Iso_F(u_0) := \{u \in L^2(F),\ u \text{ real-valued } : F(u) = F(u_0)\}$$

(the subscript F in $Iso_F(u_0)$ stands for *Fermi curve*), where $F(u_0)$ is assumed to have no singularities, which is expressed by the term *smooth* Fermi curve. To begin with, we want to specify more precisely what we mean by "determine" the isospectral set. The most ambitious way to do this would be to determine $Iso_F(u_0)$ as a set of real-valued L^2-potentials given by explicit formulas. Indeed, there are cases where this is possible. As a very important example in this context, we want to mention the work [23] by PÖSCHEL and TRUBOWITZ which deals with quite a similar problem, namely with the one-dimensional Schrödinger equation

$$-y''(x) + q(x)y(x) = \lambda y(x),$$

with eigenvalue $\lambda \in \mathbb{C}$ and real-valued potential $q \in L^2([0,1])$. Instead of a Fermi curve, in [23], the sequence of *Dirichlet eigenvalues* for some given potential is considered and the isospectral problem asks to find all real-valued potentials in $L^2([0,1])$ which share the same sequence of Dirichlet eigenvalues as the given potential. In [23, Theorem 5.2], an explicit solution of the isospectral set in terms of explicit formulas for the isospectral potentials is given.

For our case of Fermi curves of two-dimensional doubly periodic Schrödinger operators, however, things turn out to be more difficult so that we cannot expect such explicit formulas. The goal we are interested in is not to explicitly write down the elements of $Iso_F(u_0)$ but to determine its topological structure. More precisely, we want to find a topological space which can be explicitly parameterized and which is *homeomorphic* to $Iso_F(u_0)$, i.e. which shares the same topology as $Iso_F(u_0)$.

In general, Fermi curves are complex curves (i.e. Riemann surfaces which in general may have singularities) of *infinite* genus. A first step to solve the isospectral problem would be to consider the special case of *finite* genus, i.e. Fermi curves of so-called *finite type* potentials. These are Fermi curves that can be considered as *compact* complex curves in some sense (one crucial property is that their normalization can be compactified). The theory of those *finite type* Fermi curves of two-dimensional doubly periodic Schrödinger operators has been investigated in the extensive work [19]. For finite type Fermi curves, one can use the theory of compact complex curves which allows to use particular methods that can't be applied in the infinite type case in general. We don't want to go deeper into details of finite type theory of Fermi curves. For readers interested in this topic, we thus recommend the mentioned work [19]. In our work, however, we want to consider the general case of Fermi curves of *infinite* genus. The general precondition in our work is that we consider the isospectral problem for finite type Fermi curves as solved[1]. In this sense, we aim to solve the isospectral problem by determining a homeomorphism

$$\mathcal{I} : Iso(u_1) \times \widetilde{Iso}_\delta(u_0) \to Iso_F(u_0),$$

where $u_1 \in L^2(F)$ is a real-valued *finite type* potential with corresponding isospectral set $Iso(u_1)$ and the topological space $\widetilde{Iso}_\delta(u_0)$ is the "asymptotic remainder" which will be explicitly parameterized in this work. Hence, provided that $\mathcal{I}$ is a homeomorphism, $Iso(u_1) \times \widetilde{Iso}_\delta(u_0)$ is the topological space we can identify the isospectral set $Iso_F(u_0)$ with. For unbounded isospectral sets (i.e. unbounded with respect to the L^2-norm), we will get a weaker result than the homeomorphism property which will be discussed in Section 4.2. For isospectral sets with an additional boundedness condition, however, we will get the homeomorphism $\mathcal{I}$ just mentioned. Now, we want to give a short overview of what is done in the individual chapters and sections of this work.

Before attending to the isospectral problem, we have to do some preparatory work concerning properties and important assertions about Fermi curves. This will be done in Chapter 2. In Section 2.1, we give some basic facts about Fermi curves which are already well-known. As examples, we mention the free Fermi curve (associated to the potential $u \equiv 0$) and the Fermi curve for constant potentials, cf. [5, III.16] or [13, 4.2,4.4], where the citation of the work [5] by Feldman, Knörrer, Trubowitz deserves a special emphasis since it turnt out to be quite helpful for our work. We also recap that every Fermi curve $F(u)$ consists of three parts (cf. [19, Theorem 2.35]) which, at first, has been proved by Krichever (cf. [16]): Firstly, a compact part of finite arithmetic genus, then the remainder called the *asymptotic part* consisting of, secondly, two so-called *open ends* or *regular pieces*, which are isomorphic to the complex plane, where countably many open bounded sets (the so-called *excluded domains* e_ν, indexed by $\nu \in \Gamma^*$ with sufficiently large norm) are cut out, and thirdly, the so-called *handles* which connect the two regular pieces by some excluded domain in each regular piece,

[1] Unfortunately, the isospectral problem for finite type Fermi curves has not been completely solved in [19], but there have been done great steps towards a solution in [19] so that researchers interested in this topic may feel encouraged to complete it.

respectively, compare [5, II.5, III.17, III.18]. In the case that $F(u)$ is not smooth, handles may "close" to a singularity. Whereas in the compact part, all kinds of singularites may appear, in the asymptotic part, ordinary double points are the only possible kind of singularities. In order to index the excluded domains by $\nu \in \Gamma^*$ with sufficiently large norm, we introduce for $\delta > 0$ sufficiently small the asymptotic part of the dual lattice

$$\Gamma^*_\delta := \{\kappa \in \Gamma^* : |\kappa| > \delta^{-1}\}.$$

In Section 2.2, we provide important results needed for the asymptotic analysis of Fermi curves $F(u)$ which concern that part of the Fermi curve outside an arbitrarily large compact set in $\mathbb{C}^2$ of the just mentioned trisection. Whereas finite type theory considers the part of $F(u)$ within the compact set, an investigation of the infinite genus case, as it will be done in this work, crucially includes the asymptotic part of $F(u)$. In this context, we want to mention the work [13] which also dealt with Fermi curves of infinite genus and can be considered as a prequel to our work. In [13], important results for the asymptotic analysis have already been shown. Some of them can just be taken over to our work (for example the representation of $F(u)$ in the ν^{th} excluded domain e_ν, $\nu \in \Gamma^*_\delta$, as the zero set of $\det M_\nu$, where the 2×2-matrix $M_\nu = D_\nu + \mathcal{A}_\nu$ is the sum of a diagonal matrix D_ν incoding the informations of a constant potential Fermi curve and a *perturbation matrix* $\mathcal{A}_\nu$ representing the deviation of the given Fermi curve from the respective constant potential Fermi curve, cf. Theorem 2.2.6), whereas some others require a modification. One reason for this is that [13] considered another space of potentials than we do. Another important result of this section is the existence of unique $k_\nu \in e_\nu$ such that the diagonal entries of M_ν vanish at $k = k_\nu$. The corresponding off-diagonal elements of M_ν, the so-called *perturbed Fourier coefficients*, will play an important role in the sequel.

Fermi curves associated to potentials u of the Schrödinger equation are point-symmetric to $0 \in \mathbb{C}^2$, i.e. invariant with respect to the holomorphic $\mathbb{C}^2$-involution $k \mapsto -k$. If u is real-valued, $F(u)$ is even invariant with respect to complex conjugation, i.e. with respect to the antiholomorphic $\mathbb{C}^2$-involution $k \mapsto \overline{k}$. In Section 2.3, we will see how the perturbation matrix $\mathcal{A}_\nu$, the zeroes k_ν of the diagonal elements of M_ν and the perturbed Fourier coefficients transform by these involutions. It turns out that the symmetries of the Fermi curve induce certain symmetries of these objects. For instance, the second off-diagonal entry of M_ν at $k = k_\nu$ is already determined by the first off-diagonal entry of M_ν. Hence, it suffices to consider only this first off-diagonal entry denoted by $\check{u}_\nu$, the ν^{th} perturbed Fourier coefficient.

In Section 2.4, it will be shown in Theorem 2.4.2 that the map

$$l^2(\Gamma^*_\delta) \longrightarrow l^2(\Gamma^*_\delta), \qquad (\hat{u}(\nu))_{\nu \in \Gamma^*_\delta} \longmapsto (\check{u}_\nu)_{\nu \in \Gamma^*_\delta}.$$

between Fourier coefficients and perturbed Fourier coefficients is locally invertible. This assertion has already been proven in [13] for another space of potentials.

An important corollary of this theorem is the relation $\overline{\check{u}_\nu} = \check{u}_{-\nu}$, $\nu \in \Gamma_\delta^*$, for real-valued potentials $u \in L^2(F)$. By the way, until Section 2.4, we consider both cases of complex and real-valued potentials $u \in L^2(F)$. In other words, Fermi curves $F(u)$ both with and without antiholomorphic involution are considered. Moreover, $F(u)$ doesn't necessarily need to be smooth. Hence so far, we are still in quite a general setting. The reduction to exclusively real-valued potentials with smooth Fermi curve appears later.
In Section 2.5, we will determine suitable coordinates to parameterize the excluded domains. Since we will later restrict ourselves to smooth Fermi curves, we are especially interested in the parameterization of the handles. However, also in Section 2.5, Fermi curves don't necessarily need to be smooth, yet. By the new z-coordinates introduced in this section, such a handle H can be parameterized by

$$H := \{(z_1, z_2) \in \mathbb{C}^2 : z_1 \cdot z_2 = c, \quad |z_1|, |z_2| \le 1\},$$

where $c \in \mathbb{C}$ is the so-called handle quantity. To show that such coordinates exist and the determination of the corresponding handle quantities c_ν, $\nu \in \Gamma_\delta^*$, are the content of this section. Thereto, we introduce the so-called *model Fermi curve* as in [13, Lemma 4.5.53] which turns out to be a good approximation for the given Fermi curve, at least for real-valued potentials (for complex-valued potentials, we will face some difficulties). This model curve is the curve that we get by a linear approximation of the matrix M_ν. The advantage of the model curve is that the z-coordinates and the corresponding model handle quantities $\widetilde{c}_\nu$ can be immediately computed. The main effort of Section 2.5 is to yield analogous results for the actual Fermi curve by analyzing the appearing perturbation terms which have to be subjected to an asymptotic analysis. A very important tool is the so-called Quantitative Morse Lemma proved in [5, Lemma B.1, p. 245]. In order to apply this lemma, we verify its conditions in our case. Thereto, we have to make some estimates which are necessary to keep certain perturbation terms sufficiently small. In order to finally determine the handle quantities, we have to delve into the proof of the Quantity Morse Lemma since the authors in [5] justify in their proof that one can assume without restriction the case $c_\nu = 0$. For the proof in [5], this is an admissible assumption. For our aim to determine the handle quantities, however, we need to reconsider the general case without any simplifying assumptions. After having determined the handle quantities c_ν, we finally show that for real-valued potentials, they satisfy together with the model handle quantities $\widetilde{c}_\nu$ the relation

$$\frac{c_\nu}{\widetilde{c}_\nu} = 1 + o\left(\frac{1}{|\nu|^2}\right), \quad \text{as } |\nu| \to \infty,$$

on the subsequence indexed by $\nu \in \Gamma_\delta^*$ obeying $\widetilde{c}_\nu \neq 0$, cf. Theorem 2.5.9. Here, the condition that the potential is assumed to be real-valued is crucial. For

generic complex-valued potentials, the proof doesn't hold (and the assertion is even expected not to be true).
The asymptotic solution of the moduli problem has already begun in [13], but it hasn't been finished yet. In [13], the moduli space has been parameterized by introducing certain parameters, the so-called *moduli* $m(u)$ (depending on the potential u), but it hasn't been shown that these moduli indeed parameterize Fermi curves, yet, which shall be shown in this work (later in Section 4.3). In Section 2.6, we introduce these moduli $m(u) = (m_\nu(u))_{\nu\in\Gamma^*_\delta}$, indexed by $\nu \in \Gamma^*_\delta$, which are defined by

$$m_\nu(u) := -16\pi^3 \int_{A_\nu} k_1 dk_2,$$

as contour integral along the ν^{th} A-cycle of the homology basis of $F(u)$. We also introduce the respective moduli $\widetilde{m}_\nu$ for the *model* curve of Section 2.5. Later, the moduli m_ν will turn out to be invariants of the isospectral set. Hence, they are an appropriate tool in order to determine the isospectral set.
Chapters 3 and 4 are the heart of this work. All potentials in Chapter 3 are assumed to be real-valued. In Chapter 3, we consider the so-called *asymptotic isospectral set* $Iso_\delta(u_0)$ defined by the set of potentials u satisfying the property $m_\nu(u) = m_\nu(u_0)$, $\nu \in \Gamma^*_\delta$, where the first finitely many Fourier coefficients of u indexed by $\nu \in \Gamma^* \setminus \Gamma^*_\delta$ are kept fixed. Actually, we'll introduce $Iso_\delta(u_0)$ as a subset of $l^2(\Gamma^*_\delta)$ (and not of $L^2(F)$) in terms of perturbed Fourier coefficients (which serve as asymptotic coordinates due to Section 2.4). The precise relation between L^2-potentials u and perturbed Fourier coefficients will be given in that chapter.
In Section 3.1, we determine the asymptotic isospectral set $\widetilde{Iso}_\delta(u_0)$ for the *model* curve. As for the handle quantities for the model curve in Section 2.5, also the asymptotic model isospectral set can be computed explicitly. In Theorem 3.1.1, we'll parameterize the elements of $\widetilde{Iso}_\delta(u_0)$ by flows indexed by a flow multi-parameter $t = (t_\nu)_{\nu\in\Gamma^*_\delta} \in [0, 2\pi)^\infty$ (one independent parameter for each excluded domain).
In Section 3.2, we make a perturbation ansatz. More precisely, denoting by $r(\cdot) := m(\cdot) - \widetilde{m}(\cdot)$ the deviation between moduli and model moduli, we make the ansatz

$$m_\nu(u_0) = \widetilde{m}_\nu(u_t + \widetilde{v}_t) + r_\nu(u_t + v_t) \quad \text{for all } \nu \in \Gamma^*_\delta,$$

where the real-valued potential $u_t \in L^2(F)$ is an element of the isospectral flow of the model Fermi curve with flow parameter t, a real-valued $v_t \in L^2(F)$ is given and $\widetilde{v}_t$ shall be determined. In order that $\widetilde{v}_t$ can be uniquely determined, we make an additional linear ansatz for v_t and $\widetilde{v}_t$ yielding a map $v_t \mapsto \widetilde{v}_t$ [2]. We'll show

[2]For technical reasons, the map will look slightly different in Section 3.2 than mentioned here.

firstly that this map is well-defined, i.e. we have to show that it maps into the desired space and that some reality condition is fulfilled. Secondly, we'll show that this map fulfills the condition of Banach's Fixed Point Theorem. In order to achieve these two propositions, many asymptotic estimates have to be done. Eventually, the application of Banach's Fixed Point Theorem yields the existence of a unique fixed point v_t satisfying the initial ansatz

$$m_\nu(u_0) = \widetilde{m}_\nu(u_t + v_t) + r_\nu(u_t + v_t) = m_\nu(u_t + v_t) \quad \text{for all } \nu \in \Gamma^*_\delta.$$

Hence, we will have constructed isospectral flows u_t+v_t of the *actual* Fermi curve. In Section 3.3, we'll show with the results of the foregoing Section 3.2 that there exists a homeomorphism $\widetilde{Iso_\delta}(u_0) \to Iso_\delta(u_0)$.
After having solved the isospectral problem asymptotically for real-valued potentials, we want to determine $Iso(u_0)$ in Chapter 4 in the sense described at the beginning of this Section 1.3. Here, $Iso(u_0)$ is defined as the set of all real-valued potentials $u \in L^2(F)$ satisfying $m(u) = m(u_0)$.
In Section 4.1, we'll show that there exists a finite set of linear independent holomorphic 1-forms ω_j on a given Fermi curve $F(u)$ which is dual to the first finitely many A-cycles A_i of the homology basis of $F(u)$ provided that $F(u)$ is smooth, i.e. for $N \in \mathbb{N}$, there holds

$$\int_{A_i} \omega_j = \delta_{i,j} \quad \text{for } i,j \in \Gamma^*, |i|, |j| \leq N.$$

In particular, from now on, we require the smoothness of Fermi curves. A similar statement with further requirements on the respective Riemann surface has already been shown in [5, Theorem 1.17, Theorem 3.8]. The main goal of this section is to derive some submersion properties of the moduli map $u \mapsto m_f(u) := (m_\nu(u))_{\nu \in \Gamma^* \setminus \Gamma^*_\delta}$. We'll prove these properties both for complex-valued and for real-valued potentials. At first, we consider complex-valued potentials $(V,W) \in L^2(F) \times L^2(F)$ of the Dirac operator which can be seen as a generalization of the Schrödinger operator. Any Schrödinger potential $u \in L^2(F)$ can be considered as a Dirac potential $(V,W) \in L^2(F) \times L^2(F)$ by $(V,W) := (1, \frac{-u}{4})$ or $(V,W) := (\frac{-u}{4}, 1)$. The existence of holomorphic 1-forms dual to the first finitely many A-cycles together with a relation between a symplectic form on $L^2(F) \times L^2(F)$ and Serre duality proved in [27, Lemma 3.2] by M. Schmidt will yield that $u \mapsto m_f(u)$ is a submersion. The next step is to transfer this result to real-valued potentials. The smoothness of $F(u)$ is a crucial ingredient in order to prove the submersion properties of the moduli map.
Section 4.2 is the most important section of Chapter 4. An essential result will be the construction of a canonical sequence of finite type potentials $(u_n)_{n \in \mathbb{N}}$ converging to some given real-valued potential $u_0 \in L^2(F)$, where

$$m_\nu(u_n) = \begin{cases} m_\nu(u_0), & \nu \in \Gamma^*, |\nu| \leq n \\ 0, & \nu \in \Gamma^*, |\nu| > n \end{cases}$$

holds. In Chapter 3, we kept the first finitely many Fourier coefficients fixed and determined the remaining coefficients in terms of perturbed Fourier coefficients such that the respective moduli $(m_\nu(u))_{\nu\in\Gamma^*_\delta}$ were equal to the given $(m_\nu(u_0))_{\nu\in\Gamma^*_\delta}$. In that procedure, we didn't consider the first finitely many moduli. In fact, by varying the Fourier coefficients for $\nu \in \Gamma^*_\delta$, the first finitely many moduli $m_\nu(u)$, $\nu \in \Gamma^* \setminus \Gamma^*_\delta$, won't remain equal to $m_\nu(u_0)$ in general. Now in Section 4.2, we have to ensure that the moduli $m_\nu(u)$ are equal to $m_\nu(u_0)$ for *all* $\nu \in \Gamma^*$ (and not only for the asymptotic remainder). This will be done in two steps. In the first step, we determine a set (containing the isospectral set) of potentials u whose moduli $(m_\nu(u))_{\nu\in\Gamma^*_\delta}$ are equal to $(m_\nu(u_0))_{\nu\in\Gamma^*_\delta}$. In the second step, we pick out of this set those potentials u whose moduli $m_f(u)$ are also equal to $m_f(u_0)$. This finally yields a homeomorphism

$$\mathcal{I} : Iso(u_1) \times \widetilde{Iso}_\delta(u_0) \to Iso(u_0).$$

as already explained at the beginning of Section 1.3 provided that $Iso(u_0)$ satisfies some boundedness condition. If, however, $Iso(u_0)$ is unbounded, we intersect $Iso(u_0)$ with balls $B_R(u_0) \subset L^2(F)$ with arbitrarily large $R > 0$ to gain an analogous result. This result, however, will be weaker because firstly, the choice of $\delta > 0$ in $\widetilde{Iso}_\delta(u_0)$ depends on R and secondly, we'll have to take into account some more technical details induced by the intersection of $Iso(u_0)$ with $B_R(u_0)$. Finally in Section 4.3, we show the equivalence $F(u) = F(u_0) \Longleftrightarrow m(u) = m(u_0)$ yielding $Iso_F(u_0) = Iso(u_0)$, that is, the moduli constitute indeed an invariant of the isospectral set which justifies the investigations in the foregoing chapters.

Chapter 2

Fermi curves

2.1 Examples and basic properties

In this section, we summarize some basic properties of Fermi curves. As examples, we recap the Fermi curve of the zero potential $u \equiv 0$ (the so-called *free Fermi curve*) and the Fermi curve for constant potentials $u \equiv const$. Furthermore, we recap the asymptotic freeness and the trisection into a compact part, regular pieces and handles.
Let's begin with the easiest and at the same time most important example: The free Fermi curve. Due to [13, Theorem 4.2.5], the free Fermi curve $F(0)$ is given by

$$F(0) = \mathcal{R} + \Gamma^*,$$

with $\mathcal{R} := \{k = (k_1, k_2) \in \mathbb{C}^2 : k^2 = k_1^2 + k_2^2 = (k_1 - ik_2)(k_1 + ik_2) = 0\}$. Since Fermi curves $F(u)$ (for arbitrary potential) are periodic with respect to Γ^*, one usually considers the quotient $F(u)/\Gamma^*$. This quotient is well-defined if the pairs of distinct points (k_κ^-, k_κ^+), defined by

$$k_\kappa^- := \frac{1}{2}\begin{pmatrix} \kappa_1 + i\kappa_2 \\ -i\kappa_1 + \kappa_2 \end{pmatrix}, \qquad k_\kappa^+ := \frac{1}{2}\begin{pmatrix} -\kappa_1 + i\kappa_2 \\ -i\kappa_1 - \kappa_2 \end{pmatrix}, \quad \kappa = (\kappa_1, \kappa_2) \in \Gamma^*,$$

are identified to double points for all $\kappa \in \Gamma^*$ (cf. [13, Theorem 4.2.5]). For an arbitrary doubly periodic potential u , we consider its Fourier series expansion, cf. [13], p. 89:

$$u(x) = \frac{1}{\mu(F)} \sum_{\kappa \in \Gamma^*} \psi_\kappa(x)\hat{u}(\kappa) =: \underbrace{4\pi^2 \hat{u}_0}_{\text{constant part}} + \underbrace{\frac{1}{\mu(F)} \sum_{\kappa \in \Gamma^* \setminus \{0\}} \psi_\kappa(x)\hat{u}(\kappa)}_{=:\bar{u}(x) = \text{non-constant part}}, \tag{2.1}$$

where $\hat{u}_0 := \frac{\hat{u}(0)}{4\pi^2 \mu(F)}$ and $\mu(F)$ denotes the Lebesgue measure of the fundamental domain F. As to Fermi curves with constant potential, we consider only the constant part $4\pi^2\hat{u}_0$ and get the following parameterization of the corresponding Fermi curve (cf. [13, Theorem 4.4.1]):

$$F(4\pi^2\hat{u}_0) = \mathcal{R}(\hat{u}_0) + \Gamma^*,$$

where $\mathcal{R}(\hat{u}_0) := \{k \in \mathbb{C}^2 : k^2 + \hat{u}_0 = 0\}$. Again, $\mathcal{R}(\hat{u}_0)$ serves as a system of representatives of the quotient $F(4\pi^2\hat{u}_0)/\Gamma^*$ provided that the pairs of distinct points $(k_\kappa^-(\hat{u}_0), k_\kappa^+(\hat{u}_0))$, defined by

$$k_\kappa^-(\hat{u}_0) := \frac{1}{2}\begin{pmatrix} \kappa_1 + i\kappa_2\xi \\ -i\kappa_1\xi + \kappa_2 \end{pmatrix}, \quad k_\kappa^+(\hat{u}_0) := \frac{1}{2}\begin{pmatrix} -\kappa_1 + i\kappa_2\xi \\ -i\kappa_1\xi - \kappa_2 \end{pmatrix}, \quad \kappa \in \Gamma^* \setminus \{0\}, \tag{2.2}$$

with $\xi := \xi(\hat{u}_0, \kappa) := \sqrt{1 + 4\frac{\hat{u}_0}{\kappa^2}}$, are identified to double points for all $\kappa \in \Gamma^* \setminus \{0\}$. Fermi curves of constant potentials (including the free Fermi curve as a special case) therefore have a quite clear structure: They are complex curves with ordinary double points as singularities exactly at the points $k_\kappa^\pm(\hat{u}_0)$, $0 \neq \kappa \in \Gamma^*$. Fermi curves (of arbitrary potential $u \in L^2(F)$) are subvarieties of $\mathbb{C}^2$ (cf. for example [13, Theorem 4.1.3]), that is, locally, they are described as the zero locus of a holomorphic function $f : U \subset \mathbb{C}^2 \to \mathbb{C}$. *Singularities* of the Fermi curve are thus described by the zeroes of the gradient, $\nabla f = 0$, on the curve $\{f = 0\}$.
For arbitrary potentials, the corresponding Fermi curves turn out to be much more complicated than in the case of constant potentials. There can occur singularities of higher order (not only double points), for example. Fortunately, outside a sufficiently large compact set $K \subset \mathbb{C}^2$ (depending on the potential), any Fermi curve approximates the free Fermi curve. The crucial result in this context is the so-called *asymptotic freeness* of Fermi curves of arbitrary $L^2(F)$-potentials, which has been shown in [19, Theorem 2.35]. This means that for every potential $u \in L^2(F)$, the singularities of the corresponding Fermi curve $F(u)/\Gamma^*$ in $\mathbb{C}^2 \setminus K$ remain well-behaved. More precisely: There is a $\delta > 0$ (depending on u) which defines the *asymptotic part* Γ_δ^* of the lattice Γ^*,

$$\Gamma_\delta^* := \{\kappa \in \Gamma^* : |\kappa| > \delta^{-1}\}, \tag{2.3}$$

as well as an open set $V \subset \mathbb{C}^2$ (with $0 \in V$) that only depends on Γ^* such that all possible singularities outside the just mentioned compact set are contained in the so-called *excluded domains* $F(u) \cap (k_\kappa^\pm(\hat{u}_0) + V)$, $\kappa \in \Gamma_\delta^*$. Moreover, every excluded domain contains at most one singularity and double points are the only kind of singularities that can occur. If there's no singularity in $k_\kappa^\pm(\hat{u}_0) + V$, we say that the double point at $k_\kappa^\pm(\hat{u}_0)$ *splits up* to a *handle*. Such a handle can be (up to a diffeomorphism) considered as a cylinder connecting the excluded domain around $k_\kappa^-(\hat{u}_0)$ with the excluded domain around $k_\kappa^+(\hat{u}_0)$ (for the definition of handles, compare [5, II.5, (GH2)] and in particular Section 2.5 for a more detailed treatment of the handles). If, on the other hand, there is a singularity in $k_\kappa^\pm(\hat{u}_0) + V$, i.e. a double point (as in the free case), we say that the double point *doesn't split up* (or equivalently: remains *unsplit*). Outside the compact set K, the singularities of $F(u)/\Gamma^*$ are thus enumerated by $\kappa \in \Gamma_\delta^*$. We now arrive at the trisection of the Fermi curve $F(u)/\Gamma^*$ stated in [13, Corollary 4.3.9] and proven in [19, Theorem 2.35]:

- The *compact part* of $F(u)/\Gamma^*$ which is contained in a compact subset $K \subset \mathbb{C}^2$
- The *asymptotic free* part of $F(u)/\Gamma^*$ which is contained in $\mathbb{C}^2 \setminus K$ and where the excluded domains are cut off (this part of $F(u)/\Gamma^*$ corresponds to the *regular pieces* introduced in [5, II.5, (GH1)], at least to those part of the regular pieces lying in $\mathbb{C}^2 \setminus K$). Moreover, Fermi curves have two regular pieces. They can be considered as two complex planes (with corresponding domains cut off).
- The *handles* which connect the two regular pieces of the asymptotic free part to each other, i.e. which connect the excluded domain around $k_\kappa^-(\hat{u}_0)$ with the excluded domain around $k_\kappa^+(\hat{u}_0)$, $\kappa \in \Gamma_\delta^*$.

2.2 Important results for the asymptotic analysis

In many points of view, this work is a sequel of [13]. In this section, we recap some important results, often taken from [13] and partially reformulated, which are needed for our further considerations. In [13], the theorems were more generally stated for $\mathcal{F}l^{\infty,1}(\Gamma^*)$-potentials. However, many (but not all) of the results carry over to $u \in \mathcal{F}l^2(\Gamma^*)$, since $l^2(\Gamma^*) \subset l^{\infty,1}(\Gamma^*)$ (cf. [13, Proposition 2.4.3]). Although we won't use so-called *Lorentz spaces* like $l^{\infty,1}(\Gamma^*)$ in this work, some tools for their definition (the so-called *decreasing rearrangement*, for instance) will appear in some proofs of this work anyway, namely in those proofs taken from [13] which had to be modified to fit into the setting of l^2-sequences. That is why we give the definition of $l^{\infty,1}(\Gamma^*)$ (cf. [13, p. 26] and [13, Definition 2.4.1]). We also give the definition of the Lorentz space $l^{1,\infty}(\Gamma^*)$ since it will appear in Lemma 2.2.3 .

Definition 2.2.1. Let $a := (a_\nu)_{\nu\in\Gamma^*} \in l^\infty(\Gamma^*)$ be a bounded sequence of complex numbers. Then we define the *distribution function* d_a of a by

$$d_a : (0,\infty) \to \mathbb{N}_0 \cup \{\infty\}, \quad \lambda \mapsto \#\{\nu \in \Gamma^* : |a_\nu| > \lambda\}.$$

The *decreasing rearrangement* $a^* := (a_n^*)_{n\in\mathbb{N}}$ of a is defined as

$$a^* : \mathbb{N} \to [0,\infty), \quad n \mapsto a_n^* := \inf\{\lambda > 0 : d_a(\lambda) \le n-1\}.$$

The *Lorentz space* $l^{\infty,1}(\Gamma^*)$ is defined as the set of all sequences $a \in l^\infty(\Gamma^*)$ such that

$$\|a\|_{\infty,1} := \sum_{n=1}^{\infty} \frac{a_n^*}{n} < \infty.$$

The *Lorentz space* $l^{1,\infty}(\Gamma^*)$ is defined as the set of all sequences $a \in l^\infty(\Gamma^*)$ such that

$$\|a\|_{1,\infty} := \sup_{n\in\mathbb{N}}(n \cdot a_n^*) < \infty.$$

Remark. We can depict the decreasing rearrangement as follows (as has already been mentioned in [13, p. 27]): If $\lim_{n\to\infty} a_n^* = 0$, then a^* enumerates all values of $(|a_\nu|)_{\nu\in\Gamma^*}$ counting multiplicity (except possibly zero) in decreasing order.

We want to give a local description of a given Fermi curve $F(u)$, restricted to the excluded domains, in terms of the zero set of a holomorphic function $f := \det M : U \subset \mathbb{C}^2 \to \mathbb{C}$ (with some matrix-valued function $M : U \subset \mathbb{C}^2 \to \mathbb{C}^{2\times 2}$) as already seen in (1.6). We start with a definition, cf. [13, Definitions 4.5.1 and 4.5.18].

Definition 2.2.2. For all $k \in \mathbb{C}^2$, all $u \in L^2(F)$ and all $\nu \in \Gamma_\delta^*$ such that the operator

$$\mathbb{1} - (\mathbb{1} - \pi_{K_{\pm\nu}})(\Delta_{k+k_\nu^\pm(\hat{u}_0)} - 4\pi^2\hat{u}_0)^{-1}\bar{u} \tag{2.4}$$

exists and is boundedly invertible on $\mathcal{F}l^1(\Gamma^*)$, let $\mathcal{A}_{\pm,\nu}(k+k_\nu^\pm(\hat{u}_0), u)$, the so-called *perturbation matrix*, be the restriction of the operator

$$\bar{u}(\mathbb{1} - (\mathbb{1} - \pi_{K_{\pm\nu}})(\Delta_{k+k_\nu^\pm(\hat{u}_0)} - 4\pi^2\hat{u}_0)^{-1}\bar{u})^{-1} \tag{2.5}$$

to $K_{\pm\nu}$. Here, $\bar{u}$ denotes the non-constant part of u (compare the representation (2.1)), $K_{\pm\nu} := span\{\psi_0, \psi_{\pm\nu}\}$ denotes the generally (for $\nu \neq 0$) two-dimensional complex vector space generated by the Fourier modes $\psi_0 \equiv 1$ and $\psi_{\pm\nu}$ (compare (1.8)) and

$$\pi_{K_\nu} : E \to K_\nu, \quad f \mapsto \hat{f}(0)\psi_0 + \hat{f}(\nu)\psi_\nu$$

denotes the projection onto K_ν, where E denotes an arbitrary complex Banach space which contains K_ν as a closed subspace.

Remark. In the proof of [13, Proposition 4.5.15], it has been shown that the operator (2.4) is indeed boundedly invertible on $\mathcal{F}l^1(\Gamma^*)$ as required, provided that $\delta > 0$ is sufficiently small (as to the dependence on δ, note that $\nu \in \Gamma_\delta^*$ appears in the operator (2.4)).

We briefly want to comment on this definition. The perturbation matrix $\mathcal{A}_\pm(k + k_\nu^\pm(\hat{u}_0), u)$ is an operator mapping a subspace of $\mathcal{F}l^1(\Gamma^*)$ into a subspace of $\mathcal{F}l^2(\Gamma^*)$. This is due to the fact that (2.4) is boundedly invertible and $\hat{u} \in \mathcal{F}l^2(\Gamma^*)$. More precisely, the inverse of (2.4) exists and maps a subspace of $\mathcal{F}l^1(\Gamma^*)$ into $\mathcal{F}l^1(\Gamma^*)$. Multiplying with $-\bar{u} \in \mathcal{F}l^2(\Gamma^*)$ (this yields the operator (2.5)), we get as result a function in $\mathcal{F}l^2(\Gamma^*)$ because

$$\widehat{u \cdot f} = \hat{u} * \hat{f} \in l^2(\Gamma^*) * l^1(\Gamma^*) \subseteq l^2(\Gamma^*)$$

for $f \in \mathcal{F}l^1(\Gamma^*)$ due to Young's inequality for convolutions (cf. [2, p. 199, Theorem 4.2.4], for instance).

However, since the perturbation operator $\mathcal{A}_\pm(k + k_\nu^\pm(\hat{u}_0), u)$ is, by definition, restricted to $K_{\pm\nu}$, it can be considered as a 2×2- matrix (with entries depending on u and k) because $K_{\pm\nu}$ is two-dimensional (we don't consider the case $\nu = 0$ unless explicitly stated). Let's explain some indices and arguments in the term

$$\mathcal{A}_{\pm,\nu}(k + k_\nu^\pm(\hat{u}_0), u) := \pi_{K_{\pm\nu}}[\bar{u}(\mathbb{1} - (\mathbb{1} - \pi_{K_{\pm\nu}})(\Delta_{k+k_\nu^\pm(\hat{u}_0)} - 4\pi^2\hat{u}_0)^{-1}\bar{u})^{-1}]|_{K_{\pm\nu}}. \tag{2.6}$$

The subscript $\pm$ in $\mathcal{A}_{\pm,\nu}$ refers to $\pm\nu$ in $\pi_{K_{\pm\nu}}$, the argument $k + k_\nu^\pm(\hat{u}_0)$ in $\mathcal{A}_{\pm,\nu}$ refers to the corresponding subscript in the Laplacian $\Delta_{k+k_\nu^\pm(\hat{u}_0)}$. Strictly speaking, the subscript ν in $\mathcal{A}_{\pm,\nu}$ is redundant since the dependence on ν is already indicated in the argument $k + k_\nu^\pm(\hat{u}_0)$. In other words, both the subscripts ν in $\mathcal{A}_{\pm,\nu}$ and in $k_\nu^\pm(\hat{u}_0)$ always denote the same ν. Yet, we write $\mathcal{A}_{\pm,\nu}$ instead of $\mathcal{A}_\pm$ because there are, besides $k_\nu^\pm(\hat{u}_0)$, also other terms in the perturbation matrix depending on ν. The rest of the notation should be clear. We emphasize this because these indices have an effect on signs of certain ν appearing in (2.6). When we will later consider transformation properties of the perturbation matrix, the knowledge of how to read this notation will get important. We introduce the following abbreviations (as in [13], equation (4.5.22)) that will turn out to be handy in the subsequent considerations:

$$\begin{aligned} &A := \bar{u} \\ &B := (\mathbb{1} - \pi_{K_{\pm\nu}})(\Delta_{k+k_\nu^\pm(\hat{u}_0)} - 4\pi^2\hat{u}_0)^{-1} = (\Delta_{k+k_\nu^\pm(\hat{u}_0)} - 4\pi^2\hat{u}_0)^{-1}(\mathbb{1} - \pi_{K_{\pm\nu}}) \\ &\mathcal{A}_{\pm,\nu}(k + k_\nu^\pm(\hat{u}_0), u) =: \pi_{K_{\pm\nu}} A(\mathbb{1} - BA)^{-1}|_{K_{\pm\nu}}. \end{aligned} \tag{2.7}$$

The so-called *reduced resolvent* B defined above will be an important object in our considerations. The following lemma (cf. [13, Lemma 4.5.9]) deals with its Fourier transform.

Lemma 2.2.3. *There is an open neighbourhood V of $0 \in \mathbb{C}^2$ which depends only on Γ^* such that*

$$\lim_{|\nu| \to \infty} \inf_{\substack{k \in V \\ \rho \in \Gamma^* \setminus \{0, \pm\nu\}}} |(\rho + k + k_\nu^\pm(\hat{u}_0))^2 + \hat{u}_0| = \infty$$

and there is a $\delta > 0$ such that for $k \in V$

$$g(k,\cdot,\cdot) : (\nu,\rho) \mapsto \begin{cases} \frac{1}{(\rho+k+k_\nu^\pm(\hat{u}_0))^2+\hat{u}_0}, & \rho \neq 0, \pm\nu \\ 0 & \textit{otherwise} \end{cases} \tag{2.8}$$

is in $c^0(\Gamma^*_\delta) \otimes l^{1,\infty}(\Gamma^*)$, that is, $g(k,\cdot,\rho) \in c^0(\Gamma^*_\delta)$ with respect to ν and $g(k,\nu,\cdot) \in l^{1,\infty}(\Gamma^*)$ with respect to ρ. Here, $c^0(\Gamma^*)$ denotes the subspace of $l^\infty(\Gamma^*)$ of all sequences converging to zero.

We briefly comment on the just defined map (2.8). As mentioned above, the sequence (2.8) with respect to ρ (for fixed ν) is virtually the Fourier transform of the reduced resolvent

$$B := (\mathbb{1} - \pi_{K_{\pm\nu}})(\Delta_{k+k_\nu^\pm(\hat{u}_0)} - 4\pi^2\hat{u}_0)^{-1},$$

introduced in (2.7), more precisely, for $f \in \mathcal{F}l^2(\Gamma^*)$,

$$\widehat{Bf}(\rho) = -\frac{1}{4\pi^2}\frac{\hat{f}(\rho)}{(\rho+k+k_\nu^\pm(\hat{u}_0))^2+\hat{u}_0}, \quad \rho \in \Gamma^* \setminus \{0, \pm\nu\}. \tag{2.9}$$

In particular, B maps $\mathcal{F}l^2(\Gamma^*)$ into $\mathcal{F}l^1(\Gamma^*)$ due to Hölder's inequality (note that $l^{1,\infty}(\Gamma^*) \subseteq l^2(\Gamma^*)$, cf. [13, Proposition 2.4.3]). The following lemma provides an important estimate of $\|g(k,\nu,\cdot)\|_{l^2(\Gamma^*)}$.

Lemma 2.2.4. *The function g (2.8) satisfies*

$$\|g(k,\nu,\cdot)\|_{l^2(\Gamma^*)} = O\left(1/\sqrt{|\nu|}\right), \quad \textit{as } |\nu| \to \infty$$

uniformly for all $k \in V$ (with V as in Lemma 2.2.3). Moreover, there holds

$$\sup_{\rho \in \Gamma^* \setminus \{0,\pm\nu\}} |g(k,\nu,\rho)| = O\left(\frac{1}{|\nu|}\right), \quad \textit{as } |\nu| \to \infty,$$

uniformly in $k \in V$.

Remark. A reader of the work [13] may possibly ask why we don't use the estimate $g^*(k,\nu,n) \leq \frac{c}{n+|\nu|^2}$ (with some $c > 0$) which is stated in [13, Definition 4.5.37] because with this estimate, one could conclude the desired estimate $\|g(k,\nu,\cdot)\|_{l^2(\Gamma^*)} = O\left(1/\sqrt{|\nu|}\right)$, as $|\nu| \to \infty$, considerably faster than we'll do in the following proof. The answer is that firstly, this estimate in [13] is claimed without any proof and that secondly, there is evidence that this estimate is even wrong. One could at most expect an estimate like $g^*(k,\nu,n) \leq \frac{c}{n+|\nu|}$. But of course, also this weaker estimate would need to be proved. The failure of the estimate in [13] is, by the way, the crucial point why essential subsequent proofs in

[13] won't hold for $\mathcal{F}l^{\infty,1}$-potentials anymore and we thus decided to consider L^2-potentials instead of (the larger space of) $\mathcal{F}l^{\infty,1}$-potentials. Although the proof of [13, Lemma 4.5.9] (stated in this work as Lemma 2.2.3 above) already uses this wrong estimate[1] in order to prove the $c^0(\Gamma^*_\delta)$-assertion in Lemma 2.2.3, the proof of [13, Lemma 4.5.9] still holds since the estimate of Lemma 2.2.4 implies the $c^0(\Gamma^*_\delta)$-claim. Of course, this is not a circular argument since in the following proof of Lemma 2.2.4, we will only use the definition of g (2.8), not the statement of Lemma 2.2.3.

Proof. Let $\hat{\kappa}$ and $\check{\kappa}$ be two generators of Γ^*. We set $|\kappa| := \min\{|\hat{\kappa}|, |\check{\kappa}|\}$ which can be considered as a lattice constant.
Since the domain V only dependent on Γ^* is bounded, we may without loss of generality set $k = 0$ in this proof. In other words, the parameter $k \in V$ doesn't affect the decreasing behaviour of $\|g(k, \nu, \cdot)\|_{l^2(\Gamma^*)}$ with respect to $|\nu|$. Moreover, for simplicity, we only consider the signature $k^+_\nu(\hat{u}_0)$ (the other signature $k^-_\nu(\hat{u}_0)$ is treated completely analogously). With the notation $\nu^\perp := \binom{\nu_2}{-\nu_1}$ for $\nu \in \Gamma^*$, we compute with $k^\pm_\nu(\hat{u}_0)^2 + \hat{u}_0 = 0$

$$\begin{aligned} g(0, \nu, \rho) &= \frac{1}{(\rho + k^+_\nu(\hat{u}_0))^2 + \hat{u}_0} = \frac{1}{\rho^2 + 2\langle \rho, k^+_\nu(\hat{u}_0)\rangle} = \frac{1}{\rho^2 + \langle \rho, -\nu + i\xi\nu^\perp\rangle} = \\ &= \frac{1}{(\rho - \frac{\nu}{2})^2 - \frac{|\nu|^2}{4} + i\xi\langle \rho, \nu^\perp\rangle} \end{aligned} \tag{2.10}$$

for all $\nu, \rho \in \Gamma^*$ with $\rho \neq 0, \nu$. Without loss of generality[2], we may assume $\hat{u}_0 \in \mathbb{R}$. Hence, there is an $N \in \mathbb{N}$ such that $\xi \in \mathbb{R}$ for all $|\nu| \geq N$. Therefore,

$$\|g(0, \nu, \cdot)\|^2_{l^2(\Gamma^*)} = \sum_{\rho \in \Gamma^* \setminus \{0, \nu\}} \frac{1}{\left((\rho - \frac{\nu}{2})^2 - \frac{|\nu|^2}{4}\right)^2 + \xi^2 \langle \rho, \nu^\perp\rangle^2} \tag{2.11}$$

for all $|\nu| \geq N$. For the rest of the proof, all appearing ν shall fulfill $|\nu| \geq N$ (although it won't always be explicitly mentioned). We will estimate the series (2.11) in two steps. In the first step, we use an estimate by setting the second summand in the denominator in (2.11) equal to zero. In the second step, we use an estimate by setting the first summand in the denominator in (2.11) equal to zero. In both steps, we estimate the appearing series by corresponding integrals. This is admissible due to monotonicity properties of the functions f_1, f_2 defined in the following in their respective domains of consideration.

[1] This estimate actually appears in the proof of [13, Lemma 4.3.3] which the proof of [13, Lemma 4.5.9] refers to.

[2] Since $\xi \to 1$ as $|\nu| \to \infty$, cf. the definition of ξ after (2.2), a possible imaginary part of ξ can be neglected since it doesn't disturb our estimates provided $|\nu| \geq N$ for $N \in \mathbb{N}$ sufficiently large.

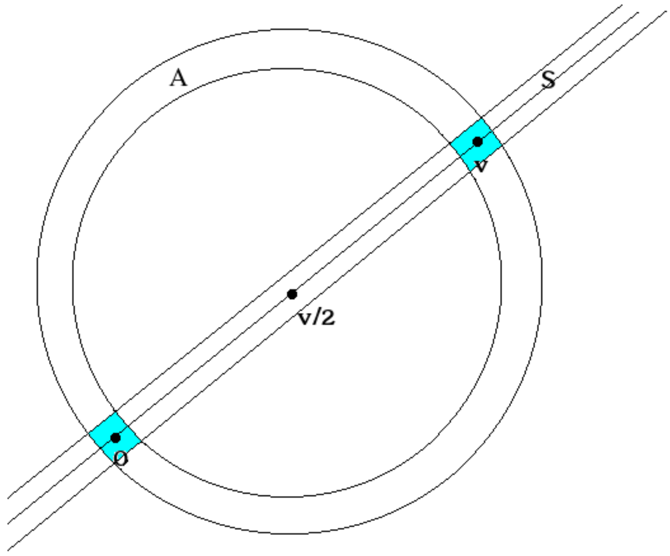

Figure 2.1: Concerning the estimate of $\|g(0,\nu,\cdot)\|_{l^2(\Gamma^*)}$

Let's begin with the first step. We consider the function

$$f_1 : \mathbb{R}^2 \setminus A \to \mathbb{R}, \quad (x,y) \mapsto \left(\left((x,y) - \frac{\nu}{2} \right)^2 - \frac{|\nu|^2}{4} \right)^{-2},$$

where $A \subset \mathbb{R}^2$ is the annulus centered at $\nu/2$ with inner radius $|\nu|/2 - |\kappa|$ and outer radius $|\nu|/2 + |\kappa|$. This annulus is depicted in Figure 2.1. Obviously, the level sets of f_1 are concentric circles with center $\nu/2$. The circle with center $\nu/2$ and radius $|\nu|/2$ is the set of singularities of f_1. Hence, f_1 is well-defined in its domain of definition where A is cut out. We compute the integral $\int_{\mathbb{R}^2 \setminus A} f_1(x,y) d(x,y)$ by using polar coordinates and a coordinate shift $(x,y) \mapsto (x,y) + \nu/2$:

$$\int_{\mathbb{R}^2 \setminus A} f_1(x,y) d(x,y) = 2\pi \int_0^{\frac{|\nu|}{2} - |\kappa|} \frac{r}{\left(r^2 - \frac{|\nu|^2}{4} \right)^2} dr + 2\pi \int_{\frac{|\nu|}{2} + |\kappa|}^{\infty} \frac{r}{\left(r^2 - \frac{|\nu|^2}{4} \right)^2} dr.$$

By the decomposition

$$\frac{r}{\left(r^2 - \frac{|\nu|^2}{4} \right)^2} = \frac{1}{2|\nu|} \left(\frac{1}{\left(r - \frac{|\nu|}{2} \right)^2} - \frac{1}{\left(r + \frac{|\nu|}{2} \right)^2} \right),$$

we get

$$\int \frac{r}{\left(r^2 - \frac{|\nu|^2}{4} \right)^2} dr = \frac{1}{2|\nu|} \left(-\frac{1}{r - \frac{|\nu|}{2}} + \frac{1}{r + \frac{|\nu|}{2}} \right) + const.$$

This yields

$$\int_0^{\frac{|\nu|}{2}-|\kappa|} \frac{r}{\left(r^2-\frac{|\nu|^2}{4}\right)^2}dr = \frac{1}{2|\nu|}\left(\frac{1}{|\kappa|}+\frac{1}{|\nu|-|\kappa|}-\frac{4}{|\nu|}\right),$$

$$\int_{\frac{|\nu|}{2}+|\kappa|}^{\infty} \frac{r}{\left(r^2-\frac{|\nu|^2}{4}\right)^2}dr = \frac{1}{2|\nu|}\left(\frac{1}{|\kappa|}-\frac{1}{|\nu|+|\kappa|}\right)$$

and hence,

$$\int_{\mathbb{R}^2\setminus A} f_1(x,y)d(x,y) = O\left(\frac{1}{|\nu|}\right), \quad \text{as } |\nu|\to\infty. \tag{2.12}$$

For the second step, we consider the function

$$f_2 : \mathbb{R}^2\setminus S \to \mathbb{R}, \quad (x,y)\mapsto \left\langle \begin{pmatrix} x\\ y\end{pmatrix}, \begin{pmatrix} \nu_2\\ -\nu_1\end{pmatrix}\right\rangle^{-2} = (\nu_2 x-\nu_1 y)^{-2},$$

where the strip $S\subset\mathbb{R}^2$ is defined as $S := \left\{t\frac{\nu}{|\nu|}+s\frac{\nu^\perp}{|\nu|} : t\in\mathbb{R}, s\in[-|\kappa|,|\kappa|]\right\}$. Since the set of all $(x,y)\in\mathbb{R}^2$ fulfilling $\nu_2 x-\nu_1 y=0$ is the line $\{t\nu : t\in\mathbb{R}\}$ through 0 and ν, the map f_2 is well-defined on $\mathbb{R}^2\setminus S$. The strip S is also depicted in Figure 2.1. Let $\Omega := \left\{t\frac{\nu}{|\nu|}+s\frac{\nu^\perp}{|\nu|} : t\in\left[-\frac{|\nu|}{2}-|\kappa|, \frac{|\nu|}{2}+|\kappa|\right], s\in\mathbb{R}\setminus[-|\kappa|,|\kappa|]\right\}$. Consider the linear map $\phi:\mathbb{R}^2\to\mathbb{R}^2$ uniquely defined by $\phi(1,0) := \nu/|\nu|$ and $\phi(0,1) := \nu^\perp/|\nu|$. Its functional determinant is equal to $\det\phi' = (-\nu_1^2-\nu_2^2)/|\nu|^2 = -1$. We thus get

$$\begin{aligned}
&\int_\Omega f_2(x,y)d(x,y) = \int_{\phi^{-1}(\Omega)} f_2(\phi(x,y))|\det\phi'|d(x,y) = \\
&= 2\int_{-\frac{|\nu|}{2}-|\kappa|}^{\frac{|\nu|}{2}+|\kappa|}\int_{|\kappa|}^{\infty} \frac{1}{\langle\phi(x,y),\nu^\perp\rangle^2}dy\,dx = \\
&= 2\int_{-\frac{|\nu|}{2}-|\kappa|}^{\frac{|\nu|}{2}+|\kappa|}\int_{|\kappa|}^{\infty} \frac{|\nu|^2}{\langle x\nu+y\nu^\perp,\nu^\perp\rangle^2}dy\,dx = 2\int_{-\frac{|\nu|}{2}-|\kappa|}^{\frac{|\nu|}{2}+|\kappa|}\int_{|\kappa|}^{\infty} \frac{1}{|\nu|^2 y^2}dy\,dx = \\
&= \frac{2}{|\nu|^2}\int_{-\frac{|\nu|}{2}-|\kappa|}^{\frac{|\nu|}{2}+|\kappa|} \frac{1}{|\kappa|}dx = \frac{2}{|\nu|^2|\kappa|}(|\nu|+2|\kappa|) = O\left(\frac{1}{|\nu|}\right), \quad \text{as } |\nu|\to\infty.
\end{aligned} \tag{2.13}$$

We now go back to (2.11). We estimate

$$\begin{aligned}
&\sum_{\rho\in\Gamma^*\setminus\{0,\nu\}} \frac{1}{\left((\rho-\frac{\nu}{2})^2-\frac{|\nu|^2}{4}\right)^2+\xi^2\langle\rho,\nu^\perp\rangle^2} \le \\
&\le \sum_{\rho\in(\mathbb{R}^2\setminus A)\cap\Gamma^*} \frac{1}{\left((\rho-\frac{\nu}{2})^2-\frac{|\nu|^2}{4}\right)^2} + \frac{1}{\xi^2}\sum_{\rho\in\Omega\cap\Gamma^*}\frac{1}{\langle\rho,\nu^\perp\rangle^2} \overset{(2.12),(2.13)}{=} O\left(\frac{1}{|\nu|}\right),
\end{aligned}$$

as $|\nu| \to \infty$. This shows $\|g(0,\nu,\cdot)\|^2_{l^2(\Gamma^*)} = O(1/|\nu|)$, as $|\nu| \to \infty$ and the first claim of the lemma is proved.
As to the second claim concerning the estimate of $\sup_{\rho\in\Gamma^*\setminus\{0,\pm\nu\}} |g(k,\nu,\rho)|$, we may again consider without loss of generality the case $k = 0$ with the same reason as before. The idea of estimating this term is essentially the same as before with the only difference that we don't estimate integrals of squares of f_1, f_2 but this time the supremum of f_1, f_2. More precisely, we get with $\kappa := |\kappa|\frac{\nu}{|\nu|}$

$$\sup_{\rho\in(\mathbb{R}^2\setminus A)\cap\Gamma^*} \frac{1}{\left|(\rho-\frac{\nu}{2})^2 - \frac{|\nu|^2}{4}\right|} = O\left(\frac{1}{\left|(\frac{\nu}{2}\pm\kappa)^2 - \frac{|\nu|^2}{4}\right|}\right) = O\left(\frac{1}{|\nu|}\right),$$

as $|\nu| \to \infty$, since

$$\left|\left(\frac{\nu}{2}\pm\kappa\right)^2 - \frac{|\nu|^2}{4}\right| = \left|\pm|\kappa||\nu| + |\kappa|^2\right| \geq |\nu|\left(|\kappa| - \frac{|\kappa|^2}{|\nu|}\right).$$

Likewise (compare Figure 2.1),

$$\sup_{\rho\in\Omega\cap\Gamma^*} \frac{1}{|\langle\rho,\nu^\perp\rangle|} \leq \sup\left\{\frac{1}{|\langle x\nu + y\nu^\perp, \nu^\perp\rangle|} : (x,y)\in\mathbb{R}^2, |y| \geq \frac{|\kappa|}{|\nu|}\right\} =$$
$$= \sup_{|y|\geq\frac{|\kappa|}{|\nu|}} \frac{1}{|y||\nu|^2} = \frac{1}{|\kappa||\nu|} = O\left(\frac{1}{|\nu|}\right),$$

as $|\nu| \to \infty$. Altogether with (2.10), we get

$$\sup_{\rho\in\Gamma^*\setminus\{0,\pm\nu\}} |g(0,\nu,\rho)| = \sup_{\rho\in\Gamma^*\setminus\{0,\pm\nu\}} \frac{1}{\sqrt{\left|(\rho-\frac{\nu}{2})^2 - \frac{|\nu|^2}{4}\right|^2 + \xi^2|\langle\rho,\nu^\perp\rangle|^2}} \leq$$
$$\leq \sup_{\rho\in(\mathbb{R}^2\setminus A)\cap\Gamma^*} \frac{1}{\left|(\rho-\frac{\nu}{2})^2 - \frac{|\nu|^2}{4}\right|} + \frac{1}{\xi}\sup_{\rho\in\Omega\cap\Gamma^*}\frac{1}{|\langle\rho,\nu^\perp\rangle|} = O\left(\frac{1}{|\nu|}\right),$$

as $|\nu| \to \infty$. This proves the second claim of the lemma. □

As already mentioned, [13] deals with $\mathcal{F}l^{\infty,1}$-potentials. In this context, one uses so-called *localised quasi-norms*. Since we are interested in $L^2(F)$-potentials, we can use the ordinary L^2-norm (or the l^2-norm in the Fourier space, respectively). In the following, we will recap some assertions of [13] (namely the subsequent Theorem 2.2.6, Lemma 2.2.7 and Theorem 2.2.8) that can be translated into the L^2-potential case without any difficulties by adapting the formulations correspondently. For this purpose, we have to prove the following lemma (the analogon of [13, Lemma 4.5.13]) which is the crucial statement for the L^2-reformulation of the subsequent assertions.

Lemma 2.2.5. *There is an open neighbourhood V of $0 \in \mathbb{C}^2$ which depends only on Γ^* such that for all $\epsilon > 0$ and all $u_0 \in L^2(F)$, there is a $\delta > 0$ and an $R > 0$ such that for all $k \in V$, all $\nu \in \Gamma^*_\delta$ and all $u \in B_R(u_0) \subset L^2(F)$, there holds*

$$\|(\mathbb{1} - \pi_{K_{\pm\nu}})(\Delta_{k+k_\nu^\pm(\hat{u}_0)} - 4\pi^2\hat{u}_0)^{-1}\bar{u}\|_{\mathcal{F}l^1(\Gamma^*)\to\mathcal{F}l^1(\Gamma^*)} \overset{(2.7)}{=} \|BA\|_{\mathcal{F}l^1(\Gamma^*)\to\mathcal{F}l^1(\Gamma^*)} < \epsilon.$$

Proof. Let $\epsilon > 0$ and $f \in \mathcal{F}l^1(\Gamma^*)$. Then due to Young's inequality for convolutions and the convolution theorem $\|\bar{u}f\|_{\mathcal{F}l^2(\Gamma^*)} = \|\hat{\bar{u}} * \hat{f}\|_{l^2(\Gamma^*)}$ (cf. [2, p. 199, Theorem 4.2.4] or [13, Theorem 2.1.10], for example), we obtain

$$\|BAf\|_{\mathcal{F}l^1(\Gamma^*)} \le \|B\|_{\mathcal{F}l^2(\Gamma^*)\to\mathcal{F}l^1(\Gamma^*)} \cdot \|\hat{\bar{u}}\|_{l^2(\Gamma^*)} \cdot \|\hat{f}\|_{l^1(\Gamma^*)}$$

We have (by using the suggestive notation $l^q(\rho)$ which shall signify $l^q(\Gamma^*)$ with respect to ρ)

$$\begin{aligned}\|B\|_{\mathcal{F}l^2(\Gamma^*)\to\mathcal{F}l^1(\Gamma^*)} &= \sup_{\|f\|_{\mathcal{F}l^2}=1} \|Bf\|_{\mathcal{F}l^1} = \sup_{\|f\|_{\mathcal{F}l^2}=1} \|(g(k,\nu,\rho)\cdot\hat{f}(\rho))_\rho\|_{l^1(\rho)} \le \\ &\le \|g(k,\nu,\rho)\|_{l^2(\rho)}\end{aligned}$$

due to Hölder's inequality. Lemma 2.2.4 yields $\|g(k,\nu,\rho)\|_{l^2(\rho)} \to 0$, as $|\nu| \to \infty$. For $u \in B_R(u_0) \subset L^2(F)$, there are suitable $0 \le r < R$, $\hat{h} \in B_1(0) \subset l^2(\Gamma^*)$ with $\hat{u} = \hat{u}_0 + r\hat{h}$ and

$$\|\hat{u}\|_{l^2(\Gamma^*)} = \|\hat{u}_0 + r\hat{h}\|_{l^2(\Gamma^*)} \le \|\hat{u}_0\|_{l^2(\Gamma^*)} + R,$$

where we made use of Parceval's identity $\|u\|_{L^2(F)} = \|\hat{u}\|_{l^2(\Gamma^*)}$. Choose for example $R = 1$ and $\delta > 0$ such that with $\|\hat{u}_0\|_{l^2(\Gamma^*)} + 1 =: C$

$$\|g(k,\nu,\rho)\|_{l^2(\rho)} < \frac{\epsilon}{C}$$

for $\nu \in \Gamma^*_\delta$. Thus, $\|BA\|_{\mathcal{F}l^1(\Gamma^*)\to\mathcal{F}l^1(\Gamma^*)} < (\epsilon/C)\cdot C = \epsilon$. This shows the assertion. □

The reason why we call the object (2.6) perturbation matrix will get clear in the following Theorem, cf. [13, Theorem 4.5.19].

Theorem 2.2.6. *There is an open neighbourhood V of $0 \in \mathbb{C}^2$ which depends only on Γ^* such that for all $u_0 \in L^2(F)$, there is a $\delta > 0$ and an $R > 0$ such that for all $k \in V$, all $\nu \in \Gamma^*_\delta$ and all $u \in B_R(u_0) \subset L^2(F)$, the local part of the Fermi curve $F(u) \cap (V + k_\nu^\pm(\hat{u}_0))$ is described by the zero locus of*

$$k \mapsto \det(M_{\pm,\nu}(k + k_\nu^\pm(\hat{u}_0), u)),$$

where $M = M_{\pm,\nu}(k + k_\nu^\pm(\hat{u}_0), u)$ *is defined by* [3]

$$M := \begin{pmatrix} 4\pi^2((k + k_\nu^\pm(\hat{u}_0))^2 + \hat{u}_0) & 0 \\ 0 & 4\pi^2((k + k_\nu^\mp(\hat{u}_0))^2 + \hat{u}_0) \end{pmatrix} + \mathcal{A}_{\pm,\nu}(k + k_\nu^\pm(\hat{u}_0), u). \tag{2.14}$$

For a constant potential $u \equiv const$, we have $\mathcal{A}_{\pm,\nu}(k + k_\nu^\pm(\hat{u}_0), u) \equiv 0$ since $\bar{u} \equiv 0$ in this case and the matrix M is equal to the diagonal matrix in formula (2.14). If we perturb the constant part of the potential by a non-constant term $\bar{u} \neq 0$, we get a perturbation term in (2.14), namely the perturbation matrix (2.6). This explains the name *perturbation matrix*.
An important lemma is the following (cf. [13, Lemma 4.5.21])

Lemma 2.2.7. *There is an open neighbourhood V of $0 \in \mathbb{C}^2$ which depends only on Γ^* such that for all $u_0 \in L^2(F)$, there is a $\delta > 0$ and an $R > 0$ such that for all $k \in V$, all $\nu \in \Gamma^*_\delta$ and all $u \in B_R(u_0) \subset L^2(F)$, the matrix $\mathcal{A}_{\pm,\nu}(k + k_\nu^\pm(\hat{u}_0), u)$ is continuously differentiable in k, and we have*

$$\lim_{|\nu| \to \infty} \left\| \frac{\partial}{\partial k} \mathcal{A}_{\pm,\nu}(k + k_\nu^\pm(\hat{u}_0), u) \right\| = 0$$

uniformly in $k \in V$ and $u \in B_R(u_0)$. Here, $\|\cdot\|$ denotes the matrix norm induced by the standard hermitian vector norm in $\mathbb{C}^2$.

Next, we want to introduce the so-called *perturbed Fourier coefficients* which will serve as some kind of asymptotic coordinates for a potential u. First, we need the following result which states that in every excluded domain, there exists a unique point in which the matrix (2.14) is off-diagonal (cf. [13, Proposition 4.5.29]).

Theorem 2.2.8. *There is an open neighbourhood V of $0 \in \mathbb{C}^2$ which depends only on Γ^* such that for all $u_0 \in L^2(F)$, there is a $\delta > 0$ and an $R > 0$ such that for all $\nu \in \Gamma^*_\delta$ and all $u \in B_R(u_0) \subset L^2(F)$, there is a unique $k_{\pm,\nu} \in V$ such that the diagonal entries of the matrix (2.14) get zero at $k = k_{\pm,\nu}$, more precisely:*

$$\begin{aligned}
\mathcal{A}_{+,\nu}(k_{+,\nu} + k_\nu^+(\hat{u}_0), u)_{11} + 4\pi^2((k_{+,\nu} + k_\nu^+(\hat{u}_0))^2 + \hat{u}_0) = 0,\\
\mathcal{A}_{+,\nu}(k_{+,\nu} + k_\nu^+(\hat{u}_0), u)_{22} + 4\pi^2((k_{+,\nu} + k_\nu^-(\hat{u}_0))^2 + \hat{u}_0) = 0,\\
\mathcal{A}_{-,\nu}(k_{-,\nu} + k_\nu^-(\hat{u}_0), u)_{11} + 4\pi^2((k_{-,\nu} + k_\nu^-(\hat{u}_0))^2 + \hat{u}_0) = 0,\\
\mathcal{A}_{-,\nu}(k_{-,\nu} + k_\nu^-(\hat{u}_0), u)_{22} + 4\pi^2((k_{-,\nu} + k_\nu^+(\hat{u}_0))^2 + \hat{u}_0) = 0.
\end{aligned}$$

[3] In [13], there occurred a small sign mistake (as the proof of [13, Theorem 4.5.19] shows): The minus sign in [13, Theorem 4.5.19] in front of the diagonal terms $-4\pi^2((k+k_\nu^\pm(\hat{u}_0))^2+\hat{u}_0)$ must be a plus sign. Equivalently, one could also use the sign as in [13] if instead, one defines the perturbation matrix with a minus sign in front of the right hand side of (2.6). We decided to define the perturbation matrix as in [13] and set plus signs in front of the diagonal terms of the diagonal matrix in M as just explained.

Now, we can define the (preliminary) *perturbed Fourier coefficients* $\check{u}_1(\pm,\nu)^\pm$, $\check{u}_2(\pm,\nu)^\pm$ for $\nu \in \Gamma^*$ by (compare [13, Definition 4.5.31])

$$\begin{aligned}\check{u}_1(\pm,\nu)^\pm &:= \mathcal{A}_{\pm,\nu}(k_{\pm,\nu} + k_\nu^\pm(\hat{u}_0), u)_{12},\\ \check{u}_2(\pm,\nu)^\pm &:= \mathcal{A}_{\pm,\nu}(k_{\pm,\nu} + k_\nu^\pm(\hat{u}_0), u)_{21}.\end{aligned} \tag{2.15}$$

Here, the first argument $\pm$ of $\check{u}_{1/2}(\pm,\nu)^\pm$ refers to the subscript $\pm$ in $k_{\pm,\nu}$ (which also corresponds to the subscript $\pm$ in $\mathcal{A}_{\pm,\nu}$), the superscript $\pm$ of $\check{u}_{1/2}(\pm,\nu)^\pm$ refers to the superscript $\pm$ in $k_\nu^\pm(\hat{u}_0)$.
This is only a preliminary definition (cf. Definition 2.3.3) because in Section 2.3, we will use some transformation properties to get rid of some sub- and superscripts (cf. equation (2.20)) which complicate the notation at the moment.

2.3 Involutions and their transformation properties

In this section, we want to consider involutions of $\mathbb{C}^2$ and the transformation behaviour of Fermi curves and their inherent objects (such as the perturbation matrix) by action of these involutions. The most important involution is the holomorphic involution

$$\sigma : \mathbb{C}^2 \to \mathbb{C}^2, \quad k \mapsto -k.$$

Due to [13, Proposition 4.5.8], σ leaves a Fermi curve $F(u)$ invariant. Moreover, this proposition shows the statement $F(u) = -F(u)$, which implies that Fermi curves for *arbitrary* potential $u \in L^2(F)$ are point-symmetric with respect to the origin $0 \in \mathbb{C}^2$. Since it has been shown in [13, Proposition 4.5.8] that $(-\Delta_k + u)^T = -\Delta_{-k} + u$ (which can be easily verified by direct calculation: $\langle \Delta_k f, g\rangle = \langle f, \Delta_{-k} g\rangle$ for all $f, g \in L^2(F)$, thus $\Delta_k^T = \Delta_{-k}$, with $\langle f, g\rangle := \int_F f(x)g(x)dx$), the involution σ acts by mapping a Schrödinger operator to its transposed one. Note that, speaking of transposed operators, we use the *euclidean* scalar product (extended to complex-valued functions), i.e. $\langle f, g\rangle = \int_F f(x)g(x)dx$ as defined above, not the hermitian form. Therefore, transposition leaves the potential u invariant (although u doesn't need to be real-valued in our present consideration). So, in any case, the potential u, considered as a multiplication operator, is symmetric ($u^T = u$) but in general not self-adjoint (since u needn't be real-valued).
Now, we want to examine how the perturbation matrix, the diagonal-zeroes $k_{\pm,\nu}$ (see Theorem 2.2.8) and the perturbed Fourier coefficients transform by action of σ. As to the perturbation matrix, we have the following theorem (cf. [13, Proposition 4.5.24]).

Theorem 2.3.1. *Let $u \in L^2(F), \nu \in \Gamma_\delta^*$ and let $0 \in V \subseteq \mathbb{C}^2$ with $V = -V$ such that $\mathcal{A}_{\pm,\nu}(k + k_\nu^\pm(\hat{u}_0), u)$ is well-defined for all $k \in V$ in the sense of Definition 2.2.2. Then*

$$\mathcal{A}_{\pm,\nu}(k + k_\nu^\pm(\hat{u}_0), u)^T = \mathcal{A}_{\mp,\nu}(-k - k_\nu^\pm(\hat{u}_0), u), \tag{2.16}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{A}_{\pm,\nu}(k + k_\nu^\pm(\hat{u}_0), u) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathcal{A}_{\mp,\nu}(k + k_\nu^\mp(\hat{u}_0), u). \tag{2.17}$$

The first equation (2.16) indicates the transformation of the perturbation matrix by action of σ, the second equation (2.17) indicates the transformation behaviour by changing the ordered base $(\psi_0, \psi_{\pm\nu})$ to $(\psi_{\pm\nu}, \psi_0)$. As is well known from linear algebra, such a base change has the effect on the matrix that not only the two off-diagonal entries permute (as it is the case in the transposition) but also the two diagonal entries permute.
The following theorem shows how the transformations in Theorem 2.3.1 affect the diagonal-zeroes $k_{\pm,\nu}$ from Theorem 2.2.8 and the perturbed Fourier coefficients.

Theorem 2.3.2. *For the diagonal-zeroes $k_{\pm,\nu}$ of Theorem 2.2.8 and the perturbed Fourier coefficients, defined in (2.15), there holds*

$$k_\nu := k_{+,\nu} = k_{-,\nu}, \tag{2.18}$$

$$k_\nu = -k_{-\nu}, \tag{2.19}$$

$$\check{u}_1(+,\nu)^+ = \check{u}_2(+,-\nu)^+ = \check{u}_1(-,-\nu)^- = \check{u}_2(-,\nu)^- \tag{2.20}$$

for $\nu \in \Gamma_\delta^*$ with suitable $\delta > 0$ depending on the potential as in the conditions of Theorem 2.2.8. If we define $\check{u}_\nu := \check{u}_1(+,\nu)^+$, the matrix M from (2.14) is at k_ν equal to

$$M_{\pm,\nu}(k_\nu + k_\nu^\pm(\hat{u}_0), u) = \begin{pmatrix} 0 & \check{u}_{\pm\nu} \\ \check{u}_{\mp\nu} & 0 \end{pmatrix}.$$

Proof. By definition of $k_{+,\nu}$, we have by Theorem 2.2.8 and Theorem 2.3.1

$$\begin{aligned} 0 &= \mathcal{A}_{+,\nu}(k_{+,\nu} + k_\nu^+(\hat{u}_0), u)_{11} + 4\pi^2((k_{+,\nu} + k_\nu^+(\hat{u}_0))^2 + \hat{u}_0) = \\ &\overset{(2.17)}{=} \mathcal{A}_{-,\nu}(k_{+,\nu} + k_\nu^-(\hat{u}_0), u)_{22} + 4\pi^2((k_{+,\nu} + k_\nu^+(\hat{u}_0))^2 + \hat{u}_0) = \\ (= 0) &= \mathcal{A}_{-,\nu}(k_{-,\nu} + k_\nu^-(\hat{u}_0), u)_{22} + 4\pi^2((k_{-,\nu} + k_\nu^+(\hat{u}_0))^2 + \hat{u}_0), \end{aligned} \tag{2.21}$$

where in the last step, we applied once again Theorem 2.2.8, this time considering the lower signature for the second diagonal entry[4]. We get equality between the second and the third line because both terms are equal to zero. Similarly, we get,

[4]Remember the different signature $k_\nu^\mp(\hat{u}_0)$ instead of $k_\nu^\pm(\hat{u}_0)$ in the corresponding second diagonal entry of (2.14)

by using exactly the same arguments (i.e. at first upper signature with equation (2.17) and then lower signature), this time applied to the other diagonal entry:

$$\begin{aligned} 0 &= \mathcal{A}_{+,\nu}(k_{+,\nu} + k_\nu^+(\hat{u}_0), u)_{22} + 4\pi^2((k_{+,\nu} + k_\nu^-(\hat{u}_0))^2 + \hat{u}_0) = \\ &\overset{(2.17)}{=} \mathcal{A}_{-,\nu}(k_{+,\nu} + k_\nu^-(\hat{u}_0), u)_{11} + 4\pi^2((k_{+,\nu} + k_\nu^-(\hat{u}_0))^2 + \hat{u}_0) = \\ (= 0) &= \mathcal{A}_{-,\nu}(k_{-,\nu} + k_\nu^-(\hat{u}_0), u)_{11} + 4\pi^2((k_{-,\nu} + k_\nu^-(\hat{u}_0))^2 + \hat{u}_0). \end{aligned} \tag{2.22}$$

Due to the equality of the second and the third line of (2.21) and (2.22), respectively, we obtain, by the uniqueness of $k_{\pm,\nu}$ due to Theorem 2.2.8, the claimed identity (2.18).

Now, set $k_\nu := k_{+,\nu}$. Again, by definition of k_ν, we have by Theorem 2.2.8 and Theorem 2.3.1

$$\begin{aligned} 0 &= \mathcal{A}_{+,\nu}(k_\nu + k_\nu^+(\hat{u}_0), u)_{11} + 4\pi^2((k_\nu + k_\nu^+(\hat{u}_0))^2 + \hat{u}_0) = \\ &\overset{(2.16)}{=} \mathcal{A}_{-,\nu}(-k_\nu - k_\nu^+(\hat{u}_0), u)_{11} + 4\pi^2((k_\nu + k_\nu^+(\hat{u}_0))^2 + \hat{u}_0). \end{aligned}$$

Since $(k_\nu + k_\nu^+(\hat{u}_0))^2 = (-(k_\nu + k_\nu^+(\hat{u}_0)))^2 = (-k_\nu + k_{-\nu}^+(\hat{u}_0))^2$ (note that $k_{-\nu}^+(\hat{u}_0) = -k_\nu^+(\hat{u}_0)$ by definition), the above is equal to

$$0 = \mathcal{A}_{-,\nu}(-k_\nu + k_{-\nu}^+(\hat{u}_0), u)_{11} + 4\pi^2((-k_\nu + k_{-\nu}^+(\hat{u}_0))^2 + \hat{u}_0).$$

If we apply Theorem 2.2.8 again, this time considering $-\nu$ instead of ν (but using the upper signature as before), we get

$$0 = \mathcal{A}_{+,-\nu}(k_{-\nu} + k_{-\nu}^+(\hat{u}_0), u)_{11} + 4\pi^2((k_{-\nu} + k_{-\nu}^+(\hat{u}_0))^2 + \hat{u}_0).$$

Thus, we have obtained

$$\begin{aligned} 0 &= \mathcal{A}_{-,\nu}(-k_\nu + k_{-\nu}^+(\hat{u}_0), u)_{11} + 4\pi^2((-k_\nu + k_{-\nu}^+(\hat{u}_0))^2 + \hat{u}_0) = \\ &= \mathcal{A}_{+,-\nu}(k_{-\nu} + k_{-\nu}^+(\hat{u}_0), u)_{11} + 4\pi^2((k_{-\nu} + k_{-\nu}^+(\hat{u}_0))^2 + \hat{u}_0) \end{aligned} \tag{2.23}$$

as a result for the first diagonal entry. Similarly, we get for the second diagonal entry

$$\begin{aligned} 0 &= \mathcal{A}_{-,\nu}(-k_\nu + k_{-\nu}^+(\hat{u}_0), u)_{22} + 4\pi^2((-k_\nu + k_{-\nu}^-(\hat{u}_0))^2 + \hat{u}_0) = \\ &= \mathcal{A}_{+,-\nu}(k_{-\nu} + k_{-\nu}^+(\hat{u}_0), u)_{22} + 4\pi^2((k_{-\nu} + k_{-\nu}^-(\hat{u}_0))^2 + \hat{u}_0). \end{aligned} \tag{2.24}$$

Again by the uniqueness of the k_ν due to Theorem 2.2.8, we get by (2.23) and (2.24) the claimed assertion (2.19) provided that $\mathcal{A}_{-,\nu} = \mathcal{A}_{+,-\nu}$ which remains to be proved. But this follows immediately if we write these terms down explicitly:

$$\mathcal{A}_{-,\nu}(-k_\nu + k_{-\nu}^+(\hat{u}_0), u) = \pi_{K_{-\nu}}[\bar{u}(\mathbb{1} - (\mathbb{1} - \pi_{K_{-\nu}})(\Delta_{-k_\nu + k_{-\nu}^+(\hat{u}_0)} - 4\pi^2\hat{u}_0)^{-1}\bar{u})^{-1}]|_{K_{-\nu}}$$
$$\mathcal{A}_{+,-\nu}(k_{-\nu} + k_{-\nu}^+(\hat{u}_0), u) = \pi_{K_{-\nu}}[\bar{u}(\mathbb{1} - (\mathbb{1} - \pi_{K_{-\nu}})(\Delta_{k_{-\nu} + k_{-\nu}^+(\hat{u}_0)} - 4\pi^2\hat{u}_0)^{-1}\bar{u})^{-1}]|_{K_{-\nu}}$$

Up to the difference that in the first term, $k = -k_\nu$, whereas in the second term $k = k_{-\nu}$, the two terms $\mathcal{A}_{-,\nu}$ and $\mathcal{A}_{+,-\nu}$ are equal as claimed.
Now, we'll show (2.20) (which has already been stated in [13, Def. 4.5.31] with slightly different notation; here, we want to give a full proof). We have

$$\begin{aligned}\check{u}_1(+,\nu)^+ = \mathcal{A}_{+,\nu}(k_\nu + k_\nu^+(\hat{u}_0), u)_{12} &\overset{(2.16)}{=} \mathcal{A}_{-,\nu}(-k_\nu + k_{-\nu}^+(\hat{u}_0), u)_{21} = \\ &\overset{(2.19)}{=} \mathcal{A}_{-,\nu}(k_{-\nu} + k_{-\nu}^+(\hat{u}_0), u)_{21} = \mathcal{A}_{+,-\nu}(k_{-\nu} + k_{-\nu}^+(\hat{u}_0), u)_{21} = \check{u}_2(+,-\nu)^+,\end{aligned}$$

as well as

$$\begin{aligned}\check{u}_2(+,-\nu)^+ &= \mathcal{A}_{+,-\nu}(k_{-\nu} + k_{-\nu}^+(\hat{u}_0), u)_{21} \overset{(2.17)}{=} \mathcal{A}_{-,-\nu}(k_{-\nu} + k_{-\nu}^-(\hat{u}_0), u)_{12} = \\ &= \check{u}_1(-,-\nu)^-\end{aligned}$$

and

$$\begin{aligned}\check{u}_1(-,-\nu)^- &= \mathcal{A}_{-,-\nu}(k_{-\nu} + k_{-\nu}^-(\hat{u}_0), u)_{12} \overset{(2.16)}{=} \mathcal{A}_{+,-\nu}(-k_{-\nu} + k_\nu^-(\hat{u}_0), u)_{21} = \\ &\overset{(2.19)}{=} \mathcal{A}_{-,\nu}(k_\nu + k_\nu^-(\hat{u}_0), u)_{21} = \check{u}_2(-,\nu)^-,\end{aligned}$$

where we made use of $\mathcal{A}_{-,\nu} = \mathcal{A}_{+,-\nu}$ as before. This proves (2.20).
Finally, by setting $\check{u}_\nu := \check{u}_1(+,\nu)^+$, we get by (2.20) $\check{u}_2(+,\nu)^+ = \check{u}_1(+,-\nu)^+ = \check{u}_{-\nu}$. Thus,

$$M_{+,\nu}(k_\nu + k_\nu^+(\hat{u}_0), u) = \begin{pmatrix} 0 & \check{u}_\nu \\ \check{u}_{-\nu} & 0 \end{pmatrix}$$

for the upper signature. Because of $\check{u}_1(-,\nu)^- = \check{u}_1(+,-\nu)^+ = \check{u}_{-\nu}$ and $\check{u}_2(-,\nu)^- = \check{u}_1(+,\nu)^+ = \check{u}_\nu$, we obtain

$$M_{-,\nu}(k_\nu + k_\nu^-(\hat{u}_0), u) = \begin{pmatrix} 0 & \check{u}_{-\nu} \\ \check{u}_\nu & 0 \end{pmatrix}$$

for the lower signature. This proves the theorem. □

Now, we define (again with suitable $\delta > 0$) the sequence of *perturbed Fourier coefficients* as it has already been done in (2.15). But this time, due to (2.20), we can restrict ourselves to the term $\check{u}_1(+,\nu)^+$.

Definition 2.3.3. Let $\delta > 0$ as in Theorem 2.2.8 (depending on the given potential $u \in L^2(F)$). Then we call the sequence $(\check{u}_\nu)_{\nu \in \Gamma_\delta^*}$ defined by

$$\check{u}_\nu := \check{u}_1(+,\nu)^+ := \mathcal{A}_{+,\nu}(k_\nu + k_\nu^+(\hat{u}_0), u)_{12}, \quad \nu \in \Gamma_\delta^*,$$

the sequence of *perturbed Fourier coefficients* of the potential u.

Remark. The name *perturbed Fourier coefficients* will become clear by Theorem 2.4.2 and its proof. There, we will see that the perturbed Fourier coeficients can indeed be considered as a perturbation of the (ordinary) Fourier coefficients $(\hat{u}(\nu))_\nu$. In order to avoid confusions between Fourier coefficients and perturbed Fourier coefficients, we will reserve the notation $\check{u}$ for the sequence of perturbed Fourier coefficients and (as usual) $\hat{u}$ for the sequence of ordinary Fourier coefficients.

Next, we introduce two further *anti-holomorphic* involutions η and τ (where the latter is just the composition of the two others, $\tau := \eta \circ \sigma$) by

$$\eta : \mathbb{C}^2 \to \mathbb{C}^2, \quad k \mapsto \bar{k}$$
$$\tau : \mathbb{C}^2 \to \mathbb{C}^2, \quad k \mapsto -\bar{k}$$

Here, $\bar{k}$ denotes the complex conjugation of k. These anti-holomorphic involutions will be important if we consider *real-valued* potentials $u : F \to \mathbb{R}$. At first, we want to see how the perturbation matrix transforms by action of η and τ. In analogy to (2.16), we now have to consider the hermitian adjunction (i.e transposition and conjugation, which shall be denoted by a $*$, i.e. $A^* := \bar{A}^T$ for a matrix A) instead of transposition.

Theorem 2.3.4. *Let $u \in L^2(F), \nu \in \Gamma^*_\delta$ and let $0 \in V \subseteq \mathbb{C}^2$ such that $\mathcal{A}_{\pm,\nu}(k + k^\pm_\nu(\hat{u}_0), u)$ (as well as its transformations appearing in the following equations* (2.25),(2.26)*) is well-defined for all $k \in V$ in the sense of Definition 2.2.2. Then*

$$\mathcal{A}_{\pm,\nu}(k + k^\pm_\nu(\hat{u}_0), u)^* = \mathcal{A}_{\pm,\nu}(\bar{k} + \overline{k^\pm_\nu}(\hat{u}_0), \bar{u}), \quad (\textit{action by } \eta) \tag{2.25}$$
$$\overline{\mathcal{A}_{\pm,\nu}(k + k^\pm_\nu(\hat{u}_0), u)} = \mathcal{A}_{\mp,\nu}(-\bar{k} - \overline{k^\pm_\nu}(\hat{u}_0), \bar{u}) \quad (\textit{action by } \tau) \tag{2.26}$$

Remark. Here, $\bar{u}$ on the right hand side of (2.25) and (2.26), respectively, denotes the complex conjugation of the potential u, not to be confused with the non-constant part of u which we denoted until now with $\bar{u}$ as well. Below, we will therefore use the definition $A := \bar{u}$ (cf. (2.7)) for the non-constant part of u in order to avoid such confusions.

Proof. We have to see how the individual terms of the perturbation matrix (2.6) behave by action of the involutions. Let's begin with the projector π_{K_ν}. In a first step, we claim that π_{K_ν} is a self-adjoint operator, i.e.

$$\pi^*_{K_\nu} = \pi_{K_\nu}.$$

Let

$$\langle f, g \rangle := \int_F f(x)\overline{g(x)}dx$$

denote the hermitian L^2- scalar product. We compute for (periodic) $f, g \in L^2(F)$ by respecting $\hat{f}(\nu) = \int_F f(x)\psi_{-\nu}(x)dx = \langle f, \psi_\nu \rangle$ and $\langle f, g \rangle = \overline{\langle g, f \rangle}$:

$$\begin{aligned}\langle \pi_{K_\nu} f, g \rangle &= \left\langle \hat{f}(0)\psi_0 + \hat{f}(\nu)\psi_\nu, g \right\rangle = \hat{f}(0)\overline{\hat{g}(0)} + \hat{f}(\nu)\overline{\hat{g}(\nu)} = \\ &= \overline{\hat{g}(0)} \langle f, \psi_0 \rangle + \overline{\hat{g}(\nu)} \langle f, \psi_\nu \rangle = \langle f, \hat{g}(0)\psi_0 + \hat{g}(\nu)\psi_\nu \rangle = \langle f, \pi_{K_\nu} g \rangle .\end{aligned}$$

This shows that π_{K_ν} is self-adjoint. Consider now the Laplacian with boundary condition (1.10) $\Delta_k = \Delta + 4\pi i(k \cdot \nabla) - 4\pi^2 k^2$. The Laplacian Δ is self-adjoint, $\Delta^* = \Delta$, which follows immediately by double integration by parts. Likewise by integration by parts, one gets $(4\pi i(k \cdot \nabla))^* = 4\pi i(\bar{k} \cdot \nabla)$ since

$$\int_F 4\pi i(k \cdot \nabla f)\bar{g} = -\int_F 4\pi i f(k \cdot \nabla \bar{g}) = \int_F f \cdot \overline{4\pi i(\bar{k} \cdot \nabla g)}.$$

Note that the boundary terms in the formula of partial integration vanish due to the periodicity of the appearing functions since we integrate over the fundamental domain F. As to the multiplication operator $-4\pi^2 k^2$, one immediately gets $(-4\pi^2 k^2)^* = -4\pi^2 \bar{k}^2$. To sum up,

$$\Delta_k^* = \Delta + 4\pi i(\bar{k} \cdot \nabla) - 4\pi^2 \bar{k}^2 = \Delta_{\bar{k}}.$$

We use the abbreviations A, B introduced in (2.7). Considering B^*, we get by what we have just shown,

$$B^* = (\mathbb{1} - \pi_{K_{\pm\nu}})(\Delta_{\bar{k}+\overline{k_\nu^\pm}(\hat{u}_0)} - 4\pi^2 \overline{\hat{u}_0})^{-1}.$$

Together with

$$(A(\mathbb{1} - BA)^{-1})^* = ((\mathbb{1} - AB)^{-1}A)^* = A^*(\mathbb{1} - B^*A^*)^{-1} = \bar{A}(\mathbb{1} - B^*\bar{A})^{-1},$$

where we used [13, Lemma 4.5.23] in the first equality, we obtain the first transformation property (2.25). The second identity (2.26) now follows immediately from the first together with (2.16) via $\overline{\mathcal{A}} = (\mathcal{A}^*)^T$. □

Analogously to Theorem 2.3.2, we can now prove how the transformations of Theorem 2.3.4 affect the diagonal-zeroes $(k_\nu)_\nu$ and the perturbed Fourier coefficients:

Theorem 2.3.5. *Let $u \in L^2(F)$ and denote by $v := \bar{u}$ the complex conjugation of u. Let the constant part $\hat{u}_0$ of u (cf. (2.1)) be real. Then for the diagonal-zeroes $(k_\nu(u))_{\nu \in \Gamma_\delta^*}$ and $(k_\nu(v))_{\nu \in \Gamma_\delta^*}$ of Theorem 2.2.8 and the perturbed Fourier coefficients $(\check{u}_\nu)_{\nu \in \Gamma_\delta^*}$ and $(\check{v}_\nu)_{\nu \in \Gamma_\delta^*}$ (cf. Def. 2.3.3), respectively, there holds*

$$-\overline{k_\nu}(u) = k_\nu(v), \qquad \overline{\check{u}_\nu} = \check{v}_{-\nu}$$

for $\nu \in \Gamma_\delta^*$ with suitable $\delta > 0$ depending on the potential as in the conditions of Theorem 2.2.8.

Remark. We require the condition $\hat{u}_0 \in \mathbb{R}$ since then $k_\nu^\pm(\hat{u}_0) = k_\nu^\pm(\hat{v}_0)$ is satisfied. The requirement $\hat{u}_0 \in \mathbb{R}$ is no severe restriction since we will later consider real-valued potentials where $\hat{u}_0 \in \mathbb{R}$ is always fulfilled.

Proof. As in the proof of Theorem 2.3.2, we use Theorem 2.2.8 and the transformation properties of the perturbation matrix induced by the corresponding involutions, this time those of the anti-holomorphic involutions shown in Theorem 2.3.4. Since the constant part $\hat{u}_0$ is real, we have $\hat{u}_0 = \hat{v}_0$ and $k_\nu^\pm(\hat{u}_0) = k_\nu^\pm(\hat{v}_0)$. We thus obtain by definition of $k_\nu(u)$ and $k_\nu(v)$, respectively

$$\begin{aligned} 0 &= \overline{\mathcal{A}_{+,\nu}(k_\nu(u) + k_\nu^+(\hat{u}_0), u)_{11}} + 4\pi^2((\overline{k_\nu}(u) + \overline{k_\nu^+}(\hat{u}_0))^2 + \hat{u}_0) = \\ &\overset{(2.26)}{=} \mathcal{A}_{-,\nu}(-\overline{k_\nu}(u) + k_\nu^-(\hat{v}_0), v)_{11} + 4\pi^2((-\overline{k_\nu}(u) + k_\nu^-(\hat{v}_0))^2 + \hat{v}_0) = \\ (= 0) &\overset{\text{per def.}}{=} \mathcal{A}_{-,\nu}(k_\nu(v) + k_\nu^-(\hat{v}_0), v)_{11} + 4\pi^2((k_\nu(v) + k_\nu^-(\hat{v}_0))^2 + \hat{v}_0), \end{aligned}$$

for the first diagonal entry of (2.14), where we used $-\overline{k_\nu^\pm(\hat{v}_0)} = k_\nu^\mp(\hat{v}_0)$ which holds by definition of $k_\nu^\pm(\hat{u}_0)$ (2.2) for potentials with real constant part $\hat{u}_0$ (note that for potentials with non-real $\hat{u}_0$, this is generally not true). As to the second diagonal entry of (2.14), we get as well

$$\begin{aligned} 0 &= \overline{\mathcal{A}_{+,\nu}(k_\nu(u) + k_\nu^+(\hat{u}_0), u)_{22}} + 4\pi^2((\overline{k_\nu}(u) + \overline{k_\nu^-}(\hat{u}_0))^2 + \hat{u}_0) = \\ &\overset{(2.26)}{=} \mathcal{A}_{-,\nu}(-\overline{k_\nu}(u) + k_\nu^-(\hat{v}_0), v)_{22} + 4\pi^2((-\overline{k_\nu}(u) + k_\nu^+(\hat{v}_0))^2 + \hat{v}_0) = \\ (= 0) &\overset{\text{per def.}}{=} \mathcal{A}_{-,\nu}(k_\nu(v) + k_\nu^-(\hat{v}_0), v)_{22} + 4\pi^2((k_\nu(v) + k_\nu^+(\hat{v}_0))^2 + \hat{v}_0). \end{aligned}$$

Again, by the uniqueness of the k_ν, we obtain the first claim $-\overline{k_\nu}(u) = k_\nu(v)$. This, together with (2.26) implies

$$\begin{aligned} \overline{\check{u}_\nu} = \overline{\check{u}_1(+,\nu)^+} = \overline{\mathcal{A}_{+,\nu}(k_\nu(u) + k_\nu^+(\hat{u}_0), u)_{12}} &\overset{(2.26)}{=} \mathcal{A}_{-,\nu}(-\overline{k_\nu}(u) - \overline{k_\nu^+}(\hat{v}_0), v)_{12} = \\ = \mathcal{A}_{-,\nu}(k_\nu(v) + k_\nu^-(\hat{v}_0), v)_{12} = \check{v}_1(-,\nu)^-. \end{aligned}$$

With (2.20), we obtain the second claim

$$\overline{\check{u}_\nu} = \check{v}_1(-,\nu)^- = \check{v}_1(+,-\nu)^+ = \check{v}_{-\nu}.$$

This proves the theorem. □

The preceding theorem immediately leads to an important assertion concerning the perturbed Fourier coefficients of *real-valued* potentials $u : F \to \mathbb{R}$:

Corollary 2.3.6. *Let $u \in L^2(F)$ be real-valued. For the diagonal-zeroes $(k_\nu)_{\nu \in \Gamma_\delta^*}$ of Theorem 2.2.8 and the perturbed Fourier coefficients $(\check{u}_\nu)_{\nu \in \Gamma_\delta^*}$ (cf. Def. 2.3.3), there holds*

$$-\overline{k_\nu} = k_\nu, \qquad \overline{\check{u}_\nu} = \check{u}_{-\nu}$$

for $\nu \in \Gamma_\delta^*$ with suitable $\delta > 0$ depending on the potential as in the conditions of Theorem 2.2.8.

Proof. Due to reality, we have $v := \bar{u} = u$. The assertion then follows from Theorem 2.3.5. □

2.4 The perturbed Fourier coefficients as coordinates

In this section, we want to prove that the map between (ordinary) Fourier coefficients $\hat{u}(\nu)$ and perturbed Fourier coefficients $\check{u}_\nu$, $\nu \in \Gamma^*_\delta$, defined by

$$l^2(\Gamma^*_\delta) \to l^2(\Gamma^*_\delta), \qquad (\hat{u}(\nu))_{\nu \in \Gamma^*_\delta} \mapsto (\check{u}_\nu)_{\nu \in \Gamma^*_\delta},$$

is for sufficiently small $\delta > 0$ locally boundedly invertible, provided that the first finitely many Fourier coefficients indexed by $\nu \in \Gamma^* \setminus \Gamma^*_\delta$ are kept constant (otherwise the above map wouldn't be well-defined). The purpose of this is that we will later use the perturbed Fourier coefficients to parameterize the potentials (at least asymptotically). The first step in proving that the map between Fourier coefficients and perturbed Fourier coefficients is locally invertible in $l^2(\Gamma^*_\delta)$ will be to show that the sequence of perturbed Fourier coefficients $(\check{u}_\nu)_{\nu \in \Gamma^*_\delta}$ is in $l^2(\Gamma^*_\delta)$, provided that $u \in L^2(F)$ (that is, $\hat{u} \in l^2(\Gamma^*)$). By definition, the perturbed Fourier coefficients are certain entries of the perturbation matrix (2.6) evaluated at $k = k_\nu$. Therefore, we show in the following theorem that the entries of the perturbation matrix are in $l^2(\Gamma^*_\delta)$ (with respect to ν) if $u \in L^2(F)$.

Theorem 2.4.1. *There is an open neighbourhood V of $0 \in \mathbb{C}^2$ which only depends on Γ^* such that for all $u_0 \in L^2(F)$, there is a $\delta > 0$ and an $R > 0$ such that for all $k \in V$ and all $u \in B_R(u_0) \subset L^2(F)$, the entries of the matrix* (2.6) *are in $l^2(\Gamma^*_\delta)$ with respect to ν. Furthermore, $(\check{u}_\nu)_{\nu \in \Gamma^*_\delta} \in l^2(\Gamma^*_\delta)$.*

Remark. This theorem is the analogon to [13, Theorem 4.5.42], where the assertion was claimed for $\mathcal{F}l^{\infty,1}(\Gamma^*)$-potentials. As already mentioned before, the proof in [13] unfortunately uses a wrong estimate for $g^*(k, \nu, n)$ so that Theorem 4.5.42 in [13] is not proved (it might even be wrong). However, not all is lost. We will adopt the main ideas of the proof, transfer them to our proof for $L^2(F)$-potentials and use then different arguments where it is necessary.

Proof. Let $u_0 \in L^2(F)$ and $\delta > 0$, $R > 0$ chosen as in Lemma 2.2.5. Let $u \in B_R(u_0) \subset L^2(F)$. Due to Neumann's Theorem (cf. [30, Satz II.1.11]) and [13, Lemma 4.5.14], the operator

$$\mathbb{1} - (\mathbb{1} - \pi_{K_{\pm\nu}})(\Delta_{k+k_\nu^\pm(\hat{u}_0)} - 4\pi^2 \hat{u}_0)^{-1} \bar{u}$$

is invertible in $\mathcal{F}l^1(\Gamma^*)$ (compare the beginning of the proof of Prop. 4.5.15 in [13], p. 98). Using the abbreviations of (2.7) again, this means that the operator

$\mathbb{1} - BA$ is invertible in $\mathcal{F}l^1(\Gamma^*)$. Thus, $(\mathbb{1} - BA)^{-1}$ maps $\mathcal{F}l^1(\Gamma^*)$ boundedly into itself. For $f \in \mathcal{F}l^1(\Gamma^*)$, we obtain

$$\mathcal{F}(A(\mathbb{1} - BA)^{-1}f) = \underbrace{\hat{\hat{u}}}_{\in l^2(\Gamma^*)} * \underbrace{\mathcal{F}((\mathbb{1} - BA)^{-1}f)}_{\in l^1(\Gamma^*)} \in l^2(\Gamma^*)$$

since $l^2 * l^1 \subseteq l^2$ due to Young's inequality. This shows

$$A(\mathbb{1} - BA)^{-1} : \mathcal{F}l^1(\Gamma^*) \to \mathcal{F}l^2(\Gamma^*). \tag{2.27}$$

Now, we enter the proof of [13, Theorem 4.5.42]. Therefrom, we have the expansion

$$A(\mathbb{1} - BA)^{-1} = A + ABA(\mathbb{1} - BA)^{-1}. \tag{2.28}$$

Now, we have to consider the restriction to $K_{\pm\nu}$, more precisely, we need to examine the Fourier transform of the term above at $\kappa = 0$ and $\kappa = \pm\nu$ with respect to ν. The entries of the first summand A (restricted to $K_{\pm\nu}$) are obviously in $l^2(\Gamma^*)$ with respect to ν with the same justification as in the proof of [13, Theorem 4.5.42]. Remember: In order to compute the entry (κ, μ) of $ABA(\mathbb{1} - BA)^{-1}$, we have to compute $\mathcal{F}(ABA(\mathbb{1} - BA)^{-1}e)(\kappa)$, where $e \in \mathcal{F}l^1(\Gamma^*)$ denotes the μ-th Fourier mode in the Fourier space, i.e. $\hat{e}(\mu) = 1$ and $\hat{e}(\lambda) = 0$ for $\lambda \neq \mu$. We are interested in tuples (κ, μ) with suitable $\kappa, \mu \in \{0, \pm\nu\}$. The Fourier transform of $ABA(\mathbb{1} - BA)^{-1}$ yields the entry at (κ, μ) (cf. [13, equation (4.5.44)])

$$\sum_{\rho \in \Gamma^*} \hat{\hat{u}}(\kappa - \rho) g(k, \nu, \rho) f(k, \nu, \rho), \tag{2.29}$$

where f denotes the Fourier transform of $A(\mathbb{1} - BA)^{-1}e$ with e as above. In the following, the index μ turns out to be immaterial, so we suppress it. In [13], p. 111, the Fourier transform (2.29) could be estimated via

$$\left| \sum_{\rho \in \Gamma^*} \hat{\hat{u}}(\kappa - \rho) g(k, \nu, \rho) f(k, \nu, \rho) \right| \leq \sum_{n=1}^{\infty} \hat{u}^*(n) g^*(k, \nu, n) f^*(k, \nu, n),$$

where f^*, $\hat{u}^*$ and g^* are the decreasing rearrangements of f, $\hat{u}$ and g, respectively, with respect to ρ (recall Definition 2.2.1). Now, we leave the proof of [13, Theorem 4.5.42] again. We have to show that[5]

$$\sum_{n=1}^{\infty} \hat{u}^*(n) g^*(k, \nu, n) f^*(k, \nu, n) \in l^2(\nu).$$

[5] As before, the suggestive notation $l^2(\nu)$ shall signify $l^2(\Gamma^*)$ with respect to ν.

Due to Hölder's inequality, we obtain by recalling that $f^* \in l^2(n)$ (cf. (2.27)) together with $\sup_{n\in\mathbb{N}} g^*(k,\nu,n) = O(1/|\nu|)$, as $|\nu| \to \infty$ (cf. Lemma 2.2.4)

$$\sum_{n=1}^{\infty} \underbrace{\hat{u}^*(n)}_{\in l^2} \underbrace{g^*(k,\nu,n)}_{\in l^{1,\infty}\subset l^\infty} \underbrace{f^*(k,\nu,n)}_{\in l^2} \leq \|\hat{u}^*\|_{l^2(n)} \cdot \|f^*\|_{l^2(n)} \cdot \sup_{n\in\mathbb{N}} g^*(k,\nu,n) = O\left(\frac{1}{|\nu|}\right),$$

as $|\nu| \to \infty$. This is an estimate for the second summand in (2.28). Here, we used that $\|\hat{u}^*(n)\|_{l^2(n)}$ is independent of ν and that $\|f^*\|_{l^2(n)}$ is in $l^\infty(\nu)$ which is a consequence of the representation $A(\mathbb{1} - BA)^{-1}$ and Lemma 2.2.4 where we showed that $\|g(k,\nu,\rho)\|_{l^2(\rho)} = O(1/|\sqrt{\nu}|)$ (this corresponds to the term B), as $|\nu| \to \infty$.
Together with the first summand A which we have examined above, we obtain

$$f(k,\nu,\rho) \in l^2(\nu) + O\left(\frac{1}{|\nu|}\right), \quad |\nu| \to \infty \tag{2.30}$$

for fixed $\rho \in \Gamma^*$. Unfortunately, $O\left(\frac{1}{|\nu|}\right)$ isn't an $l^2(\nu)$-sequence, yet (note that Γ^* is two-dimensional). The estimate thus has to be improved. Inserting f one more time into the estimate for the second summand in (2.28), we obtain again by Hölder's inequality (but this time with another decomposition) with (2.30)

$$\sum_{n=1}^{\infty} \underbrace{\hat{u}^*(n)}_{\in l^2(n)} \underbrace{g^*(k,\nu,n)}_{\in l^2(n)} \underbrace{f^*(k,\nu,n)}_{\in l^\infty(n)} \leq \|\hat{u}^*\|_{l^2(n)} \cdot \underbrace{\|f^*\|_{l^\infty(n)}}_{=O\left(\frac{1}{|\nu|}\right)} \cdot \underbrace{\|g^*(k,\nu,n)\|_{l^2(n)}}_{=O\left(\frac{1}{\sqrt{|\nu|}}\right)} =$$

$$= O\left(\frac{1}{|\nu|^{3/2}}\right) \subseteq l^2(\nu),$$

where for the estimate of $\|g^*(k,\nu,n)\|_{l^2(n)}$, we used Lemma 2.2.4. Thus, by one-time iteration, we have improved the preliminary result (2.30) to $l^2(\nu) + O\left(\frac{1}{|\nu|^{3/2}}\right) \subseteq l^2(\nu)$. Hence, we have proved that the entries of the perturbation matrix (2.6) are in $l^2(\nu)$.
It remains to be proved that the associated sequence of perturbed Fourier coefficients $(\check{u}_\nu)_{\nu\in\Gamma^*_\delta}$ is in $l^2(\nu)$, too. Here, we can copy the end of the proof of [13, Theorem 4.5.42]: We have to show that the sequence of the diagonal-zeroes $(|k_\nu|)_\nu$ (see Theorem 2.2.8) is in $l^2(\Gamma^*_\delta)$. But this is done literally as in the mentioned proof in [13] if we replace the condition used in [13] that the entries of the perturbation matrix are in $l^{\infty,1}(\nu)$ by the just proved result that the entries of the perturbation matrix are in $l^2(\nu)$. □

Before we formulate the main result of this section, we want to state more precisely how the map (introduced at the beginning of this section) between Fourier coefficients and perturbed Fourier coefficients is defined. We consider the map (compare [13, (4.5.50)])

$$l^2(\Gamma^*) \longrightarrow \mathcal{F}l^2(\Gamma^*) \longrightarrow l^2(\Gamma^*_\delta), \qquad \hat{u} \longmapsto u \longmapsto (\check{u}_\nu)_{\nu\in\Gamma^*_\delta} \tag{2.31}$$

The first map $\hat{u} \mapsto u$ is the inverse of the Fourier transform $u \mapsto \hat{u}$ which is as a linear isomorphism of vector spaces globally defined on $l^2(\Gamma^*)$. The second map $u \mapsto (\check{u}_\nu)_{\nu\in\Gamma^*_\delta}$ is also well-defined by Theorem 2.2.8 and Definition 2.3.3. If we restrict Γ^* to Γ^*_δ and require furthermore that the first finitely many Fourier coefficients indexed by $\nu \in \Gamma^* \setminus \Gamma^*_\delta$ are kept constant, we thus get a well-defined map

$$l^2(\Gamma^*_\delta) \longrightarrow l^2(\Gamma^*_\delta), \qquad (\hat{u}(\nu))_{\nu\in\Gamma^*_\delta} \longmapsto (\check{u}_\nu)_{\nu\in\Gamma^*_\delta}. \tag{2.32}$$

Keeping the first finitely many Fourier coefficients constant is necessary for the well-definition of (2.32) since by this requirement, it suffices to know the sequence $(\hat{u}(\nu))_{\nu\in\Gamma^*_\delta}$ to determine the potential u.
Since we will prove in the following theorem in particular the *holomorphy* of the map (2.32), let's briefly recall in this context that the well-known statement from the finite-dimensional case that functions which are complex differentiable in an open set are already *holomorphic* (i.e. expandable into a convergent power series) also holds in the general case of mappings between *complex Banach spaces* (maybe with infinite dimension). More precisely: A differentiable function $f : U \to F$ (with E, F complex Banach spaces and $U \subseteq E$ an open subset) is already holomorphic (in the usual definition such as [21, p. 33, Def. 5.1]). This can be seen as follows: If $f : U \to F$ is differentiable, it is in particular continuous and partially differentiable in the sense that the restriction of f to $U \cap M$ is differentiable for all finite-dimensional subspaces M of E. Then, for an arbitrary functional $\psi \in F^*$ (with F^* denoting the dual space to F), $\psi \circ f|_{U\cap M}$ is holomorphic due to [3, Theorem 3.1.7] (Osgood's Lemma). This in turn implies that $f|_{U\cap M}$ is holomorphic due to [21, Theorem 8.12(b)]. Finally, due to [21, Theorem 8.7], f is holomorphic on U.
To sum up, in order to show holomorphy, it suffices to verify complex differentiability in an open subset. Now, we can state the main result.

Theorem 2.4.2. *For all $u \in L^2(F)$, there is a $\delta > 0$ and an $R > 0$ such that the map* (2.32) *is boundedly invertible on $B_R(\hat{u}) \subset l^2(\Gamma^*_\delta)$ provided that the first finitely many Fourier coefficients indexed by $\nu \in \Gamma^* \setminus \Gamma^*_\delta$ are kept constant. Moreover, the maps* (2.31) *and* (2.32) *are holomorphic (locally in their respective domain of definition).*

Remark. The constraint of keeping the first finitely many Fourier coefficients (indexed by $\nu \in \Gamma^* \setminus \Gamma^*_\delta$) constant is not a severe restriction if we are interested in solving the *asymptotic* isospectral problem. Indeed, we will do so in Chapter

3 anyway (compare also the remark to Corollary 2.4.4). We usually choose this constant to be equal to $(\hat{u}(\nu))_{\nu\in\Gamma^*\setminus\Gamma^*_\delta}$, the first finitely many Fourier coefficients of the initial potential $\hat{u}$ (expressed in its sequence of Fourier coefficients), i.e. the center of the ball $B_R(\hat{u})$ mentioned in the theorem.

Proof. We prove the theorem with the help of the Inverse Function Theorem. Thereto, we have to show that the map (2.32) is differentiable and that its derivative is invertible in $u \in L^2(F)$. Since the perturbed Fourier coefficients are certain entries of the perturbation matrix (2.6), we have to consider the derivative of the perturbation matrix $\mathcal{A}_{\pm,\nu}(k + k^{\pm}_{\nu}(\hat{u}_0), u)$ with respect to the potential u. In [13, Theorem 4.5.25], this derivative has been calculated (this theorem also yields the holomorphy of the perturbation matrix with respect to u). In [13, Lemma 4.5.45], the derivative (evaluated at some fixed potential which is suppressed)

$$\frac{\partial}{\partial u}(A(\mathbb{1} - BA)^{-1}) \in \mathcal{L}(\mathcal{F}l^2(\Gamma^*);\ \mathcal{F}l^1(\Gamma^*) \to \mathcal{F}l^2(\Gamma^*))$$

is splitted into two summands (with the usual A, B notation, cf. (2.7)):

$$\begin{aligned} &h \mapsto \pi_{K_{\pm\nu}}(\mathbb{1} - AB)^{-1}\bar{h}(\mathbb{1} - BA)^{-1}|_{K_{\pm\nu}}, \\ &h \mapsto \pi_{K_{\pm\nu}}(\mathbb{1} - AB)^{-1}ABC(h)BA(\mathbb{1} - BA)^{-1}|_{K_{\pm\nu}}, \end{aligned} \tag{2.33}$$

where $h \in \mathcal{F}l^2(\Gamma^*)$ and $C(h)$ is the (diagonal) operator defined by

$$C(h) := diag\left(-\frac{8\pi^2 i}{\xi\nu^2}\left\langle \rho + k + k^{\pm}_{\nu}(\hat{u}_0), \begin{pmatrix}\nu_2\\ -\nu_1\end{pmatrix}\right\rangle \cdot \hat{h}(0) - 4\pi^2\hat{h}(0)\right)_{\rho\in\Gamma^*\setminus\{0,\pm\nu\}}, \tag{2.34}$$

compare [13, p. 104]. In fact, in terms of Fourier coefficients, C is the derivative with respect to u of the operator

$$\hat{E} := diag(-4\pi^2(\rho + k + k^{\pm}_{\nu}(\hat{u}_0))^2 - 4\pi^2\hat{u}_0)_{\rho\in\Gamma^*\setminus\{0,\pm\nu\}}, \tag{2.35}$$

which is the operator $E := (\mathbb{1} - \pi_{K_{\pm\nu}})(\Delta_{k+k^{\pm}_{\nu}(\hat{u}_0)} - 4\pi^2\hat{u}_0)$ in the Fourier space, compare (2.9). More precisely, (2.34) can be derived as follows: Deriving the diagonal entries of $\hat{E}$ yields for $\rho \in \Gamma^* \setminus \{0, \pm\nu\}$

$$\frac{\partial\hat{E}_{\rho\rho}}{\partial\hat{u}}(\hat{h}) = \frac{\partial\hat{E}_{\rho\rho}}{\partial\hat{u}_0}\hat{h}(0) = -4\pi^2\left(2\left\langle\rho + k + k^{\pm}_{\nu}(\hat{u}_0), \frac{\partial k^{\pm}_{\nu}(\hat{u}_0)}{\partial\hat{u}_0}\right\rangle - 1\right)\hat{h}(0),$$

where $\frac{\partial k^{\pm}_{\nu}(\hat{u}_0)}{\partial\hat{u}_0} = \frac{i}{\xi\nu^2}\binom{\nu_2}{-\nu_1}$ by definition (2.2). This shows (2.34).
We now follow the proof of [13, Lemma 4.5.45]. The first summand in (2.33) can be expanded into the four summands

$$\begin{aligned} &(\mathbb{1} - AB)^{-1}\bar{h}(\mathbb{1} - BA)^{-1} = \\ &= \bar{h} + A(\mathbb{1} - BA)^{-1}B\bar{h} + \bar{h}BA(\mathbb{1} - BA)^{-1} + A(\mathbb{1} - BA)^{-1}B\bar{h}BA(\mathbb{1} - BA)^{-1}. \end{aligned} \tag{2.36}$$

The first three summands are handled like (2.28), in particular the second and the third summand like (2.29), the fourth summand similarly (for details, see [13, p. 113], which can be translated into the $\mathcal{F}l^2$ case without any problems by replacing $\mathcal{F}l^{\infty,1}$-functions by $\mathcal{F}l^2$-functions at the correspondent positions). Altogether, the first summand in (2.33) maps into $l^2(\Gamma^*_\delta)$ with respect to ν. As to the second summand in (2.33), there appears the operator (2.34). Here, we deviate from the proof in [13], since that proof uses that C boundedly maps $l^{\infty,1}(\Gamma^*)$-left multiplications to $l^{\infty,1}(\Gamma^*)$-left multiplications. In our case, it's not clear yet whether $C(h)$ maps $\mathcal{F}l^1(\Gamma^*)$ into $\mathcal{F}l^2(\Gamma^*)$.
We compute for $f, h \in \mathcal{F}l^2(\Gamma^*)$, $\rho \in \Gamma^* \setminus \{0, \pm\nu\}$

$$\mathcal{F}(BC(h)Bf)(\rho) =$$
$$= \underbrace{g(k,\nu,\rho)\left(-\frac{8\pi^2 i}{\xi\nu^2}\left\langle \rho + k + k_\nu^\pm(\hat{u}_0), \begin{pmatrix}\nu_2\\ -\nu_1\end{pmatrix}\right\rangle \cdot \hat{h}(0) - 4\pi^2\hat{h}(0)\right)}_{\in l^\infty(\rho)} \cdot \underbrace{g(k,\nu,\rho)}_{\in l^2(\rho)} \cdot \underbrace{\hat{f}(\rho)}_{\in l^2(\rho)}.$$

Hence, due to Hölder's inequality,

$$\mathcal{F}(BC(h)Bf) \in l^1(\rho)$$

and clearly $\mathcal{F}(BC(h)Bf) \in l^\infty(\nu)$ (recall Lemma 2.2.4). Therefore and since for $f \in \mathcal{F}l^1(\Gamma^*)$,

$$\mathcal{F}(A(\mathbb{1} - BA)^{-1}f) \in l^2(\rho) \otimes l^2(\nu)$$

(cf. (2.27) and Theorem 2.4.1), we get for $f \in \mathcal{F}l^1(\Gamma^*), h \in \mathcal{F}l^2(\Gamma^*)$

$$\mathcal{F}(BC(h)B\underbrace{A(\mathbb{1} - BA)^{-1}f}_{\in l^2(\rho)}) \in l^1(\rho) \otimes l^2(\nu),$$

where for the $l^2(\nu)$-term, we used Theorem 2.4.1 since $BC(h)B$ is bounded with respect to ν. Finally, together with (2.27), we obtain for $f \in \mathcal{F}l^1(\Gamma^*), h \in \mathcal{F}l^2(\Gamma^*)$

$$\mathcal{F}(A(\mathbb{1} - BA)^{-1}\underbrace{BC(h)BA(\mathbb{1} - BA)^{-1}f}_{\in l^1(\rho)}) \in l^2(\rho) \otimes l^2(\nu), \tag{2.37}$$

where the $l^2(\nu)$-assertion follows again from Theorem 2.4.1.
By using $(\mathbb{1} - AB)^{-1}A = A(\mathbb{1} - BA)^{-1}$ (cf. [13, Lemma 4.5.23]), this yields that the second summand in (2.33) maps into $l^2(\nu)$.
As to the derivative of the perturbed Fourier coefficients $(\check{u}_\nu)_{\nu\in\Gamma^*_\delta}$ with respect to u, we obtain by setting $A_{12}(k,u) := \mathcal{A}_{+,\nu}(k + k_\nu^+(\hat{u}_0), u)_{12}$

$$\frac{d\check{u}_\nu}{du} = \frac{d}{du}A_{12}(k_\nu(u), u) = \frac{\partial A_{12}(k_\nu(u), u)}{\partial k} \cdot \frac{dk_\nu(u)}{du} + \frac{\partial A_{12}(k_\nu, u)}{\partial u}. \tag{2.38}$$

Here, the last summand $\frac{\partial}{\partial u}A_{12}(k_\nu, u)$ maps into $l^2(\nu)$ by what we have just proved. We have to show that the first summand maps into $l^2(\nu)$, too.

The derivative $\frac{\partial}{\partial k}A(\mathbb{1}-BA)^{-1}$ evaluated at some fixed k is (compare the proof of [13, Lemma 4.5.21]) equal to

$$\widetilde{k} \mapsto -A(\mathbb{1}-BA)^{-1}B\widetilde{C}(\widetilde{k})BA(\mathbb{1}-BA)^{-1}, \tag{2.39}$$

with

$$\widetilde{C}(\widetilde{k}) := -8\pi^2 diag\left(\left\langle \rho + k + k_\nu^\pm(\hat{u}_0), \widetilde{k}\right\rangle\right)_{\rho\in\Gamma^*\setminus\{0,\pm\nu\}}. \tag{2.40}$$

The derivation of (2.40) is virtually the same as the derivation of (2.34) except that this time, we derived (2.35) with respect to k instead of with respect to u. Comparing $\widetilde{C}(\widetilde{k})$ with $C(h)$ (2.34) above, we get in the same fashion as in the computations before (cf. (2.37)) that, for $f \in \mathcal{F}l^1(\Gamma^*)$,

$$\mathcal{F}(A(\mathbb{1}-BA)^{-1}B\widetilde{C}(\widetilde{k})BA(\mathbb{1}-BA)^{-1}f) \in l^2(\nu).$$

Note that we have virtually the same term as in (2.37) except that here, $\widetilde{C}(\widetilde{k})$ appears instead of $C(h)$. But both $\widetilde{C}(\widetilde{k})$ and $C(h)$ have the same behaviour in the sense that the operators $B\widetilde{C}(\widetilde{k})$ and $BC(h)$ are both bounded with respect to ν (that's the crucial property needed in the computations). In order to show our current aim that the linear operator (2.38) maps into $l^2(\nu)$ (more precisely: $(\frac{d\check{u}_\nu}{du}(h))_{\nu\in\Gamma^*_\delta}$ is in $l^2(\nu)$ for $h \in \mathcal{F}l^2(\Gamma^*)$), we have to show that $\frac{dk_\nu(u)}{du}$ is bounded (with respect to ν) which shall be sourced out into Lemma 2.4.3.
Now, the remainder of the proof has actually been done in [13, Lemma 4.5.49]. We briefly recap it. We show that the derivative of (2.32) is invertible in order to apply the Inverse Function Theorem. Due to $\frac{\partial}{\partial k}A_{12}(k_\nu(u), u) \in l^2(\nu)$ and $\left(\frac{dk_\nu(u)}{du}\right)_{\nu\in\Gamma^*_\delta} = O\left(\frac{1}{|\nu|}\right)$, as $|\nu| \to \infty$, cf. the following Lemma 2.4.3, the norm of the first summand $(\frac{\partial A_{12}(k_\nu(u),u)}{\partial k}\cdot\frac{dk_\nu(u)}{du})_{\nu\in\Gamma^*_\delta}$ in (2.38) tends to zero as $\delta \to 0$. So let's consider the second summand in (2.38). Recall that the derivative of the perturbation matrix with respect to u has been decomposed into two summands (2.33), where the norm of the second summand vanishes as $\delta \to 0$ due to Lemma 2.2.5. The first summand has been decomposed once again into four summands (2.36) where the last three summands vanish as $\delta \to 0$, again due to Lemma 2.2.5. Only the first summand of the decomposition (2.36), namely[6]

$$h \mapsto \pi_{K_{\pm\nu}}\bar{h}|_{K_{\pm\nu}} = \begin{pmatrix} 0 & \hat{h}(\pm\nu) \\ \hat{h}(\mp\nu) & 0 \end{pmatrix} \tag{2.41}$$

[6]The matrix in (2.41) is computed with respect to the ordered basis (ψ_ν, ψ_0). Taking the (maybe more self-evident) ordered basis (ψ_0, ψ_ν) for example, the matrix in (2.41) would be $\begin{pmatrix} 0 & \hat{h}(\mp\nu) \\ \hat{h}(\pm\nu) & 0 \end{pmatrix}$. Yet, we chose the basis (ψ_ν, ψ_0) in order to be consistent with the notations in [13].

does not vanish as $\delta \to 0$. In fact, the map $(\hat{h}(\nu))_\nu \mapsto (\hat{h}(\pm\nu))_\nu$ is boundedly invertible. Hence, we may apply the Inverse Function Theorem (cf. [23, p. 142], for example) and the assertion follows. $\square$

Remark. Now, we have seen the motivation for the name *perturbed Fourier coefficients*: The variation of the term $\left(\begin{smallmatrix} 0 & \check{u}_{\pm\nu} \\ \check{u}_{\mp\nu} & 0 \end{smallmatrix}\right)$ can be approximated by the term in (2.41), that is, it equals $\left(\begin{smallmatrix} 0 & \hat{u}(\pm\nu) \\ \hat{u}(\mp\nu) & 0 \end{smallmatrix}\right)$ plus the remaining "perturbation terms" which, however, vanish as $\delta \to 0$.

We owe the proof that the u-derivative of k_ν is bounded with respect to ν:

Lemma 2.4.3. *Let $(k_\nu)_{\nu\in\Gamma^*_\delta}$ be the sequence of the diagonal zeros of the matrix (2.14). Then the derivative (evaluated at some $u \in L^2(F)$)*

$$\frac{dk_\nu(u)}{du} : L^2(F) \longrightarrow \mathbb{C}^2$$

satisfies

$$\left(\frac{dk_\nu(u)}{du}\right)_{\nu\in\Gamma^*_\delta} = O\left(\frac{1}{|\nu|}\right), \quad |\nu| \to \infty,$$

locally uniform in u. In particular, the derivative $\frac{dk_\nu(u)}{du}$ is bounded with respect to ν.

Proof. Set

$$\begin{aligned}
&D_1(k) := 4\pi^2((k + k_\nu^+(\hat{u}_0))^2 + \hat{u}_0) + A_{11}(k,u),\\
&D_2(k) := 4\pi^2((k + k_\nu^-(\hat{u}_0))^2 + \hat{u}_0) + A_{22}(k,u),\\
&\text{where } A_{ij}(k,u) := \mathcal{A}_{+,\nu}(k + k_\nu^+(\hat{u}_0), u)_{ij} \quad \text{for } i,j \in \{1,2\}.
\end{aligned}$$

Define

$$D : V \subseteq \mathbb{C}^2 \to \mathbb{C}^2, \quad k \mapsto D(k,u) := \begin{pmatrix} D_1(k,u) \\ D_2(k,u) \end{pmatrix}.$$

By definition of the k_ν, we have $D(k_\nu(u), u) \equiv 0$ for all $u \in L^2(F)$. Differentiating this equation with respect to u yields

$$\begin{aligned}
\frac{\partial D_1(k_\nu(u),u)}{\partial k} \cdot \frac{dk_\nu(u)}{du} + \frac{\partial D_1(k_\nu,u)}{\partial u} = 0,\\
\frac{\partial D_2(k_\nu(u),u)}{\partial k} \cdot \frac{dk_\nu(u)}{du} + \frac{\partial D_2(k_\nu,u)}{\partial u} = 0,
\end{aligned}$$

which is equivalent to

$$\underbrace{\begin{pmatrix} \frac{\partial D_1(k_\nu(u),u)}{\partial k_1} & \frac{\partial D_1(k_\nu(u),u)}{\partial k_2} \\ \frac{\partial D_2(k_\nu(u),u)}{\partial k_1} & \frac{\partial D_2(k_\nu(u),u)}{\partial k_2} \end{pmatrix}}_{=:C} \cdot \frac{dk_\nu(u)}{du} = -\frac{\partial D(k_\nu,u)}{\partial u}.$$

The matrix $\frac{1}{8\pi^2}C$ is invertible because its determinant is equal to

$$\frac{1}{64\pi^4}\left[\frac{\partial D_1(k_\nu(u),u)}{\partial k_1}\cdot\frac{\partial D_2(k_\nu(u),u)}{\partial k_2}-\frac{\partial D_2(k_\nu(u),u)}{\partial k_1}\cdot\frac{\partial D_1(k_\nu(u),u)}{\partial k_2}\right]=$$
$$=(k_{\nu,1}+k^+_{\nu,1}(\hat u_0))(k_{\nu,2}+k^-_{\nu,2}(\hat u_0))-(k_{\nu,1}+k^-_{\nu,1}(\hat u_0))(k_{\nu,2}+k^+_{\nu,2}(\hat u_0))+o(|\nu|)=$$
$$=(k_{\nu,1}+k^+_{\nu,1}(\hat u_0))(k_{\nu,2}+k^+_{\nu,2}(\hat u_0)+\nu_2)-(k_{\nu,1}+k^+_{\nu,1}(\hat u_0)+\nu_1)(k_{\nu,2}+k^+_{\nu,2}(\hat u_0))+o(|\nu|)=$$
$$=(k_{\nu,1}+k^+_{\nu,1}(\hat u_0))\nu_2-(k_{\nu,2}+k^+_{\nu,2}(\hat u_0))\nu_1+o(|\nu|)=\left\langle k_\nu+k^+_\nu(\hat u_0),\begin{pmatrix}\nu_2\\-\nu_1\end{pmatrix}\right\rangle+o(|\nu|)=$$
$$=\left\langle k_\nu,\begin{pmatrix}\nu_2\\-\nu_1\end{pmatrix}\right\rangle+\frac{1}{2}i\xi|\nu|^2+o(|\nu|)=\frac{1}{2}i\xi|\nu|^2(1+o(1)),\quad \text{as } |\nu|\to\infty, \tag{2.42}$$

where the $o(|\nu|)$-term in the above computation is a consequence of the multiplication of terms of the form $k_i+k^\pm_{\nu,i}(\hat u_0)$ with terms $\frac{\partial A_{jj}(k_\nu,u)}{\partial k_l}$ $(i,j,l=1,2)$, cf. Lemma 2.2.7. Since C^{-1} is thus well-defined, we get

$$\frac{dk_\nu(u)}{du}=-\begin{pmatrix}\frac{\partial D_1(k_\nu(u),u)}{\partial k_1} & \frac{\partial D_1(k_\nu(u),u)}{\partial k_2}\\ \frac{\partial D_2(k_\nu(u),u)}{\partial k_1} & \frac{\partial D_2(k_\nu(u),u)}{\partial k_2}\end{pmatrix}^{-1}\cdot\frac{\partial D(k_\nu,u)}{\partial u}=$$
$$=-8\pi^2\begin{pmatrix}k_{\nu,1}+k^+_{\nu,1}(\hat u_0)+\frac{1}{8\pi^2}\frac{\partial A_{11}(k_\nu,u)}{\partial k_1}, & k_{\nu,2}+k^+_{\nu,2}(\hat u_0)+\frac{1}{8\pi^2}\frac{\partial A_{11}(k_\nu,u)}{\partial k_2}\\ k_{\nu,1}+k^-_{\nu,1}(\hat u_0)+\frac{1}{8\pi^2}\frac{\partial A_{22}(k_\nu,u)}{\partial k_1}, & k_{\nu,2}+k^-_{\nu,2}(\hat u_0)+\frac{1}{8\pi^2}\frac{\partial A_{22}(k_\nu,u)}{\partial k_2}\end{pmatrix}^{-1}\cdot$$
$$\cdot\begin{pmatrix}\frac{\partial A_{11}(k_\nu,u)}{\partial u}+O(1)\\ \frac{\partial A_{22}(k_\nu,u)}{\partial u}+O(1)\end{pmatrix},\quad\text{as } |\nu|\to\infty,$$

where the $O(1)$-terms reflect that the derivative of $4\pi^2((k+k^\pm_\nu(\hat u_0))^2+\hat u_0)$ with respect to u is bounded with respect to $|\nu|$. More precisely, deriving this term with respect to $\hat u_0$ yields $\frac{8\pi^2 i}{\xi\nu^2}\left\langle k+k^\pm_\nu(\hat u_0),\binom{\nu_2}{-\nu_1}\right\rangle+4\pi^2$, cf. (2.34), which is clearly bounded with respect to $|\nu|$. Since the entries of the matrix C are $O(|\nu|)$, the entries of its inverse are $O\left(\frac{1}{|\nu|}\right)$, as $|\nu|\to\infty$. Due to the boundedness of $\frac{\partial A_{11}}{\partial u}$ and $\frac{\partial A_{22}}{\partial u}$ with respect to ν (see the proof of Theorem 2.4.2), the assertion now follows. □

In Corollary 2.3.6, we have derived a property of real-valued potentials in terms of their perturbed Fourier coefficients: $\overline{\check u_\nu}=\check u_{-\nu}$ for all $\nu\in\Gamma^*_\delta$. With the help of Theorem 2.4.2, we are able to prove that this property is also *sufficient* for the realness of a given potential, at least in some asymptotic sense. More precisely, we have the following corollary of Theorem 2.4.2.

Corollary 2.4.4. *Let $u\in L^2(F)$ and let $\hat u=(\hat u(\nu))_{\nu\in\Gamma^*}$ denote its sequence of Fourier coefficients and let $(\check u_\nu)_{\nu\in\Gamma^*_\delta}$ denote its sequence of perturbed Fourier coefficients (for suitably small $\delta>0$). Let further*

$$\overline{\hat u(\nu)}=\hat u(-\nu)\ \text{for all}\ \nu\in\Gamma^*\setminus\Gamma^*_\delta. \tag{2.43}$$

Then there holds the equivalence

$$\overline{\hat{u}(\nu)} = \hat{u}(-\nu) \text{ for all } \nu \in \Gamma^*_\delta \iff \overline{\check{u}_\nu} = \check{u}_{-\nu} \text{ for all } \nu \in \Gamma^*_\delta. \tag{2.44}$$

Remark. Before proving the corollary, let's briefly discuss why this is a sufficient criterion for realness of u in some asymptotic sense. Let $u \in L^2(F)$ and set $v := \bar{u}$ as the complex conjugation of u. By definition of the Fourier transform (1.9), we obtain

$$\overline{\hat{v}(\nu)} = \hat{u}(-\nu) \quad \text{for all } \nu \in \Gamma^*. \tag{2.45}$$

Hence, there holds:

$$u \in L^2(F) \text{ is real-valued} \iff \overline{\hat{u}(\nu)} = \hat{u}(-\nu) \quad \text{for all } \nu \in \Gamma^*,$$

since u is uniquely determined by its Fourier coefficients $\hat{u}$ and vice versa. Now, the right hand side of (2.44) is not sufficient for reality of u since we only consider indices $\nu \in \Gamma^*_\delta$ such that the left hand side of (2.44) will not hold for all $\nu \in \Gamma^*$, at least not without the additional assumption (2.43). In this sense, Corollary 2.4.4 provides a criterion for "asymptotic reality". Moreover, we have to require the condition (2.43) for the first finitely many Fourier coefficients since in the map $\hat{u} \mapsto u \mapsto (\check{u}_\nu)_{\nu \in \Gamma^*_\delta}$, the potential u (and consequently the sequence of perturbed Fourier coefficients) is determined by the *entire* sequence $(\hat{u}(\nu))_{\nu \in \Gamma^*}$ (not only by the asymptotic remainder indexed by $\nu \in \Gamma^*_\delta$). However, the condition (2.43) is not really a severe restriction of generality since in Chapter 3, we will fix the first finitely many Fourier coefficients anyway in such a way that in particular, (2.43) is fulfilled when we will solve the asymptotic isospectral problem (compare for example (3.2) in Chapter 3).

Proof. Consider the map

$$l^2(\Gamma^*) \longrightarrow \mathcal{F}l^2(\Gamma^*) \longrightarrow l^2(\Gamma^*_\delta), \qquad \hat{u} \longmapsto u \longmapsto (\check{u}_\nu)_{\nu \in \Gamma^*_\delta}.$$

Set $v := \bar{u}$. Then $\check{v}_\nu = \overline{\check{u}_{-\nu}}$, $\nu \in \Gamma^*_\delta$ (due to Theorem 2.3.5) and $\hat{v}(\nu) = \overline{\hat{u}(-\nu)}$ for all $\nu \in \Gamma^*$ (due to (2.45)). For v, we thus have the mapping

$$l^2(\Gamma^*) \longrightarrow \mathcal{F}l^2(\Gamma^*) \longrightarrow l^2(\Gamma^*_\delta), \qquad (\overline{\hat{u}(-\nu)})_{\nu \in \Gamma^*} \longmapsto \bar{u} \longmapsto (\overline{\check{u}_{-\nu}})_{\nu \in \Gamma^*_\delta}.$$

Due to Theorem 2.4.2, the map $(\hat{u}(\nu))_{\nu \in \Gamma^*_\delta} \longmapsto (\check{u}_\nu)_{\nu \in \Gamma^*_\delta}$ is locally boundedly invertible. The assertion of the corollary follows if the sequence $(\overline{\hat{u}(-\nu)})_{\nu \in \Gamma^*_\delta}$ is inside the ball $B_R(\hat{u})$ where invertibility holds due to Theorem 2.4.2. But by choosing $\delta > 0$ sufficiently small, this can be established[7]: Choose in a first step a radius $R > 0$ and $\delta_1 > 0$ such that invertibility of the map (2.32) holds in $B_R(\hat{u}) \subset l^2(\Gamma^*_{\delta_1})$. All we have to show is, by choosing $0 < \delta_2 < \delta_1$ sufficiently small, that

[7]Compare also the choice of $\delta > 0$ on p. 72.

invertibility also holds in the ball $B_R(\hat{u}) \subset l^2(\Gamma^*_{\delta_2})$ with still the same radius R. But this follows immediately from the proof of Theorem 2.4.2. There, we showed that the derivative of the map (2.32) is equal to the identity map plus some perturbation terms whose norms, however, tend to zero (locally uniformly in u) as $|\nu| \to \infty$. If we want to answer the question how large the ball inside of which invertibility of the map (2.32) holds can be chosen, we have to recall the proof of the Inverse Function Theorem which is usually proved by applying Banach's Fixed Point Theorem: If f denotes the continuously differentiable map (between some Banach spaces) in the Inverse function Theorem whose local invertibility shall be proved, the ball B around some given point x_0 where invertibility shall hold has to satisfy for example $\|f'(x) - f'(x_0)\| \leq 1/2$ for all $x \in B$ (cf. the proof of the Inverse Function Theorem in [23, p. 142], for example). In our case, the corresponding term $\|f'(x) - f'(x_0)\|$ gets smaller the larger $|\nu|$ gets due to the limit behaviour of the perturbation terms which has just been mentioned. In this way, one sees that the radius R can be chosen fixed whereas $\delta > 0$ can be chosen more and more smaller.
Let $u \in L^2(F)$ be arbitrary. Choose $\delta = \delta_2 > 0$ small enough such that $\|\hat{u}\|_{l^2(\Gamma^*_\delta)}$ is sufficiently small, i.e. $(\hat{u}(\nu))_{\nu \in \Gamma^*_\delta}$ is in particular sufficiently close to $0 \in l^2(\Gamma^*_\delta)$. However, the radius R (belonging to the initial $\delta_1 > 0$) has not decreased such that we can achieve that $(\overline{\hat{u}(-\nu)})_{\nu \in \Gamma^*_\delta}$ is in the ball $B_R(\hat{u})$. In other words, we chose a $\delta > 0$ and a radius $R > 0$ such that the ball in $l^2(\Gamma^*_\delta)$ with center $0 \in l^2(\Gamma^*_\delta)$ and radius $\|\hat{u}\|_{l^2(\Gamma^*_\delta)}$ is contained in the ball $B_R(\hat{u}) \subset l^2(\Gamma^*_\delta)$ where invertibility of the map (2.32) holds. □

2.5 Parameterization of the handles

In Section 2.1, we roughly introduced the *handles* in the context of the trisection of a given Fermi curve. There, we defined what we mean by saying that a double point *splits up* or remains *unsplit*. By Theorem 2.2.6, the handles which are contained in the asymptotic part of a Fermi curve are described by the zero locus of holomorphic functions, the determinant of the corresponding matrices (2.14), defined in the corresponding excluded domains. Unfortunately, we are in general not able to see by looking at (2.14) whether the corresponding double point splits up or not. Therefore, we are interested in a parameterization of the Fermi curve in the excluded domains which reflects the handle properties in a more obvious way. This will require new coordinates $z = (z_1, z_2) \in \mathbb{C}^2$ instead of $k = (k_1, k_2) \in \mathbb{C}^2$. In [5, II.5, (GH2)], a model handle is defined by

$$H := \{(z_1, z_2) \in \mathbb{C}^2 : z_1 \cdot z_2 = c, \quad |z_1|, |z_2| \leq 1\} \tag{2.46}$$

with some $0 < c < \frac{1}{2}$. We allow that this c may also be complex-valued with $0 < |c| < \frac{1}{2}$. We call c the *handle quantity*. Here, the upper bound $\frac{1}{2}$ is arbitrary. Since it will turn out that the handle quantities decrease with increasing index

$\nu \in \Gamma^*$ of the corresponding excluded domain (cf. (2.49) and Theorem 2.5.9), the absolute value $|c|$ of any handle in the asymptotic part of $F(u)$ will be less than $\frac{1}{2}$ for sufficiently small $\delta > 0$ (cf. (2.3)). The aim of this section is to find such coordinates (z_1, z_2) which parameterize the Fermi curve in the excluded domains in the form of (2.46). Such a parameterization will turn out to be very helpful when later, we'll have to estimate certain contour integrals which occur in terms of the so-called *moduli*. Let's begin with an approximation, the so-called *model Fermi curve*, of the given curve $F(u)$ which is easier to handle than the actual Fermi curve. In [13, Lemma 4.5.53], it has been proved that the matrix M (2.14) may be represented as

$$M = 4\pi^2 \begin{pmatrix} 2\langle k - k_\nu, k_\nu^\pm(\hat{u}_0)\rangle & 0 \\ 0 & 2\langle k - k_\nu, k_\nu^\mp(\hat{u}_0)\rangle \end{pmatrix} + \begin{pmatrix} 0 & \check{u}_\nu \\ \check{u}_{-\nu} & 0 \end{pmatrix} + o(|k - k_\nu|), \tag{2.47}$$

as $k \to k_\nu$. If we omit the error term $o(|k - k_\nu|)$, we get a modified matrix $\widetilde{M}$ (the quantities of the *model* Fermi curve shall be indicated with a tilde). If we compute, like in Theorem 2.2.6, the zero locus of $\det \widetilde{M}$, we obtain locally a variety close to the Fermi curve in the corresponding excluded domain indexed by $\nu \in \Gamma^*_\delta$. We call this approximation the *model Fermi curve* (note that all varieties we consider are, of course, described only *locally* in the corresponding excluded domains).

For the model curve, the $\widetilde{z}$-coordinates are quite easy to calculate. A computaion of $\det \widetilde{M} = 0$ yields (as has already been done in [13, equation (4.5.57)][8])

$$\begin{aligned} &[(k_1 - k_{\nu,1})(-\nu_1 + i\nu_2\xi) + (k_2 - k_{\nu,2})(-i\nu_1\xi - \nu_2)]\cdot \\ &\qquad \cdot [(k_1 - k_{\nu,1})(\nu_1 + i\nu_2\xi) + (k_2 - k_{\nu,2})(-i\nu_1\xi + \nu_2)] = \frac{\check{u}_\nu \cdot \check{u}_{-\nu}}{16\pi^4} \end{aligned}$$

(recall the definition (2.2)). This representation already has the desired form since the left hand side is a product. Thus, the factors can be defined as $\widetilde{z}_1$ and $\widetilde{z}_2$, respectively:

$$\begin{aligned} \widetilde{z}_1 &:= (k_1 - k_{\nu,1})(-\nu_1 + i\nu_2\xi) + (k_2 - k_{\nu,2})(-i\nu_1\xi - \nu_2) \\ \widetilde{z}_2 &:= (k_1 - k_{\nu,1})(\nu_1 + i\nu_2\xi) + (k_2 - k_{\nu,2})(-i\nu_1\xi + \nu_2) \end{aligned} \tag{2.48}$$

Set

$$\widetilde{c}_\nu := \frac{\check{u}_\nu \cdot \check{u}_{-\nu}}{16\pi^4}, \qquad \nu \in \Gamma^*_\delta, \tag{2.49}$$

we obtain the desired representation (2.46) $\widetilde{z}_1 \cdot \widetilde{z}_2 = \widetilde{c}_\nu$ for the ν-th excluded domain. Note that the handle quantity $\widetilde{c}_\nu$ doesn't have the right scaling, yet,

[8] The reader may have remarked the different sign compared to [13, (4.5.57)]. This is a consequence of the (harmless) wrong sign already discussed in a footnote to (2.14).

since $|z_1|, |z_2| \leq 1$ is required in (2.46). Such a scaling is necessary if we want to compare different handles of different excluded domains with one another. For example, for the assertion that the handle quantities decrease with increasing ν, a consistent scaling is necessary (otherwise, that assertion wouldn't make any sense). However, if we consider a fixed excluded domain and only need the z-coordinates because of the desired representation $z_1 \cdot z_2 = c$, we can use an arbitrary scaling (provided that we use the corresponding correct maps $z \mapsto k$ and $k \mapsto z$ between the coordinates z and k, respectively). If we wanted to determine the quantities $\widetilde{c}_\nu$ with the correct scaling as in (2.46), we would have to multiply (2.49) with a term of dimension $O\left(1/|\nu|^2\right)$ since (2.48) is of dimension $O\left(|\nu|\right)$ as $|\nu| \to \infty$ (note that the domain V the k-coordinates reside in is bounded and independent of ν).
The map $(k - k_\nu) \mapsto \widetilde{z} = (\widetilde{z}_1, \widetilde{z}_2)$ between k- and $\widetilde{z}$-coordinates is a linear vector space isomorphism of $\mathbb{C}^2$ with inverse $\widetilde{z} \mapsto (k - k_\nu)$ defined by

$$
\begin{aligned}
k_1 - k_{\nu,1} &= \frac{1}{2i|\nu|^2\xi}[(-i\nu_1\xi + \nu_2)\widetilde{z}_1 + (i\nu_1\xi + \nu_2)\widetilde{z}_2] \\
k_2 - k_{\nu,2} &= \frac{1}{2i|\nu|^2\xi}[(-\nu_1 - i\nu_2\xi)\widetilde{z}_1 + (-\nu_1 + i\nu_2\xi)\widetilde{z}_2]
\end{aligned}
$$

(since $(-\nu_1 + i\nu_2\xi)(-i\nu_1\xi + \nu_2) - (\nu_1 + i\nu_2\xi)(-i\nu_1\xi - \nu_2) = 2i\xi|\nu|^2$).
Let's go back to the actual Fermi curve again. Here, things turn out to be much more difficult than in the model case (the map $(k - k_\nu) \mapsto z$ is far away from being a linear isomorphism, for instance). The most important tool to obtain z-coordinates will be the so-called *Quantitative Morse Lemma* (cf. [5, III. Appendix B, Lemma B.1]) which we denote, however, as a *theorem* due to its importance for our purposes:

Theorem 2.5.1 (Quantitative Morse Lemma). *Let*

$$
f(x_1, x_2) = x_1 x_2 + h(x_1, x_2) \tag{2.50}
$$

be a holomorphic function on

$$
D_r := \{(x_1, x_2) \in \mathbb{C}^2 : |x_1| \leq r, |x_2| \leq r\}, \quad r > 0,
$$

where h is a holomorphic function that fulfils the estimates

$$
\left|\frac{\partial h}{\partial x_i}(x)\right| \leq a, \quad \left\|\frac{\partial^2 h}{\partial x_i \partial x_j}(x)\right\| \leq b \tag{2.51}
$$

for $x \in D_r$ with constants $a, b > 0$ such that $a < r$, $b < 1/30$. Then f has a unique critical point $\zeta = (\zeta_1, \zeta_2)$ in D_r, and

$$
|\zeta_1| \leq a, \quad |\zeta_2| \leq a.
$$

Put $s = \max\{|\zeta_1, |\zeta_2|\}$. *Then there is a biholomorphic map* Φ *from* $D_{(r-s)(1-10b)}$ *to a neighbourhood of* ζ *in* D_r *that contains* $\{(z_1, z_2) \in \mathbb{C}^2 : |z_i - \zeta_i| < (r - s)(1 - 30b), i = 1, 2\}$ *such that*

$$f \circ \Phi(z_1, z_2) = z_1 z_2 - c \tag{2.52}$$

with a constant $c \in \mathbb{C}$ *fulfilling*[9] $|c - h(\zeta)| \leq a^2$. *The derivative* $D\Phi$ *fulfils*

$$\|D\Phi - \mathbb{1}\| \leq 12b.$$

Remark. In [5, Lemma B.1 (p. 245)], there is the additional normalization requirement that $r < 1$ (in [5], the r is denoted with δ). Since the x-coordinates in our case turn out to be $O(|\nu|)$ with respect to ν, the possibility that $r \geq 1$ in D_r may occur shouldn't be omitted. If one wants to avoid this and use $r < 1$, one can consider a correspondent scaling (by multiplying (2.50) with an appropriate factor which then affects the x-coordinates but has no effect on the Fermi curve locally described by the equation $f = 0$). But this is not absolutely necessary.

In our case, the holomorphic function f is the function $f := \det M$ with matrix M (2.14), holomorphic in k. In order to get the desired representation (2.52), we must at first find suitable *intermediate coordinates* $x = (x_1, x_2) \in \mathbb{C}^2$ such that f has the form (2.50). To this, we have to compute the zero locus of $\det M$. From now on, we restrict ourselves to the upper signatures in (2.14) in order to simplify the notation (the lower signatures are treated completely analogously, of course). Since $k_\nu^-(\hat{u}_0) - k_\nu^+(\hat{u}_0) = \nu$ for all $\nu \in \Gamma^*$, we don't lose any information. Set (by suppressing the index ν in d_1 and d_2)

$$\begin{aligned} &d_1(k) := (k + k_\nu^+(\hat{u}_0))^2 + \hat{u}_0, \quad d_2(k) := (k + k_\nu^-(\hat{u}_0))^2 + \hat{u}_0 \\ &A_{ij}^\nu(k, u) := \frac{1}{4\pi^2} \mathcal{A}_{+,\nu}(k + k_\nu^+(\hat{u}_0), u)_{ij} \quad \text{for } i, j \in \{1, 2\}. \end{aligned} \tag{2.53}$$

Thus,

$$\det M = 0 \;\Leftrightarrow\; \underbrace{(d_1(k) + A_{11}^\nu(k, u))}_{=:x_1} \underbrace{(d_2(k) + A_{22}^\nu(k, u))}_{=:x_2} = A_{12}^\nu(k, u) \cdot A_{21}^\nu(k, u). \tag{2.54}$$

We want to get x_1, x_2 into a handier form such that these coordinates can be compared with the model coordinates. We have, due to $d_1(k_\nu) = -A_{11}^\nu(k_\nu, u)$ and $d_2(k_\nu) = -A_{22}^\nu(k_\nu, u)$ (cf. Theorem 2.2.8),

$$\begin{aligned} x_1 &:= d_1(k) + A_{11}^\nu(k, u) = d_1(k) - d_1(k_\nu) + A_{11}^\nu(k, u) - A_{11}^\nu(k_\nu, u) \\ x_2 &:= d_2(k) + A_{22}^\nu(k, u) = d_2(k) - d_2(k_\nu) + A_{22}^\nu(k, u) - A_{22}^\nu(k_\nu, u). \end{aligned} \tag{2.55}$$

[9] In [5], the authors write $|c - h(0)| \leq a^2$ instead of $|c - h(\zeta)| \leq a^2$. The proof of Theroem 2.5.4 shows, however, that the term that we use here (i.e. with ζ instead of with 0 in the argument of h) is more reasonable. Probably, this is a typing error in [5] particularly since in the proof of [5, Lemma B.1], the authors consider the case $\zeta = 0$.

We compute

$$d_{1/2}(k) - d_{1/2}(k_\nu) = (k + k_\nu^\pm(\hat{u}_0))^2 - (k_\nu + k_\nu^\pm(\hat{u}_0))^2 = (k_1 + \frac{1}{2}(\mp\nu_1 + i\nu_2\xi))^2 +$$
$$+ (k_2 + \frac{1}{2}(-i\nu_1\xi \mp \nu_2))^2 - (k_{\nu,1} + \frac{1}{2}(\mp\nu_1 + i\nu_2\xi))^2 - (k_{\nu,2} + \frac{1}{2}(-i\nu_1\xi \mp \nu_2))^2 =$$
$$= (k_1 - k_{\nu,1})(\mp\nu_1 + i\nu_2\xi + k_1 + k_{\nu,1}) + (k_2 - k_{\nu,2})(-i\nu_1\xi \mp \nu_2 + k_2 + k_{\nu,2}).$$

Here, we rediscover the map $(k - k_\nu) \mapsto \widetilde{z} = (\widetilde{z}_1, \widetilde{z}_2)$ from (2.48). Together with (2.55), we obtain

$$\begin{aligned} x_1(k) &= \widetilde{z}_1(k - k_\nu) + k^2 - k_\nu^2 + A_{11}^\nu(k, u) - A_{11}^\nu(k_\nu, u), \\ x_2(k) &= \widetilde{z}_2(k - k_\nu) + k^2 - k_\nu^2 + A_{22}^\nu(k, u) - A_{22}^\nu(k_\nu, u). \end{aligned} \tag{2.56}$$

These are the intermediate coordinates $x = (x_1, x_2)$. We sometimes express (2.56) in the following matrix-vector representation

$$x(k) =: B_0 \cdot (k - k_\nu) + B_1(k), \tag{2.57}$$
$$B_0 := \begin{pmatrix} -\nu_1 + i\nu_2\xi & -i\nu_1\xi - \nu_2 \\ \nu_1 + i\nu_2\xi & -i\nu_1\xi + \nu_2 \end{pmatrix}, B_1(k) := \begin{pmatrix} k^2 - k_\nu^2 + A_{11}^\nu(k, u) - A_{11}^\nu(k_\nu, u) \\ k^2 - k_\nu^2 + A_{22}^\nu(k, u) - A_{22}^\nu(k_\nu, u) \end{pmatrix}$$

Moreover, due to (2.54), we can define the holomorphic function $h = h_\nu$ in (2.50) by

$$h_\nu(x_1, x_2) := -A_{12}^\nu(k(x), u) \cdot A_{21}^\nu(k(x), u), \tag{2.58}$$

where we used that the map $k \mapsto x(k)$ defined by (2.56) is invertible in a neighbourhood of k_ν. Then, the term $k(x)$ is well-defined. This has to be proved:

Theorem 2.5.2. *The map $k \mapsto x(k)$, locally defined by* (2.56)*, is biholomorphic in a neighbourhood V of $0 \in \mathbb{C}^2$ which only depends on Γ_δ^*.*

Remark. Remember that the map $k \mapsto x(k)$ depends on $\nu \in \Gamma_\delta^*$ (even though the index ν has been suppressed in the notation).

Proof. Let V be the neighbourhood from Lemma 2.2.7. The entries of $\frac{\partial}{\partial k}\mathcal{A}_{\pm,\nu}(k + k_\nu^\pm(\hat{u}_0), u)$ tend, due to Lemma 2.2.7, to zero as $|\nu| \to \infty$. Therefore and because of the boundedness of V, we have with $B_0^{-1} = O\left(\frac{1}{|\nu|}\right)$, $|\nu| \to \infty$, that

$$B_0^{-1} \cdot \frac{\partial}{\partial k}(k^2 - k_\nu^2 + A_{11}^\nu(k, u) - A_{11}^\nu(k_\nu, u)) \to 0 \tag{2.59}$$

uniformly in $k \in V$ as $|\nu| \to \infty$. The same holds for the other component x_2 in (2.56). Hence, due to the Inverse Function Theorem (cf. [23, p. 142], for example), the map $k \mapsto k - k_\nu + B_0^{-1}B_1(k)$ and consequently also the map

$$k \mapsto x(k) = B_0\left(k - k_\nu + B_0^{-1}B_1(k)\right)$$

is locally invertible since B_0 is invertible. It remains to be proved that the neighbourhood where the map $k \mapsto x(k)$ is invertible is *independent* of the index ν. This, however, follows from the Inverse Function Theorem, too, more precisely from its proof: One usually shows the theorem by using Banach's Fixed Point Theorem. Let's briefly recap the essential step where the domain V in which the considered function, in our case $x : U \subset \mathbb{C}^2 \to \mathbb{C}^2$, shall be invertible, is defined[10]. Without restriction, one assumes $x'(0) = \mathbb{1}$ (otherwise, consider a suitable linear change of coordinates $x \mapsto x'(0)^{-1}x$ which, however, doesn't affect the choice of V since $x'(0)$ is independent of k). For given x_0 in a neighbourhood of an element lying in the image of x, the map $V \to V$, defined by $k \mapsto k + (x_0 - x(k))$ shall be a contractive map. For this purpose, one chooses V in such a way that for instance, $\|x'(k) - \mathbb{1}\| \leq \frac{1}{2}$ for all $k \in V$. This is the crucial condition that the domain V has to satisfy. Let's go back to our proof of the theorem. Choose one neighbourhood V for fixed $\kappa \in \Gamma^*_\delta$ such that the map $k \mapsto x(k)$ is invertible in V. Then for all $\nu \in \Gamma^*_\delta$ with $|\nu| \geq |\kappa|$, the corresponding map $k \mapsto x(k)$ is invertible in the same V, a fortiori, due to (2.59) and the method of finding V which has just been explained. Therefore, V only depends on Γ^*_δ. □

In order to apply Theorem 2.5.1, we have to see that its conditions are satisfied. These conditions concern estimates of certain partial derivatives of the function h:

Lemma 2.5.3. *Let h_ν be the function defined in* (2.58). *It fulfills for $i, j \in \{1, 2\}$*

$$\frac{\partial h_\nu}{\partial x_i}(x) = o\left(\frac{1}{|\nu|}\right), \quad \text{as } |\nu| \to \infty,$$
$$\frac{\partial^2 h_\nu}{\partial x_i \partial x_j}(x) = o\left(\frac{1}{|\nu|^2}\right), \quad \text{as } |\nu| \to \infty.$$

These estimates are locally uniform in x.

Remark. Actually, one could get even better estimates. For example, one could use Theorem 2.4.1 where we showed that the entries of the perturbation matrix are l^2-sequences with respect to ν. This would lead to better estimates for the derivatives of h_ν to be considered. But this is not necessary here.

Proof. Using definition (2.58), we compute

$$\begin{aligned} -\frac{\partial h_\nu}{\partial x_i} &= \frac{\partial}{\partial k}\left(A^\nu_{12}(k,u) \cdot A^\nu_{21}(k,u)\right) \cdot \frac{\partial k}{\partial x_i} = \\ &= \left(A^\nu_{12}(k,u)\frac{\partial}{\partial k}A^\nu_{21}(k,u) + A^\nu_{21}(k,u)\frac{\partial}{\partial k}A^\nu_{12}(k,u)\right) \cdot \frac{\partial k}{\partial x_i} \end{aligned} \tag{2.60}$$

[10]Compare the proof of Corollary 2.4.4 where we already used the proof of the Inverse Function Theorem, too.

By (2.57), we get (with a prime denoting the derivative with respect to k)

$$\frac{\partial k}{\partial x}(x) = (x'(k))^{-1} = (\mathbb{1} + B_0^{-1}B_1'(k))^{-1} \cdot B_0^{-1}$$

(with $k = k(x)$) since $x'(k) = B_0 + B_1'(k) = B_0 \cdot (\mathbb{1} + B_0^{-1}B_1'(k))$. Recall that B_0 is invertible. Since $B_0^{-1} = O\left(\frac{1}{|\nu|}\right)$ and $B_0^{-1}B_1'(k) \to 0$ as $|\nu| \to \infty$ (as has already been justified in the proof of Theorem 2.5.2), we obtain

$$\frac{\partial k}{\partial x} = O\left(\frac{1}{|\nu|}\right), \quad |\nu| \to \infty. \tag{2.61}$$

Since $A_{ij}^\nu(k,u)$ and $\frac{\partial}{\partial k}A_{ij}^\nu(k,u)$ tend to zero as $|\nu| \to \infty$, respectively (Theorem 2.4.1 and Lemma 2.2.7), the first assertion of the lemma follows from (2.60).
As to the second claim, we compute

$$\begin{aligned}
-\frac{\partial^2 h_\nu}{\partial x_j \partial x_i} &= \sum_{n=1}^{2} \frac{\partial^2 k_n}{\partial x_j \partial x_i}\left(A_{12}^\nu(k,u)\frac{\partial A_{21}^\nu(k,u)}{\partial k_n} + A_{21}^\nu(k,u)\frac{\partial A_{12}^\nu(k,u)}{\partial k_n}\right) + \\
&+\sum_{n=1}^{2}\frac{\partial k_n}{\partial x_i}\left(\left\langle \frac{\partial A_{12}^\nu(k,u)}{\partial k}, \frac{\partial k}{\partial x_j}\right\rangle \frac{\partial A_{21}^\nu(k,u)}{\partial k_n} + \left\langle \frac{\partial A_{21}^\nu(k,u)}{\partial k}, \frac{\partial k}{\partial x_j}\right\rangle \frac{\partial A_{12}^\nu(k,u)}{\partial k_n}\right) + \\
&+\sum_{n=1}^{2}\frac{\partial k_n}{\partial x_i}\left(A_{12}^\nu(k,u)\left\langle \frac{\partial}{\partial k}\frac{\partial A_{21}^\nu(k,u)}{\partial k_n}, \frac{\partial k}{\partial x_j}\right\rangle + A_{21}^\nu(k,u)\left\langle \frac{\partial}{\partial k}\frac{\partial A_{12}^\nu(k,u)}{\partial k_n}, \frac{\partial k}{\partial x_j}\right\rangle\right).
\end{aligned} \tag{2.62}$$

The second summand in the middle is most easily to handle since all terms occuring there have already been estimated. Due to (2.61) and $\frac{\partial}{\partial k}A_{ij}^\nu(k,u) = o(1)$ as $|\nu| \to \infty$, the second summand is in $o\left(\frac{1}{|\nu|^2}\right)$ as $|\nu| \to \infty$. We don't know anything about the asymptotic behaviour of the terms $\frac{\partial^2 k_n}{\partial x_j \partial x_i}$ and $\frac{\partial}{\partial k}\frac{\partial A_{ij}^\nu(k,u)}{\partial k_n}$ appearing in the first and third summand, respectively, yet. These terms shall be considered now. We start with the term $\frac{\partial}{\partial k}\frac{\partial A_{ij}^\nu(k,u)}{\partial k_n}$. Thereto, we examine the second k-derivative of the operator $A(\mathbb{1} - BA)^{-1}$ (cf. (2.7)). We compute (cf. (2.39) and (2.40))

$$\begin{aligned}
&\frac{\partial^2}{\partial k^2}A(\mathbb{1} - BA)^{-1} = -\frac{\partial}{\partial k}[A(\mathbb{1} - BA)^{-1}B\widetilde{C}BA(\mathbb{1} - BA)^{-1}] = \\
&= -\frac{\partial}{\partial k}[A(\mathbb{1} - BA)^{-1}]B\widetilde{C}BA(\mathbb{1} - BA)^{-1} - A(\mathbb{1} - BA)^{-1}B\widetilde{C}B\frac{\partial}{\partial k}[A(\mathbb{1} - BA)^{-1}] - \\
&- A(\mathbb{1} - BA)^{-1}\frac{\partial}{\partial k}[B\widetilde{C}B]A(\mathbb{1} - BA)^{-1}.
\end{aligned} \tag{2.63}$$

The first two summands are $o(1)$ as $|\nu| \to \infty$ due to Lemma 2.2.7 and the boundedness of $B\widetilde{C}$ with respect to ν (as already seen in the proof of Theorem

2.4.2). As to the third summand, we compute

$$\mathbb{C}^2 \times \mathbb{C}^2 \ni (x,y) \mapsto \frac{\partial}{\partial k}[B\widetilde{C}B](x,y) =$$
$$= -B\widetilde{C}(x)B\widetilde{C}(y)B - B\widetilde{C}(y)B\widetilde{C}(x)B + B\left[\frac{\partial}{\partial k}\widetilde{C}\right](x,y)B,$$

where

$$\frac{\partial}{\partial k}\widetilde{C} : \mathbb{C}^2 \times \mathbb{C}^2 \ni (x,y) \mapsto -8\pi^2 diag(\langle x,y\rangle)_{\rho\in\Gamma^*\setminus\{0,\pm\nu\}}.$$

So, this derivative is both constant with respect to the point $k \in V$ where $\widetilde{C}$ has been evaluated and constant with respect to $\rho \in \Gamma^* \setminus \{0, \pm\nu\}$. Thus, the asymptotic behaviour of $B\left[\frac{\partial}{\partial k}\widetilde{C}\right]$ is even better than that of $B\widetilde{C}$. Four our purposes, however, it suffices that these operators have bounded operator norm with respect to ν. In every summand, the operator B whose norm tends to zero as $|\nu| \to \infty$ (cf. the norm of its Fourier transform estimated in Lemma 2.2.4) occurs. The remaining operators are uniformly bounded as is already well-known from the preceding investigations. This shows that

$$\frac{\partial^2}{\partial k^2} A(\mathbb{1} - BA)^{-1} \longrightarrow 0 \quad \text{as } |\nu| \to \infty$$

and in particular, for $i, j, n \in \{1, 2\}$

$$\frac{\partial}{\partial k}\frac{\partial A^\nu_{ij}(k,u)}{\partial k_n} \longrightarrow 0 \quad \text{as } |\nu| \to \infty. \tag{2.64}$$

It remains to estimate the term $\frac{\partial^2 k_n}{\partial x_i \partial x_j}$: Due to $\frac{\partial k_1}{\partial k} = (1,0)$, $\frac{\partial k_2}{\partial k} = (0,1)$, we get

$$(0,0) = \frac{\partial}{\partial x_i}\left(\frac{\partial k_n}{\partial k}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial k_n}{\partial x}\cdot\frac{\partial x}{\partial k}\right) = \left(\frac{\partial}{\partial x_i}\frac{\partial k_n}{\partial x}\right)\cdot\frac{\partial x}{\partial k} + \frac{\partial k_n}{\partial x}\cdot\frac{\partial}{\partial x_i}\left(\frac{\partial x}{\partial k}\right).$$
$$\Rightarrow \frac{\partial^2 k_n}{\partial x\,\partial x_i}\cdot\frac{\partial x}{\partial k} = -\frac{\partial k_n}{\partial x}\cdot\frac{\partial}{\partial x_i}\left(\frac{\partial x}{\partial k}\right). \tag{2.65}$$

Further (compare also [5, p. 236]),

$$\frac{\partial}{\partial x_i}\left(\frac{\partial k_n}{\partial x_j}\right) = \left(\frac{\partial}{\partial x}\frac{\partial k_n}{\partial x_j}\right)\cdot\frac{\partial x}{\partial k}\cdot\frac{\partial k}{\partial x_i} \overset{(2.65)}{=} -\frac{\partial k_n}{\partial x}\cdot\left(\frac{\partial}{\partial x_j}\frac{\partial x}{\partial k}\right)\cdot\frac{\partial k}{\partial x_i} =$$
$$= -\sum_{\alpha,\beta=1}^{2}\frac{\partial k_n}{\partial x_\alpha}\cdot\left(\frac{\partial}{\partial x_j}\frac{\partial x_\alpha}{\partial k_\beta}\right)\cdot\frac{\partial k_\beta}{\partial x_i} = -\sum_{\alpha,\beta,\gamma=1}^{2}\frac{\partial k_n}{\partial x_\alpha}\cdot\frac{\partial^2 x_\alpha}{\partial k_\gamma\,\partial k_\beta}\cdot\frac{\partial k_\gamma}{\partial x_j}\cdot\frac{\partial k_\beta}{\partial x_i}.$$

We show that the term $\frac{\partial^2 x_\alpha}{\partial k_\gamma \partial k_\beta}$ is bounded with respect to ν. Due to (2.61), the above computation then yields

$$\frac{\partial^2 k_n}{\partial x_i \partial x_j} = O\left(\frac{1}{|\nu|^3}\right), \quad |\nu| \to \infty. \tag{2.66}$$

By (2.56) and (2.64), we have

$$\frac{\partial^2 x_\alpha}{\partial k_\gamma^2} = 2 + \frac{\partial^2 A^\nu_{\alpha\alpha}(k,u)}{\partial k_\gamma^2} = 2 + o(1), \quad |\nu| \to \infty, \quad \text{if } \beta = \gamma$$
$$\frac{\partial^2 x_\alpha}{\partial k_\gamma \partial k_\beta} = \frac{\partial^2 A^\nu_{\alpha\alpha}(k,u)}{\partial k_\gamma \partial k_\beta} = o(1), \quad |\nu| \to \infty, \quad \text{if } \beta \neq \gamma,$$

such that in any case, $\frac{\partial^2 x_\alpha}{\partial k_\gamma \partial k_\beta}$ is bounded as $|\nu| \to \infty$. Now, by the estimates of $\frac{\partial^2 k_n}{\partial x_j \partial x_i}$ in (2.66) and $\frac{\partial}{\partial k} \frac{\partial A^\nu_{ij}(k,u)}{\partial k_n}$ in (2.64) together with (2.61) and the well-known results about the behaviour of $A^\nu_{ij}(k,u)$ and $\frac{\partial}{\partial k} A^\nu_{ij}(k,u)$ with respect to $|\nu| \to \infty$, also the first and the third summand of (2.62) are in $o\left(\frac{1}{|\nu|^2}\right)$ as $|\nu| \to \infty$. Hence, the claimed error term for $\frac{\partial^2 h_\nu}{\partial x_j \partial x_i}$ follows. □

With the lemma just proved, the conditions for the Quantitative Morse Lemma (Theorem 2.5.1) are now fulfilled. Due to that Theorem 2.5.1, there exists a biholomorphic map Φ from $D_{(r-s)(1-10b)}$ (with b, r, s as in Theorem 2.5.1) to a neighbourhood of ζ in D_r such that

$$f \circ \Phi(z_1, z_2) = z_1 z_2 - c$$

with a constant $c \in \mathbb{C}$ fulfilling $|c - h(\zeta)| \leq a^2$. With this transformation of coordinates, the (asymptotic) handles of the Fermi curve are isomorphic to the correspondent zero sets of $f \circ \Phi$, i.e.

$$\{z = (z_1, z_2) \in D_{(r-s)(1-10b)} : z_1 \cdot z_2 = c\}.$$

In the proof of [5, Lemma B.1, p. 246-248], it has been shown that it suffices to consider the special case $h(0,0) = \frac{\partial h}{\partial x_1}(0,0) = \frac{\partial h}{\partial x_2}(0,0) = 0$ without loss of generality, which leads to $c = 0$ in this special case. If we want to determine the handle quantity $c = c_\nu$ in our case, that special case is not very helpful. So let's retrace the part of the proof of [5, Lemma B.1] where the map ϕ is constructed, but this time without making any simplifying assumptions:

Theorem 2.5.4. *Let*

$$f(x_1, x_2) = x_1 x_2 + h(x_1, x_2)$$

be the function (2.50) *with* x*-coordinates* (2.56) *and the function* $h = h_\nu$ *defined in* (2.58). *Then the handle quantity* $c = c_\nu$ *obtained in* (2.52) *is equal to*

$$c_\nu = -(h_\nu(\zeta_\nu) + \zeta_{\nu,1} \cdot \zeta_{\nu,2}) = A_{12}^\nu(k(\zeta_\nu), u) \cdot A_{21}^\nu(k(\zeta_\nu), u) - \zeta_{\nu,1} \cdot \zeta_{\nu,2},$$

where $\zeta_\nu = (\zeta_{\nu,1}, \zeta_{\nu,2})$ *is the unique zero of* ∇f *in* D_r*, due to Theorem 2.5.1.*

Remark. If $\zeta_\nu = 0$, we obtain the handle quantities from the model curve (2.49), i.e. $c_\nu = \widetilde{c}_\nu$ by $k(0) = k_\nu$ (cf. (2.53), (2.56) and Defintion 2.3.3).

Proof. To simplify notations, we suppress the index ν. Set for $t \in [0, 1]$

$$f_t(x_1, x_2) := (x_1 - \zeta_1)(x_2 - \zeta_2) + t \cdot [x_1 x_2 + h(x) - (x_1 - \zeta_1)(x_2 - \zeta_2) - \zeta_1\zeta_2 - h(\zeta)].$$

$$\Rightarrow \nabla f_t(x) = \begin{pmatrix} x_2 - \zeta_2 + t\left(\frac{\partial h}{\partial x_1} + \zeta_2\right) \\ x_1 - \zeta_1 + t\left(\frac{\partial h}{\partial x_2} + \zeta_1\right) \end{pmatrix}.$$

Set further

$$\widetilde{h}(x) := x_1 x_2 + h(x) - (x_1 - \zeta_1)(x_2 - \zeta_2) - \zeta_1\zeta_2 - h(\zeta).$$

Similarly to [5, (B.1), p. 246], we search for a t-dependent holomorphic vector field $X^t : D_{r(1-4b)} \to \mathbb{C}^2$ that solves

$$\widetilde{h} + \nabla f_t \cdot X^t = 0. \tag{2.67}$$

This equation is equivalent to

$$\begin{aligned} &\widetilde{h}(x) + \left\langle \begin{pmatrix} x_2 - \zeta_2 + t\left(\frac{\partial h}{\partial x_1} + \zeta_2\right) \\ x_1 - \zeta_1 + t\left(\frac{\partial h}{\partial x_2} + \zeta_1\right) \end{pmatrix}, X^t(x) \right\rangle = 0 \Leftrightarrow \\ &\qquad \Leftrightarrow \widetilde{h}(x) + \left\langle \begin{pmatrix} x_2 - \zeta_2 + t\frac{\partial \widetilde{h}}{\partial x_1} \\ x_1 - \zeta_1 + t\frac{\partial \widetilde{h}}{\partial x_2} \end{pmatrix}, X^t(x) \right\rangle = 0 \Leftrightarrow g(y) = \left\langle \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, Y^t(y) \right\rangle, \end{aligned} \tag{2.68}$$

where

$$\begin{aligned} y = P_t(x) &:= \left(x_2 - \zeta_2 + t\left(\frac{\partial h}{\partial x_1} + \zeta_2\right), x_1 - \zeta_1 + t\left(\frac{\partial h}{\partial x_2} + \zeta_1\right)\right) \\ g(y) &:= -\widetilde{h} \circ P_t^{-1}(y), \quad Y^t(y) := X^t \circ P_t^{-1}(y). \end{aligned} \tag{2.69}$$

Note that the map $P_t : D_r \to \mathbb{C}^2$ is biholomorphic into its image due to Lemma 2.5.3 which becomes obvious by considering its Jacobian

$$D_x P_t = \begin{pmatrix} t\frac{\partial^2 h}{\partial x_1^2} & 1 + t\frac{\partial^2 h}{\partial x_2 \partial x_1} \\ 1 + t\frac{\partial^2 h}{\partial x_1 \partial x_2} & t\frac{\partial^2 h}{\partial x_2^2} \end{pmatrix}. \tag{2.70}$$

P_t doesn't map D_r onto D_r, but the image $P_t(D_r)$ contains $D_{r(1-2b)}$ (see [5, (B.6)]) Now, the last equivalence of (2.68) is obviously solved by $Y^t = (Y_1^t, Y_2^t)$ defined by

$$Y_1^t(y) := \frac{1}{y_1} g(y_1, 0), \quad Y_2^t := \frac{1}{y_2}(g(y_1, y_2) - g(y_1, 0)).$$

These functions are holomorphic in D_r as it has been shown in [5, p. 247] (or equivalently will follow from (2.78)). Thus, by setting $X^t := Y^t \circ P_t$, we obtain, according to (2.68), the desired vector field X^t defined on $P_t^{-1}(D_{r(1-2b)})$ solving (2.67). Since $P_t^{-1}(D_{r(1-2b)})$ contains $D_{r(1-4b)}$[11], this yields the claimed domain in which X^t should be defined. Now that we have constructed X^t, let's consider the initial value problem[12]

$$\frac{d}{dt}\Phi_t(x) = X^t(\Phi_t(x)), \quad \Phi_0(x) = x. \tag{2.71}$$

Since $x \mapsto X^t(x)$ is holomorphic (and continuous with respect to (t, x)), the problem (2.71) has a unique solution Φ_t by the Theorem of Picard-Lindelöf (cf. [24, Satz 2.2.2], for instance). We obtain

$$\frac{d}{dt} f_t(\Phi_t(x)) = \widetilde{h}(\Phi_t(x)) + \nabla f_t(\Phi_t(x)) \cdot X^t(\Phi_t(x)) \overset{(2.67)}{=} 0.$$

Hence, $f_t(\Phi_t(x))$ is constant with respect to t. Due to the initial value $\Phi_0(x) = x$, we therefore obtain $f_1(\Phi_1(x)) = f_0(x)$ which is equivalent to

$$f(\Phi_1(x)) - h(\zeta) - \zeta_1\zeta_2 = (x_1 - \zeta_2)(x_2 - \zeta_2).$$

Setting $z \mapsto \Phi(z) := \Phi_1(z + \zeta)$, we finally get

$$f \circ \Phi(z_1, z_2) = z_1 \cdot z_2 + \underbrace{h(\zeta) + \zeta_1 \cdot \zeta_2}_{=:-c}.$$

This proves the assertion. □

Let's recap how the map Φ looks like. Due to (2.71), we obtain $\Phi_1(z) = \Phi_0(z) + \int_0^1 X^t(\Phi_t(z))dt$ and thus, by definition of Φ and with $\Phi_0(z) = z$,

$$\begin{aligned} x = \Phi(z) = z + \zeta + \int_0^1 X^t(\Phi_t(z+\zeta))dt =: z - \epsilon(z), \\ \text{with } \epsilon(z) := -\zeta - \int_0^1 X^t(\Phi_t(z+\zeta))dt, \end{aligned} \tag{2.72}$$

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[11] See [5, p. 248], where we must use in our case the estimate $\left|\frac{d\widetilde{h}}{dx_i}\right| \leq b(|x_1 - \zeta_1| + |x_2 - \zeta_2|)$ instead of [5, (B.5)]. This estimate will be proved in (2.74) in Lemma 2.5.5. However, the exact definition of the domain, say $D_{r(1-4b)}$ or $D_{r(1-10b)}$, for instance, is finally immaterial for our purposes.

[12] Here, our proof differs from [5] since the map Φ_t in [5] obtained by just integrating X^t is not suitable.

again with suppressed index ν.
We want to investigate the map $k \mapsto x \mapsto z$ more precisely. We already know that, by $k \mapsto x$, k_ν is mapped to 0. An interesting question would be: By $x \mapsto z$, 0 is mapped to what? It would be nice if 0 was mapped to 0, but there is no evidence. Yet, we can show that, by $z \mapsto x$, 0 is mapped to the critical point ζ. In order to prove this, we need to show at first $X^t(\zeta) = 0$. This follows from the next lemma which provides even more, namely an estimate of X^t:

Lemma 2.5.5. *Using the notations of Theorem 2.5.1, there holds*

$$|X^t(x)| \leq o\left(\frac{1}{|\nu|^2}\right)(|x_1 - \zeta_1| + |x_2 - \zeta_2|), \quad \textit{as } \nu \to \infty, \tag{2.73}$$

for all $x \in D_r$.

Remark. The proof is based on the proof of [5, Lemma B.1] and uses its main ideas. Since the proof in [5] only treats the special case $\zeta = 0$, its difference to our proof is that we consider the general case including $\zeta \neq 0$ as we already did before.

Proof. We consider the map $\widetilde{h}$ defined in the proof of Theorem 2.5.4,

$$\widetilde{h}(x) := x_1 x_2 + h(x) - (x_1 - \zeta_1)(x_2 - \zeta_2) - \zeta_1\zeta_2 - h(\zeta).$$

By definition of the critical point $\zeta = (\zeta_1, \zeta_2)$, we have $\frac{\partial h(\zeta)}{\partial x_1} + \zeta_2 = 0$ as well as $\frac{\partial h(\zeta)}{\partial x_2} + \zeta_1 = 0$. Hence, $\nabla \widetilde{h}(\zeta) = (\frac{\partial h(\zeta)}{\partial x_1} + \zeta_2, \frac{\partial h(\zeta)}{\partial x_2} + \zeta_1) = (0, 0)$. By the Fundamental Theorem of Calculus, we have for $i \in \{1, 2\}$

$$\frac{\partial \widetilde{h}(x)}{\partial x_i} - \underbrace{\frac{\partial \widetilde{h}(\zeta)}{\partial x_i}}_{=0} = \int_0^1 \left(\frac{\partial^2 \widetilde{h}(\zeta + t(x - \zeta))}{\partial x_1 \partial x_i} \cdot (x_1 - \zeta_1) + \frac{\partial^2 \widetilde{h}(\zeta + t(x - \zeta))}{\partial x_2 \partial x_i} \cdot (x_2 - \zeta_2) \right) dt,$$

and hence by Lemma 2.5.3 (note that $\frac{\partial^2 \widetilde{h}(x)}{\partial x_i \partial x_j} = \frac{\partial^2 h(x)}{\partial x_i \partial x_j}$ for $i, j \in \{1, 2\}$),

$$\left| \frac{\partial \widetilde{h}(x)}{\partial x_i} \right| \leq b(|x_1 - \zeta_1| + |x_2 - \zeta_2|), \tag{2.74}$$

as $|\nu| \to \infty$, where we use (according to the notation of Theorem 2.5.1) the abbreviation b for the term $o(1/|\nu|^2)$, as $|\nu| \to \infty$. Therefore, for $i = 1, 2$ and $t \in [0, 1]$,

$$\left| \frac{\partial \widetilde{h}(\zeta + t(x - \zeta))}{\partial x_i} \right| \leq bt(|x_1 - \zeta_1| + |x_2 - \zeta_2|).$$

Again by the Fundamental Theorem of Calculus,

$$\widetilde{h}(x) - \underbrace{\widetilde{h}(\zeta)}_{=0} = \int_0^1 \left(\frac{\partial \widetilde{h}(\zeta + t(x-\zeta))}{\partial x_1} \cdot (x_1 - \zeta_1) + \frac{\partial \widetilde{h}(\zeta + t(x-\zeta))}{\partial x_2} \cdot (x_2 - \zeta_2) \right) dt,$$

this implies

$$|\widetilde{h}(x)| \leq b(|x_1 - \zeta_1| + |x_2 - \zeta_2|) \int_0^1 t dt |x_1 - \zeta_1| + b(|x_1 - \zeta_1| + |x_2 - \zeta_2|) \int_0^1 t dt |x_2 - \zeta_2| = \\ = \frac{b}{2}(|x_1 - \zeta_1| + |x_2 - \zeta_2|)^2.$$

With $y = P_t(x)$ as in (2.69), we get by (2.74)

$$\begin{aligned} |y_1| + |y_2| &\leq |x_1 - \zeta_1| + |x_2 - \zeta_2| + t \left(\left| \frac{\partial \widetilde{h}(x)}{\partial x_1} \right| + \left| \frac{\partial \widetilde{h}(x)}{\partial x_2} \right| \right) \leq \\ &\leq (1 + 2b)(|x_1 - \zeta_1| + |x_2 - \zeta_2|) \end{aligned} \tag{2.75}$$

as well as

$$\begin{aligned} |y_1| + |y_2| &\geq |x_1 - \zeta_1| + |x_2 - \zeta_2| - t \left(\left| \frac{\partial \widetilde{h}(x)}{\partial x_1} \right| + \left| \frac{\partial \widetilde{h}(x)}{\partial x_2} \right| \right) \geq \\ &\geq (1 - 2b)(|x_1 - \zeta_1| + |x_2 - \zeta_2|). \end{aligned} \tag{2.76}$$

Since $\frac{\partial^2 \widetilde{h}(x)}{\partial x_i \partial x_j} = \frac{\partial^2 h(x)}{\partial x_i \partial x_j}$ for $i, j \in \{1, 2\}$, we can use equations [5, (B.6),(B.7), p. 247], namely

$$\left\| D_x P_t - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\| = \left\| t \frac{\partial^2 h}{\partial x_i \, \partial x_j} \right\| \leq tb, \quad \left\| D_y P_t^{-1} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\| \leq \frac{tb}{1 - tb},$$

which also follow from (2.70). With g as in (2.69), we thus have together with (2.74) for $i \in \{1, 2\}$

$$\begin{aligned} \left| \frac{\partial g(y)}{\partial y_i} \right| &= \left| \frac{\partial \widetilde{h}(x)}{\partial x} \Big|_{x = P_t^{-1}(y)} \cdot \frac{\partial P_t^{-1}(y)}{\partial y_i} \right| \leq b(|x_1 - \zeta_1| + |x_2 - \zeta_2|) \cdot \left(1 + \frac{b}{1-b} \right) \leq \\ &\overset{(2.76)}{\leq} \frac{b}{(1-2b)(1-b)}(|y_1| + |y_2|) \leq b \frac{1+2b}{1-2b}(|y_1| + |y_2|), \end{aligned} \tag{2.77}$$

where in the last step, we used $\frac{1}{1-b} \leq 2$ for $b \leq \frac{1}{2}$ (this can surely be achieved for $|\nu|$ sufficiently large) which implies $\frac{1}{1-b} = 1 + \frac{b}{1-b} \leq 1 + 2b$. This yields, again together with the Fundamental Theorem of Calculus (by using $P_t(\zeta) = 0$ which implies $P_t^{-1}(0) = \zeta$ and $g(0) = -\widetilde{h}(\zeta) = 0$)

$$|g(y) - \underbrace{g(0)}_{=0}| \leq \int_0^1 |Dg(ty)y|dt \leq \int_0^1 \left(\left| \frac{\partial g(ty)}{\partial y_1} y_1 \right| + \left| \frac{\partial g(ty)}{\partial y_2} y_2 \right| \right) dt \leq$$
$$\leq b\frac{1+2b}{1-2b} \left((|y_1| + |y_2|)\frac{|y_1|}{2} + (|y_1| + |y_2|)\frac{|y_2|}{2} \right) = \frac{b}{2}\frac{1+2b}{1-2b}(|y_1| + |y_2|)^2. \quad (2.78)$$

Now, we have all estimates we need and can proceed more or less exactly as in the rest of the proof of [5, Lemma B.1]. For the sake of completeness, we don't refer to [5] but give the rest of the proof here anyway. We estimate $Y^t(y) = (Y_1^t(y), Y_2^t(y))$ defined in (2.69). We get

$$|Y_1^t(y)| = \frac{|g(y_1, 0)|}{|y_1|} \leq \frac{b}{2}\frac{1+2b}{1-2b}|y_1|.$$

For $Y_2^t(y)$, we discuss the cases $|y_2| \geq |y_1|$ and $|y_2| < |y_1|$ seperately. In the first case, it is

$$|Y_2^t(y)| \leq \frac{1}{|y_2|}(|g(y_1, y_2)| + |g(y_1, 0)|) \leq b\frac{1+2b}{1-2b}\frac{(|y_1| + |y_2|)^2}{|y_2|} =$$
$$= b\frac{1+2b}{1-2b}\left(\frac{|y_1|^2}{|y_2|} + 2|y_1| + |y_2| \right) \overset{|y_1|\leq|y_2|}{\leq} 2b\frac{1+2b}{1-2b}(|y_1| + |y_2|).$$

Now consider the case $|y_2| < |y_1|$. For fixed y_1, we apply the maximum principle (recall that $y \mapsto Y^t(y)$ is holomorphic) to $y_2 \mapsto Y_2^t(y_1, y_2)$ yielding for $|y_2| \leq |y_1|$

$$|Y_2^t(y)| \leq 2b\frac{1+2b}{1-2b}(|y_1| + |y_1|) = 4b\frac{1+2b}{1-2b}|y_1|.$$

Putting the estimates for Y_1^t and Y_2^t together yields

$$|Y^t(y)| \leq 4\sqrt{2}\, b\frac{1+2b}{1-2b}(|y_1| + |y_2|).$$

By (2.75), we finally get

$$|X^t(x)| = |Y^t(P_t(x))| \leq 4\sqrt{2}\, b\frac{(1+2b)^2}{1-2b}(|x_1 - \zeta_1| + |x_1 - \zeta_2|).$$

Since $b = o(1/|\nu|^2)$ as $|\nu| \to \infty$, the lemma is proved. □

Now let's go back to our initial question: We wanted to show that, by the map $z \mapsto x$, 0 is mapped to the critical point ζ. The statement (2.73) implies $X^t(\zeta) = 0$ for all $t \in [0, 1]$. Considering the corresponding initial value problem

$$\frac{d}{dt}\Phi_t(\zeta) = X^t(\Phi_t(\zeta)), \quad \Phi_0(\zeta) = \zeta,$$

the constant function $\Phi_t(\zeta) = \zeta$ for all $t \in [0,1]$ is obviously the (unique) solution of the problem. In particular, we have $\Phi_1(\zeta) = \zeta$ and thus $\Phi(0) = \Phi_1(\zeta) = \zeta$. To sum up, we have the following mapping

$$\begin{array}{ccccc} z & \longmapsto & x & \longmapsto & k \\ * & \longmapsto & 0 & \longmapsto & k_\nu \\ 0 & \longmapsto & \zeta_\nu & \longmapsto & * \end{array} \tag{2.79}$$

where the star $*$ shall denote corresponding values which haven't been determined, yet.
By the parameterization $z_1 \cdot z_2 = c_\nu$ of the ν-th excluded domain (up to isomorphy, of course), we can now give a characterization of an unsplit double point since that parameterization allows us to see immediately (by examining whether $c_\nu = 0$ or $c_\nu \neq 0$) if a double point splits or does not.

Corollary 2.5.6. *A double point in the ν-th excluded domain $(k_\nu^\pm(\hat{u}_0)+V) \cap F(u)$ for $\nu \in \Gamma_\delta^*$ remains* unsplit *if and only if the corresponding handle quantity c_ν vanishes, i.e.*

$$c_\nu = -h_\nu(\zeta_\nu) - \zeta_{\nu,1} \cdot \zeta_{\nu,2} = 0.$$

The double point splits up to a handle if and only if $c_\nu \neq 0$.

The handle quantities $\widetilde{c}_\nu$ (2.49) of the model curve are easily described by the product of perturbed Fourier coefficients $\check{u}_\nu \cdot \check{u}_{-\nu}$. One might ask the following question: Does $\widetilde{c}_\nu = 0$ already imply $c_\nu = 0$? In other words: Can the question whether a double point remains unsplit or not already be answered by looking at the handle quantities of the *model* curve? Indeed, for real-valued potentials, this is the case (see Theorem 2.5.9). If we require a little bit more than only the condition that the product $\check{u}_\nu \cdot \check{u}_{-\nu}$ vanishes, we can even deduce $c_\nu = 0$ for complex-valued potentials. More precisely, we have the following theorem.

Theorem 2.5.7. *Let $u \in L^2(F)$ and let $(\check{u}_\nu)_{\nu \in \Gamma_\delta^*}$ be its associated sequence of perturbed Fourier coefficients. Moreover, let $\check{u}_\nu = \check{u}_{-\nu} = 0$ for some $\nu \in \Gamma_\delta^*$. Then $c_\nu = 0$.*

Proof. By expanding the matrix M (2.14) into its Taylor series (see the proof of [13, Lemma 4.5.53]), we get with the usual notations (2.53)

$$\begin{pmatrix} 8\pi^2 \left\langle k - k_\nu, k_\nu^+(\hat{u}_0) + \frac{1}{2}\frac{\partial A_{11}^\nu(k_\nu,u)}{\partial k} \right\rangle & \check{u}_\nu + 4\pi^2 \left\langle k - k_\nu, \frac{\partial A_{12}^\nu(k_\nu,u)}{\partial k} \right\rangle \\ \check{u}_{-\nu} + 4\pi^2 \left\langle k - k_\nu, \frac{\partial A_{21}^\nu(k_\nu,u)}{\partial k} \right\rangle & 8\pi^2 \left\langle k - k_\nu, k_\nu^-(\hat{u}_0) + \frac{1}{2}\frac{\partial A_{22}^\nu(k_\nu,u)}{\partial k} \right\rangle \end{pmatrix} +$$
$$+ 4\pi^2 \begin{pmatrix} k^2 - k_\nu^2 & 0 \\ 0 & k^2 - k_\nu^2 \end{pmatrix} + O\left(|k - k_\nu|^2\right), \quad \text{as } k \to k_\nu.$$

Since $k^2 - k_\nu^2 = 2\langle k-k_\nu, k_\nu\rangle + (k-k_\nu)^2$, M is equal to

$$\begin{pmatrix} 8\pi^2\left\langle k-k_\nu, k_\nu^+(\hat{u}_0)+k_\nu+\frac{1}{2}\frac{\partial A_{11}^\nu(k_\nu,u)}{\partial k}\right\rangle & \check{u}_\nu + 4\pi^2\left\langle k-k_\nu, \frac{\partial A_{12}^\nu(k_\nu,u)}{\partial k}\right\rangle \\ \check{u}_{-\nu} + 4\pi^2\left\langle k-k_\nu, \frac{\partial A_{21}^\nu(k_\nu,u)}{\partial k}\right\rangle & 8\pi^2\left\langle k-k_\nu, k_\nu^-(\hat{u}_0)+k_\nu+\frac{1}{2}\frac{\partial A_{22}^\nu(k_\nu,u)}{\partial k}\right\rangle \end{pmatrix} + O\left(|k-k_\nu|^2\right),$$

as $k \to k_\nu$. We set $f : V \subset \mathbb{C}^2 \to \mathbb{C}$ defined by $f(k) := \det M$. Now let $\check{u}_\nu = \check{u}_{-\nu} = 0$, in particular $\widetilde{c}_\nu = 0$ by (2.49). We get by the above expansion $f(k_\nu) = 0$ (that is, k_ν is a point on the Fermi curve) as well as

$$\nabla f(k_\nu) = -4\pi^2\check{u}_\nu \cdot \frac{\partial A_{21}^\nu(k_\nu,u)}{\partial k} - 4\pi^2\check{u}_{-\nu} \cdot \frac{\partial A_{12}^\nu(k_\nu,u)}{\partial k} = 0,$$

that is, the Fermi curve has a singularity in the ν-th excluded domain at $k = k_\nu$. In other words, the corresponding double point remains unsplit and is, by the above calculation, equal to k_ν. Since

$$\nabla_x f(k(x))|_{x=k^{-1}(k_\nu)} = \nabla_k f(k)|_{k=k_\nu} \cdot \frac{dk}{dx}|_{x=k^{-1}(k_\nu)} = 0,$$

the critical point ζ_ν is mapped to k_ν by the map $x \mapsto k$. Consequently, in the diagram (2.79), the second and the third row are identical. This implies $\zeta_\nu = 0$. Hence, by the remark to Theorem 2.5.4, we get $c_\nu = \widetilde{c}_\nu = 0$, which had to be proved. □

This leads to the definition of so-called *finite type potentials*:

Definition 2.5.8. A potential $u \in L^2(F)$ is called *finite type potential* if

$$\check{u}_\nu = \check{u}_{-\nu} = 0$$

for all but finitely many $\nu \in \Gamma_\delta^*$ (with $\delta > 0$ sufficiently small such that the perturbed Fourier coefficients are well-defined, cf. Definition 2.3.3). The corresponding Fermi curve $F(u)$ is then called *Fermi curve of finite type* or simply *finite type Fermi curve.*

Due to Theorem 2.5.7, a Fermi curve $F(u)$ of a finite type potential $u \in L^2(F)$ has the property that at most a finite number of singularities splits up to a handle whereas infinitely many double points remain unsplit.
Now, we prove the converse of Theorem 2.5.7 for the case of *real-valued* potentials:

Theorem 2.5.9. *Let $u \in L^2(F)$ be real-valued, let $\delta > 0$ sufficiently small and $\nu \in \Gamma_\delta^*$. Then*

$$\widetilde{c}_\nu = 0 \iff c_\nu = 0.$$

Furthermore,

$$\frac{c_\nu}{\widetilde{c}_\nu} = 1 + o\left(\frac{1}{|\nu|^2}\right), \quad \text{as } |\nu| \to \infty$$

on the subsequence indexed by all $\nu \in \Gamma_\delta^*$ *satisfying* $\widetilde{c}_\nu \neq 0$. *This estimate is locally uniform in* u.

Proof. Let $\nu \in \Gamma_\delta^*$. Assume at first that $\widetilde{c}_\nu = 0$. By $\overline{\breve{u}_\nu} = \breve{u}_{-\nu}$ (cf. Corollary 2.3.6), this implies $\breve{u}_{-\nu} = \breve{u}_\nu = 0$, cf. (2.49). Due to Theorem 2.5.7, $c_\nu = 0$ follows.
Conversely, let $\widetilde{c}_\nu \neq 0$ (and consequently $|\breve{u}_\nu| = |\breve{u}_{-\nu}| \neq 0$, again by Corollary 2.3.6). We have to prove $c_\nu \neq 0$. Thereto, we estimate the quotient

$$\frac{c_\nu}{\widetilde{c}_\nu} = \frac{h_\nu(\zeta_\nu) + \zeta_{\nu,1} \cdot \zeta_{\nu,2}}{h_\nu(0)},$$

with respect to the limit behaviour $|\nu| \to \infty$, i.e. we consider at first arbitrary (non-fixed) $\nu \in \Gamma_\delta^*$ fulfilling $\widetilde{c}_\nu \neq 0$. Thus, in the following (even though it is not always explicitly mentioned), all sequences are indexed by $\nu \in \Gamma_\delta^*$ with $\widetilde{c}_\nu \neq 0$. These are subsequences[13]. Firstly, we estimate the critical point ζ_ν. By definition of ζ_ν, we have (cf. (2.60))

$$\begin{aligned}
\zeta_{\nu,2} &= -\frac{\partial h_\nu(\zeta_\nu)}{\partial x_1} = \\
&= \left(A_{12}^\nu(k(\zeta_\nu), u)\frac{\partial}{\partial k}A_{21}^\nu(k(\zeta_\nu), u) + A_{21}^\nu(k(\zeta_\nu), u)\frac{\partial}{\partial k}A_{12}^\nu(k(\zeta_\nu), u)\right) \cdot \frac{\partial k(\zeta_\nu)}{\partial x_1}, \\
\zeta_{\nu,1} &= -\frac{\partial h_\nu(\zeta_\nu)}{\partial x_2} = \\
&= \left(A_{12}^\nu(k(\zeta_\nu), u)\frac{\partial}{\partial k}A_{21}^\nu(k(\zeta_\nu), u) + A_{21}^\nu(k(\zeta_\nu), u)\frac{\partial}{\partial k}A_{12}^\nu(k(\zeta_\nu), u)\right) \cdot \frac{\partial k(\zeta_\nu)}{\partial x_2}.
\end{aligned} \tag{2.80}$$

By the Mean Value Theorem (cf. [30, Satz III.5.4(b)]), we have

$$|A_{12}^\nu(k(\zeta_\nu), u) - A_{12}^\nu(k(0), u)| \leq \sup_{k(x) \in V} |\nabla_x A_{12}^\nu(k(x), u)| \cdot |\zeta_\nu| = o\left(\frac{1}{|\nu|}\right) \cdot |\zeta_\nu|,$$

as $|\nu| \to \infty$, since

$$\nabla_x A_{12}^\nu(k(x), u) = \nabla_k A_{12}^\nu(k(x), u) \cdot \frac{dk(x)}{dx} = o\left(\frac{1}{|\nu|}\right), \quad \text{as } |\nu| \to \infty$$

[13]If there are only finitely many $\nu \in \Gamma_\delta^*$ with $\widetilde{c}_\nu \neq 0$, there is, of course, nothing to prove since then, the implication $c_\nu = 0 \Rightarrow \widetilde{c}_\nu = 0$ is trivially fulfilled for $\nu \in \Gamma_\delta^*$ by choosing $\delta > 0$ suitably smaller.

due to Lemma 2.2.7 and (2.61). We thus obtain

$$|A_{12}^{\nu}(k(\zeta_\nu), u)| \le o\left(\frac{1}{|\nu|}\right) \cdot |\zeta_\nu| + \frac{|\check{u}_\nu|}{4\pi^2}, \quad \text{as } |\nu| \to \infty$$

since $A_{12}^{\nu}(k(0), u) = A_{12}^{\nu}(k_\nu, u) = \check{u}_\nu / 4\pi^2$ per definitionem (cf. (2.53) and Definition 2.3.3). In the same fashion, we obtain

$$|A_{21}^{\nu}(k(\zeta_\nu), u)| \le o\left(\frac{1}{|\nu|}\right) \cdot |\zeta_\nu| + \frac{|\check{u}_{-\nu}|}{4\pi^2}, \quad \text{as } |\nu| \to \infty$$

and moreover for all $t \in [0,1]$ (note that $|\check{u}_\nu| = |\check{u}_{-\nu}|$ due to the reality of u)

$$|A_{ij}^{\nu}(k(t\zeta_\nu), u)| \le o\left(\frac{1}{|\nu|}\right) \cdot |\zeta_\nu| + \frac{|\check{u}_\nu|}{4\pi^2}, \quad \text{as } |\nu| \to \infty \tag{2.81}$$

for $i, j \in \{1, 2\}, i \neq j$. Together with (2.80) and (2.61), this yields by using once again $\frac{\partial}{\partial k} A_{ij}^{\nu}(k, u) = o(1)$ as $|\nu| \to \infty$

$$|\zeta_\nu| \le o(1) \cdot \left(o\left(\frac{1}{|\nu|}\right) \cdot |\zeta_\nu| + |\check{u}_\nu| + o\left(\frac{1}{|\nu|}\right) \cdot |\zeta_\nu| + |\check{u}_{-\nu}| \right) \cdot O\left(\frac{1}{|\nu|}\right) =$$
$$= o\left(\frac{1}{|\nu|}\right) \left(|\zeta_\nu| + |\check{u}_\nu|\right), \quad \text{as } |\nu| \to \infty.$$

Solving for $|\zeta_\nu|$ yields

$$|\zeta_\nu| \le \frac{|\check{u}_\nu|}{1 + o\left(\frac{1}{|\nu|}\right)} \cdot o\left(\frac{1}{|\nu|}\right) = |\check{u}_\nu| \cdot o\left(\frac{1}{|\nu|}\right), \quad \text{as } |\nu| \to \infty, \tag{2.82}$$

in particular $\lim_{|\nu| \to \infty} |\zeta_\nu| = 0$.

By the Fundamental Theorem of Calculus, we have

$$h_\nu(\zeta_\nu) = h_\nu(0) + \int_0^1 \nabla h_\nu(t\zeta_\nu) dt \cdot \zeta_\nu.$$

Due to (2.60), (2.61), (2.81), (2.82) and Lemma 2.2.7, we get

$$|\nabla h_\nu(t\zeta_\nu)| \le |\check{u}_\nu| \cdot o\left(\frac{1}{|\nu|}\right), \quad \text{as } |\nu| \to \infty$$

uniformly in $t \in [0,1]$. Hence, due to (2.82) and $h_\nu(0) = \widetilde{c}_\nu$ (2.49),

$$\left| \frac{h_\nu(\zeta_\nu)}{h_\nu(0)} \right| \le 1 + \sup_{t \in [0,1]} \left| \frac{\nabla h_\nu(t\zeta_\nu) \cdot \zeta_\nu}{h_\nu(0)} \right| = 1 + o\left(\frac{1}{|\nu|^2}\right), \quad \text{as } |\nu| \to \infty.$$

This shows

$$\lim_{|\nu|\to\infty} \frac{h_\nu(\zeta_\nu)}{h_\nu(0)} = 1,$$

still considered on the subsequence indexed by $\nu \in \Gamma^*_\delta$ with $\widetilde{c}_\nu \neq 0$. Further, we have, again by (2.82),

$$\frac{\zeta_{\nu,1} \cdot \zeta_{\nu,2}}{h_\nu(0)} = o\left(\frac{1}{|\nu|^2}\right) \cdot \frac{|\check{u}_\nu|^2}{|\check{u}_\nu|^2} = o\left(\frac{1}{|\nu|^2}\right), \quad \text{as } |\nu| \to \infty,$$

which shows

$$\lim_{|\nu|\to\infty} \frac{\zeta_{\nu,1} \cdot \zeta_{\nu,2}}{h_\nu(0)} = 0.$$

Hence,

$$\frac{c_\nu}{\widetilde{c}_\nu} = \frac{h_\nu(\zeta_\nu)}{h_\nu(0)} + \frac{\zeta_{\nu,1} \cdot \zeta_{\nu,2}}{h_\nu(0)} = 1 + o\left(\frac{1}{|\nu|^2}\right), \quad \text{as } |\nu| \to \infty.$$

Now, choose $\delta > 0$ so small such that

$$\frac{|h_\nu(\zeta_\nu)|}{|h_\nu(0)|} > \frac{1}{2} \quad \wedge \quad \frac{|\zeta_{\nu,1} \cdot \zeta_{\nu,2}|}{|h_\nu(0)|} < \frac{1}{2} \quad \text{for all } \nu \in \Gamma^*_\delta,\ \widetilde{c}_\nu \neq 0 \tag{2.83}$$

which is possible due to the limits computed above. Now, let $\widetilde{c}_\nu \neq 0$ for some fixed $\nu \in \Gamma^*_\delta$. Suppose $c_\nu = 0$. Then, $h_\nu(\zeta_\nu) + \zeta_{\nu,1} \cdot \zeta_{\nu,2} = 0$. Hence, $\frac{h_\nu(\zeta_\nu)}{h_\nu(0)} = \frac{-\zeta_{\nu,1}\cdot\zeta_{\nu,2}}{h_\nu(0)}$, which is a contradiction to (2.83). Therefore, $c_\nu \neq 0$, which had to be proved. □

2.6 Definition of the moduli

In this section, we introduce a data set which shall characterize a given Fermi curve and distingush two different Fermi curves from each other: the so-called *moduli*. The crucial statement in this context will be Theorem 4.3.2. Until we can prove it, there is yet a long way to go.
At first, we need to recap the *first homology group* $H_1(X, \mathbb{Z})$ of a given compact Riemann surface X of genus g (see [5, Introduction to chapter 1]) with its canonical *homology basis* of A-cycles $A_1, \dots, A_g$ and B-cycles $B_1, \dots B_g$. In [5, Def. 1.10], the concept of a canonical homology basis $A_1, B_1, A_2, B_2, A_3, B_3, \dots$ is generalized to Riemann surfase of *infinite* genus: More precisely, the cycles satisfy[14] $A_i \times B_j = \delta_{ij}$, $A_i \times A_j = B_i \times B_j = 0$ and for every submanifold $Y \subset X$ with boundary there is an $n \in \mathbb{N}$ such that the range of the canonical map $H_1(Y, \mathbb{Z}) \to H_1^\natural(X, \mathbb{Z})$ is contained in the span of A_i, B_i, $i = 1, \dots, n$,

[14] As in [5], we denote by $\gamma_1 \times \gamma_2$ the *intersection number* of two cycles $\gamma_1, \gamma_2 \in H_1(X, \mathbb{Z})$.

where $H_1^\natural(X,\mathbb{Z}) := H_1(X,\mathbb{Z})/(\text{subgroup of dividing cycles})$. Here, $\sigma \in H_1(X,\mathbb{Z})$ is called a *dividing cycle* if $\sigma \times \tau = 0$ for all $\tau \in H_1(X,\mathbb{Z})$. The authors of [5] refer to [1, Chapter 1], where it has been shown that there is a canonical homology basis for every Riemann surface. In the definition of our moduli, we only need the A-cycles so that we don't need to care about the B-cycles in this context. The A-cycles are just the countour cycles around the waists of the handles (compare [5, p. 43], in particular the representation of the handle in [5, Lemma 4.3]). If the Fermi curve is smooth, i.e. it has no singularities, we can obviously choose the A-cycles such that they are pairwise disjoint. Now, it may happen that our Fermi curves have singularities. In these cases, the waist of the corresponding handle (and thus the corresponding A-cycle) is contracted into one point[15]. In the definition of the moduli, this will be reflected in the fact that the corresponding contour integral (as integral of the locally bounded 1-form $k_1 dk_2$ over one point) will be equal to zero. Due to [27, p. 152, **4.**], where the moduli have been defined in a more general setting (namely for Fermi curves of *Dirac operators*[16] instead of Schrödinger operators), the A-cycles A_ν can be indexed by the dual lattice vectors $\nu \in \Gamma^*$. However, we have to make a small exception concerning the cycle A_ν corresponding to $\nu = 0$. If $F(u)$ is a Fermi curve of finite type, its normalization can be compactified by adding two points "at infinity", cf. [19]. It turns out that in this two-point-compactification, the cycle A_0 is homologous to zero and hence not an element of the homology basis. If the Fermi curve, however, is of *infinite* type, such a compactification is not possible. In this case, A_0 is in general not homologous to zero. As a motivation why we neglect A_0 also in this case, we only mention that A_0 is a dividing cycle, i.e. $A_0 = 0$ as an element of $H_1^\natural(X,\mathbb{Z})$. This fact won't, however, be needed in the following so that we don't prove it here. Indeed, the only parts of this work where the question whether we consider the A-cycles for all $\nu \in \Gamma^*$ or only for $\nu \in \Gamma^* \setminus \{0\}$ plays an essential role are those parts where we make use of the linear independence of the cycles A_ν. For example, in Section 4.1, it will be necessary to exclude the cycle A_0 in the corresponding considerations since in Lemma 4.1.2, for instance, it is essential that the appearing A-cycles are linearly independent.
Nevertheless, we formally define the moduli m_ν for *all* $\nu \in \Gamma^*$ despite the uselessness of the modulus m_0. We do this in order to avoid that the notations appearing in the following are too cumbersome. For instance, we will show in the sequel that the moduli are l^1-sequences. We then prefer writing $l^1(\Gamma^*)$ instead of $l^1(\Gamma^* \setminus \{0\})$ for sake of legibility.
But now, let's finally define the moduli (cf. [27, p. 152, **4.**] and [13, Def. 4.5.62]).

Definition 2.6.1. Let $F(u)$ be a Fermi curve (which may have singularities) of a given potential $u \in L^2(F)$, let $(A_\nu)_{\nu\in\Gamma^*}$ be the sequence of A-cycles (whose

[15] In the asymptotic part of the Fermi curve where double points are the only singularities which can occur, this point is just a double point.

[16] In Section 4.1, we will deal with this more general Dirac Fermi curves as well.

elements may also be contracted into one single point in the case a singularity occurs). Then the sequence of moduli $(m_\nu(u))_\nu$, indexed by $\nu \in \Gamma^*$, is defined by

$$m_\nu(u) := -16\pi^3 \int_{A_\nu} k_1 dk_2.$$

As to the *model* Fermi curve (cf. p. 47), we indicate the moduli with a tilde as we did before with the quantities related to the model curve,

$$\widetilde{m}_\nu(u) := -16\pi^3 \int_{\widetilde{A}_\nu} k_1 dk_2 \quad \text{in the model case,}$$

with a corresponding sequence of cycles $(\widetilde{A}_\nu)_{\nu \in \Gamma^*_\delta}$ (which can be considered as the waists of the handles of the model Fermi curve provided that the corresponding double points split up to handles).

We have already seen that the model Fermi curve is a lot easier to handle than the actual Fermi curve. The computation of the handle quantities $\widetilde{c}_\nu$ and the corresponding coordinates $\widetilde{z}_1, \widetilde{z}_2$ satisfying $\widetilde{z}_1 \cdot \widetilde{z}_2 = \widetilde{c}_\nu$ didn't cause any problems. The same holds for the moduli, at least for those in the asymptotic part of the Fermi curve, i.e. those with index $\nu \in \Gamma^*_\delta$ (for $\delta > 0$ sufficiently small). One obtains[17] (cf. [13, proof of Theorem 4.5.56])

$$\widetilde{m}_\nu(u) = \frac{\check{u}_\nu \cdot \check{u}_{-\nu}}{|\nu|^2 \xi} \overset{(2.49)}{=} 16\pi^4 \frac{\widetilde{c}_\nu}{|\nu|^2 \xi}, \quad \nu \in \Gamma^*_\delta, \tag{2.84}$$

which shows a connection to the handle quantities $\widetilde{c}_\nu$ (2.49). Besides, we have the obvious implications for $\nu \in \Gamma^*_\delta$

$$\begin{aligned} \widetilde{c}_\nu = 0 &\iff \widetilde{m}_\nu(u) = 0, \\ c_\nu = 0 &\implies m_\nu(u) = 0, \end{aligned} \tag{2.85}$$

where in the last implication, remember that due to an unsplit double point $(c_\nu = 0)$, the corresponding cycle A_ν degenerates to a point such that the countour integral in the definition of the modulus $m_\nu(u)$ consequently vanishes $(m_\nu(u) = 0)$. In fact, for *real-valued* potentials, all of the four identities in (2.85) are equivalent to one another: Due to Theorem 2.5.9, we have $\widetilde{c}_\nu = 0 \;\Leftrightarrow\; c_\nu = 0$. Moreover, we will see in Lemma 3.2.2 that $\widetilde{m}_\nu(u) = 0 \;\Leftrightarrow\; m_\nu(u) = 0$ holds

[17] The representation $\widetilde{m}_\nu(u) = \frac{\check{u}_\nu \cdot \check{u}_{-\nu}}{|\nu|^2 \xi}$ will be used very often in the following. This is, by the way, the reason why in Definition 2.6.1, there appears a factor $-16\pi^3$ in front of the respective contour integrals. Clearly, one could also define the moduli without this normalizing factor. But then it would permanently appear in the term for $\widetilde{m}_\nu(u)$ instead. By the way, in [13, Definitions 4.5.55, 4.5.62], there appears the factor $16\pi^3$ (without the minus sign). The reason why [13, (4.5.59)] still yields the same result as (2.84) is that in [13], there occurred a wrong sign in the computations (namely $-16\pi^4$ instead of $16\pi^4$ in [13, (4.5.54)]).

because of the estimate $m_\nu(u) = \widetilde{m}_\nu(1+o(1))$ as $|\nu| \to \infty$ which will be shown in that lemma. Thus, for *real-valued* potentials $u \in L^2(F)$, we anticipate for $\nu \in \Gamma^*_\delta$ the equivalences

$$c_\nu = 0 \iff \widetilde{c}_\nu(u) = 0 \iff \widetilde{m}_\nu = 0 \iff m_\nu(u) = 0,$$

provided that $\delta > 0$ is sufficiently small. So far, this is all we have to know about the moduli in order to investigate the asymptotic isospectral set in the next chapter. When we determine the entire isospectral set later in Chapter 4, we will get to know some more properties of the moduli.

Chapter 3

The isospectral problem I: Asymptotics

As already mentioned in Section 1.3, the main goal of this work is to determine the isospectral set

$$Iso_F(u_0) := \{u \in L^2(F),\ u \text{ real-valued } : F(u) = F(u_0)\}$$

(the subscript F in $Iso_F(u_0)$ stands for *Fermi curve*) for a given real-valued potential $u_0 \in L^2(F)$. It will turn out to be convenient to use the moduli $m(u) := (m_\nu(u))_{\nu\in\Gamma^*}$ as a data set locally characterizing the Fermi curve in the sense that $m(u) = m(u_0) \iff F(u) = F(u_0)$. This important relation between moduli and Fermi curves will be shown in Theorem 4.3.2. We are not able to show this equivalence at this point, yet, since its proof will require further properties of the moduli we will only get to know in subsequent sections.
In this chapter, we want to determine the *asymptotical* isospectral set $Iso_\delta(u_0)$. Before we define it, we want to introduce some notation. Recall the map (2.31) between potentials u, associated Fourier coefficients $\hat{u}$ and associated perturbed Fourier coefficients $\check{u}$. Let

$$P : U \subset L^2(F) \to l^2(\Gamma^*_\delta), \quad u \mapsto (\check{u}_\nu)_{\nu\in\Gamma^*_\delta} \tag{3.1}$$

be the map which assigns to every potential its associated sequence of perturbed Fourier coefficients in some neighbourhood U of the given $u_0 \in L^2(F)$. Provided that the first finitely many Fourier coefficients are kept constant, Theorem 2.4.2 implies that the map $\hat{u} \mapsto \check{u}$ (2.32) is locally boundedly invertible on $l^2(\Gamma^*_\delta)$ for $\delta > 0$ sufficiently small depending on u_0. That is, there exist neighbourhoods $\check{U}$ of $P(u_0)$ and $\hat{U}$ of $\hat{u}_0$ (the Fourier transform of u_0) such that every $\check{u} \in \check{U}$ is mapped to a $\hat{u} \in \hat{U}$. How do the corresponding potentials u look like? Since we have the embedding $l^2(\Gamma^*_\delta) \subseteq l^2(\Gamma^*)$ by setting for $(a_\nu)_\nu \in l^2(\Gamma^*_\delta)$ the first finitely many elements equal to zero, more precisely $a_\nu := 0$ for $|\nu| \le \delta^{-1}$, the map P (3.1) is locally invertible if we restrict P to

$$L^2_{\delta,0}(F) := \{u \in L^2(F) : \hat{u}(\nu) = 0 \quad \text{for all } \nu \in \Gamma^* \setminus \Gamma^*_\delta\}.$$

However, the first finitely many Fourier coefficients needn't necessarily be equal to zero for P to be locally invertible. It also suffices that the first finitely many Fourier coefficients are chosen to be constant (this constant needn't be equal to zero). Since we are interested in the isospectral set $Iso(u_0)$ for some given $u_0 \in L^2(F)$, it will turn out to be suitable to choose this constant to be equal to the first finitely many Fourier coefficients $\hat{u}_0(\nu)$, $\nu \in \Gamma^* \setminus \Gamma^*_\delta$ of the given potential u_0. More precisely, we define

$$L^2_{\delta,u_0}(F) := \{u \in L^2(F) : \hat{u}(\nu) = \hat{u}_0(\nu) \quad \text{for all } \nu \in \Gamma^* \setminus \Gamma^*_\delta\}. \tag{3.2}$$

If in the following, we write $P^{-1}(a)$ for $a \in l^2(\Gamma^*_\delta)$, we always mean $P^{-1}(a) \in L^2_{\delta,u_0}(F)$ such that the inverse P^{-1} is well-defined. Since we are interested in real isospectral sets, we introduce

$$l^2_{\mathbb{R}}(\Gamma^*_\delta) := \{(u_\nu)_\nu \in l^2(\Gamma^*_\delta) : \overline{u_\nu} = u_{-\nu} \quad \text{for all } \nu \in \Gamma^*_\delta\}$$

(compare the reality condition in Corollary 2.4.4). The *asymptotic isospectral set* $Iso_\delta(u_0)$ for given real $u_0 \in L^2(F)$ is now defined as follows:

$$Iso_\delta(u_0) := \{(u_\nu)_\nu \in l^2_{\mathbb{R}}(\Gamma^*_\delta) : m_\nu(u) = m_\nu(u_0) \text{ for all } \nu \in \Gamma^*_\delta, \quad u := P^{-1}((u_\nu)_\nu)\}, \tag{3.3}$$

where we implicitly use the convention that $(u_\nu)_\nu \notin Iso_\delta(u_0)$ if $(u_\nu)_\nu \notin \check{U}$ (since then $P^{-1}((u_\nu)_\nu)$ doesn't exist in general). Let's briefly comment on this definition (3.3): By identifying potentials with perturbed Fourier coefficients (in the usual asymptotic sense), we clearly haven't determined the potential completely, yet. In other words, finitely many degrees of freedom (namely the Fourier coefficients for $\nu \in \Gamma^* \setminus \Gamma^*_\delta$) remain. On the other hand, a Fermi curve isn't described by $(m_\nu(u))_{\nu\in\Gamma^*_\delta}$, yet. Finitely many restrictions remain to be fulfilled (namely $m_\nu(u) = m_\nu(u_0)$ for all $\nu \in \Gamma^* \setminus \Gamma^*_\delta$). Thus, we have as many open degrees of freedom as open restrictions. Note, however, that the isospectral sequences $(\check{u}_\nu)_\nu \in l^2_{\mathbb{R}}(\Gamma^*_\delta)$ determined in this chapter may vary when we take in a later step the remaining degrees of freedom and restrictions into consideration, too.

3.1 Isospectral flows of the model Fermi curve

In this section, we want to determine asymptotically the isospectral set of a given *model* Fermi curve by periodic isospectral flows. We start our considerations with the model Fermi curve since things turn out to be easier in the model case. In a later step in the next section, we will use a perturbation of the model isospectral set in order to gain the isospectral set for the actual curve. Let $u_0 \in L^2(F)$ be a given potential. Analogously to (3.3), we define the *asymptotic model isospectral set* $\widetilde{Iso}_\delta(u_0)$ by

$$\begin{aligned}\widetilde{Iso}_\delta(u_0) &:= \{(u_\nu)_\nu \in l^2_{\mathbb{R}}(\Gamma^*_\delta) : \widetilde{m}_\nu(u) = \widetilde{m}_\nu(u_0) \text{ for all } \nu \in \Gamma^*_\delta, \quad u := P^{-1}((u_\nu)_\nu)\} \\ &= \{(u_\nu)_\nu \in l^2_{\mathbb{R}}(\Gamma^*_\delta) : \quad u_\nu \cdot u_{-\nu} = \check{u}_{0,\nu} \cdot \check{u}_{0,-\nu} \quad \text{for all } \nu \in \Gamma^*_\delta\}\end{aligned} \tag{3.4}$$

Note that the quantity ξ appearing in the denominater of (2.84) remains invariant within the asymptotic model isospectral set since it only depends on the constant part (cf. (2.1)) of the given potential u_0 (compare the definition of $L^2_{\delta,u_0}(F)$). Thus, we can restrict ourselves to the numerator of (2.84). However, we must keep in mind that the homeomorphism between potentials (or their corresponding sequence of Fourier coefficients) and perturbed Fourier coefficients is defined only locally in $B_R(\hat{u}_0)$ [1] in the sense of Theorem 2.4.2. Hence, we must ensure that the isospectral flows don't leave this domain $B_R(\hat{u}_0)$. The following theorem gives a parameterization of $\widetilde{Iso}_\delta(u_0)$ for *real-valued* potentials:

Theorem 3.1.1. *Let $u_0 \in L^2(F)$ be a given real-valued potential. Then*

$$\widetilde{Iso}_\delta(u_0) = \times_{\nu \in \Gamma^*_\delta/\sigma} \{(e^{it}\check{u}_{0,\nu}, e^{-it}\check{u}_{0,-\nu}) : t \in [0, 2\pi)\}.$$

That is, $\widetilde{Iso}_\delta(u_0)$ is in general [2] *isomorphic to an infinite-dimensional torus since it can be described as an infinite Cartesian product where every factor consists of a pair of circles (each one isomorphic to the circle S^1) that are run through in opposite directions.*

Remark. Here, the quotient of the lattice with respect to the involution σ, namely Γ^*_δ/σ, has the following meaning: $\nu, \kappa \in \Gamma^*_\delta$ are equivalent in Γ^*_δ/σ if and only if $\nu = \kappa$ or $\nu = \sigma(\kappa) = -\kappa$. We divide by the involution σ in order not to count the pairs $(\nu, -\nu)$ doubly.

Proof. At first, let's recall the model moduli (2.84)

$$\widetilde{m}_\nu(u) = \frac{\check{u}_\nu \cdot \check{u}_{-\nu}}{|\nu|^2\xi}, \quad \nu \in \Gamma^*_\delta.$$

Since the sequence of perturbed Fourier coefficients $(\check{u}_\nu)_\nu$ is in $l^2(\Gamma^*_\delta)$ because of Theorem 2.4.1, it follows $(\widetilde{m}_\nu(u))_\nu \in l^1(\Gamma^*_\delta)$ due to Hölder's inequality. Now, let the potential $u \in L^2(F)$ be real-valued. Due to Corollary 2.3.6, we have $\overline{\check{u}_\nu} = \check{u}_{-\nu}$ for all $\nu \in \Gamma^*_\delta$. Therefore, $\check{u}_\nu \cdot \check{u}_{-\nu} = |\check{u}_\nu|^2 \geq 0$. Furthermore, $\xi = \sqrt{1 + 4\frac{\hat{u}_0}{\nu^2}} > 0$ for $\delta > 0$ sufficiently small since $\hat{u}_0$ is real for real-valued potential u and the radicand of ξ is positive for $\delta > 0$ sufficiently small. This shows that the model moduli $(\widetilde{m}_\nu(u))_\nu$ are a sequence of non-negative real numbers. Moreover, they are *even*, that is, $\widetilde{m}_\nu(u) = \widetilde{m}_{-\nu}(u)$ for all $\nu \in \Gamma^*_\delta$ (the evenness obviously also holds for non-real potentials). To sum up,

$$(\widetilde{m}_\nu(u))_\nu \in \frac{1}{|\nu|^2\xi} \cdot l^1_{+,e}(\Gamma^*_\delta) \subseteq l^1_{+,e}(\Gamma^*_\delta), \tag{3.5}$$

[1] Here, $\hat{u}_0$ denotes the Fourier transform of the given potential $u_0 \in L^2(F)$, not to be confused with the constant part of a potential defined in (2.1).

[2] Of course, if all but a finite number of $\check{u}_{0,\nu}$, $\nu \in \Gamma^*_\delta$ are equal to zero, the torus is only finite-dimensional. Likewise, if all $\check{u}_{0,\nu}$ are equal to zero, we wouldn't speak of a torus, at all.

where the subscripts e and $+$ denote evenness and non-negativity[3], respectively, and $(\widetilde{m}_\nu)_\nu \in \frac{1}{|\nu|^2\xi} \cdot l^1(\Gamma^*_\delta)$ shall signify $(\widetilde{m}_\nu \cdot |\nu|^2\xi)_\nu \in l^1(\Gamma^*_\delta)$. The map

$$l^2(\Gamma^*_\delta) \to l^1_{+,e}(\Gamma^*_\delta), \quad (u_\nu)_\nu \mapsto (u_\nu \cdot u_{-\nu})_\nu$$

is onto. Indeed, for $(b_\nu)_\nu \in l^1_{+,e}(\Gamma^*_\delta)$, define the sequence $(a_\nu)_\nu \in l^2(\Gamma^*_\delta)$ by $a_\nu := \sqrt{b_\nu}$ for all $\nu \in \Gamma^*_\delta$. Since the b_ν are non-negative, the sequence of the a_ν is well-defined. We obtain $a_\nu \cdot a_{-\nu} = \sqrt{|b_\nu|^2} = b_\nu$ for all $\nu \in \Gamma^*_\delta$, which shows that the considered map is onto. Now, we want to determine the complete fibre, that is, for given $(b_\nu)_\nu \in l^1_{+,e}(\Gamma^*_\delta)$ and given $\nu \in \Gamma^*_\delta$, find all $a_\nu \in \mathbb{C}$ fulfilling $\overline{a_\nu} = a_{-\nu}$ [4], such that $a_\nu \cdot a_{-\nu} = b_\nu$, or equivalently $|a_\nu|^2 = b_\nu$. Hence, we search for all $a_\nu \in \mathbb{C}$ fulfilling $|a_\nu| = \sqrt{b_\nu}$. That is the definition of a circle around the origin $0 \in \mathbb{C}$ with (non-negative) radius $\sqrt{b_\nu}$. Thus, the set of a_ν we are looking for is described by $e^{it}\sqrt{b_\nu}$, where $t \in [0, 2\pi)$. Due to the relation $\overline{a_\nu} = a_{-\nu}$, the fibre corresponding to $b_{-\nu}$ is parameterized by $e^{-it}\sqrt{b_{-\nu}}$ with the same $t \in [0, 2\pi)$. Therefore, we can combine them as a pair of circles $(e^{it}\sqrt{b_\nu}, e^{-it}\sqrt{b_{-\nu}})$, $t \in [0, 2\pi)$. If now, we choose for $b_\nu := \check{u}_{0,\nu} \cdot \check{u}_{0,-\nu}$ the product of the corresponding product of pairs of perturbed Fourier coefficients for given $\nu \in \Gamma^*_\delta$ and given potential $u_0 \in L^2(F)$, we get due to $\check{u}_{0,\nu} \cdot \check{u}_{0,-\nu} = |\check{u}_{0,\nu}|^2$

$$(e^{it}|\check{u}_{0,\nu}|, e^{-it}|\check{u}_{0,\nu}|), \quad t \in [0, 2\pi) \tag{3.6}$$

for the fibre corresponding to the pair $(-\nu, \nu)$. In this fashion, we can proceed for all $\nu \in \Gamma^*_\delta/\sigma$. With this parameterization of the model flows, we obtain for $t = 0$ the sequence $(|\check{u}_{0,\nu}|, |\check{u}_{0,\nu}|)_{\nu \in \Gamma^*_\delta/\sigma}$. However, we would like to choose a parameterization such that the potential corresponding to $t = 0$ is equal to the initial potential u_0. By reparametrization, we can easily obtain that the sequence corresponding to $t = 0$ equals $(\check{u}_{0,\nu}, \check{u}_{0,-\nu})_{\nu \in \Gamma^*_\delta/\sigma}$, namely by setting

$$(e^{it}\check{u}_{0,\nu}, e^{-it}\check{u}_{0,-\nu}), \quad t \in [0, 2\pi) \tag{3.7}$$

for all $\nu \in \Gamma^*_\delta/\sigma$. Obviously, this is just a reparametrization which parameterizes the same set as the preliminary flows in (3.6). This proves the theorem. □

We want to remark why we considered real-valued potentials instead of arbitrary *complex-valued* potentials in the foregoing theorem. Formally, the asymptotic model isospectral set for complex-valued potentials $u_0 \in L^2(F)$ can be computed similarly. However, in the complex case, the condition $\overline{\check{u}_\nu} = \check{u}_{-\nu}$ for all $\nu \in \Gamma^*_\delta$ is not satisfied anymore which has the consequence that the flow parameters are $t \in \mathbb{C}$ instead of $t \in [0, 2\pi)$. The analog result for complex-valued potentials $u_0 \in L^2(F)$ would then be given by

$$\widetilde{Iso}_\delta(u_0) = \times_{\nu \in \Gamma^*_\delta/\sigma}\{(e^{it}\sqrt{\check{u}_{0,\nu} \cdot \check{u}_{0,-\nu}}, e^{-it}\sqrt{\check{u}_{0,\nu} \cdot \check{u}_{0,-\nu}}) : t \in \mathbb{C}\}.$$

[3]Clearly, $l^1_{+,e}(\Gamma^*_\delta)$ isn't a vector space. It is merely a subset of the vector space $l^1_e(\Gamma^*_\delta)$.

[4]Compare Corollary 2.4.4 where $\overline{a_\nu} = a_{-\nu}$ for all $\nu \in \Gamma^*_\delta$ is justified as some kind of reality condition for the asymptotic analysis.

There occurs an essential problem. For $t \in \mathbb{C}$, the term $e^{it}\sqrt{\check{u}_{0,\nu} \cdot \check{u}_{0,-\nu}}$ is not bounded anymore. If we want to describe potentials by their associated sequence of perturbed Fourier coefficients, the corresponding flow (in terms of perturbed Fourier coefficients) may not leave the domain where the map (2.32) is invertibe due to Theorem 2.4.2. Hence, for unbounded flows, it's not clear anymore whether we may describe our potentials by perturbed Fourier coefficients. The main problem which occurs with complex-valued potentials is that $\check{u}_\nu$ and $\check{u}_{-\nu}$ needn't have the same absolute value anymore. In other words, the ratio of $\check{u}_\nu$ and $\check{u}_{-\nu}$ may get arbitrarily large. For real-valued potentials, however, the situation remains clearer. Indeed, the model isospectral flows don't leave the domain $B_R(\hat{u})$ (defined in Theorem 2.4.2) provided $\delta > 0$ is chosen sufficiently small: We can argue as in the proof of Corollary 2.4.4. There, we chose a $\delta > 0$ and a radius $R > 0$ such that the ball in $l^2(\Gamma^*_\delta)$ with center $0 \in l^2(\Gamma^*_\delta)$ and radius $\|\hat{u}\|_{l^2(\Gamma^*_\delta)}$ is contained in the ball $B_R(\hat{u}) \subset l^2(\Gamma^*_\delta)$ where invertibility of the map (2.32) holds. Although in general, (2.32) doesn't map $0 \in l^2(\Gamma^*_\delta)$ to $0 \in l^2(\Gamma^*_\delta)$, we can nevertheless achieve (by choosing δ suitably small) that there exists an $R_1 > 0$ with $\|\check{u}\|_{l^2(\Gamma^*_\delta)} < R_1$ such that the image of $B_R(\hat{u})$ under (2.32) contains the ball $B_{R_1}(0) \subset l^2(\Gamma^*_\delta)$. This is due to the holomorphy of (2.32) and the fact that the derivative of (2.32) is equal to the identity plus some perturbation terms whose norms tend to zero (locally uniformly in u) as $|\nu| \to \infty$, cf. also the remark after the proof of Theorem 2.4.2. Hence, $\widetilde{Iso}_\delta(u_0)$ is contained in the respective image of (2.32) where invertibility holds.
In the sequel, we will often make use of this choice of $\delta > 0$.

3.2 An ansatz via perturbation of the model flows

After having examined the asymptotic isospectral set $\widetilde{Iso}_\delta(u_0)$ of the *model* Fermi curve in Section 3.1, we now want to consider the asymptotic isospectral set of the *actual* Fermi curve. This shall be done by perturbing the *isospectral flows of the model curve*:

$$\check{u}^\nu_t := e^{it}\check{u}_{0,\nu}, \quad t \in [0, 2\pi),\ \nu \in \Gamma^*_\delta. \tag{3.8}$$

We write $\check{u}_t := (\check{u}^\nu_t)_{\nu\in\Gamma^*_\delta}$ for the flow in terms of perturbed Fourier coefficients and u_t for the flow in terms of $L^2_{\delta,u_0}(F)$-potentials, respectively. Note that the parameter t in $\check{u}_t$ is strictly speaking a "multi-parameter" since every flow indexed by $\nu \in \Gamma^*_\delta$ has its own parameter $t = t_\nu$ depending on ν (with $t_{-\nu} = -t_\nu$, cf. (3.7)). Writing $\check{u}_t$, one should keep in mind that t contains the information of the entire sequence $(t_\nu)_{\nu\in\Gamma^*_\delta}$ of parameters. We indicate this by writing $t \in [0, 2\pi)^\infty$ in the sequel. If we consider fixed elements of sequences, such as $\check{u}^\nu_t$ for fixed $\nu \in \Gamma^*_\delta$, we nevertheless write t instead of t_ν in order to keep the notation as simple as possible. There shouldn't occur any confusions since it will be clear from the context when t is a parameter in $[0, 2\pi)$ and when it is a multi-parameter in

$[0, 2\pi)^\infty$. For example, for sequences $a_t := (a_t^\nu)_{\nu\in\Gamma_\delta^*}, b_t := (b_t^\nu)_{\nu\in\Gamma_\delta^*}$ depending on the multi-parameter t, we will often use the notation $a_t \cdot b_t := (a_t^\nu \cdot b_t^\nu)_{\nu\in\Gamma_\delta^*}$. Clearly, on the left hand side, $t \in [0, 2\pi)^\infty$ is a multi-parameter, whereas on the right hand side, $t \in [0, 2\pi)$ is a scalar parameter.
Again, we consider *real-valued* potentials. Let $u_0 \in L^2(F)$ be a given real-valued potential and consider its sequence of moduli $m(u_0) := (m_\nu(u_0))_{\nu\in\Gamma_\delta^*}$ represented as

$$m(u_0) = \widetilde{m}(u_0) + r(u_0),$$

where $\widetilde{m}(u_0) := (\widetilde{m}_\nu(u_0))_{\nu\in\Gamma_\delta^*}$ are the model moduli and $r(\cdot) := m(\cdot) - \widetilde{m}(\cdot)$ denotes the *deviation* of the moduli from the model moduli. The *asymptotic isospectral set* $Iso_\delta(u_0)$ is defined by (recall (3.3))

$$Iso_\delta(u_0) := \{(u_\nu)_\nu \in l^2_{\mathbb{R}}(\Gamma_\delta^*) : m_\nu(u) = m_\nu(u_0) \text{ for all } \nu \in \Gamma_\delta^*, \quad u := P^{-1}((u_\nu)_\nu)\}.$$

In contrast to the model isospectral set (3.4), we don't have a handy characterization of the moduli by perturbed Fourier coefficients.
The *perturbation ansatz* is now as follows. For every $t \in [0, 2\pi)^\infty$, we look for real-valued $v_t, \widetilde{v}_t \in L^2(F)$ with assosiated $\check{v}_t := (\check{v}_t^\nu)_{\nu\in\Gamma_\delta^*} \in l^2_{\mathbb{R}}(\Gamma_\delta^*)$ and $\check{\widetilde{v}}_t := (\check{\widetilde{v}}_t^\nu)_{\nu\in\Gamma_\delta^*} \in l^2_{\mathbb{R}}(\Gamma_\delta^*)$ such that

$$m_\nu(u_0) = \widetilde{m}_\nu(u_t + \widetilde{v}_t) + r_\nu(u_t + v_t) \quad \text{for all } \nu \in \Gamma_\delta^*. \tag{3.9}$$

The motivation for this ansatz is that we would like to have $v_t = \widetilde{v}_t$ so that we could then define $(\check{u}_t^\nu + \check{v}_t^\nu)_{\nu\in\Gamma_\delta^*}$ as isospectral flow. Thereto, we will consider a map[5] $v_t \mapsto \widetilde{v}_t$ (defined on a suitable domain) and show by Banach's Fixed Point Theorem that this map has a fixed point. In a first step, we will construct this map and show in a second step the desired properties. Let's begin with the first step.
There are many possibilities to construct the *perturbation flows* v_t and $\widetilde{v}_t$. We make a linear ansatz:

$$\begin{aligned} \check{v}_t &:= a_t \cdot \check{u}_t := (a_t^\nu \cdot \check{u}_t^\nu)_{\nu\in\Gamma_\delta^*} \\ \check{\widetilde{v}}_t &:= \widetilde{a}_t \cdot \check{u}_t := (\widetilde{a}_t^\nu \cdot \check{u}_t^\nu)_{\nu\in\Gamma_\delta^*} \end{aligned} \tag{3.10}$$

with[6] $a_t := (a_t^\nu)_{\nu\in\Gamma_\delta^*}$, $\widetilde{a}_t := (\widetilde{a}_t^\nu)_{\nu\in\Gamma_\delta^*} \in l_e^\infty(\Gamma_\delta^*)$, that is, we assume in particular that a_t and $\widetilde{a}_t$ are even (we will see later that this assumption is admissible). We get due to Hölder's inequality

$$\check{u}_t + \check{v}_t = (1 + a_t)\check{u}_t \in l^2(\Gamma_\delta^*).$$

[5]The map we will actually consider will be in terms of perturbed Fourier coefficients, not in terms of potentials.

[6]Although a_t is a *sequence* and not a potential in some L^p-space, we deliberately write a_t (instead of $\check{a}_t$). The notation $\check{a}_t$ would suggest that $\check{a}_t$ is the sequence of perturbed Fourier coefficients of some L^2-potential. This, however, needn't be true since a_t is in l^∞ and (generally) not in l^2.

The moduli $m(\cdot)$ are functions of $L^2(F)$-potentials (and not of $l^2(\Gamma^*_\delta)$-sequences). So, we have to consider corresponding preimages under the map P (3.1). We set

$$\begin{aligned} u_t + v_t &= P^{-1}((1+a_t)\check{u}_t) \in L^2_{\delta,u_0}(F), \\ u_t + \widetilde{v}_t &= P^{-1}((1+\widetilde{a}_t)\check{u}_t) \in L^2_{\delta,u_0}(F) \end{aligned} \tag{3.11}$$

with the meaning of P^{-1} and $L^2_{\delta,u_0}(F)$ explained at the beginning of this chapter. Let $\nu \in \Gamma^*_\delta$ be fixed for the moment. Then, with $r_\nu = r_\nu(P^{-1}((1+a_t)\cdot\check{u}_t))$ (respect the notation $(1+a_t)\cdot\check{u}_t := ((1+a^\nu_t)\check{u}^\nu_t)_{\nu\in\Gamma^*_\delta}$), (3.9) yields

$$m_\nu(u_0) = \frac{(1+\widetilde{a}^\nu_t)(1+\widetilde{a}^{-\nu}_t)|\check{u}_{0,\nu}|^2}{|\nu|^2\xi} + r_\nu.$$

Setting $a := \widetilde{a}^\nu_t = \widetilde{a}^{-\nu}_t$ for the moment, we obtain

$$(2a+a^2)|\check{u}_{0,\nu}|^2 = \underbrace{(m_\nu(u_0) - r_\nu)|\nu|^2\xi - |\check{u}_{0,\nu}|^2}_{=:R_\nu}$$

and thus the equation

$$a^2 + 2a - \frac{R_\nu}{|\check{u}_{0,\nu}|^2} = 0,$$

which has the two solutions $a_\pm = -1 \pm \sqrt{1+\frac{R_\nu}{|\check{u}_{0,\nu}|^2}}$. Choosing the positive sign, this motivates the following map with parameter $t \in [0,2\pi)^\infty$

$$\Psi_t : U \to l^\infty_{r,e}(\Gamma^*_\delta), \quad a_t \mapsto \widetilde{a}_t = (\widetilde{a}^\nu_t)_{\nu\in\Gamma^*_\delta} := \left[-1 + \sqrt{\frac{m_\nu(u_0) - r_\nu(P^{-1}((1+a_t)\cdot\check{u}_t))}{\widetilde{m}_\nu(u_0)}}\right]_{\nu\in\Gamma^*_\delta}, \tag{3.12}$$

where $U \subseteq l^\infty_{r,e}(\Gamma^*_\delta)$ is a neighbourhood of $0 \in l^\infty_{r,e}(\Gamma^*_\delta)$. Here, the subscript e denotes (as before) *evenness* and the subscript r shall denote that we consider sequences of *real* numbers, i.e.

$$(a_\nu)_\nu \in l^\infty_{r,e}(\Gamma^*_\delta) :\Longleftrightarrow [(a_\nu)_\nu \in l^\infty(\Gamma^*_\delta),\ a_{-\nu} = a_\nu \in \mathbb{R} \text{ for all } \nu \in \Gamma^*_\delta].$$

In particular, $l^\infty_{r,e}(\Gamma^*_\delta)$ is considered as a *real* vector space[7].
We will have to prove that the map Ψ_t is well-defined. For this, there is a lot to do so that we will split the proof of well-definition into several lemmata. The main effort will be to prove that the deviation term $(r_\nu)_\nu$ is small with respect to the model moduli $(\widetilde{m}_\nu)_\nu$ (see Lemma 3.2.2).

[7]Note that $l^\infty_{r,e}(\Gamma^*_\delta)$ is indeed a vecor space, in contrast to $l^1_{+,e}(\Gamma^*_\delta)$ considered before which is only a subset of the vector space $l^1_{r,e}(\Gamma^*_\delta)$.

In the above definition, we made use of $\widetilde{m}_\nu(u_0) = \widetilde{m}_\nu(u_t) = \frac{|\check{u}_{0,\nu}|^2}{|\nu|^2\xi}$ for all $t \in [0,2\pi)^\infty$ (note that the flow u_t is isospectral with respect to the model curve, by definition). The above definition (3.12) only makes sense if $\widetilde{m}_\nu(u_0) \neq 0$, i.e. $\check{u}_{0,\nu} \neq 0$ for all $\nu \in \Gamma^*_\delta$. Assume $\check{u}_{0,\nu} = 0$ for some fixed $\nu \in \Gamma^*_\delta$. Consequently $\check{u}_{0,-\nu} = 0$ and $\widetilde{m}_\nu(u_t) = 0$ for all $t \in [0,2\pi)^\infty$. It follows by Theorem 2.5.7 and (2.85) that $m_\nu(u_t) = 0$ for all $t \in [0,2\pi)^\infty$ as well. Hence, $\widetilde{m}_\nu(u_t) = m_\nu(u_t) = 0$ for all $t \in [0,2\pi)^\infty$ and $t \mapsto \check{u}^\nu_t$ thus leaves the ν^{th} modulus m_ν invariant (note that $m(u_t) = m(u_0)$ for all $t \in [0,2\pi)^\infty$ is just the criterion for being an isospectral flow with respect to the *actual* curve). Recall that the aim of the following procedure is constructing isospectral flows by applying Banach's Fixed Point Theorem to $a_t \mapsto \widetilde{a}_t$. In the singular case (i.e. $\check{u}_{0,\nu} = 0$ holds), we trivially have $a^\nu_t = \widetilde{a}^\nu_t = 0$ for all $t \in [0,2\pi)^\infty$ for the considered ν and we're done. The deviation term r_ν thus vanishes in this case. More precisely: Exclude the subsequence of $(\widetilde{m}_\nu(u_0))_\nu$ indexed by all ν fulfilling $\widetilde{m}_\nu(u_0) = 0$ and apply the following procedure to the remainder. Let, for the moment, α_t denote the fixed point of $a_t \mapsto \widetilde{a}_t$ (which will lead to the desired isospectral flow) restricted to the subsequence defined by $\widetilde{m}_\nu(u_0) \neq 0$. The desired perturbation flow $\check{v}_t$ will then be defined by $\check{v}^\nu_t := 0$ if $\nu \in \{\nu \in \Gamma^*_\delta : \widetilde{m}_\nu(u_0) = 0\}$ and $\check{v}^\nu_t := \alpha^\nu_t \cdot \check{u}^\nu_t$ otherwise.
In order to prove that Ψ_t is well-defined, we have to show that the radicand of the square root in (3.12) is real and non-negative. Furthermore, evenness and boundedness of $(\widetilde{a}^\nu_t)_{\nu\in\Gamma^*_\delta}$ have to be verified.
We start with the estimate of the deviation term $(r_\nu)_\nu$. Thereto, we have to recall the z-coordinates introduced in Section 2.5. Recall the diagram (2.79) and in particular the map $\Phi : z \mapsto x$ between z- and x-coordinates. Due to (2.56) and (2.72), we have the following representation of z-coordinates

$$\begin{aligned} z_1(k) &= \widetilde{z}_1(k-k_\nu) + k^2 - k_\nu^2 + A^\nu_{11}(k,u) - A^\nu_{11}(k_\nu,u) + \epsilon_{\nu,1}(k), \\ z_2(k) &= \widetilde{z}_2(k-k_\nu) + k^2 - k_\nu^2 + A^\nu_{22}(k,u) - A^\nu_{22}(k_\nu,u) + \epsilon_{\nu,2}(k). \end{aligned}$$

Here, $\epsilon = \epsilon_\nu = (\epsilon_{\nu,1}, \epsilon_{\nu,2})$ is the "deviation term" between x- and z-coordinates introduced in (2.72). Originally, ϵ is a function of z, but due to the biholomorphic map $z \overset{\Phi}{\mapsto} x \mapsto k$, we can consider ϵ as a function of k by $\epsilon(k) := \epsilon(z(x(k)))$. Analogously to (2.57), we can write the z-coordinates in a matrix-vector representation[8]:

$$\begin{aligned} z(k) &=: B_0 \cdot (k-k_\nu) + B_2(k) =: B(k), \\ B_0 &:= \begin{pmatrix} -\nu_1 + i\nu_2\xi & -i\nu_1\xi - \nu_2 \\ \nu_1 + i\nu_2\xi & -i\nu_1\xi + \nu_2 \end{pmatrix}, \\ B_2(k) &:= \begin{pmatrix} k^2 - k_\nu^2 + A^\nu_{11}(k,u) - A^\nu_{11}(k_\nu,u) + \epsilon_{\nu,1}(k) \\ k^2 - k_\nu^2 + A^\nu_{22}(k,u) - A^\nu_{22}(k_\nu,u) + \epsilon_{\nu,2}(k) \end{pmatrix}, \end{aligned} \tag{3.13}$$

[8]In order to keep the notation simple, we mostly suppress the dependence on u (as we already did in (2.57)) if the dependence on u is not explicitly needed. Note, however, that the deviation term ϵ_ν for instance also depends on the potential u.

where B_0 is the same as in (2.57) and $B_2(k)$ differs from $B_1(k)$ (2.57) by the additional ϵ-term. We compute $z'(k) = B_0 + B_2'(k) = B_0 \cdot (\mathbb{1} + B_0^{-1}B_2'(k))$ as we already did in the proof of Lemma 2.5.3 for $x'(k)$. However, we don't know yet whether $B_0^{-1}B_2'(k) = o(1)$ as $|\nu| \to \infty$. Note that, in the proof of Lemma 2.5.3, we could use Lemma 2.2.7, i.e. $\lim_{|\nu|\to\infty} \frac{\partial}{\partial k} A_{ij}^{\nu}(k_\nu, u) = 0$. Yet, in the quantity $B_2'(k)$, there occur the terms $\frac{\partial \epsilon_{\nu,i}}{\partial k_j}$ for $i, j \in \{1, 2\}$. These have to be estimated at first:

Lemma 3.2.1. *For the error term $\epsilon = \epsilon_\nu$ introduced in* (2.72)*, there holds*

$$\frac{d\epsilon_\nu(k)}{dk} = o\left(\frac{1}{|\nu|}\right), \quad \text{as } |\nu| \to \infty.$$

Proof. We suppress the index ν and write $\epsilon = \epsilon_\nu$. By setting $\epsilon(k) := \epsilon(z(x(k)))$, we compute

$$\frac{d\epsilon}{dk} = \frac{d\epsilon}{dz} \cdot \frac{dz}{dx} \cdot \frac{dx}{dk}. \tag{3.14}$$

We must estimate the three factors of this product.
As to the first factor $\frac{d\epsilon}{dz}$: By the initial value problem (2.71), we get for $t \in [0, 1]$

$$\Phi_t(z) = z + \int_0^t X^s(\Phi_s(z))ds.$$

By (2.73), we have $|X^t(x)| \leq o(1/|\nu|^2)(|x_1 - \zeta_1| + |x_2 - \zeta_2|)$ for all $x \in D_r$ and all $t \in [0, 1]$, as $|\nu| \to \infty$. The radius r of D_r is of dimension $O(|\nu|)$ (this is due to the fact that the bounded domain V the k-coordinates reside in is mapped by $k \mapsto x$ onto a domain of dimension $O(|\nu|)$, cf. (2.56)). Hence,

$$|\Phi_t(z) - z| \leq \sup_{s\in[0,1]} |X^s(\Phi_s(z))| = o\left(\frac{1}{|\nu|}\right), \quad \text{as } |\nu| \to \infty.$$

Since the bound on the right hand side is independent of z and t, $\Phi_t(z)$ converges to z as $|\nu| \to \infty$, uniformly with respect to z and t. Since $\Phi_t(z)$ is holomorphic with respect to z, we may, due to the Weierstrass convergence theorem, interchange derivatives and limits, that is,

$$\frac{d}{dz}\Phi_t(z) \to \mathbb{1}, \quad \text{as } |\nu| \to \infty,$$

uniformly with respect to z and t. Hence, $\frac{d}{dz}\Phi_t(z)$ is uniformly bounded with respect to z and t as $|\nu| \to \infty$. This fact gets important now. By (2.72), we have

$$\Phi(z) = z - \epsilon(z) = z + \zeta + \int_0^1 X^t(\Phi_t(z + \zeta))dt.$$

$$\Rightarrow \frac{d}{dz}\Phi(z) = \mathbb{1} + \int_0^1 \frac{d}{dz}[X^t(\Phi_t(z + \zeta))]dt.$$

Here,

$$\frac{d}{dz}[X^t(\Phi_t(z+\zeta))] = \frac{d}{dx}X^t(x)|_{x=\Phi_t(z+\zeta)} \cdot \frac{d}{dz}\Phi_t(z+\zeta).$$

The second factor has just been estimated as uniformly bounded. As to the first factor, we have $\left|\frac{\partial}{\partial x_i}X^t(x)\right| \leq 8b = o(1/|\nu|^2)$, as $|\nu| \to \infty$ [9] for $i = 1, 2$ due to [5, p. 246, (B.3)] (where we used Lemma 2.5.3, i.e. $b = o(1/|\nu|^2)$, for the estimate of the term b appearing in Theorem 2.5.1). We therefore obtain

$$\frac{d}{dz}\Phi(z) = \mathbb{1} + o\left(\frac{1}{|\nu|^2}\right), \quad \text{as } |\nu| \to \infty. \tag{3.15}$$

On the other hand, we have by definition $\frac{d}{dz}\Phi(z) = \mathbb{1} - \frac{d}{dz}\epsilon(z)$, which yields

$$\frac{d\epsilon}{dz} = o\left(\frac{1}{|\nu|^2}\right), \quad \text{as } |\nu| \to \infty.$$

For the second factor $\frac{dz}{dx}$ in (3.14), we obtain by (3.15)

$$\frac{dz(x)}{dx} = (D\Phi)^{-1}(x) = \mathbb{1} + o(1), \quad \text{as } |\nu| \to \infty. \tag{3.16}$$

Finally, the last factor $\frac{dx}{dk}$ in (3.14) has already been estimated in the proof of Lemma 2.5.3:

$$\frac{dx}{dk} = x'(k) = B_0 + B_1'(k) = B_0 \cdot (\mathbb{1} + B_0^{-1}B_1'(k)) = O(|\nu|), \quad \text{as } |\nu| \to \infty$$

with a prime denoting the derivative with respect to k. Hence, all factors of (3.14) are estimated, so that we get

$$\frac{d\epsilon}{dk} = o\left(\frac{1}{|\nu|^2}\right) \cdot (\mathbb{1} + o(1)) \cdot O(|\nu|) = o\left(\frac{1}{|\nu|}\right), \quad \text{as } |\nu| \to \infty.$$

This proves the lemma. □

The following lemma states that the deviation term $(r_\nu)_\nu$ is small with respect to the model moduli $(\widetilde{m}_\nu)_\nu$:

[9] This estimate can also be concluded from the estimates shown in the proof of Lemma 2.5.5. Some parts of the proof of Lemma 2.5.5 differed from the proof of [5, Lemma B.1] (since the proof in [5] only deals with the special case $\zeta = 0$ - a fact already discussed before) which made us give a full proof of the estimate (2.73) in this work. The proof of the estimate of $\frac{\partial}{\partial x_i}X^t(x)$, however, can be copied one-to-one from the proof of [5, Lemma B.1] because all tools needed for this estimate, namely (2.77) and (2.78), are already provided in the proof of Lemma 2.5.5 and are equal to the corresponding estimates in [5]. In other words, at this point, there is no difference between the cases $\zeta = 0$ and $\zeta \neq 0$ anymore such that there is no need to literally copy a proof already given in another work.

Lemma 3.2.2. *Let $u \in L^2(F)$ be real-valued and $r_\nu(u) := m_\nu(u) - \widetilde{m}_\nu(u)$, $\nu \in \Gamma^*_\delta$, denote the deviation term. Then*

$$r_\nu(u) = O\left(\frac{1}{|\nu|}\right) \cdot \widetilde{m}_\nu(u), \quad \text{as } |\nu| \to \infty. \tag{3.17}$$

This estimate holds locally uniformly in $u \in L^2(F)$, u real-valued. Moreover, $(r_\nu)_\nu \in l^1_{r,e}(\Gamma^*_\delta)$ is even and real with

$$\forall_{\epsilon>0}\, \exists_{\delta>0}\, \|(r_\nu)_\nu\|_{l^1_{r,e}(\Gamma^*_\delta)} \le \epsilon \cdot \|(\widetilde{m}_\nu)_\nu\|_{l^1_{r,e}(\Gamma^*_\delta)}. \tag{3.18}$$

Proof. In Definition 2.6.1, we introduced the (model) moduli $\widetilde{m}_\nu$ and m_ν, respectively, as contour integrals around the A-cycle A_ν (and $\widetilde{A}_\nu$, respectively) over the form $k_1 dk_2$. In z-coordinates, a parameterization of the cycle is, due to the representations $\widetilde{z}_1 \cdot \widetilde{z}_2 = \widetilde{c}_\nu$ and $z_1 \cdot z_2 = c_\nu$, simply given by

$$\widetilde{z} = \sqrt{\widetilde{c}_\nu} \cdot \begin{pmatrix} e^{i\varphi} \\ e^{-i\varphi} \end{pmatrix}, \quad z = \sqrt{c_\nu} \cdot \begin{pmatrix} e^{i\varphi} \\ e^{-i\varphi} \end{pmatrix}, \quad \varphi \in [0, 2\pi), \tag{3.19}$$

respectively. The branch of the square root $\sqrt{c_\nu}$ may be chosen arbitrarily as long as the choice is consistent. Consider at first the subsequence of $(c_\nu)_{\nu \in \Gamma^*_\delta}$ indexed by all $\nu \in \Gamma^*_\delta$ such that $c_\nu = 0$. Due to Theorem 2.5.9, there also holds $\widetilde{c}_\nu = 0$ for the corresponding indices ν. We thus have $\widetilde{m}_\nu(u) = m_\nu(u) = r_\nu(u) = 0$ on this subsequence and the assertion of the lemma is fulfilled. So let's exclude this subsequence from now on, i.e. we may assume $c_\nu \neq 0$ and consequently $\widetilde{c}_\nu \neq 0$ for all $\nu \in \Gamma^*_\delta$, again by Theorem 2.5.9. In particular, the quotioent $c_\nu / \widetilde{c}_\nu$ is well-defined. With the notation (3.13), we compute with the parameterization (3.19) (a prime denoting the derivative with respect to k)

$$\begin{aligned}
&(B'(k))^{-1} z - B_0^{-1} \widetilde{z} = \left(\sqrt{\frac{c_\nu}{\widetilde{c}_\nu}} (B'(k))^{-1} - B_0^{-1}\right) \widetilde{z} = \\
&= \left(\sqrt{\frac{c_\nu}{\widetilde{c}_\nu}} \cdot \left(\mathbb{1} + B_0^{-1} B_2'(k)\right)^{-1} B_0^{-1} - B_0^{-1}\right) \widetilde{z} = \\
&= \left[\sqrt{\frac{c_\nu}{\widetilde{c}_\nu}} \cdot \left(\mathbb{1} + B_0^{-1} B_2'(k)\right)^{-1} - \left(\mathbb{1} + B_0^{-1} B_2'(k)\right) \cdot \left(\mathbb{1} + B_0^{-1} B_2'(k)\right)^{-1}\right] B_0^{-1} \widetilde{z} = \\
&= \left(\left(\sqrt{\frac{c_\nu}{\widetilde{c}_\nu}} - 1\right) \cdot \mathbb{1} - B_0^{-1} B_2'(k)\right) \left(\mathbb{1} + B_0^{-1} B_2'(k)\right)^{-1} B_0^{-1} \widetilde{z},
\end{aligned} \tag{3.20}$$

Note that the appearing inverse operators are all well-defined (for sufficiently small $\delta > 0$ as always) since $B_0^{-1} B_2'(k) \to 0$ as $|\nu| \to 0$: this is due to the well-known limit behaviour $B_0^{-1} B_1'(k) \to 0$ as $|\nu| \to 0$ (cf. the proof of Lemma 2.5.3, for example). In Lemma 3.2.1, we also estimated the additional term $d\epsilon/dk$, which yields the claimed limit behaviour $B_0^{-1} B_2'(k) \to 0$ as $|\nu| \to 0$. We now compute the difference of the terms $k(z) - k(0)$ (defined on the *actual* curve) and

$\widetilde{k}(\widetilde{z}) - \underbrace{\widetilde{k}(0)}_{=k_\nu}$ (defined on the *model* curve, compare (2.48)). By the Fundamental Theorem of Calculus, we obtain due to $dk(z)/dz = [B'(k(z))]^{-1}$:

$$\begin{aligned}
&(k(z) - k(0)) - (\widetilde{k}(\widetilde{z}) - k_\nu) = \\
&= \int_0^1 [B'(k(tz))]^{-1} z\, dt - \int_0^1 B_0^{-1}\widetilde{z}\, dt = \int_0^1 \left([B'(k(tz))]^{-1} z - B_0^{-1}\widetilde{z}\right) dt = \\
&\overset{(3.20)}{=} \int_0^1 \left(\left(\sqrt{\frac{c_\nu}{\widetilde{c}_\nu}} - 1\right) \cdot \mathbb{1} - B_0^{-1}B_2'(k(tz))\right) \left(\mathbb{1} + B_0^{-1}B_2'(k(tz))\right)^{-1} dt \cdot B_0^{-1}\widetilde{z}.
\end{aligned} \tag{3.21}$$

Differentiating the parameterization (3.19) with respect to φ, i.e.

$$\frac{d\widetilde{z}}{d\varphi} = i\sqrt{\widetilde{c}_\nu} \cdot \begin{pmatrix} e^{i\varphi} \\ -e^{-i\varphi} \end{pmatrix}, \quad \frac{dz}{d\varphi} = i\sqrt{c_\nu} \cdot \begin{pmatrix} e^{i\varphi} \\ -e^{-i\varphi} \end{pmatrix}, \quad \varphi \in [0, 2\pi),$$

we obtain, again by using $dk(z)/dz = (B'(k(z)))^{-1} = \left(\mathbb{1} + B_0^{-1}B_2'(k(z))\right)^{-1} B_0^{-1}$,

$$\begin{aligned}
dk &= \frac{dk(z)}{dz} \cdot \frac{dz}{d\varphi} d\varphi = \sqrt{\frac{c_\nu}{\widetilde{c}_\nu}} \cdot \frac{dk(z)}{dz} \cdot \frac{d\widetilde{z}}{d\varphi} d\varphi = \\
&= \sqrt{\frac{c_\nu}{\widetilde{c}_\nu}} \cdot \left(\mathbb{1} + B_0^{-1}B_2'(k(z))\right)^{-1} B_0^{-1} \cdot \frac{d\widetilde{z}}{d\varphi} d\varphi, \\
dk - d\widetilde{k} &= \frac{dk(z)}{dz} \cdot \frac{dz}{d\varphi} d\varphi - \frac{d\widetilde{k}(z)}{d\widetilde{z}} \cdot \frac{d\widetilde{z}}{d\varphi} d\varphi = \left(\sqrt{\frac{c_\nu}{\widetilde{c}_\nu}} \cdot \frac{dk(z)}{dz} - \frac{d\widetilde{k}(z)}{d\widetilde{z}}\right) \cdot \frac{d\widetilde{z}}{d\varphi} d\varphi = \\
&\overset{(3.20)}{=} \left(\left(\sqrt{\frac{c_\nu}{\widetilde{c}_\nu}} - 1\right) \cdot \mathbb{1} - B_0^{-1}B_2'(k(z))\right) \left(\mathbb{1} + B_0^{-1}B_2'(k(z))\right)^{-1} B_0^{-1} \cdot \frac{d\widetilde{z}}{d\varphi} d\varphi.
\end{aligned} \tag{3.22}$$

Further, we have

$$\begin{aligned}
&(k_1(z) - k_1(0))dk_2(z) - (\widetilde{k}_1(\widetilde{z}) - k_{\nu,1})d\widetilde{k}_2(\widetilde{z}) = \\
&= [(k_1(z) - k_1(0)) - (\widetilde{k}_1(\widetilde{z}) - k_{\nu,1})]dk_2(z) + (\widetilde{k}_1(\widetilde{z}) - k_{\nu,1})d(k_2(z) - \widetilde{k}_2(\widetilde{z})).
\end{aligned} \tag{3.23}$$

After these preparations, we can estimate the deviation term $r_\nu(u)$. Firstly, note that for any contant $c \in \mathbb{C}$, we have $\int_{A_\nu}(k_1 - c)dk_2 = \int_{A_\nu} k_1 dk_2$ due to Cauchy's Integral Theorem. Keeping this in mind, we obtain due to (3.21), (3.22), (3.23):

$$|r_\nu(u)| = |m_\nu(u) - \widetilde{m}_\nu(u)| = 16\pi^3 \left| \int_{A_\nu} k_1 dk_2 - \int_{\widetilde{A}_\nu} \widetilde{k}_1 d\widetilde{k}_2 \right| \leq 16\pi^3(S_1 + S_2),$$

where

$$S_1 := \sup_{k\in V}\left\|\left(\left(\sqrt{\frac{c_\nu}{\widetilde{c}_\nu}}-1\right)\cdot \mathbb{1} - B_0^{-1}B_2'(k)\right)\left(\mathbb{1}+B_0^{-1}B_2'(k)\right)^{-1}\cdot B_0^{-1}\right\|\cdot|\widetilde{z}|\cdot$$
$$\cdot \sup_{k\in V}\left\|\left(\mathbb{1}+B_0^{-1}B_2'(k)\right)^{-1}B_0^{-1}\right\|\cdot\left|\sqrt{\frac{c_\nu}{\widetilde{c}_\nu}}\right|\cdot\left|\frac{d\widetilde{z}}{d\varphi}\right|\cdot 2\pi,$$
$$S_2 := \|B_0^{-1}\|\cdot|\widetilde{z}|\cdot$$
$$\cdot\sup_{k\in V}\left\|\left(\left(\sqrt{\frac{c_\nu}{\widetilde{c}_\nu}}-1\right)\cdot \mathbb{1} - B_0^{-1}B_2'(k)\right)\left(\mathbb{1}+B_0^{-1}B_2'(k)\right)^{-1}\cdot B_0^{-1}\right\|\cdot\left|\frac{d\widetilde{z}}{d\varphi}\right|\cdot 2\pi.$$

Now, we estimate the individual terms appearing in S_1 and S_2. Firstly, Theorem 2.5.9 yields $\sqrt{c_\nu/\widetilde{c}_\nu} = 1 + o(1/|\nu|^2)$ as $|\nu|\to\infty$. Further, $B_0^{-1} = O(1/|\nu|)$ as well as $B_0^{-1}B_2'(k) = O(1/|\nu|)$ which implies $\left(\mathbb{1}+B_0^{-1}B_2'(k)\right)^{-1}\cdot B_0^{-1} = O(1/|\nu|)$ as $|\nu|\to\infty$. Together with (cf. (3.19))

$$|\widetilde{z}|\cdot\left|\frac{d\widetilde{z}}{d\varphi}\right| = \sqrt{2}\cdot\sqrt{|\widetilde{c}_\nu|}\cdot\sqrt{2}\cdot\sqrt{|\widetilde{c}_\nu|} = 2|\widetilde{c}_\nu| \overset{(2.84)}{=} \frac{|\nu|^2\xi}{8\pi^4}\cdot\widetilde{m}_\nu(u),$$

we obtain for $i = 1, 2$

$$S_i = O\left(\frac{1}{|\nu|^3}\right)\cdot\frac{|\nu|^2\xi}{8\pi^4}\cdot\widetilde{m}_\nu(u) = O\left(\frac{1}{|\nu|}\right)\cdot\widetilde{m}_\nu(u), \quad \text{as } |\nu|\to\infty.$$

This proves the claim (3.17). The assertion concerning local uniformity with respect to u follows immediately since all estimates above are locally uniform in u (provided u is real-valued).
Since $(\widetilde{m}_\nu(u))_\nu \in l^1_{+,e}(\Gamma^*_\delta)$ (cf. (3.5)) and due to Hölder's inequality, we also have $(r_\nu(u))_\nu \in l^1(\Gamma^*_\delta)$ which, at the same time, shows the l^1-estimate (3.18). It remains to prove that $(r_\nu(u))_\nu$ is even and real. We show these assertions at first for the moduli $(m_\nu(u))_\nu$. Due to $r_\nu = m_\nu - \widetilde{m}_\nu$, evenness and reality immediately follow for the deviation term r_ν, too.
Evenness of the moduli follows from the point-symmetry of Fermi curves with respect to $0\in\mathbb{C}^2$, i.e. $F(u) = -F(u)$ (this property is due the holomorphic involution σ). More precisely, we obtain for $\nu\in\Gamma^*_\delta$

$$m_\nu(u)/(-16\pi^3) = \int_{A_\nu} k_1 dk_2 \overset{\sigma}{=} \int_{A_{-\nu}} (-k_1)d(-k_2) = \int_{A_{-\nu}} k_1 dk_2 = m_{-\nu}(u)/(-16\pi^3) \tag{3.24}$$

by using the involution $\sigma: k\mapsto -k$ in the second equality (recall that $-k^\pm_\nu(\hat{u}_0) = k^\pm_{-\nu}(\hat{u}_0)$). This shows the evenness claim. Since our potential is assumed to be real-valued, we also have the antiholomorphic involution $\eta: k\mapsto\bar{k}$. Using this, together with $\overline{k^\pm_\nu(\hat{u}_0)} = k^\mp_{-\nu}(\hat{u}_0)$ and the fact that $k^\pm_\nu(\hat{u}_0) = k^\mp_\nu(\hat{u}_0)$ modulo Γ^*, we obtain

$$\overline{m_\nu(u)}/(-16\pi^3) = \int_{A_\nu} \bar{k}_1 d\bar{k}_2 \overset{\eta}{=} \int_{A_{-\nu}} k_1 dk_2 = \int_{A_\nu} k_1 dk_2 = m_\nu(u)/(-16\pi^3). \tag{3.25}$$

Thus, $\overline{m_\nu(u)} = m_\nu(u)$ for all $\nu \in \Gamma^*_\delta$. This shows that the moduli $(m_\nu(u))_{\nu\in\Gamma^*_\delta}$ are real provided u is real-valued. Hence, we showed that $(m_\nu(u))_\nu$ and consequently $(r_\nu(u))_\nu$ is even and real. Thus, the lemma is proved. □

As a by-product of the above proof, we see that the properties of the model moduli such as evenness, reality, non-negativity (for $\nu \in \Gamma^*_\delta$, respectively) and being an $l^1(\Gamma^*)$-sequence are inherited by the actual moduli. We state this in the following corollary.

Corollary 3.2.3. *Let $u \in L^2(F)$ be real-valued. Then $(m_\nu(u))_\nu \in l^1_{+,e}(\Gamma^*_\delta)$ for sufficiently small $\delta > 0$.*

Proof. The assertion $(m_\nu(u))_\nu \in l^1(\Gamma^*)$ follows from $(\widetilde{m}_\nu(u))_\nu, (r_\nu(u))_\nu \in l^1(\Gamma^*_\delta)$ proved in Lemma 3.2.2. Evenness and reality (for $\nu \in \Gamma^*_\delta$) have already been proved in Lemma 3.2.2, too. Non-negativity of $m_\nu(u)$ for $\nu \in \Gamma^*_\delta$ follows from (3.17) and the non-negativity of the model moduli, that is,

$$m_\nu(u) = \widetilde{m}_\nu(u) + r_\nu(u) = \widetilde{m}_\nu(1 + o(1)), \quad \text{as } |\nu| \to \infty$$

(considered on the subsequence indexed by ν with $\widetilde{c}_\nu \neq 0$) is non-negative, at least for $\delta > 0$ sufficiently small. The case $c_\nu = 0$ has already been discussed in the proof of Lemma 3.2.2 yielding $m_\nu(u) = 0$, too. □

Now, we are able to proof that the map Ψ_t (3.12) is well-defined:

Theorem 3.2.4. *Let $u_0 \in L^2(F)$ be real-valued. Then the map $\Psi_t : U \to l^\infty_{r,e}(\Gamma^*_\delta)$ (3.12) is well-defined for all $t \in [0, 2\pi)^\infty$ for a sufficiently small neighbourhood $U \subseteq l^\infty_{r,e}(\Gamma^*_\delta)$ of $0 \in l^\infty_{r,e}(\Gamma^*_\delta)$. That is, its image consists of real and even sequences: $\widetilde{a}^\nu_t = \widetilde{a}^{-\nu}_t \in \mathbb{R}$ for all $\nu \in \Gamma^*_\delta$. Moreover, the perturbation flows v_t and $\widetilde{v}_t$ defined in (3.11) satisfy (3.9).*

Proof. Let $t \in [0, 2\pi)^\infty$ and $U \subseteq l^\infty_{r,e}(\Gamma^*_\delta)$ be a neighbourhood of $0 \in l^\infty_{r,e}(\Gamma^*_\delta)$. Recall the map $\Psi_t : U \to l^\infty_{r,e}(\Gamma^*_\delta)$, defined by

$$a_t \mapsto \widetilde{a}_t = (\widetilde{a}^\nu_t)_{\nu\in\Gamma^*_\delta} := \left[-1 + \sqrt{\frac{m_\nu(u_0) - r_\nu(P^{-1}((1 + a_t) \cdot \check{u}_t))}{\widetilde{m}_\nu(u_0)}}\right]_{\nu\in\Gamma^*_\delta},$$

where without restriction $\widetilde{m}_\nu(u_0) \neq 0$ for all $\nu \in \Gamma^*_\delta$ as has already been discussed after the definition of (3.12). We show at first $\widetilde{a}^\nu_t \in \mathbb{R}$ for all $\nu \in \Gamma^*_\delta$. As to the radicand $[m_\nu(u_0) - r_\nu(P^{-1}((1+a_t)\cdot \check{u}_t))]/\widetilde{m}_\nu(u_0)$ of (3.12), we obtain by Lemma 3.2.2

$$\frac{m_\nu(u_0)}{\widetilde{m}_\nu(u_0)} = 1 + \frac{r_\nu(u_0)}{\widetilde{m}_\nu(u_0)} \to 1, \quad \text{as } |\nu| \to \infty \tag{3.26}$$

as well as

$$\frac{r_\nu(P^{-1}((1+a_t)\cdot \check{u}_t))}{\widetilde{m}_\nu(u_0)} = (1+a_t^\nu)^2 \cdot \frac{r_\nu(P^{-1}((1+a_t)\cdot \check{u}_t))}{\widetilde{m}_\nu(P^{-1}((1+a_t)\check{u}_t))} = O\left(\frac{1}{|\nu|}\right), \text{ as } |\nu| \to \infty,$$

where we used the evenness and boundedness of $(a_t^\nu)_\nu \in U$ (with respect to ν). This shows

$$\frac{m_\nu(u_0) - r_\nu(P^{-1}((1+a_t)\cdot \check{u}_t))}{\widetilde{m}_\nu(u_0)} = 1 + O\left(\frac{1}{|\nu|}\right), \quad \text{as } |\nu| \to \infty. \tag{3.27}$$

Now, $\widetilde{m}_\nu(u_0)$ and $m_\nu(u_0)$ are real due to the reality of u_0 (cf. (3.5) and Corollary 3.2.3). Moreover, $(1+a_t)\check{u}_t$ corresponds to a real-valued potential, too. This follows from the reality and evenness of a_t and Corollary 2.4.4 since $\overline{(1+a_t^\nu)\check{u}_t^\nu} = (1+a_t^{-\nu})\check{u}_t^{-\nu}$. By Lemma 3.2.2, $r_\nu(P^{-1}((1+a_t)\cdot \check{u}_t))$ is thus real. This shows that the radicand of (3.12) is real and, due to (3.27), non-negative for $\delta > 0$ sufficiently small. This in turn shows $\widetilde{a}_t^\nu \in \mathbb{R}$ for all $\nu \in \Gamma_\delta^*$.
The evenness of $(\widetilde{a}_t^\nu)_\nu$ follows immediately from the evenness of $(\widetilde{m}_\nu)_\nu$, $(m_\nu)_\nu$ and $(r_\nu)_\nu$. Boundedness with respect to ν follows from (3.27). Thus, we have proved that Ψ_t maps U into $l^\infty_{r,e}(\Gamma_\delta^*)$.
Finally, if we define the perturbation flows $\check{v}_t$ and $\check{\widetilde{v}}_t$ by (3.10) and $v_t, \widetilde{v}_t$ by (3.11), the computations leading to the definition (3.12) show that (3.9) is fulfilled if we retrace those computations. This is admissible since all steps leading to (3.12) were equivalence transformations as long as the denominator $\widetilde{m}_\nu(u_0)$ doesn't vanish. But this special case has already been discussed and was excluded before. Thus, the theorem is proved. □

In the following lemma, we prove that for given $\nu \in \Gamma_\delta^*$, the map $a_t \mapsto r_\nu(P^{-1}((1+a_t)\check{u}_t))$ appearing in the map Ψ_t is smooth. This together with an estimate of the respective derivative will be needed in Theorem 3.2.8 to prove that Ψ_t is contractive. Besides, we prove that m_ν and r_ν are holomorphic with respect to u.

Lemma 3.2.5. *Let $\nu \in \Gamma_\delta^*$ be fixed. Let m_ν denote the ν^{th} modulus with corresponding deviation term r_ν. Then, for every $u_0 \in L^2(F)$, there exists a neighbourhood $B(u_0) \subset L^2(F)$ of u_0 such that the maps*

$$B(u_0) \longrightarrow \mathbb{C}, \quad u \longmapsto m_\nu(u),$$
$$B(u_0) \longrightarrow \mathbb{C}, \quad u \longmapsto r_\nu(u)$$

are holomorphic. Here, the neighbourhood $B(u_0)$ can be chosen independent of ν. In particular, for real-valued $u_0 \in L^2(F)$, $\check{u} \in \widetilde{Iso}_\delta(u_0)$ and $U \subseteq l^\infty_{r,e}(\Gamma^*_\delta)$ the ball defined in Theorem 3.2.4, the map

$$U \longrightarrow \mathbb{R}, \quad a \longmapsto r_\nu(P^{-1}((1+a)\check{u}))$$

is smooth.

Remark. Speaking of holomorphic maps, the potentials u appearing in the first part of the lemma concerning the holomorphy assertion are of course arbitrary *complex-valued* potentials.

Proof. Let $u_0 \in L^2(F)$ and $\nu \in \Gamma^*_\delta$ be fixed. The corresponding modulus m_ν is defined by

$$m_\nu(u) = -16\pi^3 \int_{A_\nu} k_1 dk_2.$$

We need to understand the dependence of this quantity on the potential $u \in L^2(F)$. Thereto, we consider at first the following parameterization of the ν^{th} A-cycle (differing from (3.19)):

$$z = \begin{pmatrix} c_\nu \cdot e^{i\varphi} \\ e^{-i\varphi} \end{pmatrix}, \quad \varphi \in [0, 2\pi). \tag{3.28}$$

We firstly note that we may choose A-cycles arbitrarily up to homology since the contour integral m_ν only depends on the respective homology class of A_ν. Obviously, (3.28) describes a cycle homologous to the cycle parameterized in (3.19). Secondly, the parameterization (3.28) is contained in the image of $V \subset \mathbb{C}^2$ (the domain the k-coordinates reside in) under the map $k \mapsto z$ which can be seen by (3.13), for instance. The parameterization (3.28) is thus admissible. Let's briefly motivate why we consider (3.28) instead of (3.19) in this proof. The advantage of (3.19) was that the parameterization of A_ν in model coordinates $\widetilde{z}$ is proportional to the parameterization in non-model coordinates z. In fact they only differ by a factor $\sqrt{\frac{c_\nu}{\widetilde{c}_\nu}}$. This was an esssential ingredient in estimating the deviation term r_ν in Lemma 3.2.2. In this proof, however, we are interested in proving the holomorphy of $u \mapsto m_\nu(u)$. Since the square root $\sqrt{c_\nu}$ in (3.19) is in general not holomorphic (for instance, $u \in B(u_0)$ with $c_\nu(u) = 0$ would cause problems in this context), the parameterization (3.28) turns out to be more appropriate. Now, we define with the matrix-vector representation (3.13) and the parameterization (3.28) for $\varphi \in [0, 2\pi)$

$$\begin{aligned} F_\varphi(k,u) &:= B_0 \cdot (k - k_\nu) + B_2(k,u) - z = \\ &= B_0 \cdot (k - k_\nu) + \begin{pmatrix} k^2 - k_\nu^2 + A^\nu_{11}(k,u) - A^\nu_{11}(k_\nu,u) + \epsilon_{\nu,1}(k,u) \\ k^2 - k_\nu^2 + A^\nu_{22}(k,u) - A^\nu_{22}(k_\nu,u) + \epsilon_{\nu,2}(k,u) \end{pmatrix} - \begin{pmatrix} c_\nu \cdot e^{i\varphi} \\ e^{-i\varphi} \end{pmatrix}, \end{aligned} \tag{3.29}$$

where we write $B_2(k,u)$ (instead of $B_2(k)$ as before) because now, the dependence on the potential u is essential. The same holds for the error terms $\epsilon_{\nu,1}, \epsilon_{\nu,2}$. Let's briefly outline the procedure of the proof: We'll apply the Implicit Function Theorem to the equation $F_\varphi(k,u) = 0$ by showing that $\partial F_\varphi(k,u)/\partial k$ is invertible. Hence, we can deduce that the locally defined map $u \mapsto k(u)$, which maps the potential u to the parameterization of the cycle A_ν in k-coordinates, is holomorphic provided that $F_\varphi(k,u)$ is holomorphic with respect to k and u. From this, we will be able to deduce the holomorphy of $k_1 dk_2$ (more precisely, its parameterization along the cycle A_ν) with respect to u.
Let's continue with the proof. At first, we show that the map $(k,u) \mapsto F_\varphi(k,u)$ is holomorphic, with $k \in V$ (with V the usual domain of definition for k which only depends on Γ^*) and u in a neighbourhood in $L^2(F)$ of the given potential u_0. The holomorphy in $k \in V$ is obvious since the only non-trivial terms depending on k are the entries of the perturbation matrix $A^\nu_{ii}(k,u)$, $i = 1,2$, and the error term ϵ_ν whose derivative with respect to k has even been estimated in Lemma 2.2.7 and Lemma 3.2.1, respectively. As to the holomorphy with respect to u, there occur besides the entries of the perturbation matrix and the term ϵ_ν some other terms depending on u, namely $k_\nu = k_\nu(u)$ and $c_\nu = -(h_\nu(\zeta_\nu) + \zeta_{\nu,1} \cdot \zeta_{\nu,2})$, where the critical point $\zeta_\nu = \zeta_\nu(u)$ depends on u, too. The holomorphy of $A^\nu_{ii}(k,u)$, $i = 1,2$ with respect to u was shown in [13, Theorem 4.5.25]. In Lemma 2.4.3, we estimated the derivative of k_ν with respect to u. In particular, $k_\nu(u)$ is holomorphic in u as the proof of Lemma 2.4.3 shows[10]. As to the holomorphy of $\zeta_\nu = \zeta_\nu(u)$, consider for $x \in D_r$ (where D_r denotes the polydisc defined in Theorem 2.5.1) and u again in a neighbourhood of u_0

$$G(x,u) := x + \begin{pmatrix} \frac{\partial h(x,u)}{\partial x_2} \\ \frac{\partial h(x,u)}{\partial x_1} \end{pmatrix},$$

where ν is suppressed and $h = h_\nu$ is the map (2.58) (obviously depending on u). By definition, the critical point $\zeta = \zeta_\nu$ is determined by the equation $G(\zeta(u), u) = 0$. We obtain by Lemma 2.5.3

$$\frac{\partial G(x,u)}{\partial x} = \mathbb{1} + o\left(\frac{1}{|\nu|^2}\right), \quad |\nu| \to \infty.$$

Due to the Implicit Function Theorem (cf. [23, p. 144], for example), the locally defined map $u \mapsto \zeta_\nu(u)$ is holomorphic because $G(x,u)$ is obviously holomorphic with respect to u and x (recall the definition of h_ν and the holomorphy of the entries of the perturbation matrix which has already been shown). Hence, also the handle quantity $c_\nu = -(h_\nu(\zeta_\nu) + \zeta_{\nu,1} \cdot \zeta_{\nu,2})$ is holomorphic with respect to u. It remains to show the holomorphy with respect to u of the term $\epsilon = \epsilon_\nu$ defined

[10] In fact, the proof of Lemma 2.4.3 already uses an "Implicit Function Theorem" argument analogous to what we show next.

in (2.72) by

$$\epsilon = -\zeta - \int_0^1 X^t(\Phi_t(z+\zeta))dt,$$

where ν is again suppressed. Since the holomorphy of $u \mapsto \zeta(u)$ has already been shown above, it remains to show that the vector field X^t is holomorphic with respect to u. Then, Φ_t as the solution of the initial value problem (2.71) also depends holomorphically on the "parameter" u, as is well-known from the theory of ordinary differential equations. We now have to understand the holomorphy of X^t with respect to u. Let's recap how X^t was defined: By definition, $X^t(x) := Y^t \circ P_t(x)$, $x \in D_r$, where (cf. the proof of Theorem 2.5.4)

$$\begin{aligned} Y_1^t(y) &:= \frac{1}{y_1} g(y_1, 0), \quad Y_2^t := \frac{1}{y_2}(g(y_1,y_2) - g(y_1,0)), \\ g(y) &:= -\widetilde{h} \circ P_t^{-1}(y), \\ y = P_t(x) &:= \left(x_2 - \zeta_2 + t\left(\frac{\partial h}{\partial x_1} + \zeta_2\right), x_1 - \zeta_1 + t\left(\frac{\partial h}{\partial x_2} + \zeta_1\right)\right), \\ \widetilde{h}(x) &:= x_1 x_2 + h(x) - (x_1 - \zeta_1)(x_2 - \zeta_2) - \zeta_1\zeta_2 - h(\zeta). \end{aligned}$$

The holomorpy of $\widetilde{h}$ and P_t with respect to u is obvious (mainly due to the well-known holomorphy with respect to u of the entries of the perturbation matrix and their k-derivatives). As to the holomorphy of the inverse P_t^{-1} with respect to u, we can use the Implicit Function Theorem, this time applied to $H(x,u) := P_t(x) - y$, where $y \in P_t(D_r)$ is a given point. Deriving $H(x,u)$ with respect to x yields the Jacobian

$$\frac{\partial H}{\partial x} = D_x P_t = \begin{pmatrix} t\frac{\partial^2 h}{\partial x_1^2} & 1 + t\frac{\partial^2 h}{\partial x_2 \partial x_1} \\ 1 + t\frac{\partial^2 h}{\partial x_1 \partial x_2} & t\frac{\partial^2 h}{\partial x_2^2} \end{pmatrix},$$

which is invertible due to Lemma 2.5.3 for $\delta > 0$ sufficiently small. Hence, the holomorphy of $u \mapsto x(u) = x = P_t^{-1}(y)$ follows from the Implicit Function Theorem since $H(x,u)$ is holomorphic in both variables. This shows that the above terms $g(y)$ and $Y^t(y)$ are holomorphic with respect to u and finally also X^t is holomorphic with respect to u. Together with the above arguments, this sows that the term ϵ is holomorphic with respect to u. Hence, we have proved that $F_\varphi(k,u)$ (3.29) is holomorphic in both variables k and u. We thus can apply the Implicit Function Theorem to the equation $F_\varphi(k,u) = 0$. Deriving F_φ with respect to k yields

$$\frac{\partial F_\varphi(k,u)}{\partial k} = B_0 \cdot \left(\mathbb{1} + O\left(\frac{1}{|\nu|}\right)\right), \quad \text{as } |\nu| \to \infty,$$

since k lives in the *bounded* domain V and the derivatives of $A_{ij}^\nu(k,u)$ and $\epsilon_\nu(k,u)$ with respect to k both tend to zero as $|\nu| \to \infty$ (see Lemma 2.2.7 and Lemma 3.2.1). Hence, due to the Implicit Function Theorem, there exists a neigbourhood

$B(u_0)$ of u_0 such that the map $u \mapsto k(u)$ defined on $B(u_0)$ is holomorphic. Recall that the notation $k(u)$ signifies the parameterization of the cycle A_ν in k-coordinates. Moreover, the neighbourhood $B(u_0)$ is independent of ν. In fact, for all terms in the preceding considerations, there were no restrictions concerning $B(u_0)$ with respect to $\nu \in \Gamma^*_\delta$ (such as having to choose $B(u_0)$ smaller the larger $|\nu|$ gets, for instance) so that $B(u_0)$ is indeed independent of $\nu \in \Gamma^*_\delta$ (provided -as always- that $\delta > 0$ has been chosen sufficiently small).
Further, we have in analogy to (3.22) (this time with (3.28) instead of (3.19))

$$dk = \frac{dk(z)}{dz} \cdot \frac{dz}{d\varphi} d\varphi = \left(\mathbb{1} + B_0^{-1} B_2'(k(z))\right)^{-1} B_0^{-1} \cdot i \begin{pmatrix} c_\nu \cdot e^{i\varphi} \\ -e^{-i\varphi} \end{pmatrix} d\varphi.$$

Along the parameterization (3.28), the form dk is thus holomorphic with respect to u since for all appearing terms in dk, we have proved above holomorphy with respect to u. This shows that the contour integral $\int_{A_\nu} k_1 dk_2$ is holomorphic with respect to u. Thus, the ν^{th} modulus $m_\nu(u)$ as well as $r_\nu(u) = m_\nu(u) - \widetilde{m}_\nu(u)$ are holomorphic in u (note that the holomorphy of $u \mapsto \widetilde{m}_\nu(u)$ is obvious due to the representation (2.84)). The smoothness of $a \longmapsto r_\nu(P^{-1}((1+a)\check{u}))$ now follows immediately since P^{-1} is biholomorphic in the respective domain of definition. The lemma is proved. □

In the next lemma, we estimate the derivative of $a \mapsto r_\nu(P^{-1}((1+a)\check{u}))$, as a preperation for the proof that Ψ_t (3.12) is contractive.

Lemma 3.2.6. *Let $u_0 \in L^2(F)$ be real-valued, $\check{u} \in \widetilde{Iso}_\delta(u_0)$ and $U \subseteq l^\infty_{r,e}(\Gamma^*_\delta)$ be the ball defined in Theorem 3.2.4. Then for $\nu \in \Gamma^*_\delta$ with $\check{u}_\nu \neq 0$, the derivative of the map*

$$U \longrightarrow \mathbb{R}, \quad a \longmapsto r_\nu(P^{-1}((1+a)\check{u}))$$

satisfies

$$\left\| \frac{d}{da} r_\nu(P^{-1}((1+a) \cdot \check{u})) \right\| = o(1) \cdot |\widetilde{m}_\nu(u_0)|, \quad \text{as } |\nu| \to \infty,$$

where the estimate by the error term $o(1)$ is uniform in $a \in U$. Here, $\| \cdot \| := \| \cdot \|_{l^\infty_{r,e}(\Gamma^*_\delta) \to \mathbb{R}}$ denotes the corresponding operator norm. Moreover, there holds

$$\left\| \frac{d}{du} r_\nu(u) \right\|_{L^2 \to \mathbb{R}} = o(1) \cdot \frac{|\check{u}_\nu|}{|\nu|^2}, \quad \text{as } |\nu| \to \infty$$

for all real $u \in L^2(F)$ in a (sufficiently small) neighbourhood of u_0, again for $\nu \in \Gamma^*_\delta$ with $\check{u}_\nu \neq 0$.

Remark. The additional requirement in the conditions of the lemma that the perturbed Fourier coefficients have to satisfy $\check{u}_\nu \neq 0$ is needed in the proof in order that quotients like $c_\nu/\widetilde{c_\nu}$ are well-defined. In Lemma 3.2.2, we didn't have to make this additional assumption since $\check{u}_\nu = 0$ implies $m_\nu(u) = \widetilde{m}_\nu(u) = r_\nu(u) = 0$ due to Theorem 2.5.9 and (2.85). We don't know, however, whether $\frac{d}{du} r_\nu(u) = 0$ if $\check{u}_\nu = 0$. Therefore, we require the condition $\check{u}_\nu \neq 0$. In fact, this is no severe restriction since later in Theorem 3.2.8, we will apply this lemma to such subsequences of $(\check{u}_\nu)_\nu$ fulfilling this condition (compare the exclusion of the case $\widetilde{m}_\nu(u_0) = 0$ in the discussion after the definition of the map Ψ_t in (3.12)). In later chapters, we will restrict ourselves to smooth Fermi curves anyway where the condition that the perturbed Fourier coefficients don't vanish is always fulfilled.

Proof. For $\check{u} \in \widetilde{Iso}_\delta(u_0)$, we consider potentials of the form

$$u = P^{-1}((1+a)\cdot \check{u})$$

with $a \in U$. Some terms in r_ν can be derived easily with respect to a (since they already appear in terms of perturbed Fourier coefficients), whereas for others, we use the chain rule $\frac{d}{da} = \frac{d\cdot}{du}\cdot\frac{du}{da}$, where (with a prime denoting the derivative of P)

$$\frac{du}{da} = \frac{d}{da} P^{-1}((1+a)\cdot\check{u}) = \left[b \mapsto \left(P'|_{P^{-1}((1+a)\check{u})}\right)^{-1}(\check{u}\cdot b)\right]$$

as a linear operator mapping from $l^\infty_{r,e}(\Gamma^*_\delta)$ into $L^2_{\delta,u_0}(F)$. In Theorem 2.4.2, we have proved that the derivative P' approximates the identity for sufficiently small $\delta > 0$ by identifying L^2-potentials with their associated sequence of l^2-Fourier coefficients. More precisely, we have shown in that theorem that for $\nu \in \Gamma^*_\delta$, there holds

$$\frac{d\check{u}_\nu}{du} : L^2(F) \to \mathbb{C}, \quad v \mapsto \hat{v}(\nu) + o(1), \quad \text{as } |\nu| \to \infty, \tag{3.30}$$

where the error term $o(1)$ encodes an operator whose norm tends to zero as $\delta \to 0$. We thus have

$$\left\| \frac{du}{da} \right\|_{l^\infty_{r,e}(\Gamma^*_\delta)\to L^2(F)} \leq \|(\check{u}_\nu)_\nu\|_{l^2(\Gamma^*_\delta)} \cdot (1+o(1)), \quad \text{as } \delta \to 0.$$

In the following, we sometimes only need rough estimates where $\|du/da\| = O(1)$ (as $\delta \to 0$) already suffices. In these cases, we will just derive the respective terms with respect to u (instead of deriving with respect to a) without mentioning it always explicitly. For the proof of the second claim of the lemma concerning the estimate for $\frac{d}{du} r_\nu(u)$, we will only need the derivative with respect to u, anyway. We use the parameterization (3.28) for both model curve and actual curve:

$$\widetilde{z} = \begin{pmatrix} \widetilde{c}_\nu \cdot e^{i\varphi} \\ e^{-i\varphi} \end{pmatrix}, \quad z = \begin{pmatrix} c_\nu \cdot e^{i\varphi} \\ e^{-i\varphi} \end{pmatrix}, \quad \varphi \in [0, 2\pi). \tag{3.31}$$

Let $\nu \in \Gamma^*_\delta$. In a first step, we consider the derivative dz/da. Thereto, we have to estimate the derivative dc_ν/du. Due to Theorem 2.5.4, we have $-c_\nu = h_\nu(\zeta_\nu) + \zeta_{\nu,1} \cdot \zeta_{\nu,2}$. The function $h_\nu(\zeta_\nu)$ depends twice on the potential: firstly, because of the explicit u-dependence in the off-diagonal entries of the perturbation matrix A^ν_{ij}, secondly, because of the implicit u-dependence in the critical point $\zeta_\nu = \zeta_\nu(u)$. Since ζ_ν satisfies $\nabla(h_\nu(x) + x_1 x_2)|_{x=\zeta_\nu} = 0$ by definition, we obtain

$$\begin{aligned} -\frac{dc_\nu}{du} &= \frac{dh_\nu(\zeta_\nu)}{du} + \zeta_{\nu,1}\frac{d\zeta_{\nu,2}}{du} + \frac{d\zeta_{\nu,1}}{du}\zeta_{\nu,2} = \\ &= \underbrace{\frac{\partial h_\nu(\zeta_\nu)}{\partial x} \cdot \frac{d\zeta_\nu}{du}}_{=-\zeta_{\nu,2}\frac{d\zeta_{\nu,1}}{du} - \zeta_{\nu,1}\frac{d\zeta_{\nu,2}}{du}} + \frac{\partial h_\nu(\zeta_\nu)}{\partial u} + \zeta_{\nu,1}\frac{d\zeta_{\nu,2}}{du} + \frac{d\zeta_{\nu,1}}{du}\zeta_{\nu,2} = \frac{\partial h_\nu(\zeta_\nu)}{\partial u}, \end{aligned} \tag{3.32}$$

that is, the u-derivatives of ζ_ν have been cancelled out and we have to compute $\partial h_\nu/\partial u$ evaluated at ζ_ν. This makes some of the following computations a lot easier since ζ_ν doesn't have to be derived and can virtually be considered as constant with respect to u in this sense. As in the proof of Theorem 2.5.9, we get by the Fundamental Theorem of Calculus

$$h_\nu(\zeta_\nu) = h_\nu(0) + \int_0^1 \nabla h_\nu(t\zeta_\nu)dt \cdot \zeta_\nu. \tag{3.33}$$

We would like to estimate the difference $\frac{\partial h_\nu(\zeta_\nu)}{\partial a} - \frac{\partial h_\nu(0)}{\partial a}$. Due to (2.60), we have for all $t \in [0,1]$

$$\begin{aligned} &-\nabla h_\nu(t\zeta_\nu) = \\ &= \left(A^\nu_{12}(k(t\zeta_\nu),u)\frac{\partial}{\partial k}A^\nu_{21}(k(t\zeta_\nu),u) + A^\nu_{21}(k(t\zeta_\nu),u)\frac{\partial}{\partial k}A^\nu_{12}(k(t\zeta_\nu),u)\right) \cdot \frac{dk(t\zeta_\nu)}{dx}. \end{aligned} \tag{3.34}$$

Let's consider at first the term $\frac{d}{du}\frac{dk}{dx}$. We claim that we have for $n = 1, 2$

$$\frac{d}{du}\frac{dk_n}{dx} = \frac{d^2 k_n}{dx^2}\frac{dx}{du} = O\left(\frac{1}{|\nu|^3}\right), \quad \text{as } |\nu| \to \infty. \tag{3.35}$$

Due to (2.66), we only have to show $dx/du = O(1)$ as $|\nu| \to \infty$. By definition of the x-coordinates (2.56), we have

$$\begin{aligned} \frac{dx_1}{du} =& -\frac{dk_{\nu,1}}{du}(-\nu_1 + i\nu_2\xi) - \frac{dk_{\nu,2}}{du}(-i\nu_1\xi - \nu_2) - 2k_\nu \cdot \frac{dk_\nu}{du} + \\ &+ \frac{\partial A_{11}(k)}{\partial u} - \left(\frac{\partial A_{11}(k_\nu)}{\partial u} + \frac{\partial A_{11}(k_\nu)}{\partial k} \cdot \frac{dk_\nu}{du}\right) = O(1), \quad \text{as } |\nu| \to \infty \end{aligned}$$

due to $\frac{d}{du}k_\nu = O(1/|\nu|)$ (cf. Lemma 2.4.3), $\frac{\partial}{\partial u}A_{11} = O(1)$ (cf. the proof of Theorem 2.4.2, in particular (2.33) and the fact that the operator BC is bounded

and AB, BA both tend to zero as $|\nu| \to \infty$, cf. Lemma 2.2.5, in the respective operator norms) and $\frac{\partial}{\partial k} A_{11} = o(1)$ (cf. Lemma 2.2.7), as $|\nu| \to \infty$, respectively. The estimate for dx_2/du is completely analogous. This proves (3.35).
Again by the Fundamental Theorem of Calculus, we have with $k(0) = k_\nu$ (with respect to the map $x \mapsto k(x)$, recall (2.79)) for all $t \in [0,1]$

$$\begin{aligned} A_{12}^\nu(k(t\zeta_\nu), u) &= A_{12}^\nu(k(0), u) + \int_0^1 \nabla_x A_{12}^\nu(k(st\zeta_\nu), u) ds \cdot t\zeta_\nu = \\ &= \check{u}_\nu(1 + a_\nu) + \int_0^1 \nabla_k A_{12}^\nu(k(st\zeta_\nu), u) \frac{dk}{dx}|_{x=st\zeta_\nu} ds \cdot t\zeta_\nu, \end{aligned}$$

as $|\nu| \to \infty$. Virtually the same holds for the other entry $A_{21}^\nu(k(t\zeta_\nu), u)$ which is shown completely analogously. This together with (3.35) and $\frac{\partial^2}{\partial u\, \partial k} A_{ij}^\nu(k,u) = o(1)$ as $|\nu| \to \infty$ (we postpone the proof of this assertion into the next Lemma 3.2.7 in order not to make the structure of this proof too confusing) implies by deriving (3.34) and using (2.61), (2.82) (recall that ζ_ν doesn't have to be derived as explained before: we firstly derive h_ν and then evaluate at ζ_ν)

$$\frac{\partial}{\partial a} \nabla h_\nu(k(t\zeta_\nu)) = |\check{u}_\nu| \cdot o\left(\frac{1}{|\nu|}\right), \quad \text{as } |\nu| \to \infty. \tag{3.36}$$

Further, $h_\nu(0) = \widetilde{c}_\nu = (1 + a_\nu)^2 |\check{u}_\nu|^2 / (16\pi^4)$. Hence,

$$\left\| \frac{d\widetilde{z}}{da} \right\| = \left\| \frac{d\widetilde{c}_\nu}{da} \cdot \begin{pmatrix} e^{i\varphi} \\ 0 \end{pmatrix} \right\| = \left\| \frac{\partial}{\partial a} h_\nu(0) \right\| = \frac{2|1 + a_\nu| \cdot |\check{u}_\nu|^2}{16\pi^4} = \frac{2}{1 + a_\nu} |\widetilde{c}_\nu| \begin{cases} \leq 4|\widetilde{c}_\nu| \\ \geq \frac{4}{3} |\widetilde{c}_\nu| \end{cases} \tag{3.37}$$

if we choose without restriction the ball U such that $\|a\|_{l^\infty} \leq 1/2$ for all $a \in U$. Summing up, we obtain by the above estimates (3.36), (3.37) together with (2.82) and (3.33) the estimates

$$\begin{aligned} \left\| \frac{\partial h_\nu(\zeta_\nu)}{\partial a} - \frac{\partial h_\nu(0)}{\partial a} \right\| &= |\widetilde{c}_\nu| \cdot o\left(\frac{1}{|\nu|^2}\right), \quad \text{as } |\nu| \to \infty, \\ \left\| \frac{\partial h_\nu(\zeta_\nu)}{\partial a} \right\| &= \frac{2}{1 + a_\nu} |\widetilde{c}_\nu| (1 + o(1)), \quad \text{as } |\nu| \to \infty \end{aligned}$$

and consequently with (3.32)

$$\left\| \frac{dz}{da} - \frac{d\widetilde{z}}{da} \right\| = \left\| \left(\frac{dc_\nu}{da} - \frac{d\widetilde{c}_\nu}{da} \right) \cdot \begin{pmatrix} e^{i\varphi} \\ 0 \end{pmatrix} \right\| = |\widetilde{c}_\nu| \cdot o\left(\frac{1}{|\nu|^2}\right), \quad \text{as } |\nu| \to \infty. \tag{3.38}$$

This implies with $\widetilde{c}_\nu = h_\nu(0)$, the parameterization (3.31) and (3.32)

$$\begin{aligned} \left\| \frac{dz}{da} \right\| &= \left\| \frac{dc_\nu}{da} \cdot \begin{pmatrix} e^{i\varphi} \\ 0 \end{pmatrix} \right\| = \frac{2}{1 + a_\nu} |\widetilde{c}_\nu| (1 + o(1)) \overset{(3.37)}{=} \left\| \frac{\partial}{\partial a} h_\nu(0) \right\| (1 + o(1)) = \\ &= \left\| \frac{d\widetilde{z}}{da} \right\| (1 + o(1)), \quad \text{as } |\nu| \to \infty. \end{aligned} \tag{3.39}$$

In a second step, we consider a decomposition analogous to (3.23) and derive the individual terms with respect to a, more precisely, we consider

$$k_1(z)dk_2(z) - \widetilde{k}_1(\widetilde{z})d\widetilde{k}_2(\widetilde{z}) = [k_1(z) - \widetilde{k}_1(\widetilde{z})]dk_2(z) + \widetilde{k}_1(\widetilde{z})d[k_2(z) - \widetilde{k}_2(\widetilde{z})]. \tag{3.40}$$

As already mentioned in the proof of Lemma 3.2.2, it makes no difference to add terms which are constant in k since due to Cauchy's Integral Theorem, they vanish anyway by computing the contour integral m_ν. We can thus consider as well (3.23) itself, namely

$$\begin{aligned}&(k_1(z) - k_1(0))dk_2(z) - (\widetilde{k}_1(\widetilde{z}) - k_{\nu,1})d\widetilde{k}_2(\widetilde{z}) = \\ &\quad = [(k_1(z) - k_1(0)) - (\widetilde{k}_1(\widetilde{z}) - k_{\nu,1})]dk_2(z) + (\widetilde{k}_1(\widetilde{z}) - k_{\nu,1})d(k_2(z) - \widetilde{k}_2(\widetilde{z})).\end{aligned}$$

Which decomposition we will actually use depends on the respective computations. We start with the term $k_1(z) - \widetilde{k}_1(\widetilde{z})$. Deriving with respect to a yields with the usual representation (3.13) analogously to (3.20)

$$\begin{aligned}&\frac{dk}{da} - \frac{d\widetilde{k}}{da} = \frac{dk(z)}{dz}\cdot\frac{dz}{da} - \frac{d\widetilde{k}(z)}{d\widetilde{z}}\cdot\frac{d\widetilde{z}}{da} = (B'(k(z)))^{-1}\cdot\frac{dz}{da} - B_0^{-1}\cdot\frac{d\widetilde{z}}{da} = \\ &= B_0^{-1}\left(\mathbb{1} + B_2'(k(z))B_0^{-1}\right)^{-1}\cdot\frac{dz}{da} - B_0^{-1}\cdot\frac{d\widetilde{z}}{da} = \\ &= B_0^{-1}\left[\left(\mathbb{1} + B_2'(k(z))B_0^{-1}\right)^{-1}\cdot\frac{dz}{da} - \left(\mathbb{1} + B_2'(k(z))B_0^{-1}\right)^{-1}\left(\mathbb{1} + B_2'(k(z))B_0^{-1}\right)\frac{d\widetilde{z}}{da}\right] = \\ &= B_0^{-1}\cdot\left(\mathbb{1} + B_2'(k(z))B_0^{-1}\right)^{-1}\left(\frac{dz}{da} - \frac{d\widetilde{z}}{da} - B_2'(k(z))B_0^{-1}\cdot\frac{d\widetilde{z}}{da}\right) = O\left(\frac{1}{|\nu|^2}\right)|\widetilde{c}_\nu|,\end{aligned} \tag{3.41}$$

as $|\nu| \to \infty$, due to (3.37),(3.38) and the well-known estimates for $B_0^{-1}, B_2'(k)$ (see the proof of Lemma 3.2.2, for instance).
We continue with the term $dk_2(z)$. We have $dk_2 = \frac{dk_2(z)}{dz}\cdot\frac{dz}{d\varphi}d\varphi$. The derivative of the term

$$\frac{dz}{d\varphi} = i\begin{pmatrix} c_\nu \cdot e^{i\varphi} \\ -e^{-i\varphi}\end{pmatrix}$$

with respect to a is virtually estimated as dz/da. More precisely,

$$\frac{d}{da}\frac{dz}{d\varphi} = O(|\widetilde{c}_\nu|), \quad \text{as } |\nu| \to \infty \tag{3.42}$$

due to (3.37) and (3.39). As to the term $\frac{d}{da}\frac{dk_2(z)}{dz}$, we have

$$\frac{d}{da}\frac{dk_2(z)}{dz} = \frac{d^2k_2}{dz^2}\cdot\frac{dz}{da}.$$

We estimate the term $\frac{d^2k_2}{dz^2}$. We have for $i, j \in \{1, 2\}$

$$\frac{d^2k_2}{dz_i dz_j} = \frac{d}{dz_i}\left(\frac{dk_2}{dx} \cdot \frac{dx}{dz_j}\right) = \left(\frac{dx}{dz_i}\right)^T \cdot \frac{d^2k_2}{dx^2} \cdot \frac{dx}{dz_j} + \frac{dk_2}{dx} \cdot \frac{d^2x}{dz_i dz_j}. \tag{3.43}$$

Due to (2.66), the entries of the Hessian $\frac{d^2k_2}{dx^2}$ are $O(1/|\nu|^3)$ as $|\nu| \to \infty$. Furthermore, $dx/dz = 1 + o(1)$ (cf. (3.16)) and $dk/dx = O(1/|\nu|)$ (cf. (2.61)) as $|\nu| \to \infty$, respectively. As to the remaining term $\frac{d^2x}{dz_i dz_j}$, we argue as in the proof of Lemma 3.2.1, where we deduced the estimate for dx/dz by showing that the solution $\Phi_t(z)$ of the initial value problem (2.71) converges to z as $|\nu| \to \infty$, uniformly with respect to z and t. Hence, the second partial derivatives $\frac{d^2}{dz_i dz_j}\Phi_t(z)$ converge to zero as $|\nu| \to \infty$, also uniformly with respect to z and t. This shows

$$\frac{d^2x}{dz_i dz_j} = \frac{d^2\Phi_1(z+\zeta)}{dz_i dz_j} = o(1), \quad \text{as } |\nu| \to \infty.$$

This together with the estimates before yields

$$\frac{d}{da}\left(\frac{dk_2(z)}{dz} \cdot \frac{dz}{d\varphi}\right) = \left(\frac{d}{da}\frac{dk_2(z)}{dz}\right)\frac{dz}{d\varphi} + \frac{dk_2(z)}{dz} \cdot \frac{d}{da}\frac{dz}{d\varphi} = |\widetilde{c}_\nu| \cdot O\left(\frac{1}{|\nu|}\right), \tag{3.44}$$

as $|\nu| \to \infty$.

Next, we want to derive $d(k_2(z) - \widetilde{k}_2(\widetilde{z}))$ with respect to a. We would like to use a representation analogous to (3.22). In the former parameterization (3.19), we had $z = \sqrt{\frac{c_\nu}{\widetilde{c}_\nu}} \cdot \widetilde{z}$. With the new ("square-avoiding") parameterization (3.31), we have

$$z = C_\nu \cdot \widetilde{z}, \quad \text{where } C_\nu := \begin{pmatrix} \frac{c_\nu}{\widetilde{c}_\nu} & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{C}^{2\times 2}.$$

The matrix C_ν will now take the place of the matrix $\sqrt{\frac{c_\nu}{\widetilde{c}_\nu}} \cdot \mathbb{1}$ we considered before. Note that due to Theorem 2.5.9,

$$C_\nu - \mathbb{1} = o\left(\frac{1}{|\nu|^2}\right), \quad \text{as } |\nu| \to \infty. \tag{3.45}$$

We obtain completely analogously to (3.20) and (3.21) (simply by replacing $\sqrt{\frac{c_\nu}{\widetilde{c}_\nu}} \cdot \mathbb{1}$ by C_ν in the respective computations)

$$(B'(k))^{-1}z - B_0^{-1}\widetilde{z} = \left(C_\nu - \mathbb{1} - B_0^{-1}B_2'(k)\right)\left(\mathbb{1} + B_0^{-1}B_2'(k)\right)^{-1} B_0^{-1}\widetilde{z},$$

and

$$\begin{aligned}(k(z) - k(0)) - (\widetilde{k}(\widetilde{z}) - k_\nu) &= \\ = \int_0^1 \left(C_\nu - \mathbb{1} - B_0^{-1}B_2'(k(tz))\right)&\left(\mathbb{1} + B_0^{-1}B_2'(k(tz))\right)^{-1} dt \cdot B_0^{-1}\widetilde{z},\end{aligned} \tag{3.46}$$

respectively. The analogon to (3.22) is then

$$\begin{aligned} dk &= C_\nu \cdot \left(\mathbb{1} + B_0^{-1} B_2'(k(z))\right)^{-1} B_0^{-1} \cdot \frac{d\widetilde{z}}{d\varphi} d\varphi, \\ dk - d\widetilde{k} &= \left(C_\nu - \mathbb{1} - B_0^{-1} B_2'(k)\right) \left(\mathbb{1} + B_0^{-1} B_2'(k)\right)^{-1} B_0^{-1} \cdot \frac{d\widetilde{z}}{d\varphi} d\varphi. \end{aligned} \tag{3.47}$$

We have

$$dk - d\widetilde{k} = \frac{dk(z)}{dz} \cdot \frac{dz}{d\varphi} d\varphi - \frac{d\widetilde{k}(z)}{d\widetilde{z}} \cdot \frac{d\widetilde{z}}{d\varphi} d\varphi = \left(\frac{dk(z)}{dz} \cdot C_\nu - \frac{d\widetilde{k}(z)}{d\widetilde{z}} \right) \cdot \frac{d\widetilde{z}}{d\varphi} d\varphi. \tag{3.48}$$

Again, we derive the appearing terms with respect to a. We start with the first factor. We have

$$\frac{d}{da} \left(\frac{dk(z)}{dz} \cdot C_\nu - \frac{d\widetilde{k}(z)}{d\widetilde{z}} \right) = \left(\frac{d}{da} \frac{dk(z)}{dz} \right) \cdot C_\nu + \frac{dk(z)}{dz} \cdot \frac{dC_\nu}{da} - \underbrace{\frac{d}{da} B_0^{-1}}_{=0}. \tag{3.49}$$

Here, $\left(\frac{d}{da} \frac{dk(z)}{dz} \right) \cdot C_\nu = |\widetilde{c}_\nu| \cdot o(1/|\nu|)$, as $|\nu| \to \infty$, due to the above estimates (cf. the estimates of the first summand in (3.44), in particular (3.43)[11]) and (3.45). As to the second summand in (3.49), we have to estimate dC_ν/da. Since $\frac{d}{da} C_\nu = \frac{d}{da} \frac{c_\nu}{\widetilde{c}_\nu} \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, we estimate $\frac{d}{da} \frac{c_\nu}{\widetilde{c}_\nu}$. We have (by denotong the derivative with respect to a with a prime) due to Theorem 2.5.9

$$\left(\frac{c_\nu}{\widetilde{c}_\nu} \right)' = \frac{1}{\widetilde{c}_\nu} \left(c_\nu' - \widetilde{c}_\nu' \cdot \frac{c_\nu}{\widetilde{c}_\nu} \right) = \frac{1}{\widetilde{c}_\nu} \left((c_\nu' - \widetilde{c}_\nu') - \widetilde{c}_\nu' \cdot o\left(\frac{1}{|\nu|^2} \right) \right), \quad \text{as } |\nu| \to \infty. \tag{3.50}$$

By (3.38), we have $c_\nu' - \widetilde{c}_\nu' = |\widetilde{c}_\nu| \cdot o(1/|\nu|^2)$, as $|\nu| \to \infty$. Together with (3.37), we obtain by (3.50) $(\frac{c_\nu}{\widetilde{c}_\nu})' = o(1/|\nu|^2)$, as $|\nu| \to \infty$. Hence, we have (together with $dk/dz = O(1/|\nu|)$)

$$\frac{dk(z)}{dz} \cdot \left(\frac{c_\nu}{\widetilde{c}_\nu} \right)' = o\left(\frac{1}{|\nu|^3} \right), \quad \text{as } |\nu| \to \infty$$

and consequently, due to $\frac{dC_\nu}{da} = (\frac{c_\nu}{\widetilde{c}_\nu})' \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ [12]

$$\begin{aligned} \left(\frac{dk(z)}{dz} \cdot \frac{dC_\nu}{da} \right) \cdot \frac{d\widetilde{z}}{d\varphi} &= \begin{pmatrix} \frac{dk_1}{dz_1} \cdot \left(\frac{c_\nu}{\widetilde{c}_\nu} \right)' & 0 \\ \frac{dk_2}{dz_1} \cdot \left(\frac{c_\nu}{\widetilde{c}_\nu} \right)' & 0 \end{pmatrix} \cdot \frac{d\widetilde{z}}{d\varphi} = \\ &= ie^{i\varphi} \cdot \widetilde{c}_\nu \cdot \frac{dk}{dz_1} \cdot \left(\frac{c_\nu}{\widetilde{c}_\nu} \right)' = |\widetilde{c}_\nu| \cdot o\left(\frac{1}{|\nu|^3} \right), \quad \text{as } |\nu| \to \infty. \end{aligned}$$

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[11] In (3.44), we only considered k_2. Clearly, the same estimate holds for k_1, too.

[12] Note that in the parameterization (3.31), $\widetilde{z}$ as well as $d\widetilde{z}/d\varphi$ are not $O(|\widetilde{c}_\nu|)$ as $|\nu| \to \infty$ because -in contrast to (3.19)- the second entry $e^{-i\varphi}$ is independent of $\widetilde{c}_\nu$. In this context, it is essential that the second column of $\frac{dC_\nu}{da}$ vanishes.

Together with the estimate of the first summand in (3.49) and with (3.42) and (3.47), we finally obtain

$$\frac{d}{da}\left[\left(\frac{dk(z)}{dz}\cdot C_\nu - \frac{d\widetilde{k}(z)}{d\widetilde{z}}\right)\cdot\frac{d\widetilde{z}}{d\varphi}\right] = |\widetilde{c}_\nu|\cdot o\left(\frac{1}{|\nu|}\right), \quad \text{as } |\nu|\to\infty. \tag{3.51}$$

Now, we have estimated everything we need. In a last step, we now sum up all these estimates to gain the desired estimate for dr_ν/da. For $v\in\mathbb{C}^2$, we use the usual notation v_1 and v_2 for the first and the second component, respectively. This explains the terms $[\cdot]_1$ and $[\cdot]_2$ in the following.
We have with $u = P^{-1}((1+a)\cdot\breve{u})$, the decompositions (3.23) and (3.40), respectively, as well as by (3.37), (3.41), (3.44), (3.45), (3.46), (3.47), (3.48), (3.51) and the well-known estimates for $B_0^{-1}, B_2'(k)$ (see the proof of Lemma 3.2.2, for instance):

$$\begin{aligned}
&\frac{-1}{16\pi^3}\frac{d}{da}r_\nu(u) = \frac{d}{da}\left(\int_{A_\nu} k_1 dk_2 - \int_{\widetilde{A}_\nu}\widetilde{k}_1 d\widetilde{k}_2\right) = \\
&= \int_0^{2\pi}\left(\frac{dk_1}{da}-\frac{d\widetilde{k}_1}{da}\right)\frac{dk_2}{dz}\cdot\frac{dz}{d\varphi}d\varphi + \\
&+\int_0^{2\pi}(k_1(z)-k_1(0)-(\widetilde{k}_1(\widetilde{z})-k_{\nu,1}))\frac{d}{da}\left(\frac{dk_2}{dz}\cdot\frac{dz}{d\varphi}\right)d\varphi + \\
&+\int_0^{2\pi}\frac{d\widetilde{k}_1}{da}\frac{d}{d\widetilde{z}}(k_2-\widetilde{k}_2)\cdot\frac{d\widetilde{z}}{d\varphi}d\varphi + \int_0^{2\pi}\widetilde{k}_1\frac{d}{da}\left(\frac{d}{d\widetilde{z}}(k_2-\widetilde{k}_2)\cdot\frac{d\widetilde{z}}{d\varphi}\right)d\varphi = \\
&\overset{(3.47)}{=}\int_0^{2\pi}\left(\frac{dk_1}{da}-\frac{d\widetilde{k}_1}{da}\right)\left[C_\nu\cdot\left(\mathbb{1}+B_0^{-1}B_2'(k(z))\right)^{-1}B_0^{-1}\cdot\frac{d\widetilde{z}}{d\varphi}\right]_2 d\varphi + \\
&\overset{(3.46)}{+}\int_0^{2\pi}\left[\int_0^1\left(C_\nu-\mathbb{1}-B_0^{-1}B_2'(k(tz))\right)\left(\mathbb{1}+B_0^{-1}B_2'(k(tz))\right)^{-1}dt\cdot B_0^{-1}\widetilde{z}\right]_1\cdot \\
&\qquad\cdot\frac{d}{da}\left(\frac{dk_2(z)}{dz}\cdot\frac{dz}{d\varphi}\right)d\varphi + \\
&\overset{(3.47)}{+}\int_0^{2\pi}\left[B_0^{-1}\frac{d\widetilde{z}}{da}\right]_1\left[\left(C_\nu-\mathbb{1}-B_0^{-1}B_2'(k)\right)\left(\mathbb{1}+B_0^{-1}B_2'(k)\right)^{-1}B_0^{-1}\cdot\frac{d\widetilde{z}}{d\varphi}\right]_2 d\varphi + \\
&\overset{(3.48)}{+}\int_0^{2\pi}\left[B_0^{-1}\widetilde{z}\right]_1\cdot\left[\frac{d}{da}\left(\left(\frac{dk(z)}{dz}\cdot C_\nu-\frac{d\widetilde{k}(z)}{d\widetilde{z}}\right)\cdot\frac{d\widetilde{z}}{d\varphi}\right)\right]_2 d\varphi = \\
&= o\left(\frac{1}{|\nu|^2}\right)|\widetilde{c}_\nu| \overset{(2.84)}{=} o(1)\cdot|\widetilde{m}_\nu|, \quad \text{as } |\nu|\to\infty.
\end{aligned} \tag{3.52}$$

This proves the first claim of the lemma.
As to the second claim, we can essentially use the results just proven with some

exceptions. At first, we get by the chain rule

$$\frac{d}{da} r_\nu(P^{-1}((1+a)\cdot \check{u})) = \frac{d}{du} r_\nu(u)|_{u=P^{-1}((1+a)\cdot\check{u})} \cdot \frac{d}{da} P^{-1}((1+a)\cdot \check{u}).$$

In most of the terms above, we have virtually already estimated $\frac{d}{du} r_\nu(u)$ and used for the second factor the rough estimate $\frac{d}{da} P^{-1}((1+a)\cdot \check{u}) = O(1)$, as $\delta \to 0$ such that the above estimates carry over. In other words, we already derived these terms with respect to u (instead of with respect to a). Things are different for terms which are already explicitly given in terms of perturbed Fourier coefficients. For the term $(1+a_\nu)^2|\check{u}_\nu|^2$, for instance, we didn't use the above chain rule but derived explicitly with respect to a_ν yielding $2(1+a_\nu)|\check{u}_\nu|^2$. Deriving the corresponding term with respect to u, however yields

$$\frac{d}{du}|\check{u}_\nu|^2 = \frac{d}{d\check{u}_\nu}(\check{u}_\nu \check{u}_{-\nu}) \cdot \frac{d\check{u}_\nu}{du} \overset{(3.30)}{=} \check{u}_{-\nu}(\mathbb{1} + o(1)) = O(|\check{u}_\nu|), \quad \text{as } |\nu| \to \infty.$$

In other words, we can carry out virtually the same estimates as above with the restriction that by deriving $|\check{u}_\nu|^2$, we only get a term $O(|\check{u}_\nu|)$ instead of the stronger estimate $O(|\check{u}_\nu|^2)$ as before when we derived $(1+a_\nu)^2|\check{u}_\nu|^2$ with respect to a_ν, i.e we get a reduction of 1 in the power of $|\check{u}_\nu|$. If we retrace the above proof, we see that the equations affected by this are (3.36), (3.37), (3.38), (3.41), (3.42), (3.44) and (3.51). More precisely, the right hand sides of all of these equations contain either a term of order $16\pi^4|\widetilde{c}_\nu| = |\check{u}_\nu|^2$ or $|\check{u}_\nu|$. These terms then reduce to terms of order $|\check{u}_\nu|$ or $|\check{u}_\nu|^0 = 1$, respectively, in the sense just explained. If we plug the corresponding estimates into the big computation (3.52), the corresponding estimate in (3.52) then yields that $\frac{d}{du} r_\nu(u)$ equals only $\frac{|\check{u}_\nu|}{|\nu|^2} o(1)$ (instead of $|\widetilde{m}_\nu(u)| o(1) = \frac{|\check{u}_\nu|^2}{|\nu|^2} o(1)$) as $|\nu| \to \infty$ [13]. This shows the desired estimate and the lemma is proved. □

We owe the proof of the following assertion used in the foregoing lemma.

Lemma 3.2.7.

$$\frac{\partial^2}{\partial u\, \partial k} A^\nu_{ij}(k,u) = o(1), \quad \textit{as } |\nu| \to \infty.$$

Proof. In order to determine $\frac{\partial^2}{\partial u \partial k} A(\mathbb{1} - BA)^{-1}$, we use virtually the same expression as in (2.63) where we computed $\frac{\partial^2}{\partial k^2} A(\mathbb{1} - BA)^{-1}$. This time, of course,

[13] The reason for this is that each summand of $\frac{d}{du} r_\nu(u)$ in the corresponding computation analogous to (3.52) is due to the product rule a product of a term which is derived with respect to u (where the mentioned reduction of the power appears) and another term which is not derived and stays the same as before.

the second derivative is with respect to u (instead of with respect to k). More precisely, with the operator $\widetilde{C}$ defined in (2.40),

$$\frac{\partial^2}{\partial u\,\partial k}A(\mathbb{1}-BA)^{-1} = -\frac{\partial}{\partial u}[A(\mathbb{1}-BA)^{-1}B\widetilde{C}BA(\mathbb{1}-BA)^{-1}] =$$
$$= -\frac{\partial}{\partial u}[A(\mathbb{1}-BA)^{-1}]B\widetilde{C}BA(\mathbb{1}-BA)^{-1} - A(\mathbb{1}-BA)^{-1}B\widetilde{C}B\frac{\partial}{\partial u}[A(\mathbb{1}-BA)^{-1}]-$$
$$- A(\mathbb{1}-BA)^{-1}\frac{\partial}{\partial u}[B\widetilde{C}B]A(\mathbb{1}-BA)^{-1}. \tag{3.53}$$

Let's consider the three summands of this expression seperately. By the proof of Theorem 2.4.2 (and the respective comment in the proof of the foregoing Lemma 3.2.6), we know that the operators $\frac{\partial}{\partial u}[A(\mathbb{1}-BA)^{-1}]$ and $B\widetilde{C}$ are bounded with respect to $|\nu|$. Moreover, by Lemma 2.2.5, $\|BA\| = o(1)$ as $|\nu| \to \infty$. Hence, the first summand of (3.53) is $o(1)$ as $|\nu| \to \infty$. Due to [13, Lemma 4.5.23], we have $A(\mathbb{1}-BA)^{-1} = (\mathbb{1}-AB)^{-1}A$. Hence, the second summand of (3.53) is $o(1)$ as $|\nu| \to \infty$, as well. As to the third summand, we compute using the operator C defined in (2.34)

$$\mathbb{C}^2 \times L^2(F) \ni (x,h) \mapsto \frac{\partial}{\partial u}[B\widetilde{C}B](x,h) =$$
$$= -BC(h)B\widetilde{C}(x)B - B\widetilde{C}(x)BC(h)B + B\left[\frac{\partial}{\partial u}\widetilde{C}\right](x,h)B,$$

where for $x \in \mathbb{C}^2, h \in L^2(F)$,

$$\frac{\partial}{\partial u}\widetilde{C}(x,h) = -\frac{8\pi^2 i}{\xi\nu^2}diag\left(\left\langle\begin{pmatrix}\nu_2\\-\nu_1\end{pmatrix},x\right\rangle\cdot\hat{h}(0)\right)_{\rho\in\Gamma^*\setminus\{0,\pm\nu\}}$$

due to $\frac{\partial k_\nu^\pm(\hat{u}_0)}{\partial\hat{u}_0} = \frac{i}{\xi\nu^2}\begin{pmatrix}\nu_2\\-\nu_1\end{pmatrix}$ by definition (2.2)
(recall $\widetilde{C}(x) = -8\pi^2 diag\left(\langle\rho+k+k_\nu^\pm(\hat{u}_0),x\rangle\right)_{\rho\in\Gamma^*\setminus\{0,\pm\nu\}}$, (2.40)). By definition of $B, C, \widetilde{C}$, we immediately see (which we have already seen in the proof of Theorem 2.4.2) that the operators BC and $B\widetilde{C}$ are bounded with respect to $|\nu|$. Moreover, by the above calculation, the operator $B\left[\frac{\partial}{\partial u}\widetilde{C}\right]$ is bounded with respect to $|\nu|$, too. By Lemma 2.2.4, $\|B\| = o(1)$ as $|\nu| \to \infty$. Hence, the third summand of (3.53) is $o(1)$ as $|\nu| \to \infty$, too. This proves the lemma. □

In the following theorem, we state the existence of perturbation flows by applying Banach's Fixed Point Theorem.

Theorem 3.2.8. *Let $u_0 \in L^2(F)$ be real-valued. Then for all $t \in [0,2\pi)^\infty$, the map (3.12) $\Psi_t : U \to U$ with a sufficiently small closed ball $U \subseteq l^\infty_{r,e}(\Gamma^*_\delta)$ centered* *at* $0 \in l^\infty_{r,e}(\Gamma^*_\delta)$ *is contractive. Hence, there exists a unique* $a_t = (a_t^\nu)_{\nu\in\Gamma^*_\delta} \in U$ *such that for all* $\nu \in \Gamma^*_\delta$ *and for all* $t \in [0, 2\pi)^\infty$

$$m_\nu(u_0) = \widetilde{m}_\nu(u_t + v_t) + r_\nu(u_t + v_t),$$

with definition (3.11) $u_t + v_t := P^{-1}((1 + a_t) \cdot \check{u}_t)$.

Proof. Let $t \in [0, 2\pi)^\infty$ and let $U \subset l^\infty_{r,e}(\Gamma^*_\delta)$ be a ball centered at $0 \in l^\infty_{r,e}(\Gamma^*_\delta)$ (how small U has finally to be chosen will become clear in the following). We show at first that Ψ_t maps U into U. Because of (3.27), we have

$$\Psi_t(a_t) = O\left(\frac{1}{|\nu|}\right), \quad \text{as } |\nu| \to \infty,$$

uniformly in $a_t \in U$. Thus, by choosing $\delta > 0$ sufficiently small, one can achieve that Ψ_t maps U into U, even into a subset whose closure is still contained in U, so that we may choose U to be closed. Note that, in fact, the choice of $\delta > 0$ only depends on u_0 since the model flows $\check{u}_t$ and even the perturbed flows $(1 + a_t)\check{u}_t$ (by choosing U correspondently small) are contained in a ball in $l^2(\Gamma^*_\delta)$ around $0 \in l^2(\Gamma^*_\delta)$ where the map (2.32) is invertible (compare the choice of δ discussed on p. 72). Next, we show that Ψ_t is contractive. To this, we have to show that there is a constant $0 \le L < 1$ such that for all $a_t, b_t \in U$, there holds

$$\|\Psi_t(b_t) - \Psi_t(a_t)\|_{l^\infty(\Gamma^*_\delta)} \le L \cdot \|b_t - a_t\|_{l^\infty(\Gamma^*_\delta)}.$$

Let $a_t, b_t \in U$. We begin with the estimate of the radicand of (3.12). We have

$$\left|\frac{m_\nu(u_0) - r_\nu(P^{-1}((1 + b_t) \cdot \check{u}_t))}{\widetilde{m}_\nu(u_0)} - \frac{m_\nu(u_0) - r_\nu(P^{-1}((1 + a_t) \cdot \check{u}_t))}{\widetilde{m}_\nu(u_0)}\right| =$$
$$= \frac{1}{|\widetilde{m}_\nu(u_0)|}\left|r_\nu(P^{-1}((1 + b_t) \cdot \check{u}_t)) - r_\nu(P^{-1}((1 + a_t) \cdot \check{u}_t))\right|.$$

Now, $a \longmapsto r_\nu(P^{-1}((1+a)\check{u}))$ is smooth due to Lemma 3.2.5. We thus obtain by the Mean Value Theorem (cf. [30, Satz III.5.4(b)])

$$\left|r_\nu(P^{-1}((1 + b_t) \cdot \check{u}_t)) - r_\nu(P^{-1}((1 + a_t) \cdot \check{u}_t))\right| \le \sup_{a\in U}\left\|\frac{d}{da} r_\nu(P^{-1}((1 + a) \cdot \check{u}_t))\right\| \cdot$$
$$\cdot \|b_t - a_t\|_{l^\infty(\Gamma^*_\delta)}.$$

By choosing $\delta > 0$ sufficiently small, we can achieve by Lemma 3.2.6

$$\sup_{a\in U}\left\|\frac{d}{da} r_\nu(P^{-1}((1 + a) \cdot \check{u}_t))\right\| \le \frac{1}{2} \cdot |\widetilde{m}_\nu(u_0)|, \quad \nu \in \Gamma^*_\delta.$$

Now, for x, y in a sufficiently small neighbourhood of $0 \in \mathbb{R}$, we have

$$|\sqrt{1+y} - \sqrt{1+x}| = \left|\frac{(\sqrt{1+y} - \sqrt{1+x})(\sqrt{1+y} + \sqrt{1+x})}{\sqrt{1+y} + \sqrt{1+x}}\right| = \frac{|y - x|}{|\sqrt{1+y} + \sqrt{1+x}|}.$$

Setting

$$x := \frac{m_\nu(u_0) - r_\nu(P^{-1}((1+a_t)\cdot \check{u}_t))}{\widetilde{m}_\nu(u_0)} - 1, \quad y := \frac{m_\nu(u_0) - r_\nu(P^{-1}((1+b_t)\cdot \check{u}_t))}{\widetilde{m}_\nu(u_0)} - 1,$$

we obtain by (3.27) $x, y = O(1/|\nu|)$ as $|\nu| \to \infty$. Now, the above estimates imply with the abbreviations x and y for all $\nu \in \Gamma_\delta^*$

$$|\Psi_t^\nu(b_t) - \Psi_t^\nu(a_t)| = \frac{|y-x|}{|\sqrt{1+y} + \sqrt{1+x}|} \le \frac{1}{2}\frac{\|b_t - a_t\|_{l^\infty(\Gamma_\delta^*)}}{|\sqrt{1+y} + \sqrt{1+x}|} \le \frac{1}{2}\|b_t - a_t\|_{l^\infty(\Gamma_\delta^*)},$$

where we chose $\delta > 0$ (respecting the above choices) sufficiently small such that both $\sqrt{1+x} \ge 1/2$ and $\sqrt{1+y} \ge 1/2$ (still uniformly in $a_t, b_t \in U$). We obtain

$$\|\Psi_t(b_t) - \Psi_t(a_t)\|_{l^\infty(\Gamma_\delta^*)} \le \frac{1}{2}\|b_t - a_t\|_{l^\infty(\Gamma_\delta^*)}, \tag{3.54}$$

which shows that Ψ_t is contractive since all estimates were uniform in $a_t, b_t \in U$. Therefore, by Banach's Fixed Point Theorem, Ψ_t has for each $t \in [0, 2\pi)^\infty$ a unique fixed point a_t in U. Together with Theorem 3.2.4, this yields

$$m_\nu(u_0) = \widetilde{m}_\nu(u_t + v_t) + r_\nu(u_t + v_t),$$

for all $\nu \in \Gamma_\delta^*$ and for all $t \in [0, 2\pi)^\infty$ where v_t is defined analogously to (3.11), i.e. $v_t = P^{-1}((1+a_t)\check{u}_t) - u_t$ (here, a_t denotes the fixed point, whereas in (3.11), $a_t \in U$ was still arbitrary).

□

3.3 The asymptotic isospectral set

In this section, we want to state a homeomorphism I between the asymptotic model isospectral set $\widetilde{Iso}_\delta(u_0)$ (3.4) and the asymptotic isospectral set $Iso_\delta(u_0)$ (3.3), where $u_0 \in L^2(F)$ is a given real-valued potential. We define the map I as follows:

$$I : \widetilde{Iso}_\delta(u_0) \longrightarrow Iso_\delta(u_0), \quad (\check{u}_\nu)_{\nu \in \Gamma_\delta^*} \longmapsto (\check{u}_\nu \cdot (1 + a_\nu))_{\nu \in \Gamma_\delta^*} =: \check{u} \cdot (1 + a). \tag{3.55}$$

Let's explain this definition more precisely: Let $(\check{u}_\nu)_{\nu \in \Gamma_\delta^*} \in \widetilde{Iso}_\delta(u_0)$ be given. Then, for each pair of indices $\{\nu, -\nu\}$ with $\nu \in \Gamma_\delta^*$, there is a flow parameter $t = t_\nu \in [0, 2\pi)$ such that for all $\nu \in \Gamma_\delta^*$ (cf. (3.7), (3.8))

$$(\check{u}_\nu, \check{u}_{-\nu}) = (e^{it}\check{u}_{0,\nu}, e^{-it}\check{u}_{0,-\nu}) = (\check{u}_t^\nu, \check{u}_t^{-\nu}).$$

In this sense, we may write $(\check{u}_\nu)_{\nu \in \Gamma_\delta^*} = (\check{u}_t^\nu)_{\nu \in \Gamma_\delta^*} =: \check{u}_t$ with some $t \in [0, 2\pi)^\infty$. Inserting this model flow $\check{u}_t$ into (3.12), we get a map Ψ_t which has a unique fixed point $a_t = (a_\nu^t)_{\nu \in \Gamma_\delta^*} =: (a_\nu)_{\nu \in \Gamma_\delta^*}$ due to Theorem 3.2.8. This yields the image $(\check{u}_\nu \cdot (1 + a_\nu))_{\nu \in \Gamma_\delta^*}$ of $(\check{u}_\nu)_{\nu \in \Gamma_\delta^*}$ by the map I [14]. In the following lemma, we

[14]To be more precise, one could decorate (3.55) with the parameter t in order that it becomes clearer where the fixed point $(a_\nu)_{\nu \in \Gamma_\delta^*}$ comes from.

prove that the map I is bijective.

Lemma 3.3.1. *The map* $I : \widetilde{Iso}_\delta(u_0) \longrightarrow Iso_\delta(u_0)$ *is bijective.*

Proof. We prove at first that I is one-to-one. To this, let $\check{u}, \check{v} \in \widetilde{Iso}_\delta(u_0)$ with $\check{u} \neq \check{v}$. Then there are multi-parameters $t, \tilde{t} \in [0, 2\pi)^\infty$ with $\check{u} = (\check{u}_t^\nu)_{\nu \in \Gamma_\delta^*}$ and $\check{v} = (\check{u}_{\tilde{t}}^\nu)_{\nu \in \Gamma_\delta^*}$. Since $\check{u} \neq \check{v}$, there is a $\nu \in \Gamma_\delta^*$ with $\check{u}_t^\nu \neq \check{u}_{\tilde{t}}^\nu$ (with corresponding parameters in $[0, 2\pi)$). This implies $\arg(\check{u}_t^\nu) \neq \arg(\check{u}_{\tilde{t}}^\nu)$ [15] due to $|\check{u}_t^\nu| = |\check{u}_{\tilde{t}}^\nu|$ by definition of the model flows. Since the corresponding fixed points a_t and $a_{\tilde{t}}$ (cf. Theorem 3.2.8) are real due to Theorem 3.2.4, we thus obtain

$$\arg(\check{u}_t^\nu \cdot (1 + a_t^\nu)) = \arg(\check{u}_t^\nu) \neq \arg(\check{u}_{\tilde{t}}^\nu) = \arg(\check{u}_{\tilde{t}}^\nu \cdot (1 + a_{\tilde{t}}^\nu)),$$

in particular $\check{u}_t^\nu \cdot (1 + a_t^\nu) \neq \check{u}_{\tilde{t}}^\nu \cdot (1 + a_{\tilde{t}}^\nu)$. Thus $I(\check{u}) \neq I(\check{v})$. This shows that I is one-to-one.

We now prove that I is onto. To this, let $\check{w} \in Iso_\delta(u_0)$. As usual, we write $\check{w} = (\check{w}_\nu)_{\nu \in \Gamma_\delta^*}$. Consider an arbitrary $\nu \in \Gamma_\delta^*$. Set $t := \arg(\check{w}_\nu) \in [0, 2\pi)$. In Theorem 3.2.8, we have proved that the map Ψ_t has a unique fixed point a_t in the ball U. Again by Theorem 3.2.8, this assertion is equivalent to saying that there exists a unique $a_t \in U$ such that with $\check{v}_t := \check{u}_t \cdot a_t$ and (3.11)

$$m_\nu(u_0) = m_\nu(u_t + v_t) \quad \text{for all } \nu \in \Gamma_\delta^*$$

holds, in particular for our ν fixed above. In other words, the perturbation flow v_t is *locally* unique, i.e. unique for $a_t \in U$ provided that the ansatz (3.10) is fulfilled. More precisely, to the closed ball U (in Theorem 3.2.8), there corresponds a ball $U_\nu \subset \mathbb{C}$ centered at $0 \in \mathbb{C}$ with radius not larger than the radius of U (in the l^∞-norm). Denote this radius of U by $R > 0$. Consider the segment S_ν in $\mathbb{C}$ defined by the intersection of U_ν with the half-line starting at 0 and going through $\check{u}_t^\nu$. The uniqueness of the fixed point of Ψ_t in U now states that there exists a unique $a_t \in U$ fulfilling the fixed point condition such that $a_t^\nu \cdot \frac{\check{u}_t^\nu}{|\check{u}_t^\nu|} \in S_\nu$ for all $\nu \in \Gamma_\delta^*$. We now have to see that, by choosing $\delta > 0$ sufficiently small, we can ensure that for every $\check{u} \in Iso_\delta(u_0)$, there holds

$$\check{u}_\nu \in Ann_\nu := \{e^{is} \cdot \check{u}_{0,\nu} \cdot (1 + a_\nu) : \quad a_\nu \in \mathbb{R}, |a_\nu| \leq R, s \in [0, 2\pi)\} \tag{3.56}$$

for all $\nu \in \Gamma_\delta^*$ (with $R > 0$ the radius of U as mentioned above). That is, every component $\check{w}_\nu$ of our given $\check{w} \in Iso_\delta(u_0)$ shall lie in the annulus Ann_ν. If we have proved this, we are done since elements of $Iso_\delta(u_0)$ whose components are contained in the annuli Ann_ν, $\nu \in \Gamma_\delta^*$, lie in the image of I by the preceding argument of the uniqueness of the considered fixed point.

[15] For the argument function which is generally multi-valued (i.e. only unique mod 2π), we choose that branch such that arg takes values in $[0, 2\pi)$. This makes arg unique.

To prove (3.56), recall at first that due to (3.17), we have for real-valued $u \in L^2(F)$

$$m_\nu(u) = \widetilde{m}_\nu(u) \cdot \left(1 + O\left(\frac{1}{|\nu|}\right)\right) = \frac{|\check{u}_\nu|^2}{\xi|\nu|^2} \cdot \left(1 + O\left(\frac{1}{|\nu|}\right)\right),$$

as $|\nu| \to \infty$. Solving this equation for $|\check{u}_\nu|$, and replacing $m_\nu(u)$ by $m_\nu(u_0)$, we get by $m_\nu(u_0)\xi|\nu|^2 = \frac{m_\nu(u_0)}{\widetilde{m}_\nu(u_0)}|\check{u}_{0,\nu}|^2$ the following estimate for the elements $(\check{u}_\nu)_{\nu\in\Gamma^*_\delta} \in Iso_\delta(u_0)$:

$$|\check{u}_\nu| = \sqrt{m_\nu(u_0)\xi} \cdot |\nu| \cdot \left(1 + O\left(\frac{1}{|\nu|}\right)\right) = \sqrt{\frac{m_\nu(u_0)}{\widetilde{m}_\nu(u_0)}} \cdot |\check{u}_{0,\nu}| \cdot \left(1 + O\left(\frac{1}{|\nu|}\right)\right) =$$
$$= |\check{u}_{0,\nu}| \cdot \left(1 + O\left(\frac{1}{|\nu|}\right)\right), \quad \text{as } |\nu| \to \infty, \tag{3.57}$$

where in the last step, we used (3.17) (compare also (3.26)). In other words: Choosing δ sufficiently small ensures that the ν^{th} component of any element in $Iso_\delta(u_0)$ lies in a sufficiently small annulus neighbourhood of the circle centered at $0 \in \mathbb{C}$ with radius $|\check{u}_{0,\nu}|$. This annulus neighbourhood is just determined by the error term $1+O(1/|\nu|)$ in (3.57). Hence, by choosing $\delta > 0$ sufficiently small, we obtain $\check{w}_\nu \in Ann_\nu$. This holds for all $\nu \in \Gamma^*_\delta$. Due to the uniqueness of the fixed point explained above, we now obtain $\check{w}_\nu = \check{u}^\nu_t(1 + a^\nu_t)$. This holds for arbitrary $\nu \in \Gamma^*_\delta$. This proves that I is onto. □

We now prove that $\widetilde{Iso}_\delta(u_0)$ is homeomorphic to $Iso_\delta(u_0)$.

Theorem 3.3.2. *The map $I : \widetilde{Iso}_\delta(u_0) \longrightarrow Iso_\delta(u_0)$ is a homeomorphism.*

Proof. We have to show that both I and its inverse I^{-1} (which exists due to Lemma 3.3.1) are continuous. Since $\widetilde{Iso}_\delta(u_0)$ and $Iso_\delta(u_0)$ are by definition subsets of $l^2_{\mathbb{R}}(\Gamma^*_\delta)$, we endow them with the relative topology induced by the topology of $l^2_{\mathbb{R}}(\Gamma^*_\delta)$. As to the continuity of I, we have to show

$$\forall_{\check{u}\in\widetilde{Iso}_\delta(u_0)} \forall_{\epsilon>0} \exists_{\eta>0} \forall_{\substack{\check{v}\in\widetilde{Iso}_\delta(u_0)\\ \|\check{v}-\check{u}\|_{l^2}<\eta}} : \quad \|I(\check{u}) - I(\check{v})\|_{l^2} < \epsilon. \tag{3.58}$$

To this, we show that the fixed point a appearing in the map I (3.55) depends continuously on $\check{u}$. We write $a = a(\check{u})$ in $I(\check{u}) = \check{u} \cdot (1 + a(\check{u}))$ to emphasize this dependence. Therefore, we rewrite the map $\Psi = \Psi_t$ (3.12) as

$$\Psi : \widetilde{Iso}_\delta(u_0) \times U \to U, \quad \Psi(\check{u}, a) := \left[-1 + \sqrt{\frac{m_\nu(u_0) - r_\nu(P^{-1}((1+a)\check{u}))}{\widetilde{m}_\nu(u_0)}}\right]_{\nu\in\Gamma^*_\delta},$$

where this time, t is suppressed. In Lemma 3.2.5, we proved that $u \mapsto r_\nu(u)$ is holomorphic. In particular, this map is continuous. Together with the decreasing behaviour of r_ν (cf. (3.17)) and the continuity of the maps P^{-1} and

$$\widetilde{Iso}_\delta(u_0) \times U \longrightarrow l^2_{\mathbb{R}}(\Gamma^*_\delta), \quad (\breve{u}, a) \longmapsto \breve{u} \cdot (1 + a)$$

(continuity immediately follows from
$\|\breve{u}(1+a)-\breve{v}(1+b)\|_{l^2} \leq \|\breve{u}-\breve{v}\|_{l^2}+\|a\|_{l^\infty}\|\breve{u}-\breve{v}\|_{l^2}+\|\breve{v}\|_{l^2}\|a-b\|_{l^\infty}$, $\breve{u}, \breve{v} \in \widetilde{Iso}_\delta(u_0)$, $a, b \in U$), this implies that Ψ is continuous. In particular, for $\breve{u} \in \widetilde{Iso}_\delta(u_0)$ fixed, the map $\Psi(\breve{u}, \cdot)$ is Lipschitz continuous (cf. Theorem 3.2.8). We now show that the fixed point $a(\breve{u})$ continuously depends on $\breve{u}$. To this, let $\breve{u} \in \widetilde{Iso}_\delta(u_0)$ and let $\epsilon > 0$. Since $\Psi(\cdot, a(\breve{u}))$ is continuous, there exists an $\eta > 0$ such that for all $\breve{w} \in \widetilde{Iso}_\delta(u_0)$ with $\|\breve{u} - \breve{w}\|_{l^2} < \eta$, there holds

$$\|\Psi(\breve{u}, a(\breve{u})) - \Psi(\breve{w}, a(\breve{u}))\|_{l^\infty} < \frac{\epsilon}{2}.$$

We therefore obtain together with the property that for all $\breve{w} \in \widetilde{Iso}_\delta(u_0)$, the map $\Psi(\breve{w}, \cdot)$ is contractive with Lipschitz constant $\frac{1}{2}$ (cf. (3.54)), that for all $\breve{w} \in \widetilde{Iso}_\delta(u_0)$ with $\|\breve{u} - \breve{w}\|_{l^2} < \eta$, there holds

$$\begin{aligned}\|a(\breve{u}) - a(\breve{w})\|_{l^\infty} &= \|\Psi(\breve{u}, a(\breve{u})) - \Psi(\breve{w}, a(\breve{w}))\|_{l^\infty} \leq \\ &\leq \|\Psi(\breve{u}, a(\breve{u})) - \Psi(\breve{w}, a(\breve{u}))\|_{l^\infty} + \|\Psi(\breve{w}, a(\breve{u})) - \Psi(\breve{w}, a(\breve{w}))\|_{l^\infty} < \\ &< \frac{\epsilon}{2} + \frac{1}{2} \cdot \|a(\breve{u}) - a(\breve{w})\|_{l^\infty}.\end{aligned}$$

Hence, $\|a(\breve{u}) - a(\breve{w})\|_{l^\infty} < \epsilon$. This proves that

$$a : \widetilde{Iso}_\delta(u_0) \longrightarrow U, \quad \breve{w} \longmapsto a(\breve{w}).$$

is continuous.
Now, let again $\breve{u} \in \widetilde{Iso}_\delta(u_0)$ and $\epsilon > 0$. Choose $\eta < \epsilon/3$ small enough such that $\|\breve{u}\|_{l^2} \cdot \|a(\breve{u}) - a(\breve{v})\|_{l^\infty} < \epsilon/2$ for all $\breve{v} \in \widetilde{Iso}_\delta(u_0)$ with $\|\breve{v} - \breve{u}\|_{l^2} < \eta$. This is possible due to the just proved continuity of $a \mapsto a(\breve{u})$. Now, let $\breve{v} \in \widetilde{Iso}_\delta(u_0)$ with $\|\breve{v} - \breve{u}\|_{l^2} < \eta$. We may assume that $\|a(\breve{v})\|_{l^\infty} \leq 1/2$ (otherwise choose the ball U in Theorem 3.2.8 smaller which maybe requires choosing $\delta > 0$ suitably smaller[16]). Then we obtain

$$\begin{aligned}\|I(\breve{u}) - I(\breve{v})\|_{l^2} &= \|\breve{u} \cdot (1 + a(\breve{u})) - \breve{v} \cdot (1 + a(\breve{v}))\|_{l^2} \leq \\ &\leq \|\breve{u} - \breve{v}\|_{l^2} + \|\breve{u} \cdot a(\breve{u}) - \breve{u} \cdot a(\breve{v})\|_{l^2} + \|\breve{u} \cdot a(\breve{v}) - \breve{v} \cdot a(\breve{v})\|_{l^2} \leq \\ &\leq \underbrace{\|\breve{u} - \breve{v}\|_{l^2}}_{<\eta} + \underbrace{\|\breve{u}\|_{l^2} \cdot \|a(\breve{u}) - a(\breve{v})\|_{l^\infty}}_{<\epsilon/2} + \underbrace{\|a(\breve{v})\|_{l^\infty}}_{<1/2} \cdot \underbrace{\|\breve{u} - \breve{v}\|_{l^2}}_{<\eta} < \epsilon.\end{aligned}$$

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[16]Note that, as also mentioned a few times before in a similar context (for example in the proof of Theorem 3.2.8 where we assured that Ψ_t maps U into U), that this choice of δ is admissible and only depends on the initial potential u_0.

This proves (3.58). The inverse map I^{-1} is given by

$$I^{-1} : Iso_\delta(u_0) \longrightarrow \widetilde{Iso}_\delta(u_0), \quad \check{w} \longmapsto \left(\frac{\check{w}_\nu}{|\check{w}_\nu|} |\check{u}_{0,\nu}| \right)_{\nu \in \Gamma_\delta^*}$$

since for $\check{u} = (\check{u}_\nu)_{\nu \in \Gamma_\delta^*} \in \widetilde{Iso}_\delta(u_0)$, we have due to $|\check{u}_{0,\nu}| = |\check{u}_\nu|$, $\nu \in \Gamma_\delta^*$ (cf. (3.8)),

$$I^{-1}(I(\check{u})) = \left(\frac{\check{u}_\nu \cdot (1 + a_\nu)}{|\check{u}_\nu \cdot (1 + a_\nu)|} |\check{u}_{0,\nu}| \right)_{\nu \in \Gamma_\delta^*} = \left(\frac{\check{u}_\nu}{|\check{u}_{0,\nu}|} |\check{u}_{0,\nu}| \right)_{\nu \in \Gamma_\delta^*} = \check{u}.$$

Clearly, this map I^{-1} is only well-defined for $\check{u}_{0,\nu} \neq 0$ (which implies $\check{w}_\nu = \check{u}_\nu \cdot (1 + a_\nu) \neq 0$ by definition of I), $\nu \in \Gamma_\delta^*$. If we want to prove continuity, we may assume without loss of generality that $\check{u}_{0,\nu} \neq 0$ and thus $\check{w}_\nu \neq 0$ holds for all $\nu \in \Gamma_\delta^*$, $(\check{w}_\nu)_\nu \in Iso_\delta(u_0)$ (otherwise consider the corresponding subsequence indexed by all $\nu \in \Gamma_\delta^*$ fulfilling $\check{u}_{0,\nu} \neq 0$) by the same reasons as we excluded the case $\widetilde{m}_\nu(u_0) = 0$ by considering the map (3.12) (see the discussion after (3.12)) since I maps components of $\check{u}$ which are equal to zero trivially to components equal to zero.
To begin with, we note that for all $v, w \in \mathbb{C} \setminus \{0\}$, there holds

$$\left| \frac{w}{|w|} - \frac{v}{|v|} \right| = \left| \frac{w|v| - v|w|}{|v||w|} \right| = \left| \frac{w(|v| - |w|) + |w|(w - v)}{|v||w|} \right| \leq 2 \frac{|w - v|}{|v|}. \quad (3.59)$$

Now let $\check{v} \in Iso_\delta(u_0)$ and $\epsilon > 0$ be given. Since $\sum_{|\nu| \geq n} |\check{u}_{0,\nu}|^2 \to 0$ as $n \to \infty$, there is a $0 < \delta_1 < \delta$ such that $\|\check{u}_0\|^2_{l^2(\Gamma^*_{\delta_1})} < \epsilon^2/8$. Because $\Gamma_\delta^* \setminus \Gamma^*_{\delta_1}$ has only finitely many elements, the number $m := \min\{|\check{v}_\nu| : \nu \in \Gamma_\delta^* \setminus \Gamma^*_{\delta_1}\} > 0$ is well-defined. Now choose $\eta := \frac{m\epsilon}{2\sqrt{2}\|\check{u}_0\|_{l^2(\Gamma_\delta^* \setminus \Gamma^*_{\delta_1})}} > 0$, where we use the natural convention $\|\check{u}_0\|^2_{l^2(\Gamma_\delta^* \setminus \Gamma^*_{\delta_1})} := \sum_{\nu \in \Gamma_\delta^* \setminus \Gamma^*_{\delta_1}} |\check{u}_{0,\nu}|^2$ (which is a finite sum) for the respective term in the denominator. Then for all $\check{w} \in Iso_\delta(u_0)$ with $\|\check{w} - \check{v}\|_{l^2(\Gamma_\delta^*)} < \eta$, there holds

$$\|I^{-1}(\check{w}) - I^{-1}(\check{v})\|^2_{l^2(\Gamma_\delta^*)} = \|I^{-1}(\check{w}) - I^{-1}(\check{v})\|^2_{l^2(\Gamma_\delta^* \setminus \Gamma^*_{\delta_1})} + \|I^{-1}(\check{w}) - I^{-1}(\check{v})\|^2_{l^2(\Gamma^*_{\delta_1})} \leq$$

$$\overset{(3.59)}{\leq} \left(2 \sup_{\nu \in \Gamma_\delta^* \setminus \Gamma^*_{\delta_1}} \frac{|\check{w}_\nu - \check{v}_\nu|}{|\check{v}_\nu|} \right)^2 \cdot \|\check{u}_0\|^2_{l^2(\Gamma_\delta^* \setminus \Gamma^*_{\delta_1})} + \sup_{\nu \in \Gamma^*_{\delta_1}} \underbrace{\left| \frac{\check{w}_\nu}{|\check{w}_\nu|} - \frac{\check{v}_\nu}{|\check{v}_\nu|} \right|^2}_{\leq (1+1)^2 = 4} \|\check{u}_0\|^2_{l^2(\Gamma^*_{\delta_1})} <$$

$$< \frac{4\eta^2}{m^2} \|\check{u}_0\|^2_{l^2(\Gamma_\delta^* \setminus \Gamma^*_{\delta_1})} + 4\frac{\epsilon^2}{8} = \frac{\epsilon^2}{2} + \frac{\epsilon^2}{2} = \epsilon^2,$$

hence $\|I^{-1}(\check{w}) - I^{-1}(\check{v})\|_{l^2(\Gamma_\delta^*)} < \epsilon$. This proves the continuity of I^{-1} and the theorem is proved. □

Chapter 4

The isospectral problem II: The solution

In this chapter, we want to determine the isospectral set $Iso(u_0)$ for a given real-valued potential $u_0 \in L^2(F)$ with the help of the moduli in an analogous way as we already did in Chapter 3 when we determined the asymptotical isospectral set $Iso_\delta(u_0)$. An a priori manifest way to define $Iso(u_0)$ would be

$$\{u \in L^2(F),\ u \text{ real-valued } : m(u) = m(u_0)\}.$$

This definition, however, has some shortcoming: Whereas in the asymptotic part of the Fermi curve $F(u_0)$, there exists a well-defined enumeration of the moduli $(m_\nu(u))_{\nu \in \Gamma^*_\delta}$ (with $\delta > 0$ as in Chapter 3) by the enumeration of the A-cycles in the excluded domains indexed by $\nu \in \Gamma^*_\delta$, in the compact part, however, there doesn't exist such a natural enumeration of the moduli $m_\nu(u)$ for $\nu \in \Gamma^* \setminus \Gamma^*_\delta$. Hence, it is not clear which one of the $m_\nu(u)$ we mean when we speak of the ν^{th} modulus. The reason for this problem is that we don't have those excluded domains suggesting a natural enumeration of the A-cycles in the compact part of the Fermi curve as we have in the asymptotic part. The question is how to find a suitable enumeration of the first finitely many moduli. We proceed as follows: For u_0, we can just choose an arbitrary but fixed enumeration of the first finitely many moduli. As long as we consider potentials $u \in L^2(F)$ such that the associated Fermi curve $F(u)$ is only a slight deformation of the initial curve $F(u_0)$, the A-cycles of $F(u)$ are only slight deformations of the A-cycles of $F(u_0)$ as well such that we can assign the ν^{th} A-cycle of $F(u)$ to the ν^{th} A-cycle of $F(u_0)$ for $\nu \in \Gamma^* \setminus \Gamma^*_\delta$. In other words, the chosen enumeration of the first finitely many moduli of $F(u_0)$ carries over to the enumeration of the first finitely many moduli of $F(u)$ provided that $F(u) \subseteq \mathcal{T}$, where $\mathcal{T} \subset \mathbb{C}^2$ is a sufficiently small tubular neighbourhood of $F(u_0)$ only depending on u_0. The following definition of the isospectral set is thus appropriate and well-defined:

$$Iso(u_0) := \{u \in L^2(F),\ u \text{ real-valued } : m(u) = m(u_0) \text{ and } F(u) \subseteq \mathcal{T}\}. \quad (4.1)$$

Later, when we show the equivalence $m(u) = m(u_0) \iff F(u) = F(u_0)$ and thus determine the isospectral set $Iso_F(u_0) = \{u \in L^2(F),\ u$ real-valued $: F(u) = F(u_0)\}$ we are actually interested in, the additional requirement $F(u) \subseteq \mathcal{T}$ will turn out to be redundant anyway and can be dropped since $F(u_0) \subseteq \mathcal{T}$ is always fulfilled by definition. As long as we haven't proved this, however, the definition (4.1) is the appropriate one in order to guarantee the well-definition of the appearing moduli. Whenever, in the sequel, we will use the moduli $m(u)$, we tacitly remember that $m(u)$ is only well-defined for potentials u with $F(u) \subseteq \mathcal{T}$ without always explicitly mentioning it.

4.1 Submersion properties of the moduli

The first aim of this section is to show that for given $u_0 \in L^2(F)$ and associated Fermi curve $X := F(u)$, there exist for every $N \in \mathbb{N}$ holomorphic 1-forms ω_j on X, $j \in \Gamma^*, 0 < |j| \leq N$ such that

$$\int_{A_i} \omega_j = \delta_{i,j} \quad \text{for}\, i,j \in \Gamma^*, 0 < |i|, |j| \leq N. \tag{4.2}$$

In other words, we want to construct a "partial basis" of 1-forms which is dual to the A-cycles as it has also been done by Feldmann, Knörrer, Trubowitz in the first chapter of [5], for instance. In contrast to [5], we don't require further properties of X as it has been done in [5, Theorem 1.17, Theorem 3.8]. On the other hand, we don't construct a complete basis since (4.2) shall only hold for finitely many $i, j \in \Gamma^*$. As in [5], we assume from now on that X is smooth, i.e. X has no singularities. In the following propositions, this will be explicitly mentioned by formulations like "let $u \in L^2(F)$ with smooth Fermi curve".
The second and main goal of this section is to derive with the help of (4.2) certain submersion properties of the moduli both in the case of complex-valued Schrödinger potentials and in the case of real-valued Schrödinger potentials, i.e. we will prove that the derivative of the moduli is onto in the respective cases.
To begin with, we make an excursion to Fermi curves of periodic *Dirac operators*. The reason for considering this more general setting will become clear in the subsequent investigations. One important tool will be the equation (4.20) where the total residue of some differential form is related to some symplectic form Ω that will be defined in (4.19). Restricting Ω to Schrödinger potentials, Ω turns out to be useless since $\Omega \equiv 0$ in that case. This is one aspect which justifies the following investigations of the more general Dirac equation. We shall see that the Schrödinger equation is just a special case of the Dirac equation and that the results obtained in the Dirac case imply the desired results in the Schrödinger case. We start with some notations and facts based on the work [27] by M. Schmidt.

Let $V, W \in L^2(F)$ and let $\hat{\kappa}, \check{\kappa} \in \Gamma^*$ be two generators of the dual lattice Γ^*. For $p_1, p_2 \in \mathbb{C}$, we define the Dirac operator by (cf. [27, p. 42])

$$\widetilde{D}(V, W, p_1) := \begin{pmatrix} \frac{p_1\pi(\hat{\kappa}_2 - i\hat{\kappa}_1) - \bar{\partial}}{\check{\kappa}_2 - i\check{\kappa}_1} & \frac{W}{\check{\kappa}_2 - i\check{\kappa}_1} \\ \frac{V}{\check{\kappa}_2 + i\check{\kappa}_1} & \frac{p_1\pi(\hat{\kappa}_2 + i\hat{\kappa}_1) + \partial}{\check{\kappa}_2 + i\check{\kappa}_1} \end{pmatrix} \tag{4.3}$$

with the Wirtinger operators $\partial := \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$ and $\bar{\partial} := \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$. The Fermi curve $F(V, W)$ can then be defined as

$$F(V, W) := \{k = p_1\hat{\kappa} + p_2\check{\kappa} \in \mathbb{C}^2 \mid -p_2\pi \text{ is an eigenvalue of } \widetilde{D}(V, W, p_1)\}.$$

In the following, there might appear both coordinates $k = (k_1, k_2) \in \mathbb{C}^2$ and $(p_1, p_2) \in \mathbb{C}^2$ in one equation. Even if it's not always explicitly mentioned, the crucial relation they satisfy is always $k = p_1\hat{\kappa} + p_2\check{\kappa}$.
In the Dirac case, the moduli can be defined in the same way as in Definition 2.6.1 by

$$m_\nu(V, W) := -16\pi^3 \int_{A_\nu} k_1 dk_2,$$

where the contour integral is taken again over the ν^{th} cycle A_ν of the Fermi curve $F(V, W)$.
The above Dirac operator (4.3) is actually some modified Dirac operator. For the original Dirac operator, we have in fact completely analogously to the Schrödinger case discussed in Section 1.2 two possibilities of definition depending on the point of view: We can either consider the boundary condition $k \in \mathbb{C}^2$ already included in the operator with corresponding periodic eigenfunctions (compare Δ_k and (1.11)) or we can consider the operator without boundary condition where the parameter $k \in \mathbb{C}^2$ then appears in the quasi-periodicity of the eigenfunctions (compare (1.1) and (1.3) in the Schrödinger case). We now define (cf. [27, p. 15, p. 17]) analogously to the Laplace operators Δ and Δ_k

$$D(V, W) := \begin{pmatrix} V & \partial \\ -\bar{\partial} & W \end{pmatrix}, \qquad D_k(V, W) := \begin{pmatrix} V & \partial_k \\ -\bar{\partial}_k & W \end{pmatrix} \tag{4.4}$$

with $\partial_k := \partial + \pi i k_1 + \pi k_2$ and $\bar{\partial}_k := \bar{\partial} + \pi i k_1 - \pi k_2$ as in [13, p. 79]. Let

$$\psi_k(x) := \exp(2\pi i \langle k, x \rangle), \quad k \in \mathbb{C}^2, x \in \mathbb{R}^2. \tag{4.5}$$

As in the Schrödinger case, one can easily check that $\widetilde{\psi}$ is an eigenfunction of $D_k(V, W)$ if and only if $\psi = \psi_k \widetilde{\psi}$ is a (quasi-periodic) eigenfunction of $D(V, W)$ fulfilling $\psi(x + \gamma) = e^{2\pi i \langle k, \gamma \rangle}\psi(x)$ (for $\gamma \in \Gamma, x \in \mathbb{R}^2$), each with eigenvalue zero. Furthermore, $\widetilde{\psi}$ is an eigenfunction of $\widetilde{D}(V, W, p_1)$ with eigenvalue $-\pi p_2$ if and only if $\widetilde{\psi}$ is an eigenfunction of $D_k(V, W)$ with eigenvalue zero, where k and

$p = (p_1, p_2)$ are related by $k = p_1\hat{\kappa} + p_2\check{\kappa}$. The latter statement immediately follows from the equivalences

$$\left(p_1\pi(\hat{\kappa}_2 - i\hat{\kappa}_1) + \pi p_2(\check{\kappa}_2 - i\check{\kappa}_1) - \bar{\partial}\right)\widetilde{\psi}_1 + W\widetilde{\psi}_2 = 0 \Longleftrightarrow -\bar{\partial}_k\widetilde{\psi}_1 + W\widetilde{\psi}_2 = 0$$

and

$$\left(p_1\pi(\hat{\kappa}_2 + i\hat{\kappa}_1) + \pi p_2(\check{\kappa}_2 + i\check{\kappa}_1) + \partial\right)\widetilde{\psi}_2 + V\widetilde{\psi}_1 = 0 \Longleftrightarrow \partial_k\widetilde{\psi}_2 + V\widetilde{\psi}_1 = 0,$$

where we used $\pi i(k_1 \pm ik_2) = \pi ip_1(\hat{\kappa}_1 \pm i\hat{\kappa}_2) + \pi ip_2(\check{\kappa}_1 \pm i\check{\kappa}_2)$.
This motivates the above definition $F(V, W)$ if we compare it to the definition of $F(u)$ for Schrödinger potentials in Section 1.1. In the following lemma, we see in which sense Schrödinger potentials are a special case of Dirac potentials where we use the operator $D_k(V, W)$ in (4.4). For all subsequent considerations, however, we will mostly use the (modified) operator $\widetilde{D}(V, W, p_1)$ since it will turn out to be the appropriate operator for our purposes.

Lemma 4.1.1. *For $u \in L^2(F)$, there holds*[1]

$$F(u) = F\left(1, \tfrac{-u}{4}\right) = F\left(\tfrac{-u}{4}, 1\right).$$

In particular, $m_\nu(u) = m_\nu\left(1, \frac{-u}{4}\right) = m_\nu\left(\frac{-u}{4}, 1\right)$ for all $\nu \in \Gamma^*$.

Proof. Setting $(V, W) := (1, \frac{-u}{4})$, we have with (1.11) and (4.4) the equivalences

$$\begin{pmatrix} 1 & \partial_k \\ -\bar{\partial}_k & \frac{-u}{4} \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0 \Longleftrightarrow \begin{cases} \psi_1 + \partial_k\psi_2 = 0 \\ -\bar{\partial}_k\psi_1 - \frac{u}{4}\psi_2 = 0 \end{cases} \Longleftrightarrow -\Delta_k\psi_2 + u\psi_2 = 0.$$

This shows $F(u) = F\left(1, \frac{-u}{4}\right)$. The case $(V, W) := (\frac{-u}{4}, 1)$ is considered completely analogously. Finally, $m_\nu(u) = m_\nu\left(1, \frac{-u}{4}\right) = m_\nu\left(\frac{-u}{4}, 1\right)$ follows for all $\nu \in \Gamma^*$ since by definition of the moduli, equal Fermi curves have equal moduli. □

In [27, Lemma 3.2, p. 60], the following assertion has been shown (its proof can be found in Lemma A.1 in the Appendix A of this work): Given a meromorphic function g with finitely many poles on some open subset of $F(V, W)/\Gamma^*$, there exist a meromorphic function A_g^{sing} (the meaning of the superscript *sing* which stands for *singular* will soon get clear) mapping from the complex plane $p_1 \in \mathbb{C}$ into the bounded operators from the Banach spaces $L^p(F) \times L^p(F)$ into $L^{p'}(F) \times L^{p'}(F)$ (for all $1 < p' < p < \infty$) as well as unique functions $v_g, w_g \in L^2(F)$ such

[1]We implicitly use the self-explanatory convention that $F(u)$ with one argument u signifies the Fermi curve of the Schrödinger potential u, whereas $F(V, W)$ with two arguments V and W signifies the Fermi curve of the Dirac potential (V, W).

that the commutator $[A_g^{sing}(p_1), \widetilde{D}(V,W,p_1)]$ does not depend on p_1 and is equal to

$$[A_g^{sing}(p_1),\ \widetilde{D}(V,W,p_1)] = \begin{pmatrix} 0 & \frac{w_g}{\breve{\kappa}_2 - i\breve{\kappa}_1} \\ \frac{v_g}{\breve{\kappa}_2 + i\breve{\kappa}_1} & 0 \end{pmatrix}. \tag{4.6}$$

Now let an A-cycle A_ν, $\nu \in \Gamma^* \setminus \{0\}$, be given. In local coordinates, the intersection of some small neighbourhood of A_ν in $\mathbb{C}^2$ with the Fermi curve can be represented as an annular domain in $\mathbb{C}$. We denote the image of A_ν in this local coordinates by $\hat{A}_\nu$, i.e. $\hat{A}_\nu$ can be considered as a circle in this annular domain (by choosing the local coordinate appropriately). If in the following, we consider objects (sets, functions,...) both on the Fermi curve and in local coordinates, we will sometimes (whenever it helps to avoid confusions) use the hat symbol $\hat{}$ in order to point out that the respective object is considered in local coordinates.
For $\hat{z}_0 \in \hat{A}_\nu$, we define a meromorphic function $g_{\hat{z}_0}$ on this annular domain by $g_{\hat{z}_0}(\hat{z}) := \frac{1}{\hat{z}-\hat{z}_0}$. Let g_{z_0} be the respective function on the Fermi curve. In order that g_{z_0} is not only meromorphic in a neighbourhood in $F(V,W)$ but also in $F(V,W)/\Gamma^*$, we simply define $g_{z_0+\kappa} := g_{z_0}$ on the corresponding neighbourhood shifted by $\kappa \in \Gamma^*$. Since these shifted neighbourhoods by dual lattice vectors are pairwise disjoint, we don't have any problems concerning well-definition.
So far, for each $\nu \in \Gamma^* \setminus \{0\}$ and for $z_0 \in A_\nu$ [2], the function g_{z_0} is only defined in a neighbourhood U_ν of A_ν. Taking the union of these neighbourhoods $\bigcup_{\nu \in \Gamma^* \setminus \{0\}} U_\nu$, we would like to extend g_{z_0} onto this union. Since we can choose the neighbourhoods U_ν pairwise disjoint, we can define for $z_0 \in A_\nu$ and $\kappa \in \Gamma^* \setminus \{0\}$

$$\widetilde{g}_{z_0}(z) := \delta_{\kappa,\nu} g_{z_0}, \quad z \in U_\kappa, \tag{4.7}$$

again without having any problems of well-definition. In the sequel, we write g_{z_0} instead of $\widetilde{g}_{z_0}$ since we will use this extension of g_{z_0} from now on. We now define the $L^2(F)$-potentials

$$v(A_\nu) := \int_{A_\nu} v_{g_z} dz, \quad w(A_\nu) := \int_{A_\nu} w_{g_z} dz \tag{4.8}$$

(with v_g, w_g for a meromorphic function g as defined in (4.6)). In a first step, we want to prove that for $|\nu| \leq N$, $N \in \mathbb{N}$, these functions are linearly independent.

Lemma 4.1.2. *Let $u \in L^2(F)$ and $(V,W) := (1, \frac{-u}{4})$ with smooth Fermi curve $F(V,W)/\Gamma^*$. Then for all $N \in \mathbb{N}$, the potentials $(v(A_\nu), w(A_\nu))$, $0 < |\nu| \leq N$, defined in* (4.8) *(associated to $F(V,W)/\Gamma^*$), are in $L^2(F) \times L^2(F)$ linearly independent over $\mathbb{C}$.*

[2]By slight abuse of notation, we write for simplicity $z_0 \in A_\nu$ instead of the correct notation $z_0 \in \mathrm{supp}(A_\nu)$.

Remark. As we will see, the major part of the proof also holds for arbitrary potentials $(V,W) \in L^2(F) \times L^2(F)$. There is only one part in the proof, where we use the asymptotics for Fermi curves, more precisely the trisection of Fermi curves and the asymptotic freeness, cf. the end of Section 2.1, which have been shown only for Fermi curves $F(u)$ associated to *Schrödinger potentials* $u \in L^2(F)$ and not in the general Dirac case. This is the only reason why we consider $(V,W) := (1, \frac{-u}{4})$ or $(V,W) := (\frac{-u}{4}, 1)$. Since later, we will go back to Schrödinger potentials anyway, this is not a grave restriction.

Proof. Let $X := F(V,W)/\Gamma^*$ be smooth. Let $N \in \mathbb{N}$, $c_\nu \in \mathbb{C}$, $0 < |\nu| \le N$ and set

$$\sum_{0<|\nu|\le N} c_\nu (v(A_\nu), w(A_\nu)) = 0.$$

We have to show that $c_\nu = 0$ for all $0 < |\nu| \le N$. By (4.6) and the bilinearity of the commutator, we get for any $p_1 \in \mathbb{C}$

$$\left[\sum_{0<|\nu|\le N} c_\nu \int_{A_\nu} A^{sing}_{g_z}(p_1) dz,\ \widetilde{D}(V,W,p_1) \right] = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right). \tag{4.9}$$

Note that g_z clearly depends on ν. In order not to make the notation too complicated, the index ν is suppressed here. Before we can continue, we have to take a look into the proof of [27, Lemma 3.2., p. 60] (or equivalently, cf. the proof of Lemma A.1 in the Appendix A) in order to see how A^{sing}_g (with meromorphic g as mentioned above) is defined: To this, we introduce the projector P which maps functions in $L^2(F) \times L^2(F)$ to eigenfunctions of the Dirac operator (4.3) as follows: For $k \in X$, we have (up to multiplication with a scalar) unique eigenfunctions $\psi(k)$ of $D(V,W)$ (4.4) and $\phi(k)$ of the transposed operator $D^T(V,W)$, respectively (cf. [27, p. 32]). The operator $\mathsf{P}(k) : L^2(F) \times L^2(F) \to L^2(F) \times L^2(F)$ is then defined by (cf. [27, p. 41])

$$(\mathsf{P}(k))(\chi) := \frac{\langle\langle \phi([k]), \psi_k \chi \rangle\rangle}{\langle\langle \phi([k]), \psi([k]) \rangle\rangle} \psi_{-k} \psi([k]), \tag{4.10}$$

with ψ_k again as in (4.5). The bracket $[k]$ is defined as the equivalence class $[k] := \{k + \nu \,|\, \nu \in \Gamma^*\}$. Furthermore, the bilinear form $\langle\langle \cdot, \cdot \rangle\rangle$ is not the usual euclidean bilinear form on $L^2(F) \times L^2(F)$ (as in [27, p. 29]), but some modified bilinear form defined by

$$\langle\langle \phi, \psi \rangle\rangle := \langle \phi_2, \psi_1 \rangle + \langle \phi_1, \psi_2 \rangle := \int_F (\phi_2(x)\psi_1(x) + \phi_1(x)\psi_2(x)) d^2x, \tag{4.11}$$

cf. [27, p. 36] [3]. All we need to know is that with this modified bilinear form,

[3] Actually, in [27, p. 36], one defines for $\gamma = \binom{\gamma_1}{\gamma_2} \in \mathbb{C}^2$ the bilinear form $\langle\langle \phi, \psi \rangle\rangle_\gamma := (\gamma_1 + i\gamma_2) \langle \phi_2, \psi_1 \rangle + (\gamma_1 - i\gamma_2) \langle \phi_1, \psi_2 \rangle$. Since in the proof of [27, Lemma 3.2], this γ serves as one of the two generators of Γ, we can choose without restriction $\gamma = \binom{1}{0}$ for one of the generators by a suitable choice of the coordinate system.

$\mathsf{P}(k)$ (4.10) is the suitable spectral projector for the modified Dirac operator (4.3) (for deeper background information, we refer the reader to [27]). More precisely, the essential property we need is the following: If we consider X locally as a (say n-sheeted) covering space over $p_1 \in \mathbb{C}$, i.e. n points $p_{2,1}, \ldots, p_{2,n}$ lie over p_1 with respect to this covering, we have with $k_i := p_1\hat{\kappa} + p_{2,i}\check{\kappa}$, $i = 1, \ldots, n$, the projector property $\mathsf{P}(k_i)\widetilde{\psi}(k_j) = \delta_{i,j}\widetilde{\psi}(k_j)$, where the function $\widetilde{\psi}(k_j)$ is the eigenfunction of $\widetilde{D}(V, W, p_1)$ with eigenvalue $-\pi p_{2,j}$. This implies that for $k \in X$, we have $\mathsf{F}(p_1)\widetilde{\psi}(k) = g(k) \cdot \widetilde{\psi}(k)$, where $p_1 \mapsto \mathsf{F}(p_1)$ denotes (as in the beginning of the proof of [27, Lemma 3.2]) the local sum of $g \cdot \mathsf{P}$ over all sheets of X. Hence, $\mathsf{F}(p_1)$ and $\widetilde{D}(V, W, p_1)$ have the same eigenfunction $\widetilde{\psi}(k)$.
Now, we want to see how eigenfunctions transform if k is shifted by some $\nu \in \Gamma^*$ to $k + \nu$. At first recall that, as already mentioned, $\widetilde{\psi}(k) := \psi_{-k}\psi(k)$ is an eigenfunction of (4.3) with eigenvalue $-\pi p_2$ if and only if $\psi(k)$ is an eigenfunction of $D(V, W)$ (4.4) with eigenvalue zero with the relation $k = p_1\hat{\kappa} + p_2\check{\kappa}$. As already discussed in Section 1.2, not both ψ and $\widetilde{\psi}$ can be periodic in $x \in \mathbb{R}^2$ with respect to Γ. As the definition of P already suggests, we are in the following setting: ψ, ϕ are quasiperiodic in $x \in \mathbb{R}^2$ with respect to Γ and periodic in k with respect to Γ^* (this explains why ϕ, ψ are functions of equivalence classes $[k]$), whereas $\widetilde{\psi}, \widetilde{\phi}$ are periodic in $x \in \mathbb{R}^2$ with respect to Γ, cf. Section 1.2 or as well [27, Fundamental domain 2.1, Trivialization 2.2, p. 15 f.]. For $n = (n_1, n_2) \in \mathbb{Z}^2$, we thus have with $k(n) := k + n_1\hat{\kappa} + n_2\check{\kappa}$

$$\mathsf{P}(k(n))(\psi_{-n_1\hat{\kappa}-n_2\check{\kappa}}\widetilde{\psi}(k)) = \frac{\left\langle\left\langle \phi([k]), \psi_k\widetilde{\psi}(k)\right\rangle\right\rangle}{\langle\langle \phi([k]), \psi([k])\rangle\rangle}\psi_{-k(n)}\psi(k) = \psi_{-n_1\hat{\kappa}-n_2\check{\kappa}}\widetilde{\psi}(k).$$

Therefore, $\psi_{-n_1\hat{\kappa}-n_2\check{\kappa}}\widetilde{\psi}(k)$ is the[4] eigenfunction of $\mathsf{P}(k(n))$ with eigenvalue 1. This implies the quasiperiodic condition in k with respect to Γ^*:

$$\widetilde{\psi}(k(n)) = \psi_{-n_1\hat{\kappa}-n_2\check{\kappa}}\widetilde{\psi}(k).$$

Together with $\mathsf{F}(p_1)\widetilde{\psi}(k) = g(k) \cdot \widetilde{\psi}(k)$, we get

$$\mathsf{F}(p_1 + n_1)\psi_{-n_1\hat{\kappa}-n_2\check{\kappa}}\widetilde{\psi}(k) = g(k(n))\psi_{-n_1\hat{\kappa}-n_2\check{\kappa}}\widetilde{\psi}(k)$$

and hence

$$(\psi_{n_1\hat{\kappa}+n_2\check{\kappa}}\mathsf{F}(p_1 + n_1)\psi_{-n_1\hat{\kappa}-n_2\check{\kappa}})(\widetilde{\psi}(k)) = g(k(n))\widetilde{\psi}(k).$$

Thus, the operator defined by

$$A_g(p_1) := \sum_{n\in\mathbb{Z}^2} \psi_{n_1\hat{\kappa}+n_2\check{\kappa}}\mathsf{F}(p_1 + n_1)\psi_{-n_1\hat{\kappa}-n_2\check{\kappa}}$$

[4] If we sometimes speak of *the* eigenfunction (which suggests uniqueness), we always mean uniqueness up to multiplication with a scalar.

has the eigenvalue $\sum_{n\in\mathbb{Z}^2} g(k(n))$ with eigenfunction $\widetilde{\psi}(k)$. Note that so far, the appearing series over $n \in \mathbb{Z}^2$ are only formal series; we haven't made any considerations about their convergence, yet. Now, choose for A_ν and $z \in A_\nu$ the function g_z (the dependence on ν is suppressed as before) as in the definition of (4.8), defined in aneighbourhood of A_ν. We then define analogously to (4.8)

$$A_{A_\nu}(p_1) := \int_{A_\nu} A_{g_z}(p_1)dz.$$

Thus, $\sum_{0<|\nu|\leq N} c_\nu A_{A_\nu}(p_1)$ has the eigenvalue $\sum_{0<|\nu|\leq N} c_\nu \sum_{n\in\mathbb{Z}^2} \int_{A_\nu} g_z(k(n))dz$ with eigenfunction $\widetilde{\psi}(k)$. Now, we can define the operator $A_g^{sing}(p_1)$ we are actually interested in: To this, let F^{sing} be the singular part of the (in p_1) meromorphic operator F in its Laurent expansion. Then, F^{sing} is meromorphic in the entire plane $\mathbb{C}$. We now define (cf. [27, p. 60] or Lemma A.1)

$$A_g^{sing}(p_1) := \sum_{n\in\mathbb{Z}^2} \psi_{n_1\hat{\kappa}+n_2\check{\kappa}} \mathsf{F}^{sing}(p_1+n_1)\psi_{-n_1\hat{\kappa}-n_2\check{\kappa}}$$

which is meromorphic in the entire plane $p_1 \in \mathbb{C}$ as well (in contrast to A_g which is not necessarily globally defined). The convergence of this series (in the strong operator topology) has been proved in [27, Lemma 3.2(i)] and can also be retraced in Lemma A.1 in the Appendix A of this work. Analogously to A_{A_ν}, we define

$$A_{A_\nu}^{sing}(p_1) := \int_{A_\nu} A_{g_z}^{sing}(p_1)dz. \tag{4.12}$$

With this notation, (4.9) reads as

$$\left[\sum_{0<|\nu|\leq N} c_\nu A_{A_\nu}^{sing}(p_1),\ \widetilde{D}(V,W,p_1)\right] = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right).$$

This yields

$$\widetilde{D}(V,W,p_1)\left(\sum_{0<|\nu|\leq N} c_\nu A_{A_\nu}^{sing}(p_1)\right)\widetilde{\psi}(k) = \left(\sum_{0<|\nu|\leq N} c_\nu A_{A_\nu}^{sing}(p_1)\right)\widetilde{D}(V,W,p_1)\widetilde{\psi}(k) =$$
$$= -\pi p_2 \left(\sum_{0<|\nu|\leq N} c_\nu A_{A_\nu}^{sing}(p_1)\right)\widetilde{\psi}(k).$$

Consequently, $\sum_{0<|\nu|\leq N} c_\nu A_{A_\nu}^{sing}(p_1)\widetilde{\psi}(k)$ is an eigenfunction of $\widetilde{D}(V,W,p_1)$ with eigenvalue $-\pi p_2$. Since the eigenfunctions of $\widetilde{D}(V,W,p_1)$ are (except for isolated points) unique up to multiplication with a scalar, $\sum_{0<|\nu|\leq N} c_\nu A_{A_\nu}^{sing}(p_1)\widetilde{\psi}(k)$ is in the span of $\widetilde{\psi}(k)$. In other words, $\widetilde{\psi}(k)$ is an eigenfunction of $\sum_{0<|\nu|\leq N} c_\nu A^{sing}_{A_\nu}(p_1)$. We denote the corresponding eigenvalue by $f_{sing}(k)$, i.e.

$$\left(\sum_{0<|\nu|\leq N} c_\nu A^{sing}_{A_\nu}(p_1)\right)\widetilde{\psi}(k) = f_{sing}(k)\widetilde{\psi}(k).$$

To sum up, $\sum_{0<|\nu|\leq N} c_\nu A^{sing}_{A_\nu}(p_1)$ and $\widetilde{D}(V,W,p_1)$ share the same eigenfunction $\widetilde{\psi}(k)$. Since $\sum_{0<|\nu|\leq N} c_\nu A_{A_\nu}(p_1)$ also has the eigenfunction $\widetilde{\psi}(k)$ as shown above, the operator $\sum_{0<|\nu|\leq N} c_\nu A^{hol}_{A_\nu}(p_1)$ with $A^{hol}_{A_\nu}(p_1) := A_{A_\nu}(p_1) - A^{sing}_{A_\nu}(p_1)$ has the eigenfunction $\widetilde{\psi}(k)$, too. Summing up, with

$$f(k) := \sum_{0<|\nu|\leq N} c_\nu \sum_{n\in\mathbb{Z}^2} \int_{A_\nu} g_z(k(n))dz, \tag{4.13}$$

we get

$$\left(\sum_{0<|\nu|\leq N} c_\nu A^{sing}_{A_\nu}(p_1)\right)\widetilde{\psi}(k) = \sum_{0<|\nu|\leq N} c_\nu (A_{A_\nu}(p_1) - A^{hol}_{A_\nu}(p_1))\widetilde{\psi}(k) =$$
$$= \left(\sum_{0<|\nu|\leq N} c_\nu \sum_{n\in\mathbb{Z}^2} \int_{A_\nu} g_z(k(n))dz - \sum_{0<|\nu|\leq N} c_\nu A^{hol}_{A_\nu}(p_1)\right)\widetilde{\psi}(k) = (f(k) - f_{hol}(k))\widetilde{\psi}(k),$$

where $f_{hol}(k) := f(k) - f_{sing}(k)$ denotes the corresponding eigenvalue of the operator $\sum_{0<|\nu|\leq N} c_\nu A^{hol}_{A_\nu}(p_1)$. We owe the proof of the convergence of the appearing series over $n \in \mathbb{Z}^2$. The crucial terms we have to consider are $A_{A_\nu}(p_1)$, $A^{sing}_{A_\nu}(p_1)$, $A^{hol}_{A_\nu}(p_1)$, $f(k)$. It suffices to prove convergence of $A_{A_\nu}(p_1)$ and $f(k)$ since the convergence of $A^{hol}_{A_\nu}(p_1)$ follows from the convergence of $A_{A_\nu}(p_1)$ (to be proved) and $A^{sing}_{A_\nu}(p_1)$ (follows from the convergence of $A^{sing}_{g}(p_1)$ proved in [27, Lemma 6.2(i)] or Lemma A.1, respectively).

As to the convergence of $A_{A_\nu}(p_1)$, we see that this term is defined by integration over a fixed cycle A_ν. This cycle is one element of the homology basis of the Fermi curve X. The summands in the infinite sum over $n \in \mathbb{Z}^2$ are exactly the shifts of the respective argument k by dual lattice vectors to $k(n) = k + n_1\hat{\kappa} + n_2\check{\kappa}$, where $n = (n_1, n_2)$ runs through $\mathbb{Z}^2$. Since we integrate over exactly *one* cycle A_ν (and not over infinitely many cycles $\{A_\nu + \kappa \,|\, \kappa \in \Gamma^*\}$ which we would have to do if we considered $F(V,W)$ instead of $F(V,W)/\Gamma^*$), all but one of the arguments $k(n)$ are outside of the cycle A_ν, i.e. A_ν has winding number zero with respect to $k(n)$ for all but one $n \in \mathbb{Z}^2$ (recall the definition of g_z, namely $g_{\hat{z}}(\hat{w}) := \frac{1}{\hat{w}-\hat{z}}$ in local coordinates). Therefore, all summands but one vanish so that the infinite sum over $n \in \mathbb{Z}^2$ has in fact only one summand.

As to the convergence of $f(k)$, we can argue completely analogously with the only

difference that we don't integrate over only one fixed cycle A_ν but over finitely many cycles A_ν, $0 < |\nu| \leq N$. This proves the convergence of the considered terms. The terms $f_{sing}(k)$ and $f_{hol}(k)$ are, by the way, well-defined due to the well-definition of $A^{sing}_{A_\nu}(p_1)$ and $A^{hol}_{A_\nu}(p_1)$, respectively.
After having justified the convergence of the sums, we go back to the actual proof. If we cut X along the cycles A_ν, $0 < |\nu| \leq N$, we get a complex curve with boundary denoted by $\widetilde{X}$, where each cycle A_ν decomposes into two boundary curves denoted by A^+_ν and A^-_ν. Since $\widetilde{\psi}$ is a global meromorphic eigenfunction of $\widetilde{D}(\cdot, V, W)$ on X and $\sum_{0<|\nu|\leq N} c_\nu A^{sing}_{A_\nu}(p_1)$ is defined on almost the entire plane $p_1 \in \mathbb{C}$ (except for those $p_1 \in \{p_1 \in \mathbb{C} : \exists k \in (A_\nu)_{0<|\nu|\leq N}\ \exists p_2 \in \mathbb{C} \text{ such that } k = p_1\hat{\kappa}+p_2\check{\kappa}\}$), the corresponding eigenvalue function f_{sing} is a *global* meromorphic function on $\widetilde{X}$. Note that neither f nor f_{hol} need to be global functions on $\widetilde{X}$. Furthermore, f_{sing} is in general not continuously extendable onto X as we shall see now. Let $0 < |\nu| \leq N$. In local coordinates $k \mapsto z(k)$ in a neighbourhood of A_ν, we can assume without restriction that A_ν is parameterized by the unit circle $\{z : |z| = 1\}$ with sufficiently smooth parameterization $z : [0, 1] \to \mathbb{C}$, $t \mapsto z(t)$. This yields with the definition of g_z

$$\int_{A_\nu} g_z(k)dz = \int_0^1 g_{z(t)}(z(k))\dot{z}(t)dt = \int_{|z|=1} \frac{1}{z(k)-z}dz = \begin{cases} 0, & \text{if } |z(k)| > 1, \\ -2\pi i, & \text{if } |z(k)| < 1. \end{cases} \tag{4.14}$$

In particular, $\int_{A_\nu} g_z(k)dz = 0$ for $k \in U_\kappa$ with $\kappa \neq \nu$, again by the definition of g_{z_0}, cf. (4.7). Let the circle $\{z : |z| = 1\}$ considered as the (inner) boundary of $\{z : |z| > 1\}$ be A^+_ν in local coordinates and let $\{z : |z| = 1\}$ considered as the (outer) boundary of $\{z : |z| < 1\}$ be A^-_ν in local coordinates[5]. With this convention, for given $k^+ \in A^+_\nu$ and $k^- \in A^-_\nu$, there holds $f(k) = 0$ for all k in some neighbourhood of k^+ in $\widetilde{X}$ and $f(k) = -2\pi i c_\nu$ for all k in some neighbourhood of k^- in $\widetilde{X}$ (cf. (4.13) and the fact discussed above that only one element in $\{k(n) \,|\, n \in \mathbb{Z}^2\}$ lies *within* the circle of A_ν). Since A^-_ν and A^+_ν are compact, we get $f \equiv 0$ in a neighbourhood of A^+_ν in $\widetilde{X}$ and $f \equiv -2\pi i c_\nu$ in a neighbourhood of A^-_ν in $\widetilde{X}$. Consequently, $df \equiv 0$ in both of these neighbourhoods.
As to the function f_{hol}, we don't have a discontinuity as in (4.14) since there aren't any singularities on the path of integration if we integrate $A_{g_z}(p_1) - A^{sing}_{g_z}(p_1)$ along $z \in A_\nu$. Hence, the integral remains well-defined even if k (corresponding to p_1 in the usual relation) lies in A_ν. Consequently, $f_{hol}|_{A^-_\nu} \equiv f_{hol}|_{A^+_\nu}$.
The next step is to prove that $f_{sing} \equiv 0$ on $\widetilde{X}$ (and consequently extendable to zero onto X). To this, we will show at first that $df_{sing} \equiv 0$ on $\widetilde{X}$ by showing that the norm $\|df_{sing}\|_{\widetilde{X}}$ is equal to zero, where we define for a meromorphic 1-form λ

[5] Of course, one could also interchange A^+_ν and A^-_ν. Which one of these two circles is denoted by A^+_ν and which one by A^-_ν is eventually immaterial for the following arguments.

on $\widetilde{X}$ the norm $\|\lambda\|^2_{\widetilde{X}} := \int_{\widetilde{X}} \lambda \wedge \overline{*\lambda}$ with $*\lambda := -i\lambda$ [6] (for details, see [5, I.1], in particular [5, Remark 1.14]).
We calculate at first the following integral (which will appear later again when computing the norm $\|df_{sing}\|_{\widetilde{X}}$): For $0 < |\nu| \leq N$, we have

$$
\begin{aligned}
&\int_{A_\nu^+} f_{sing} \cdot \overline{*df_{sing}} - \int_{A_\nu^-} f_{sing} \cdot \overline{*df_{sing}} = \\
&\overset{df|_{A_\nu}=0}{=} - \int_{A_\nu^+} (f - f_{hol}) \cdot \overline{*df_{hol}} + \int_{A_\nu^-} (f - f_{hol}) \cdot \overline{*df_{hol}} = \\
&= - \int_{A_\nu^+} \underbrace{f}_{=0} \cdot \overline{*df_{hol}} + \int_{A_\nu^-} \underbrace{f}_{=-2\pi i c_\nu} \cdot \overline{*df_{hol}} + \underbrace{\int_{A_\nu^+} f_{hol} \cdot \overline{*df_{hol}} - \int_{A_\nu^-} f_{hol} \cdot \overline{*df_{hol}}}_{=0} = \\
&= -2\pi i c_\nu \int_{A_\nu} \overline{*df_{hol}} \overset{df|_{A_\nu}=0}{=} 2\pi c_\nu \overline{\int_{A_\nu} df_{sing}} = 0. \qquad (4.15)
\end{aligned}
$$

Next, we want to show that for every $\epsilon > 0$, there is an $r > 0$ such that $|df_{sing}(k)| < \epsilon$ for all $k \in \widetilde{X} \cap (\mathbb{C}^2 \setminus B_r(0))$ where $B_r(0)$ denotes the ball in $\mathbb{C}^2$ with center 0 and radius r. In other words, df_{sing} shall asymptotically tend to zero. F^{sing} obviously fulfills this property since it is defined as the singular part of the Laurent expansion of some meromorphic function. The same holds for A_g^{sing} and $\sum_{0<|\nu|\leq N} c_\nu A_{A_\nu}^{sing}$ (the appearing sum over $n \in \mathbb{Z}^2$ in these terms doesn't disturb the asymptotic behaviour because of the same reasons as discussed before when we justified that at most one element in $\{k(n) \,|\, n \in \mathbb{Z}^2\}$ lies within the circle of a given A_ν). Since singular parts of Laurent expansions of meromorphic functions (here in the variable $p_1 \in \mathbb{C}$) are finite linear combinations of terms like p_1^{-j}, with finitely many natural numbers j, such a singular part always fulfills the estimate $O(1/|p_1|)$ as $|p_1| \to \infty$. Now,

$$
\begin{aligned}
|f_{sing}(k)| \cdot \|\widetilde{\psi}(k)\| = \|f_{sing}(k)\widetilde{\psi}(k)\| = \|\sum_{0<|\nu|\leq N} c_\nu A_{A_\nu}^{sing}(p_1)\widetilde{\psi}(k)\| \leq \\
\leq \|\sum_{0<|\nu|\leq N} c_\nu A_{A_\nu}^{sing}(p_1)\| \cdot \|\widetilde{\psi}(k)\|,
\end{aligned}
$$

where the appearing norms are the $L^2(F) \times L^2(F)$-norm and the corresponding operator norm on $L^2(F) \times L^2(F) \to L^2(F) \times L^2(F)$, respectively. Thus, f_{sing} has the desired asymptotic behaviour. Since f_{sing} is meromorphic on $\widetilde{X}$, this asymptotic behaviour carries over to df_{sing} as claimed above: For every $\epsilon > 0$ there is an $r > 0$ such that $|df_{sing}(k)| < \epsilon$ for all $k \in \widetilde{X} \cap (\mathbb{C}^2 \setminus B_r(0))$. By the asymptotic behaviour of the singular part of the Laurent expansion explained

[6] A priori, it is not clear whether $\|\lambda\|_{\widetilde{X}} < \infty$ for the considered 1-forms λ. In the following, we will, however, apply the just defined norm of the respective λ under consideration to a compact surface with boundary where $\|\lambda\| < \infty$ will be satisfied.

above, we even get $f_{sing} = O(1/r)$ and $df_{sing} = O(1/r^2)$ as $r \to \infty$. We now define a compact curve with boundary denoted by $\widetilde{X}(r)$ by intersecting $\widetilde{X}$ with a ball $\overline{B_r(0)} \subset \mathbb{C}^2$ with sufficiently large $r > 0$ such that $\partial B_r(0)$ lies in the asymptotic part of the Fermi curve. We know due to the asymptotic freeness that X looks there like two complex planes $\cong \mathbb{C}$ that are connected to each other by handles (cf. the end of Section 2.1). The intersection of such a complex plane with $\overline{B_r(0)}$ is bounded by a circle with radius r. Without restriction, we may assume that this circle intersects no excluded domain. Otherwise, consider $B_r(0) \setminus \{$excluded domains having non-trivial intersection with $\partial B_r(0)\}$ instead of $B_r(0)$. In any case, the boundary $\partial\widetilde{X}(r) =: \partial\widetilde{X}(r)^{in} \cup \partial\widetilde{X}(r)^{out}$ consists of two parts, namely the "inner" cycles $A_\nu^{\pm}$, $0 < |\nu| \leq N$ we also considered before $(\partial\widetilde{X}(r)^{in})$ and the "outer" boundary $\partial\widetilde{X}(r)^{out}$ whose length is $O(r)$, as $r \to \infty$, since this boundary consists of two circles with radius r (or a small deviation from a circle by possibly circumnavigating the mentioned excluded domains) in the two complex planes. The essential property is that $\widetilde{X}(r)$ is compact. We now get by applying Stokes' Theorem (cf. [18, Theorem 9.6], for instance) that for all $\epsilon > 0$ there is an $r_0 > 0$ such that for all $r \geq r_0$

$$\|df_{sing}\|^2_{\widetilde{X}(r)} = \int_{\widetilde{X}(r)} df_{sing} \wedge \overline{*df_{sing}} = \int_{\widetilde{X}(r)} d(f_{sing} \cdot \overline{*df_{sing}}) = \int_{\partial\widetilde{X}(r)} f_{sing} \cdot \overline{*df_{sing}} =$$

$$= \sum_{0<|\nu|\leq N} \underbrace{\left(\int_{A_\nu^+} f_{sing} \cdot \overline{*df_{sing}} - \int_{A_\nu^-} f_{sing} \cdot \overline{*df_{sing}} \right)}_{=0,\, \text{cf. } (4.15)} + \int_{\partial\widetilde{X}(r)^{out}} f_{sing} \cdot \overline{*df_{sing}} \leq$$

$$\leq \mu(\partial\widetilde{X}(r)^{out}) \cdot \sup_{\partial\widetilde{X}(r)^{out}} (|f_{sing}| \cdot |df_{sing}|) = O(r) \cdot O\left(\frac{1}{r^3}\right) < \epsilon,$$

where $\mu(\partial\widetilde{X}(r)^{out})$ denotes the measure of $\partial\widetilde{X}(r)^{out}$. Hence, $\|df_{sing}\|_{\widetilde{X}} = 0$. Thus $df_{sing} \equiv 0$ and f_{sing} is constant on $\widetilde{X}$. Due to the asymptotic behavior of f_{sing}, we get $f_{sing} \equiv 0$ on $\widetilde{X}$. Therefore, $f = f_{hol}$. Due to $f_{hol}|_{A_\nu^-} = f_{hol}|_{A_\nu^+}$ and $f|_{A_\nu^-} = -2\pi i c_\nu$, $f|_{A_\nu^+} = 0$ for all $0 < |\nu| \leq N$, we get $c_\nu = 0$ for all $0 < |\nu| \leq N$ which had to be proved. □

As a side note, we remark that in the above proof, we made use of the assumed smoothness of the corresponding Fermi curve X in essentially two aspects: Firstly, we made use of the existence of a local coordinate in a neighbourhood of each cycle A_ν, $0 < |\nu| \leq N$, and secondly, we applied Stokes' Theorem for compact smooth manifolds with boundary.

For our next investigations, we have to recap some facts of Fermi curves and their holomorphic 1-forms in the Dirac setting proved in [27].

The Fermi curve $F(V, W)/\Gamma^*$ can locally be described by an equation of the form $R(p, V, W) = 0$ with some holomorphic function R (cf. [27, p. 58] and [27, Theorem 2.3]), where $p := (p_1, p_2)$. Since this equation holds in a neighbourhood of the given (V, W), we obtain by computing the directional derivative of $R = R(p, V, W)$ in (V, W) in direction of some $(\delta v, \delta w) \in L^2(F) \times L^2(F)$ (with a dot denoting the directional derivative with respect to $(\delta v, \delta w)$), cf. [27, p. 58]:

$$R(p, V, W) = 0 \Rightarrow \frac{\partial R}{\partial(V, W)}(\delta v, \delta w) + \frac{\partial R}{\partial p_1}\dot{p}_1 + \frac{\partial R}{\partial p_2}\dot{p}_2 = 0.$$

$$\Rightarrow \frac{\frac{\partial R}{\partial(V,W)}(\delta v, \delta w)}{\frac{\partial R}{\partial p_1}} dp_2 = -\dot{p}_1 dp_2 - \frac{\frac{\partial R}{\partial p_2}}{\frac{\partial R}{\partial p_1}}\dot{p}_2 dp_2. \tag{4.16}$$

Here, we briefly have to comment on the well-definition of the appearing directional derivatives $\dot{p}_1$ and $\dot{p}_2$ (an issue which has already been discussed in [27, p. 58]). A priori, it is not clear in which sense they are well-defined since there is no unique function $(V, W) \mapsto (p_1(V, W), p_2(V, W))$. In other words, by varying for example p_1 in direction $(\delta v, \delta w)$, there is no unique $p_1(V + \delta v, W + \delta w)$. We can circumnavigate this problem if we require either $\dot{p}_2 = 0$ or $\dot{p}_1 = 0$. In the first case, $p_1 = p_1(p_2, V, W)$ is well-defined. The same holds for $p_2 = p_2(p_1, V, W)$ in the second case. These are only two examples of choices in order to define unique directional derivatives $\dot{p}_1$ and $\dot{p}_2$. Which choice we take is eventually immaterial as long as it is consistent. As we will see in a moment, we won't deal any longer with the directional derivatives $\dot{p}_1$ and $\dot{p}_2$ anyway. They appear only here in the intermediate computations in order to derive the equation (4.18) we are actually interested in. In equation (4.18), the terms $\dot{p}_1$ and $\dot{p}_2$ finally won't appear anymore.
Let's continue our computations. For fixed $(V, W) \in L^2(F) \times L^2(F)$, we have

$$\frac{\partial R}{\partial p_1} dp_1 + \frac{\partial R}{\partial p_2} dp_2 = 0,$$

which implies

$$dp_1 = -\frac{\frac{\partial R}{\partial p_2}}{\frac{\partial R}{\partial p_1}} dp_2 \quad \left(\Longleftrightarrow \ \frac{dp_1}{\frac{\partial R}{\partial p_2}} = -\frac{dp_2}{\frac{\partial R}{\partial p_1}} \right).$$

Plugging this into the equation (4.16), yields

$$\frac{\frac{\partial R}{\partial(V,W)}(\delta v, \delta w)}{\frac{\partial R}{\partial p_1}} dp_2 = -\dot{p}_1 dp_2 + \dot{p}_2 dp_1. \tag{4.17}$$

From now on, a holomorphic 1-form in the expression of the left hand side of (4.17) shall be denoted by $\omega(V, W, \delta v, \delta w)$, more precisely

$$\omega(V, W, \delta v, \delta w) := \frac{\frac{\partial R(p,V,W)}{\partial(V,W)}(\delta v, \delta w)}{\frac{\partial R(p,V,W)}{\partial p_1}} dp_2. \tag{4.18}$$

With the symplectic form $\Omega : (L^2(F) \times L^2(F))^2 \to \mathbb{C}$ defined by

$$\Omega((v,w),(v',w')) := \frac{1}{2\pi^2 i} \int_F (vw' - wv') d^2x, \tag{4.19}$$

and with (4.17), we obtain with a meromorphic function g defined in an open neighbourhood $\mathcal{U}$ in $F(V,W)/\Gamma^*$ by [27, Lemma 3.2(iv)] the relation

$$\Omega((\delta v, \delta w),(v_g, w_g)) = \sum_{\zeta \in \mathcal{U}} \operatorname{res}_\zeta(g \cdot \omega(V,W,\delta v,\delta w)) \tag{4.20}$$

with v_g, w_g from (4.6). Here, res_ζ denotes the residue at the point ζ. The proof of the relation (4.20) can also be retraced in Lemma A.1(iii) in the Appendix A of this work. With this preliminaries, we can now prove the following theorem.

Theorem 4.1.3. *Let $u \in L^2(F)$ and $(V,W) := (1, \frac{-u}{4})$ with smooth Fermi curve $F(V,W)/\Gamma^*$. Then for all $N \in \mathbb{N}$, there exist holomorphic 1-forms ω_κ, $|\kappa| \le N$, on $F(V,W)/\Gamma^*$ such that for all $\nu \in \Gamma^*$ with $0 < |\nu| \le N$, there holds*

$$\int_{A_\nu} \omega_\kappa = \delta_{\kappa,\nu}. \tag{4.21}$$

Furthermore, these ω_κ can be chosen to be of the form (4.18) *with suitable respective directions $(\delta v, \delta w) \in L^2(F) \times L^2(F)$.*

Proof. Let $u \in L^2(F)$, $(V,W) := (1, \frac{-u}{4})$ with smooth Fermi curve $F(V,W)/\Gamma^*$, $N \in \mathbb{N}$ and set

$$2g := \#\{\nu \in \Gamma^* : 0 < |\nu| \le N\}.$$

This notation is motivated by the fact that we have an even number of lattice vectors $\nu \in \Gamma^* \setminus \{0\}$, $|\nu| \le N$ since there corresponds to each $\nu \in \Gamma^* \setminus \{0\}$, $|\nu| \le N$ the lattice vector $-\nu$ satisfying $|-\nu| \le N$ as well. Due to Lemma 4.1.2, the potentials $(v(A_\nu), w(A_\nu))$, $0 < |\nu| \le N$ generate a complex vector space of dimension $2g$. Let $\mathcal{V}$ be the complex vector space generated by the $4g$ potentials $(v(A_\nu), w(A_\nu)), (\overline{w(A_\nu)}, -\overline{v(A_\nu)})$, $0 < |\nu| \le N$ whose dimension $m := \dim_{\mathbb{C}}(\mathcal{V})$ equals $2g \le m \le 4g$. Let b_i, $i = 1 \ldots, m$, be a basis of $\mathcal{V}$ whose first g elements shall be $(v(A_\nu), w(A_\nu))$, $0 < |\nu| \le N$. We restrict the symplectic form Ω (4.19) onto $\mathcal{V}$ and claim that this form $\Omega : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ is still symplectic. The properties of being bilinear and alternating obviously carry over to the restriction to $\mathcal{V}$. We have to show that the form is still nondegenerate. To this, let $0 \ne (v,w) \in \mathcal{V}$. Then $(v',w') := (-\overline{w}, \overline{v})$ is also a vector in $\mathcal{V}$ by definition of $\mathcal{V}$. We get $\Omega((v',w'),(v,w)) = \frac{-1}{2\pi^2 i} \int_F (|v|^2 + |w|^2) d^2x \ne 0$. Hence, Ω is nondegenerate on $\mathcal{V}$ and thus symplectic on $\mathcal{V}$. Now define the linear map

$$F : \mathcal{V} \to \mathbb{C}^m, \quad (v,w) \mapsto (\Omega((v,w),b_i))_{i=1,\ldots,m}. \tag{4.22}$$

We show that $\ker(F) = \{0\}$. To this, let $(v, w) \in \mathcal{V}$ with $F(v, w) = 0$, i.e. $\Omega((v, w), b_i) = 0$ for all $i = 1, \dots, m$. We assume that $(v, w) \neq 0$. Since Ω is nondegenerate, there is a $(v', w') =: \sum_{i=1}^m \lambda_i b_i \in \mathcal{V}$ ($\lambda_i \in \mathbb{C}$) such that

$$0 \neq \Omega((v, w), (v', w')) = \sum_{i=1}^m \lambda_i \Omega((v, w), b_i) = 0,$$

a contradiction. This proves $\ker(F) = \{0\}$. Together with $\dim(\mathcal{V}) = m$, this implies that F is an isomorphism. Consequently, for every $j = 1, \dots, m$, there exists a unique $(v_j, w_j) \in \mathcal{V}$ such that $F(v_j, w_j) = e_j$, i.e. $\Omega((v_j, w_j), b_i) = \delta_{ij}$ for all $i, j = 1, \dots, m$ (here, $e_j := (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{C}^m$ denotes the j^{th} canonical unity vector). In particular by the definition of the first $2g$ vectors $b_1, \dots, b_{2g}$, there exist potentials $(v_\kappa, w_\kappa) \in \mathcal{V}$ such that

$$\Omega((v_\kappa, w_\kappa), (v(A_\nu), w(A_\nu))) = \delta_{\kappa,\nu} \quad \text{for all } 0 < |\kappa|, |\nu| \leq N.$$

Together with (4.20) and the definition of $v(A_\nu)$, $w(A_\nu)$ (4.8), this implies for $0 < |\kappa|, |\nu| \leq N$

$$\begin{aligned}\delta_{\kappa,\nu} &= \Omega((v_\kappa, w_\kappa), (v(A_\nu), w(A_\nu))) = \Omega\left((v_\kappa, w_\kappa), \int_{A_\nu} (v_{g_z}, w_{g_z}) dz\right) = \\ &= \int_{A_\nu} \Omega\left((v_\kappa, w_\kappa), (v_{g_z}, w_{g_z})\right) dz = \int_{A_\nu} \sum_{\zeta \in \mathcal{U}} \operatorname{res}_\zeta (g_z \cdot \omega(V, W, v_\kappa, w_\kappa)) dz = \\ &= \int_{A_\nu} \operatorname{res}_z \left(\xi \mapsto \frac{\omega(V, W, v_\kappa, w_\kappa)|_\xi}{\xi - z}\right) dz = \int_{A_\nu} \omega(V, W, v_\kappa, w_\kappa). \qquad (4.23)\end{aligned}$$

The theorem is proved. □

Since in the following, we will consider pairs $(\nu, -\nu)$, we recall the notation Γ^*/σ already well-known from Theorem 3.1.1, more precisely: $\nu, \kappa \in \Gamma^*$ are equivalent in Γ^*/σ if and only if $\nu = \kappa$ or $\nu = \sigma(\kappa) = -\kappa$. Moreover, we set

$$\Gamma^*_N/\sigma := \{\nu \in \Gamma^*/\sigma : 0 < |\nu| \leq N\}.$$

In the foregoing theorem we didn't make use of the property that $(V, W) = (1, -\frac{u}{4})$ was assumed to be a Schrödinger potential in its full entirety, yet. Indeed, by using the symmetry of the Fermi curve with respect to the holomorphic involution $\sigma : k \mapsto -k$, we get a sharper version stating that the first component δv of the variation $(\delta v, \delta w)$ can be chosen to be equal to zero. This will be proved in Theorem 4.1.5. Before, we prove a lemma that will be needed in the proof of Theorem 4.1.5.

Lemma 4.1.4. *Let A, B be two closed subsets of $L^2(F) \times L^2(F)$ and denote by $A^\perp$ and $B^\perp$ the orthogonal complements with respect to the symplectic form Ω (4.19). Then*

$$A^\perp + B^\perp = (A \cap B)^\perp.$$

Proof. Let A be an arbitrary closed subset of $L^2(F) \times L^2(F)$. We show at first that $(A^\perp)^\perp = A$, where $A^\perp := \{x \in L^2(F) \times L^2(F) : \forall_{a \in A}\ \Omega(x,a) = 0\}$ denotes the orthogonal complement with respect to the symplectic form Ω (4.19). Let $a \in A$. Then, by definition of the orthogonal complement, $\Omega(x,a) = 0$ for all $x \in A^\perp$. Again, by definition of the orthogonal complement, we get $a \in (A^\perp)^\perp$. This proves $A \subseteq (A^\perp)^\perp$. Conversely, assume that there is an $a \in (A^\perp)^\perp \setminus A$. Let $U := span\{A, a\}$. Define a bounded linear functional f by

$$f : U \to \mathbb{R}, \quad f(x) := \begin{cases} 1, & \text{if } x = a \\ 0, & \text{if } x \in A. \end{cases}$$

This functional f is well-defined since linear maps are already uniquely defined if they are defined on a basis. Since A is closed, a cannot be in the closure of A due to $a \in (A^\perp)^\perp \setminus A$. By Hahn-Banach's Theorem, cf. [30, Theorem III.1.5], there exists a bounded linear functional $F : L^2(F) \times L^2(F) \to \mathbb{R}$ such that $F|_U = f$. We remark the relation

$$\langle x, y\rangle = 2\pi^2 i \Omega\left((\bar{y}_2, -\bar{y}_1), (x_1, x_2)\right) \text{ for all } x = (x_1, x_2), y = (y_1, y_2) \in L^2(F) \times L^2(F)$$

between the symplectic form Ω (4.19) and the canonical hermitian scalar product $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{L^2(F)\times L^2(F)}$. By Riesz's Representation Theorem, cf. [4, Satz 2.25], there is a $z \in L^2(F) \times L^2(F)$ such that $F(x) = \Omega(x,z)$ for all $x \in L^2(F) \times L^2(F)$. Since by definition, $F(x) = f(x) = 0$ for all $x \in A$, we have $z \in A^\perp$. Again by definition of f, we have $\Omega(a,z) = f(a) = 1$. Hence, $a \notin (A^\perp)^\perp$, a contradiction to our assumption $a \in (A^\perp)^\perp$. This proves $(A^\perp)^\perp = A$.
In a next step, we show that for closed $A, B \subseteq L^2(F) \times L^2(F)$, there holds

$$(A + B)^\perp = A^\perp \cap B^\perp. \tag{4.24}$$

Since $A \subseteq A + B$, we have $(A+B)^\perp \subseteq A^\perp$. Likewise, since $B \subseteq A + B$, we have $(A+B)^\perp \subseteq B^\perp$. This proves the inclusion "$\subseteq$" in (4.24). Conversely, let $x \in A^\perp \cap B^\perp$ be given. Hence, $\Omega(x,b) = \Omega(x,a) = 0$ for all $a \in A$ and all $b \in B$. By the linearity of Ω, this implies $\Omega(x, a+b) = 0$ for all $a \in A$ and all $b \in B$. Therefore, $x \in (A+B)^\perp$. This proves (4.24). This together with $(A^\perp)^\perp = A$ for all closed subsets $A \subseteq L^2(F) \times L^2(F)$ implies for all closed $A, B \subseteq L^2(F) \times L^2(F)$

$$A^\perp + B^\perp = (A^\perp + B^\perp)^{\perp\perp} \overset{(4.24)}{=} (A^{\perp\perp} \cap B^{\perp\perp})^\perp = (A \cap B)^\perp.$$

This proves the lemma. □

Theorem 4.1.5. *Let $u \in L^2(F)$ and $(V, W) := (1, \frac{-u}{4})$ with smooth Fermi curve $F(V,W)/\Gamma^*$. Then for all $N \in \mathbb{N}$, there exist holomorphic 1-forms ω_κ, $\kappa \in \Gamma^*_N/\sigma$, on $F(V,W)/\Gamma^*$ such that for all $\nu \in \Gamma^*$ with $\nu \in \Gamma^*_N/\sigma$, there holds*

$$\int_{A_\nu} \omega_\kappa = \delta_{\kappa,\nu}.$$

Furthermore, these ω_κ *can be chosen to be of the form* (4.18) *with suitable respective directions* $(0, \delta w) \in L^2(F) \times L^2(F)$. *In particular, the direction* δv *in* (4.18) *can be chosen to be zero.*

Remark. In the Appendix B, we give an alternative proof of this theorem which we have found earlier. Since the proof given in the following is, however, much more elegant, we transferred the former proof into the appendix.

Proof. Let $u \in L^2(F)$ with $(V, W) := (1, \frac{-u}{4})$ be given. The main effort of the proof will be to show the identity

$$\begin{aligned} \Omega_+ &:= \{\omega = \omega(V, W, \delta v, \delta w) : (\delta v, \delta w) \in L^2(F) \times L^2(F) \wedge \omega \circ \sigma = \omega\} = \\ &\{\omega = \omega(V, W, 0, \delta w) : \delta w \in L^2(F)\} =: \Omega_0. \end{aligned} \tag{4.25}$$

Since holomorphic differential forms of the form (4.18) with $\delta v = 0$ of Fermi curves of Schrödinger potentials $(1, \frac{-u}{4})$ are invariant with respect to the involution σ, it remains to prove the inclusion "$\subseteq$" in (4.25).
We introduce the following subspaces of $L^2(F) \times L^2(F)$. Let

$$U := \overline{span\left\{\begin{pmatrix} v_g \\ w_g \end{pmatrix} : g \text{ is a meromorphic function on an open subset of } F(u)/\Gamma^*\right\}},$$

$$U^\pm := \overline{span\left\{\begin{pmatrix} v_g \\ w_g \end{pmatrix} : g \circ \sigma = \pm g\right\}} \subset U$$

with v_g, w_g as defined in (4.6). We call functions g with $g = g \circ \sigma$ symmetric and functions g with $g = -g \circ \sigma$ anti-symmetric. For given $(\delta v, \delta w) \in L^2(F) \times L^2(F)$, due to (4.20), we have the following equivalences for $\omega = \omega(V, W, \delta v, \delta w)$:

$$\begin{aligned} \omega \circ \sigma = \omega &\Longleftrightarrow \sum_{\zeta \in \mathcal{U}} \mathrm{res}_\zeta(g \cdot \omega(V, W, \delta v, \delta w)) = 0 \text{ for } g \text{ anti-symmetric} \\ &\Longleftrightarrow \Omega((\delta v, \delta w), (v_g, w_g)) = 0 \text{ for } g \text{ anti-symmetric.} \end{aligned} \tag{4.26}$$

Denoting by $\Omega(X)$ the space of holomorphic 1-forms on $X := F(u)/\Gamma^*$, we consider the map

$$\omega : L^2(F) \times L^2(F) \to \Omega(X), \quad (\delta v, \delta w) \mapsto \omega(V, W, \delta v, \delta w). \tag{4.27}$$

Due to (4.26), the image of $(U^-)^\perp$ under ω equals Ω_+, where the orthogonal complement $(U^-)^\perp$ is taken with respect to the symplectic form Ω. Hence, by the isomorphism theorem, we have $\Omega_+ \cong (U^-)^\perp / \ker \omega$. We claim that $U^\perp = \ker \omega$: If $(\delta v, \delta w) \in \ker \omega$, then $\omega(V, W, \delta v, \delta w) = 0$. Due to (4.20), this implies $\Omega((\delta v, \delta w), (v_g, w_g)) = 0$ for all meromorphic functions g in open neighbourhoods of X. Hence, $(\delta v, \delta w) \in U^\perp$. Conversely, let $(\delta v, \delta w) \in U^\perp$ be given, i.e. $\Omega((\delta v, \delta w), (v_g, w_g)) = 0$ for all meromorphic functions g in open neighbourhoods of X. Choose g such that it has one pole at some point of X. Due to (4.20),

$\omega(V, W, \delta v, \delta w)$ vanishes at this point. Now consider an open neighbourhood of this point. By the same argument, for all points p in this neighbourhood with g chosen such that it has exactly one pole at p, we conclude that $\omega(V, W, \delta v, \delta w)$ vanishes on this neighbourhood. Since ω is holomorphic, it thus vanishes on the whole of X. Therefore, $(\delta v, \delta w) \in \ker \omega$ which proves the claim. Hence, $\Omega_+ \cong (U^-)^\perp / U^\perp$ and we have thus described the space Ω_+.
Now, we want to describe the space Ω_0. The image of $\{0\} \times L^2(F)$ under the map (4.27) equals Ω_0. Again, by the isomorphism theorem and due to $U^\perp = \ker \omega$, we get $\Omega_0 \cong (\{0\} \times L^2(F))/(U^\perp \cap (\{0\} \times L^2(F)))$. In order to prove the identity (4.25), namely $\Omega_+ = \Omega_0$, we have to show that the linear map

$$\alpha : (\{0\} \times L^2(F))/(U^\perp \cap (\{0\} \times L^2(F))) \to (U^-)^\perp / U^\perp, \quad (0, \delta w) \mapsto (0, \delta w) \tag{4.28}$$

(modulo the respective subspaces) is an isomorphism. At first, we show that α is well-defined. By the property of the symplectic form that $\Omega((0, \delta w), (0, \delta w')) = 0$ for all $\delta w, \delta w' \in L^2(F)$ and due to Lemma A.2 proven in the appendix, we get the inclusions

$$U^- \subseteq \{0\} \times L^2(F) \subseteq (U^-)^\perp. \tag{4.29}$$

This proves that α is well-defined. Next, we show that α is one-to-one. Thereto, let $(0, \delta w) \in \ker \alpha$ be given. That is, $(0, \delta w) \in U^\perp$. In particular, $(0, \delta w) \in U^\perp \cap (\{0\} \times L^2(F))$. Hence, $\ker \alpha$ is trivial and thus, α is one-to-one.
Now, we prove that α is surjective. By definition of α, we have to show

$$(U^-)^\perp = (\{0\} \times L^2(F)) + U^\perp \tag{4.30}$$

Due to $U^\perp \subseteq (U^-)^\perp$ and (4.29), the inclusion "$\supseteq$" is fulfilled so that we have to show $(U^-)^\perp \subseteq (\{0\} \times L^2(F)) + U^\perp$. Due to $(\{0\} \times L^2(F))^\perp = \{0\} \times L^2(F)$, Lemma 4.1.4 and the closedness of U, U^-, the identity (4.30) is equivalent to

$$(\{0\} \times L^2(F)) \cap U = U^-. \tag{4.31}$$

The inclusion "$\supseteq$" is again trivial and follows from (4.29). So let $(0, w_g) \in (\{0\} \times L^2(F)) \cap U$ be given. We decompose $g = \frac{1}{2}(g + g \circ \sigma) + \frac{1}{2}(g - g \circ \sigma)$ into its symmetric and anti-symmetric part. We denote the symmetric part of g by $g_s := \frac{1}{2}(g + g \circ \sigma)$. Due to (A.11) proven in the Appendix A, v_g is a linear combination of functions of the form $\psi_2(k, x)\psi_2(-k, x)$ over certain points k on the Fermi curve (with $\psi_2(k, \cdot)$ eigenfunction of the Schrödinger equation at k). Hence, if $v_g = 0$, then also $v_{g \circ \sigma} = 0$ since $v_{g \circ \sigma}$ is then a linear combination of functions of the form $\psi_2(-k, x)\psi_2(k, x)$. Therefore $v_{g_s} = 0$. Due to the second part of Lemma A.2 proven in the Appendix A, it follows for the symmetric g_s that if $v_{g_s} = 0$, then also $w_{g_s} = 0$. Therefore, the commutator (4.6) vanishes for

the symmetric g_s. By the proof of Lemma 4.1.2, this can only hold if $g_s = 0$. Therefore, g is anti-symmetric which implies $(0, w_g) \in U^-$.
In a final step, we choose in Theorem 4.1.3 a dual basis of holomorphic 1-forms ω_κ satisfying (4.21) for $0 < |\kappa|, |\nu| \leq N$. We define $\widetilde{\omega}_\kappa := \omega_\kappa + \omega_\kappa \circ \sigma$ for $\kappa \in \Gamma^*_N/\sigma$. These forms fulfill $\widetilde{\omega}_\kappa = \widetilde{\omega}_\kappa \circ \sigma$ for all $\kappa \in \Gamma^*_N/\sigma$. Furthermore, by the duality relation (4.21), the matrix with entries

$$\int_{A_\nu} \widetilde{\omega}_\kappa = \int_{A_\nu} \omega_\kappa + \int_{\sigma(A_\nu)} \omega_\kappa = \int_{A_\nu + A_{-\nu}} \omega_\kappa$$

at (κ, ν) with $\kappa, \nu \in \Gamma^*_N/\sigma$ has full rank. Here, we defined $A_{-\nu} := \sigma(A_\nu)$ for all $\nu \in \Gamma^*/\sigma$ which is possible if A_ν is not homologous to $\sigma(A_\nu)$. The proof of the latter non-homology statement is postponed into the next Lemma 4.1.6.
Due to (4.25), these $\widetilde{\omega}_\kappa$ can be chosen to be of the form (4.18) with suitable respective directions $(0, \delta w) \in L^2(F) \times L^2(F)$. By a possible linear transformation, they finally satisfy the duality relation (4.21) for all $\kappa, \nu \in \Gamma^*_N/\sigma$. This proves the theorem. □

Next, we want to prove that for $N \in \mathbb{N}$, the locally defined map $u \mapsto (m_\nu(u))_{\nu \in \Gamma^*_N/\sigma} = -16\pi^3 \left(\int_{A_\nu} k_1 dk_2\right)_{\nu \in \Gamma^*_N/\sigma}$, cf. Definition 2.6.1, mapping Schrödinger potentials to the first finitely many moduli is a submersion. To simplify the notation, we set in the following

$$m_N(u) := (m_\nu(u))_{\nu \in \Gamma^*_N/\sigma}. \tag{4.32}$$

Furthermore, we use again $2g := \#\{\nu \in \Gamma^* : 0 < |\nu| \leq N\}$ and thus $g = \#\Gamma^*_N/\sigma$ as in the beginning of the proof of Theorem 4.1.3. This notation will be used several times in this work again (without always explicitly recalling its definition).
In (4.32), we consider only the half of the a priori $2g$ moduli $m_\nu(u)$ indexed by $0 < |\nu| \leq N$. We want to explain why the remaining neglected g moduli are redundant. In the asymptotic analysis of Chapter 3, we've already seen this. More precisely, in (3.24), we saw that $m_\nu = m_{-\nu}$ for all $\nu \in \Gamma^*_\delta$. By defining $A_{-\nu}$ to be the image of A_ν under σ for all $\nu \in \Gamma^*/\sigma$, we get by virtually the same computation as in (3.24) $m_\nu = m_{-\nu}$ for all $\nu \in \Gamma^* \setminus \{0\}$. In order to define $A_{-\nu} := \sigma(A_\nu)$ for all $\nu \in \Gamma^*/\sigma$, we have to show that $\sigma(A_\nu)$ and A_ν are not homologous to each other. This is justified in the following lemma.

Lemma 4.1.6. *For all $\nu \in \Gamma^* \setminus \{0\}$, the cycles $A_\nu - \sigma(A_\nu)$ are not homologous to zero.*

Proof. We consider at first the special case for *finite type* Fermi curves. In [19, Theorem 4.23] combined with [19, Lemma 4.13], it has been shown that the two points "at infinity" Q^+ and Q^- of the two-point compactification of the (normalized) finite type Fermi curve are the *only* fixed points of the involution

σ. Let the normalized Fermi curve be denoted by X. In particular, X is a compact Riemann surface. As in [19, Proposition A.1], we define the quotient space $X_\sigma := X/\sim_\sigma$, where $k \sim_\sigma k'$ if and only if $k = k'$ or $k = \sigma(k') = -k'$ for all $k, k' \in X$. Denote by $\pi_\sigma : X \to X_\sigma$ the natural projection. In the proof of [19, Proposition A.1], it is shown that π_σ is a two-sheeted covering whose branch points are exactly the fixed points of σ. By the above, π_σ has exactly the two branch points Q^+ and Q^-. In [19, Proposition A.9], it has been shown that in this case, the A-cycles $A_\nu - \sigma(A_\nu)$ together with the corresponding B-cycles $B_\nu - \sigma(B_\nu)$ constitute a homology basis of $H_1(X, \mathbb{Z})_- := \{\gamma \in H_1(X, \mathbb{Z}) : \sigma(\gamma) = -\gamma\}$ with $\dim H_1(X, \mathbb{Z})_- = 2g_\sigma$, where g_σ denotes the genus of X_σ. In particular, the cycles $A_\nu - \sigma(A_\nu)$, $\nu \in \Gamma^*_N/\sigma$ are not homologous to zero.
Now, consider the case of infinite type Fermi curves. By Theorem 2.4.2, in every neighbourhood in $L^2(F)$ of some potential $u \in L^2(F)$, there are potentials v with the property that all but finitely many of their perturbed Fourier coefficients are equal to zero. In other words, the finite type potentials are dense in $L^2(F)$ and there exists a sequence of finite type potentials $(u_n)_{n\in\mathbb{N}}$ converging to u [7]. Hence, by finite type approximation, the assertion of the lemma follows for all cycles A_ν whose support is contained in a sufficiently large compact subset of $\mathbb{C}^2$ by the corresponding assertion for finite type Fermi curves proved before. Together with the already well-known fact from the asymptotic analysis that the corresponding cycles outside this compact set, namely $A_\nu + A_{-\nu}$ indexed by $\nu \in \Gamma^*_\delta$ (with corresponding $\delta > 0$ sufficiently small), are not homologous to zero, the assertion finally holds for all cycles. □

Due to the choice $A_{-\nu} := \sigma(A_\nu)$ implying $m_\nu(u) = m_{-\nu}(u)$ for $\nu \in \Gamma^*_N/\sigma$, the moduli m_N lie in the space

$$\widetilde{\mathbb{C}}^g := \{(v_{-g}, \ldots, v_g) \in \mathbb{C}^{2g} : v_{-j} = v_j \text{ for all } j \in \{1, \ldots, g\}\}$$

which is obviously isomorphic to $\mathbb{C}^g$ (this explains the notation $\widetilde{\mathbb{C}}^g$). Now, we can prove the announced submersion property of m_N which is a corollary of Theorem 4.1.5.

Corollary 4.1.7. *Let $u \in L^2(F)$ with smooth Fermi curve. Then for all $N \in \mathbb{N}$, the derivative of m_N at u, i.e. the linear map*

$$dm_N|_u : L^2(F) \to \widetilde{\mathbb{C}}^g$$

is onto.

Proof. As before, we use the coordinates $p = (p_1, p_2) \in \mathbb{C}^2$. Recall that they are related to the coordinates $k = (k_1, k_2) \in \mathbb{C}^2$ by $k = p_1\hat{\kappa} + p_2\check{\kappa}$, that is $k = (\hat{\kappa}, \check{\kappa}) \cdot p$

[7] In Lemma 4.2.7, the construction of some canonical sequence of finite type potentials $(u_n)_{n\in\mathbb{N}}$ converging to u will be carried out more explicitly.

with the invertible matrix $(\hat{\kappa}, \check{\kappa})$ whose columns are the two generators $\hat{\kappa}$ and $\check{\kappa}$ of the lattice Γ^* (cf. the beginning of this section).
For $(\delta v, \delta w) \in L^2(F) \times L^2(F)$, we denote with a dot the directional derivative of p with respect to $(\delta v, \delta w)$ as we already did in (4.16), for instance. Let γ be an arbitrary A-cycle on $F(V,W)/\Gamma^*$. Due to $0 = \int_\gamma d(p_1\dot{p}_2) = \int_\gamma p_1 d\dot{p}_2 + \int_\gamma \dot{p}_2 dp_1$, we get $-\int_\gamma p_1 d\dot{p}_2 = \int_\gamma \dot{p}_2 dp_1$ [8]. Together with (4.17), this yields

$$\int_\gamma \frac{\frac{\partial R}{\partial(V,W)}(\delta v, \delta w)}{\frac{\partial R}{\partial p_1}} dp_2 = -\int_\gamma \dot{p}_1 dp_2 + \int_\gamma \dot{p}_2 dp_1 = -\int_\gamma \dot{p}_1 dp_2 - \int_\gamma p_1 d\dot{p}_2 =$$
$$= -\frac{d}{d(V,W)}\left(\int_\gamma p_1 dp_2\right)|_{(V,W)}(\delta v, \delta w).$$

Now, Theorem 4.1.5 (or more precisely equation (4.23) with $v_\kappa = 0$) implies together with (4.18) that there is a $w_\kappa \in L^2(F)$, $\kappa \in \Gamma_N^*/\sigma$, such that[9]

$$4\frac{d}{du}\left(\int_{A_\nu} p_1 dp_2\right)|_u(w_\kappa) = -\frac{d}{d(V,W)}\left(\int_{A_\nu} p_1 dp_2\right)|_{(V,W)=(1,-\frac{u}{4})}(0, w_\kappa) =$$
$$= \int_{A_\nu} \omega(1, \tfrac{-u}{4}, 0, w_\kappa) = \delta_{\kappa,\nu} \tag{4.33}$$

for all $\kappa, \nu \in \Gamma_N^*/\sigma$. Consequently, the linear map $\frac{d}{du}\left(\int_{A_\nu} p_1 dp_2\right)_{\nu \in \Gamma_N^*/\sigma}|_u$ has full rank g. Since the coordinates p and k are mapped to each other by a linear invertible map as mentioned in the beginning of the proof, the same holds for $\frac{d}{du}\left(\int_{A_\nu} k_1 dk_2\right)_{\nu \in \Gamma_N^*/\sigma}|_u$. Therefore, $dm_N|_u$ is onto which had to be proved. □

In the sequel, we will often make use of the continuity of $u \mapsto m(u)$ which immediately follows from the Definition 2.6.1 of the moduli and the fact that Fermi curves $F(u)$ continuously depend on the potential u. What is not clear a priori is that the derivative of the map $u \mapsto m(u)$ is continuous, too. We prove in the following lemma that this map is even smooth. We will need this assertion later. We already prove it here because it fits well in the context of the assertions we have just proved.

Lemma 4.1.8. *Let $u_0 \in L^2(F)$ with smooth Fermi curve and $O \subset L^2(F)$ a neighbourhood of u_0 such that $m(u)$ is well-defined for all $u \in O$. Then for all $N \in \mathbb{N}$, the map $O \to \widetilde{\mathbb{C}}^g$, $u \mapsto m_N(u)$ is smoooth. In particular, the derivative function*

$$O \to \mathcal{L}(L^2(F); \widetilde{\mathbb{C}}^g), \quad u \mapsto dm_N|_u$$

[8] As to the well-definition of the directional derivatives $\dot{p}_2$ and $d\dot{p}_2$, compare the explanation on p. 114. As before, these directional derivatives of the p-coordinates are only used temporarily. In the next equation, we already get rid of them again.

[9] By slight abuse of notation, we denote the ν^{th} A-cycle by the same symbol A_ν in both p- and k-coordinates.

is continuous.

Proof. Let $N \in \mathbb{N}$, $u_0 \in L^2(F)$ and $(V_0, W_0) := (1, -\frac{u_0}{4})$ be given. In the proof of Corollary 4.1.7, we showed for all $\nu \in \Gamma_N^*/\sigma$ and for all $(\delta v, \delta w) \in L^2(F) \times L^2(F)$

$$\int_{A_\nu} \frac{\frac{\partial R}{\partial(V,W)}(\delta v, \delta w)}{\frac{\partial R}{\partial p_1}} dp_2 = -\frac{d}{d(V,W)} \left(\int_{A_\nu} p_1 dp_2 \right) |_{(V,W)}(\delta v, \delta w) \tag{4.34}$$

for all $(V, W) \in U$ with U a suitable neighbourhood of (V_0, W_0) in $L^2(F) \times L^2(F)$. Since $R = R(p, V, W)$ is as a holomorphic function smooth in (V, W), the operator $\frac{\frac{\partial R}{\partial(V,W)}}{\frac{\partial R}{\partial p_1}} : L^2(F) \times L^2(F) \to \mathbb{C}$ is smooth in (V, W), too. Note that although at first sight, the denominator of the latter term might have zeroes, this term is still bounded which can be seen by (4.17), for instance. Because the cycles $A_\nu = A_\nu(V, W)$ smoothly depend on (V, W), the operator $\int_{A_\nu} \frac{\frac{\partial R}{\partial(V,W)}}{\frac{\partial R}{\partial p_1}} dp_2$ is thus smooth in (V, W) as well. By (4.34), $(V, W) \mapsto dm_N|_{(V,W)}$ is then smooth, in particular continuous. Clearly, this continuity is preserved when we restrict ourselves to Schrödinger potentials, i.e. all considered potentials are of the form $(V, W) = (1, -\frac{u}{4})$ with $u \in L^2(F)$. This proves the lemma. □

In the following, we will identify potentials $u \in L^2(F)$ with their associated sequence of Fourier coefficients $\hat{u} \in l^2(\Gamma^*)$. Our next step is to prove that the surjectivity statement of Corollary 4.1.7 remains true if we reduce the domain of definition $L^2(F)$ of $dm_N|_u$ to[10]

$$L_N^2(F) := \{v \in L^2(F) : \hat{v}(\nu) = 0 \text{ for all } |\nu| > N\} \cong \mathbb{C}^{2g+1}.$$

In order to prove this, we have to show the following lemma.

Lemma 4.1.9. *Let $u \in L^2(F)$ with smooth Fermi curve and consider for each $\nu \in \Gamma^* \setminus \{0\}$ the derivative of m_ν at u, i.e. the linear map $dm_\nu|_u : L^2(F) \to \mathbb{C}$. Then the operators $dm_\nu|_u$, $\nu \in (\Gamma^* \setminus \{0\})/\sigma$ are linearly independent over $\mathbb{C}$.*

Remark. Linear independence means here that $\sum_{\nu \in (\Gamma^* \setminus \{0\})/\sigma} c_\nu dm_\nu|_u = 0$ (with a sequence $c = (c_\nu)_\nu \subset \mathbb{C}$ such that the series converges in the corresponding operator norm) implies $c = 0$.

Proof. At first, with the notation $\widetilde{x} := \binom{\overline{x_2}}{-\overline{x_1}}$ for $x = \binom{x_1}{x_2} \in L^2(F) \times L^2(F)$, we remark the relation

$$\langle x, y \rangle = 2\pi^2 i \Omega\,(\widetilde{y}, x) \text{ for all } x, y \in L^2(F) \times L^2(F) \tag{4.35}$$

[10]Note that $\#\{\nu \in \Gamma^* : |\nu| \leq N\} = 2g + 1$ since the Fourier coefficient corresponding to $\nu = 0$ is included.

between the symplectic form Ω (4.19) and the canonical hermitian scalar product $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{L^2(F)\times L^2(F)}$.
Let $u \in L^2(F)$ and set $(V,W) := (1, \frac{-u}{4})$. With the potentials $v(A_\nu)$, $w(A_\nu)$ associated to the Fermi curve $F(u)$, cf. (4.8), we introduce the notations $a_\nu := \binom{v(A_\nu)}{w(A_\nu)}$ for $\nu \in (\Gamma^* \setminus \{0\})/\sigma$ and $\mathcal{A} := \{a_\nu : \nu \in (\Gamma^* \setminus \{0\})/\sigma\}$. We consider the closed subspace $\overline{span\mathcal{A}} \subseteq L^2(F) \times L^2(F)$. Since any arbitrarily large finite subset M of $\mathcal{A}$ is linearly independent by Lemma 4.1.2, and any other $a \in \mathcal{A}\setminus M$ fulfills $a \notin span(M)$, we may successively apply the Gram-Schmidt orthogonalization. This yields linearly independent potentials $b_\nu \in L^2(F)\times L^2(F)$, $\nu \in (\Gamma^*\setminus\{0\})/\sigma$, inductively defined by $b_{\nu'} := a_{\nu'}$ for some arbitrary fixed $\nu' \in \Gamma^* \setminus \{0\}$ and

$$b_\nu := a_\nu - \sum_j \frac{\langle b_j, a_\nu\rangle}{\langle b_j, b_j\rangle} b_j, \quad \nu \in (\Gamma^* \setminus \{0, \nu'\})/\sigma, \tag{4.36}$$

(where the sum runs over those $j \in (\Gamma^*\setminus\{0\})/\sigma$ for which the orthogonal vectors b_j have already been constructed) which fulfill with $\mathcal{B} := \{b_\nu : \nu \in (\Gamma^* \setminus \{0\})/\sigma\}$ the relations $\overline{span\mathcal{B}} = \overline{span\mathcal{A}}$ and $\Omega(\widetilde{b}_\kappa, b_\nu) = 0$, $\Omega(\widetilde{b}_\kappa, b_\kappa) \neq 0$ for all $\kappa,\nu \in (\Gamma^* \setminus \{0\})/\sigma$ with $\kappa \neq \nu$, cf. (4.35). By a suitable normalization, the b_ν can be chosen such that $\Omega(\widetilde{b}_\kappa, b_\nu) = \delta_{\kappa,\nu}$ for all $\kappa,\nu \in (\Gamma^* \setminus \{0\})/\sigma$. Due to (4.36) (whose right hand side has to be multiplied with respective complex numbers $\neq 0$ according to the just mentioned normalization), the transformation map T which transforms a vector in $\mathcal{A}$-coordinates into the respective vector represented in $\mathcal{B}$-coordinates can be considered as an (infinite-dimensional) quadratic upper triangular matrix whose diagonal entries are unequal to zero. By construction, the map $G := (\Omega(\widetilde{b}_\kappa,\cdot))_{\kappa\in(\Gamma^*\setminus\{0\})/\sigma}$ is the identity in $\mathcal{B}$-coordinates. Hence, by composing G (in $\mathcal{B}$-coordinates) with T (yielding G in $\mathcal{A}$-coordinates), the entries of T are equal to $\Omega(\widetilde{b}_\kappa, a_\nu)$, $\kappa,\nu \in (\Gamma^* \setminus \{0\})/\sigma$ where κ indexes the rows and ν indexes the columns of T. The matrix whose κ^{th} column is defined by $\Omega(\widetilde{b}_\kappa, a_\nu)_{\nu\in\Gamma^*}$ is therefore the transpose of T, i.e. a lower triangular matrix with diagonal entries unequal to zero. Hence, $\{\Omega(\widetilde{b}_\kappa, a_\nu)_{\nu\in(\Gamma^*\setminus\{0\})/\sigma} : \kappa \in (\Gamma^* \setminus \{0\})/\sigma\}$ is a set of linearly independent vectors. Together with (4.23) and (4.33) which relate Ω and $dm|_{(V,W)}$ to each other, this yields that $dm|_{(V,W)}(\widetilde{b}_\kappa)$, $\kappa \in (\Gamma^*\setminus\{0\})/\sigma$ are linearly independent vectors. Since moreover, the $L^2(F) \times L^2(F)$-potentials $\widetilde{b}_\kappa$ are of the form $\widetilde{b}_\kappa = (0, c_\kappa)$ for some $c_\kappa \in L^2(F)$, the two equations (4.23) and (4.33) even show that $dm|_u(c_\kappa)$, $\kappa \in (\Gamma^*\setminus\{0\})/\sigma$ are linearly independent. Since the infinite matrix with entries $dm_\nu|_u(c_\kappa)$ for $\kappa,\nu \in (\Gamma^* \setminus \{0\})/\sigma$ is triangular with diagonal entries unequal to zero, the assertion of the lemma follows. □

In the following theorem, we show the announced submersion property of the moduli restricted to $L^2_N(F)$. Since in the proof of that theorem, we use results from the asymptotic analysis of Chapter 3 proven for *real-valued* potentials, we assume the given potential u in the theorem to be real-valued.

Theorem 4.1.10. *Let $u \in L^2(F)$ be real-valued with smooth Fermi curve. Then there exists an $N \in \mathbb{N}$ sufficiently large (dependent on u) such that the linear map*

$$dm_N|_u : L^2_N(F) \to \widetilde{\mathbb{C}}^g$$

is onto.

Proof. We choose $N \in \mathbb{N}$ sufficiently large such that we are for $|\nu| > N$ in the asymptotic setting of the Chapters 2 and 3. In the course of the proof, N might be chosen even larger in order to guarantee certain asymptotic estimates. For $|\nu| > N$, the moduli can thus be approximated by the model moduli $\xi\widetilde{m}_\nu(u) = \frac{\check{u}_\nu \check{u}_{-\nu}}{|\nu|^2}$ (2.84) via $m_\nu(u) = (1 + O(1/|\nu|))\xi\widetilde{m}_\nu(u)$, as $|\nu| \to \infty$, cf. Lemma 3.2.2 and the definition of ξ in (2.2). If, for $|\nu| > N$, we derive $\xi\widetilde{m}_\nu(u)$ with respect to $\check{u}_{-\nu}$ and $\check{u}_\nu$, respectively, we get

$$\frac{d(\xi\widetilde{m}_\nu(u))}{d\check{u}_{-\nu}} = \frac{\check{u}_\nu}{|\nu|^2} =: a_\nu, \qquad \frac{d(\xi\widetilde{m}_\nu(u))}{d\check{u}_\nu} = \frac{\check{u}_{-\nu}}{|\nu|^2} = \frac{\overline{\check{u}_\nu}}{|\nu|^2} = \overline{a_\nu} \tag{4.37}$$

We consider the matrix representation

$$M := \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{4.38}$$

where the respective blocks A, B, C, D are defined as follows:

$$\begin{aligned}
A &= (A_{\nu,\kappa}) \in \mathbb{C}^{g\times(2g+1)}, \quad A_{\nu,\kappa} := \frac{dm_\nu(u)}{d\hat{u}(\kappa)}, \quad |\nu|, |\kappa| \le N,\\
B &= (B_{\nu,\kappa}) \in \mathbb{C}^{g\times\infty}, \quad B_{\nu,\kappa} := \frac{dm_\nu(u)}{d\check{u}_\kappa}, \quad |\nu| \le N, |\kappa| > N,\\
C &= (C_{\nu,\kappa}) \in \mathbb{C}^{\infty\times(2g+1)}, \quad C_{\nu,\kappa} := \frac{d(\xi\widetilde{m}_\nu(u))}{d\hat{u}(\kappa)}, \quad |\nu| > N, |\kappa| \le N,\\
D &= (D_{\nu,\kappa}) \in \mathbb{C}^{\infty\times\infty}, \quad D_{\nu,\kappa} := \frac{d(\xi\widetilde{m}_\nu(u))}{d\check{u}_\kappa}, \quad |\nu|, |\kappa| > N,
\end{aligned}$$

where for all blocks $\nu \in (\Gamma^* \setminus \{0\})/\sigma$ [11] and $\kappa \in \Gamma^*$. Our aim is to prove with the help of Lemma 4.1.9 (stating that $dm|_u$ has full rank) that the block A has full rank g possibly by choosing N even larger than we already did. We now use the following conventions: The first column of M contains the derivative with respect to the zeroth Fourier coefficient $\hat{u}(0)$. All other columns contain derivatives with respect to higher Fourier coefficients. Of course, there is no unique canonical numeration of these columns. We only demand that the norm

[11] Note that, considering for example the block A, not all $2g$ lattice vectors fulfilling $0 < |\nu| \le N$ have to be considered due to $m_\nu = m_{-\nu}$ which explains that A has g rows instead of $2g$ rows. Analogous statements hold for the other blocks constituting M.

$|\kappa|$ shall be monotonous in the column number, i.e. for two columns c_i and c_j of M $(i, j \in \Gamma^*)$ corresponding to some κ_i and some κ_j there shall hold: If $i < j$, then $|\kappa_i| \leq |\kappa_j|$. In particular, all entries of the block C are equal to zero, i.e.

$$C = 0,$$

since $\xi\widetilde{m}_\nu(u)$ is independent of $\hat{u}(\kappa)$, $|\kappa| \leq N$. Furthermore, derivatives with respect to $\hat{u}(\kappa)$ and $\hat{u}(-\kappa)$ shall be neighbouring columns. For example, if the second column of M contains the derivative with respect to $\hat{u}(\kappa)$ (for some $|\kappa| \leq N$), then the third column of M contains the derivative with respect to $\hat{u}(-\kappa)$ and so on. In particular, the rows of the block D have the following form: All entries except the neighbouring entries a_κ and $\overline{a_\kappa}$ (cf.(4.37)) corresponding to some $|\kappa| > N$ are equal to zero, i.e. D has the form

$$D = \begin{pmatrix} a_{\nu_1} & \overline{a_{\nu_1}} & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & a_{\nu_2} & \overline{a_{\nu_2}} & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & a_{\nu_3} & \overline{a_{\nu_3}} & 0 & \dots \\ \vdots & & & & & \ddots & \ddots & \end{pmatrix}. \tag{4.39}$$

In a first step, we show that Lemma 4.1.9 implies that the matrix M (4.38) has full rank. The matrix M differs from $dm|_u$ in essentially two aspects: $dm|_u$ is defined for potentials $u \in L^2(F)$ which can be identified with their respective sequence of Fourier coefficients $\hat{u} \in l^2(\Gamma^*)$, whereas M is defined for potentials in the representation $(\hat{u}_N, \check{u}) := ((\hat{u}(\nu))_{|\nu| \leq N}, (\check{u}_\nu)_{|\nu| > N})$. Due to Theorem 2.4.2, however, which provides a local isomorphism $\hat{u} \mapsto (\hat{u}_N, \check{u})$, the rank of $dm|_u$ remains invariant if we derive with respect to $(\hat{u}_N, \check{u})$ instead of with respect to u. In other words, multiplying $dm|_u$ with the inverse of the invertible operator $\frac{d(\hat{u}_N, \check{u})}{d\hat{u}} = \left(\begin{smallmatrix} \mathbb{1} & 0 \\ * & I_2 \end{smallmatrix}\right)$, where I_2 is the invertible matrix with entries $\frac{d\check{u}_\nu}{d\hat{u}(\kappa)}$ indexed by $|\kappa|, |\nu| > N$, doesn't change the rank of $dm|_u$.

The second aspect in which $dm|_u$ differs from M is that in M, the moduli m_ν indexed by $|\nu| > N$ are replaced by the model moduli $\widetilde{m}_\nu$. In order to deduce from the full rank of $dm|_u$ provided by Lemma 4.1.9 that also M has full rank, we have to show that $(dm_N, d\widetilde{m}_\delta) := ((dm_\nu|_u)_{|\nu| \leq N}, (d\widetilde{m}_\nu|_u)_{|\nu| > N})$ has full rank if $dm|_u$ has full rank. Since $d\widetilde{m}_\nu|_u$ is only defined for $|\nu| > N$, we consider $m_\delta := (m_\nu)_{|\nu| > N}$ and likewise $\widetilde{m}_\delta$. Because $dm|_u$ has full rank due to Lemma 4.1.9, we may reduce the domain of definition of $dm|_u$ to a subspace $V \subseteq L^2(F)$ such that the restriction of $dm|_u$ to V is invertible. We consider again the difference $r_\delta(\cdot) = m_\delta(\cdot) - \widetilde{m}_\delta(\cdot)$ between moduli and model moduli. We can thus write $dm_\delta = d\widetilde{m}_\delta + dr_\delta$. Hence, $dm|_u$ is a bounded (i.e. continuous) operator due to Lemma 3.2.6 and since $d\widetilde{m}_\delta$ is obviously bounded, compare the representation D (4.39). We set $T := ((dm|_u)|_V)^{-1}$. Then T is a bounded operator, i.e. with bounded operator norm $\|T\| < \infty$, since T is the inverse of a linear bijective bounded operator, cf. [30, Korollar IV.3.4]. Lemma 3.2.6 now implies

$$\|T(dm_N, d\widetilde{m}_\delta) - \mathbb{1}\| = \|T(dm_N, d\widetilde{m}_\delta) - T dm|_u\| \leq \|T\| \cdot \|dr_\delta\| = \|T\| \cdot o(1),$$

as $|\nu| \to \infty$, where the respective operators are considered as restrictions onto V. Therefore, for $\delta > 0$ sufficiently small (or equivalently N sufficiently large), we have $\|T(dm_N, d\widetilde{m}_\delta) - \mathbb{1}\| < \frac{1}{2}$ proving that $T(dm_N, d\widetilde{m}_\delta)$ and also $(dm_N, d\widetilde{m}_\delta)$ is invertible on V due to Neumann's Theorem (cf. [30, Satz II.1.11]). Hence, we have shown that the full rank (provided by Lemma 4.1.9) of $dm|_u$ as a linear operator defined on $L^2(F)$ implies that M (4.38) has full rank if N is chosen sufficiently large.

In the next step, we use the full rank of M to deduce that the block A has full rank g. Due to the form of D (4.39), it is clear that the matrix $\binom{B}{D}$ has at least those linearly independent columns which correspond to a_ν, $|\nu| > N$ in the representation (4.39) (note that $a_\nu \neq 0$, $|\nu| > N$, due to the smoothness of the Fermi curve). We ask if it is possible that $\binom{B}{D}$ has more than those linearly independent columns. These additional linearly independent columns (if they exist) must then necessarily be columns corresponding to some $\overline{a_\nu}$, $|\nu| > N$ in the representation (4.39).

At first, we consider the **case 1** that this is never possible. More precisely, we consider the case that the following holds: Any set of columns of M with the property that this set contains at least one pair of neighbouring columns of $\binom{B}{D}$ corresponding to a_κ and $\overline{a_\kappa}$ (for some $|\kappa| > N$) is linearly dependent. Therefore, we can cancel the columns of $\binom{B}{D}$ corresponding to $\overline{a_\nu}$, $|\nu| > N$ without changing the rank of M. The thus modified matrix shall be denoted by $\widetilde{M}$. It has the form

$$\widetilde{M} := \begin{pmatrix} A & \widetilde{B} \\ 0 & \widetilde{D} \end{pmatrix},$$

where $\widetilde{D}$ is a quadratic invertible block. Since the rows of $\widetilde{M}$ are linearly independent as shown above, this implies that the g rows of A are linearly independent, too. This proves the theorem in the considered case.

Now, we consider the **case 2** that there exists a linearly independent set $\mathcal{C}$ of columns of M with $\binom{\widetilde{B}}{\widetilde{C}} \subseteq \mathcal{C}$ [12] containing linearly independent neighbouring columns of $\binom{B}{D}$ corresponding to a_ν and $\overline{a_\nu}$ (for some $|\nu| > N$). Hence, we may write $\binom{\widetilde{B}}{\widetilde{D}} \cup E \subseteq \mathcal{C}$, where $E \neq \emptyset$ is a set of columns of $\binom{B}{D}$ corresponding to some $\overline{a_\nu}$, $|\nu| > N$. If E is finite, where the last one of the finitely many columns constituting E corresponds to some $\widetilde{\kappa}$, choose $N \geq |\widetilde{\kappa}|$ and go back to case 1.

It remains to consider the case that E contains infinitely many columns. We show by contradiction that this case cannot occur. So assume that E contains infinitely many columns. We proceed as follows. Choose $N' \geq N$ large enough such that with associated $2g' + 1 := \#\{\nu \in \Gamma^* : |\nu| \leq N'\}$, the block $\binom{B'}{D'}$ within

[12] In the notation $\binom{\widetilde{B}}{\widetilde{C}} \subset \mathcal{C}$, we consider the matrix $\binom{\widetilde{B}}{\widetilde{C}}$ as the set of its columns.

the $g' \times (2g'+1)$-matrix

$$M' := \begin{pmatrix} A & B' \\ 0 & D' \end{pmatrix},$$

contains $g+1$ columns in E. Here, the blocks B', D' are the blocks B, D in (4.38) whose rows and columns are cut off for $|\kappa|, |\nu| > N'$, more precisely:

$$\begin{aligned} B' &= (B'_{\nu,\kappa}) \in \mathbb{C}^{g\times 2(g'-g)}, \quad B'_{\nu,\kappa} := B_{\nu,\kappa}, \quad |\nu| \leq N,\ N < |\kappa| \leq N', \\ D' &= (D'_{\nu,\kappa}) \in \mathbb{C}^{(g'-g)\times 2(g'-g)}, \quad D'_{\nu,\kappa} := D_{\nu,\kappa}, \quad N < |\nu|, |\kappa| \leq N'. \end{aligned}$$

Due to the representation of D in (4.39), the $g' \times 2(g'-g)$-matrix $\binom{B'}{D'}$ has $g'-g$ linearly independent columns that are not contained in E. Together with the additional $g+1$ linearly independent columns in E, we have in total $(g'-g)+(g+1) = g'+1$ columns. These are linearly independent since $\binom{\widetilde{B}}{\widetilde{D}} \cup E \subseteq \mathcal{C}$. This, however, is a contradiction since M' has only g' rows and thus cannot have $g'+1$ linearly independent columns. Hence, the theorem is proved. □

From now on, the subspace of $L^2(F)$ of real-valued potentials shall be denoted by $L^2_{\mathbb{R}}(F)$. Analogously to $L^2_N(F)$, we set for $N \in \mathbb{N}$

$$L^2_{N,\mathbb{R}}(F) := \{v \in L^2_{\mathbb{R}}(F) : \hat{v}(\nu) = 0 \text{ for all } |\nu| > N\} \cong \mathbb{R}^{2g+1}.$$

At first, we recall that the Fermi curve $F(u)$ with $u \in L^2_{\mathbb{R}}(F)$ has the two further anti-holomorphic involutions η and τ introduced in Section 2.3. We've seen in (3.25) that the reality condition implies $m_\nu \in \mathbb{R}$ for all $\nu \in \Gamma^*_\delta$. It's not clear if this also holds for $\nu \in \Gamma^* \setminus \Gamma^*_\delta$, i.e for $|\nu| \leq N$ (for some $N \in \mathbb{N}$ sufficiently large) because we don't know whether A_ν is mapped to $A_{-\nu}$ by η for $|\nu| \leq N$. If this is the case, we can argue as in (3.25) and conclude that $m_\nu \in \mathbb{R}$ for this ν. In general, however, we merely know that A_ν is mapped to some linear combination of A-cycles A_κ by η with $|\kappa| \leq N$. We want to justify that we can choose the homology basis such that η maps A_ν to one A_κ with $|\kappa| \leq N$, i.e. that the just mentioned linear combination can be chosen to be equal to *one* cycle A_κ of the homology basis. In the case of finite type Fermi curves, this has been shown in [19, Lemma 6.43] together with [19, Definition 6.42]. In order to show the same result for Fermi curves of *infinite type*, we proceed exactly as in the end of the proof of Lemma 4.1.6, i.e. we combine the finite type result with the result for the asymptotic A-cycles (indexed by Γ^*_δ) by using an approximation of finite type potentials. To sum up, this choice of the homology basis yields that for each $\nu \in \Gamma^*$, there is a $\kappa \in \Gamma^*$ such that $\eta(A_\nu) = A_\kappa$. The analogous computation of (3.25) then yields $\overline{m}_\nu = m_\kappa$ for this pair ν, κ.
In any case, the dimension $2g$ of $\widetilde{\mathbb{C}}^g$ (considered as a *real* vector space) the moduli m_N reside in is halved such that real-valued potentials are mapped by $u \mapsto m_N(u)$

into a vector space of real dimension g which shall be denoted by $\widetilde{\mathbb{R}}^g$. We will see that the real dimension g is the crucial property. Whether this space is even equal to $\mathbb{R}^g$ or not, won't be needed. We now prove the real analogon to Theorem 4.1.10.

Theorem 4.1.11. *Let $u \in L^2_{\mathbb{R}}(F)$ with smooth Fermi curve. Then there exists an $N \in \mathbb{N}$ sufficiently large (dependent on u) such that the linear map*

$$dm_N|_u : L^2_{N,\mathbb{R}}(F) \to \widetilde{\mathbb{R}}^g,$$

is onto.

Proof. Let $u \in L^2_{\mathbb{R}}(F)$ and $N \in \mathbb{N}$. We use the notation $\alpha := dm_N|_u$. Due to Theorem 4.1.10, there are g potentials $v_j \in L^2_N(F)$, $j \in \{1, \dots, g\}$, such that the vectors $\alpha(v_j) \in \widetilde{\mathbb{C}}^g$, $j \in \{1, \dots, g\}$ are linearly independent and constitute a basis of $\widetilde{\mathbb{C}}^g$ which shall be denoted by $\mathcal{B}$.
Let $n \in \{1, \dots, g\}$. Then there are unique coefficients $\lambda_j \in \mathbb{C}$ such that

$$\alpha(\overline{v}_n) = \sum_{j=1}^{g} \lambda_j \alpha(v_j).$$

If $\lambda_n \neq -1$, we replace $\alpha(v_n)$ by $\alpha(v_n + \overline{v}_n)$ in $\mathcal{B}$. Because of

$$\alpha(v_n + \overline{v}_n) = (1 + \lambda_n)\alpha(v_n) + \sum_{\substack{j=1 \\ j \neq n}}^{g} \lambda_j \alpha(v_j)$$

the thus modified $\mathcal{B}$ is still a basis of $\widetilde{\mathbb{C}}^g$.
If $\lambda_n = -1$, i.e.

$$\alpha(\overline{v}_n) = -\alpha(v_n) + \sum_{\substack{j=1 \\ j \neq n}}^{g} \lambda_j \alpha(v_j),$$

we replace $\alpha(v_n)$ by $\alpha(\frac{v_n - \overline{v}_n}{i})$ in $\mathcal{B}$. Because of

$$\alpha(\tfrac{v_n - \overline{v}_n}{i}) = -2i\alpha(v_n) + i \sum_{\substack{j=1 \\ j \neq n}}^{g} \lambda_j \alpha(v_j),$$

the thus modified $\mathcal{B}$ is still a basis of $\widetilde{\mathbb{C}}^g$. We carry out this procedure for all $n \in \{1, \dots, g\}$, that is, we replace v_j by $w_j \in \{v_j + \overline{v}_j, \frac{v_j - \overline{v}_j}{i}\}$, $j \in \{1, \dots, g\}$ such that the thus modified $\mathcal{B}$ is still a basis of $\widetilde{\mathbb{C}}^g$. By construction, $w_j \in L^2_{N,\mathbb{R}}(F)$ for all $j \in \{1, \dots, g\}$ and $\alpha(w_1), \dots, \alpha(w_g)$ are linearly independent over $\mathbb{R}$ and thus a basis of $\widetilde{\mathbb{R}}^g$. This proves the theorem. □

4.2 Construction of the map $Iso(u_1) \times \widetilde{Iso}_\delta(u_0) \to Iso(u_0)$

In this section, we would like to construct a map from the Cartesian product of some finite type isospectral set $Iso(u_1)$ and some asymptotic model isospectral set $\widetilde{Iso}_\delta(u_0)$ into the isospectral set $Iso(u_0)$ of some given real-valued potential u_0. More precisely, for given $u_0 \in L^2_{\mathbb{R}}(F)$, we want to construct a map

$$\mathcal{I} : Iso(u_1) \times \widetilde{Iso}_\delta(u_0) \to Iso(u_0). \tag{4.40}$$

The desired aim would be to show that the map (4.40) can be constructed in such a way that it is a homeomorphism. We will see in this section that in the case of arbitrary generally *unbounded* isospectral sets, there occur some problems concerning the choice of some uniform $\delta > 0$ on the whole of $Iso(u_0)$ such that we can merely give a weaker result than the desired homeomorphism property just mentioned. If, however, we assume some additional boundedness condition on $Iso(u_0)$, we will finally be able to prove that there exists a homeomorphism (4.40). In this section, both the case of *unbounded* isospectral sets (Theorem 4.2.10) and the special case of isospectral sets with additional *boundedness* condition (Corollary 4.2.11) shall be treated.
To begin with, let's recap and state more precisely how the appearing objects in (4.40) are defined: $Iso(u_0)$ has been defined in (4.1) and $\widetilde{Iso}_\delta(u_0)$ has been defined in (3.4). The associated $\delta > 0$ if not stated otherwise is chosen sufficiently small due to the asymptotic analysis in Chapter 3. We note already here that in the case of unbounded isospectral sets, we will later have to deal with different values of δ since δ sensibly depends on the norm of the respective potential. The moduli of the finite type potential $u_1 \in L^2_{\mathbb{R}}(F)$ shall satisfy

$$m_\nu(u_1) = \begin{cases} m_\nu(u_0), & \nu \in \Gamma^* \setminus \Gamma^*_\delta, \\ 0, & \nu \in \Gamma^*_\delta. \end{cases} \tag{4.41}$$

So far, it's not clear whether such a potential $u_1 \in L^2_{\mathbb{R}}(F)$ exists at all. This will be proved later in Lemma 4.2.7. Clearly in general, (4.41) doesn't uniquely determine the potential u_1, yet.
We now introduce some notations. Whereas in Chapter 3, we dealt with the asymptotic part Γ^*_δ of the dual lattice, we now have to consider its complement in Γ^* as well. We denote the *finite* part of the dual lattice by

$$\Gamma^*_f := \Gamma^* \setminus \Gamma^*_\delta,$$

that is $\Gamma^* = \Gamma^*_f \cup \Gamma^*_\delta$ [13]. For $u \in L^2_{\mathbb{R}}(F)$, we decompose the sequence of the associated Fourier coefficients into finite and asymptotic part by $\hat{u} = (\hat{u}_f, \hat{u}_\delta)$

[13] Clearly, Γ^*_f depends on δ. However, the suppression of δ in this notation should not lead to confusions since we consider only one fixed δ. If we should consider different values $\delta, \delta' > 0$ at the same time, the notation will be suitably adapted to Γ^*_f and $\Gamma^*_{f'}$, respectively.

with

$$\hat{u}_f := (\hat{u}(\nu))_{\nu \in \Gamma_f^*}, \quad \hat{u}_\delta := (\hat{u}(\nu))_{\nu \in \Gamma_\delta^*}.$$

Likewise, for the perturbed Fourier coefficients (which are only defined for $\nu \in \Gamma_\delta^*$), we set $\check{u}_\delta := (\check{u}_\nu)_{\nu \in \Gamma_\delta^*}$. We would like to have the decomposition $\hat{u} = (\hat{u}_f, \hat{u}_\delta)$ for all $u \in Iso(u_0)$ with the same $\delta > 0$. If $Iso(u_0)$ is unbounded in $L^2_{\mathbb{R}}(F)$, there is no evidence that such a uniform $\delta > 0$ exists since we would have to ensure that we can choose in Theorem 2.4.2 for all $u \in Iso(u_0)$ the same δ. Since the choice of $\delta > 0$ essentially depended on the norm of u as the proof of Theorem 2.4.2 showed (the larger the norm of u gets, the smaller δ has to be chosen), cf. also the end of the proof of Corollary 2.4.4 or the discussion on p. 72 concerning the choice of δ, such a uniform choice of $\delta > 0$ doesn't seem to be possible (for a more detailed discussion of this problem, see Chapter 5). If, for some given $R > 0$, however, we restrict ourselves to $Iso(u_0) \cap \overline{B_R(u_0)}$ (as before, $B_R(u_0)$ denotes the open ball in $L^2_{\mathbb{R}}(F)$ centerd at u_0 with radius R), there exists a uniform $\delta > 0$ for all $u \in Iso(u_0) \cap \overline{B_R(u_0)}$. Clearly, this δ depends on R. If not stated otherwise, each $\delta > 0$ in the following shall be *associated to the respective given* $R > 0$ in the sense just explained.

In the discussion on p. 72, we also justified that $0 \in l^2(\Gamma_\delta^*)$ is contained in the image of the map $\hat{u} \mapsto \check{u}$ provided $\delta > 0$ is chosen sufficiently small. We will implicitly make use of this property in the following definition of the set $S_R(u_0)$. For the given $u_0 \in L^2_{\mathbb{R}}(F)$ and for $R > 0$, we define with the line segment $[0, \check{u}_{0,\delta}] := \{t \cdot \check{u}_{0,\delta} \,|\, t \in [0,1]\} \subset l^2_{\mathbb{R}}(\Gamma_\delta^*)$ this set by

$$S_R(u_0) := \{u \in \overline{B_R(u_0)} \;:\; \check{u}_\delta \in [0, \check{u}_{0,\delta}],\ \hat{u}_f = \hat{u}_{0,f}\}, \tag{4.42}$$

which can be seen as an "asymptotic line segment" in some sense. This line connects a finite type potential (sufficiently close to u_0) with the given potential u_0. Later, we will not only consider the isospectral set $Iso(u_0) \cap \overline{B_R(u_0)}$ but also isospectral sets for potentials $u \in S_R(u_0)$ along this line. This will be needed when we will identify some projection of $Iso(u_0) \cap \overline{B_R(u_0)}$ onto the finite-dimensional space spanned by the first finitely many Fourier coefficients with some finite type isospectral set. One important property we will make use of later is that $S_R(u_0)$ is compact.

In (3.3), we defined the asymptotic isospectral set $Iso_\delta(u_0)$ as a subset of $l^2_{\mathbb{R}}(\Gamma_\delta^*)$. Since it turned out that $Iso_\delta(u_0)$ could be parameterized by $l^2_{\mathbb{R}}(\Gamma_\delta^*)$-sequences, namely via the homeomorphism between $Iso_\delta(u_0)$ and $\widetilde{Iso}_\delta(u_0)$ (cf. Theorem 3.3.2), the definition (3.3) was appropriate to that former situation. In this chapter, however, we are not interested in the asymptotics alone anymore but in the parameterization of the *entire* isospectral set $Iso(u_0)$ as a subset of $L^2_{\mathbb{R}}(F)$. At first, we transfer the definition of $L^2_{\delta,u_0}(F)$ (3.2) to real-valued potentials: For $v \in L^2_{\mathbb{R}}(F)$, we set

$$L^2_{\mathbb{R},\delta,v}(F) := \{u \in L^2_{\mathbb{R}}(F) : \hat{u}(\nu) = \hat{v}(\nu) \quad \text{for all } \nu \in \Gamma^* \setminus \Gamma_\delta^*\}.$$

With this notation, we define for $u \in L^2_{\mathbb{R},\delta,v}(F)$ the following asymptotic isospectral sets

$$Iso_{\delta,v}(u) := \{w \in L^2_{\mathbb{R},\delta,v}(F) : m_\nu(w) = m_\nu(u) \text{ for all } \nu \in \Gamma^*_\delta\},$$
$$\widetilde{Iso}_{\delta,v}(u) := \{w \in L^2_{\mathbb{R},\delta,v}(F) : \widetilde{m}_\nu(w) = \widetilde{m}_\nu(u) \text{ for all } \nu \in \Gamma^*_\delta\}.$$

In order not to use same symbols for different objects, however, the former isospectral sets $Iso_\delta(u)$ and $\widetilde{Iso}_\delta(u)$ (without additional subscript v) shall still be defined as in (3.3) and (3.4), i.e. as subsets of $l^2_{\mathbb{R}}(\Gamma^*_\delta)$.
Furthermore, using the notation $2g+1 := \#\Gamma^*_f$ already well-known from Section 4.1, we introduce the vector space

$$\mathbb{C}^{2g+1}_{\mathbb{R}} := \{(x_{-g}, \ldots, x_g) \in \mathbb{C}^{2g+1} : \overline{x_j} = x_{-j} \text{ for all } j \in \{-g, \ldots, g\}\},$$

i.e. the vector space $\mathbb{C}^{2g+1}$ with "reality condition" (which is isomorphic to $\mathbb{R}^{2g+1}$ as a real vector space).
In Chapter 3, we kept the first finitely many Fourier coefficients fixed (namely equal to $\hat{u}_{0,f}$) and determined the remaining coefficients in terms of perturbed Fourier coefficients such that the respective moduli $(m_\nu(u))_{\nu\in\Gamma^*_\delta}$ were equal to the given $(m_\nu(u_0))_{\nu\in\Gamma^*_\delta}$. In that procedure, we didn't consider the first finitely many moduli. In fact, by varying the Fourier coefficients for $\nu \in \Gamma^*_\delta$, the first finitely many moduli $m_\nu(u)$, $\nu \in \Gamma^*_f$, won't remain equal to $m_\nu(u_0)$ in general.
In this chapter, we have to ensure that the moduli $m_\nu(u)$ are equal to $m_\nu(u_0)$ for *all* $\nu \in \Gamma^*$ (and not only for the asymptotic remainder). This will be done in two steps. In the first step, we determine a set (containing $Iso(u_0) \cap \overline{B_R(u_0)}$) of potentials u whose moduli $(m_\nu(u))_{\nu\in\Gamma^*_\delta}$ are equal to $(m_\nu(u_0))_{\nu\in\Gamma^*_\delta}$. In the second step, we pick out of this set those potentials u whose moduli $(m_\nu(u))_{\nu\in\Gamma^*_f}$ are also equal to $(m_\nu(u_0))_{\nu\in\Gamma^*_f}$. The following Lemma 4.2.1 realizes the first step. Before we formulate it, we make a remark on the correspondence between elements in $\mathbb{C}^{2g+1}_{\mathbb{R}}$ and elements in $L^2_{\mathbb{R}}(F)$: Given $u \in L^2_{\mathbb{R}}(F)$, we've already defined the associated $\hat{u}_f \in \mathbb{C}^{2g+1}_{\mathbb{R}}$ in the decomposition $\hat{u} = (\hat{u}_f, \hat{u}_\delta)$ introduced above. Conversely, given an element $\hat{u} \in \mathbb{C}^{2g+1}_{\mathbb{R}}$ which can be considered as an element $\hat{u} \in l^2_{\mathbb{R}}(\Gamma^*)$ by defining the elements of the sequence $\hat{u}$ indexed by $\nu \in \Gamma^*_\delta$ to be equal to zero, we associate the potential $u \in L^2_{\mathbb{R}}(F)$ by the inverse Fourier transform of $\hat{u}$. In the following, we will often implicitly make use of this correspondence. Moreover, for $u_0 \in L^2_{\mathbb{R}}(F)$, we denote by $B_R(u_0) \subset L^2_{\mathbb{R}}(F)$ and $B_R(\hat{u}_{0,f}) \subset \mathbb{C}^{2g+1}_{\mathbb{R}}$ the balls with radius $R > 0$ in the respective spaces, as usual. In order not to make the notation too confusing by using too many indices (especially in cases where it is not really necessary), we will often suppress some indices when we simply write $\hat{v} \in B_R(\hat{u}_{0,f})$ instead of $\hat{v}_f \in B_R(\hat{u}_{0,f})$ whenever it is clear from the context that $\hat{v}$ denotes an element in $\mathbb{C}^{2g+1}_{\mathbb{R}}$ (and not an element in $l^2(\Gamma^*)$). In cases where we consider both elements in $l^2(\Gamma^*)$ and in $\mathbb{C}^{2g+1}_{\mathbb{R}}$ at the same time, we clearly distinguish between $\hat{v}$ and $\hat{v}_f$ in our notation.

Lemma 4.2.1. *Let $u_0 \in L^2_{\mathbb{R}}(F)$ with smooth Fermi curve, $R > 0$ and $u \in S_R(u_0)$ (4.42). Then there exists a homeomorphism*

$$h : \overline{B_R(\hat{u}_{0,f})} \times \widetilde{Iso}_\delta(u) \to \bigcup_{\hat{v} \in \overline{B_R(\hat{u}_{0,f})}} Iso_{\delta,v}(u), \tag{4.43}$$

with the following property: For $\hat{v} \in \overline{B_R(\hat{u}_{0,f})}$ and $\widetilde{u}_\delta \in \widetilde{Iso}_\delta(u)$ [14]*, there holds $\hat{h}_f(\hat{v}, \widetilde{u}_\delta) = \hat{v}$ as well as an immediate consequence of* (4.43)

$$m_\nu(h(\hat{v}, \widetilde{u}_\delta)) = m_\nu(u) \quad \textit{for all } \nu \in \Gamma^*_\delta. \tag{4.44}$$

Moreover, there exists a natural continuous extension of h (denoted by h as well)

$$h : \overline{B_R(\hat{u}_{0,f})} \times \bigcup_{u \in S_R(u_0)} \widetilde{Iso}_\delta(u) \to \bigcup_{\substack{\hat{v} \in \overline{B_R(\hat{u}_{0,f})} \\ u \in S_R(u_0)}} Iso_{\delta,v}(u). \tag{4.45}$$

Proof. Let $u_0 \in L^2_{\mathbb{R}}(F)$ and $R > 0$. Firstly, we may assume that $\frac{|\hat{u}(0)|}{\pi^2 \mu(F) \nu^2} < \frac{1}{2}$ for all $\nu \in \Gamma^*_\delta$ and all $u \in \overline{B_R(u_0)}$ by choosing $\delta > 0$ (associated to R) small enough. This yields $\frac{1}{2} < \xi(u, \nu) < 2$ for all $\nu \in \Gamma^*_\delta$ and all $u \in \overline{B_R(u_0)}$ with $\xi(u, \nu) := \sqrt{1 + \frac{\hat{u}(0)}{\pi^2 \mu(F) \nu^2}}$ defined in (2.2).
Now, let $u \in S_R(u_0)$, $v \in \overline{B_R(u_0)}$ and $w \in L^2_{\mathbb{R},\delta,v}(F) \cap \overline{B_R(u_0)}$. We have the equivalences

$$\forall_{\nu \in \Gamma^*_\delta} : \widetilde{m}_\nu(w) = \widetilde{m}_\nu(u) \iff \frac{|\check{w}_\nu|^2}{|\nu|^2 \xi(v, \nu)} = \frac{|\check{u}_\nu|^2}{|\nu|^2 \xi(u, \nu)} \iff |\check{w}_\nu| = |\check{u}_\nu| \sqrt{\frac{\xi(v, \nu)}{\xi(u, \nu)}}.$$

Hence, the map

$$\widetilde{Iso}_\delta(u) \to \widetilde{Iso}_{\delta,v}(u), \quad \check{w} \mapsto w',$$

where $w' \in L^2_{\mathbb{R}}(F)$ is uniquely defined by $\hat{w}'_f := \hat{v}_f$ and $\check{w}'_\nu := \check{w}_\nu \sqrt{\frac{\xi(v,\nu)}{\xi(u,\nu)}}$, $\nu \in \Gamma^*_\delta$, is a homeomorphism since $\widetilde{Iso}_\delta(u)$ is obviously homeomorphic to the image of the injective map $\widetilde{Iso}_\delta(u) \to l^2_{\mathbb{R}}(\Gamma^*_\delta)$ defined by $\check{w}_\nu \mapsto \check{w}_\nu \sqrt{\frac{\xi(v,\nu)}{\xi(u,\nu)}}$, $\nu \in \Gamma^*_\delta$ (recall $\frac{1}{2} < \xi(u, \nu), \xi(v, \nu) < 2$ for all $\nu \in \Gamma^*_\delta$). But this image is homeomorphic[15] to

[14] Since the elements of $\widetilde{Iso}_\delta(u)$ are perturbed Fourier coefficients (i.e. $l^2(\Gamma^*_\delta)$-sequences) and not L^2-functions, we should actually write $\check{\widetilde{u}}_\delta$ instead of $\widetilde{u}_\delta$. In order not to make the notation too confusing, we yet write $\widetilde{u}_\delta$.

[15] Recall that we defined $\widetilde{Iso}_{\delta,v}(u)$ as a subspace of $L^2_{\mathbb{R}}(F)$. If we defined $\widetilde{Iso}_{\delta,v}(u)$ as a subspace of $l^2_{\mathbb{R}}(\Gamma^*_\delta)$ in the same way as $\widetilde{Iso}_\delta(u)$, the mentioned image would be *equal* (not only homeomorphic) to $\widetilde{Iso}_{\delta,v}(u)$.

$\widetilde{Iso}_{\delta,v}(u)$ by definition of $\widetilde{Iso}_{\delta,v}(u)$ and the form of asymptotic model isospectral sets shown in Theorem 3.1.1 (whether the first finitely many Fourier coefficients are fixed equal to $\hat{u}_{0,f}$ or $\hat{v}_f$ is immaterial in this context).
In Theorem 3.3.2, we showed (by denoting homeomorphy with the symbol $\cong$) that $\widetilde{Iso}_\delta(u) \cong Iso_\delta(u)$. The proof is the same if the first finitely many constant Fourier coefficients are equal to $\hat{v}_f$ instead of equal to $\hat{u}_{0,f}$. Therefore, $\widetilde{Iso}_{\delta,v}(u) \cong Iso_{\delta,v}(u)$ as well. Together with $\widetilde{Iso}_\delta(u) \cong \widetilde{Iso}_{\delta,v}(u)$ shown above, we thus get $\widetilde{Iso}_\delta(u) \cong Iso_{\delta,v}(u)$ for all $v \in \overline{B_R(u_0)}$ or as well (using the correspondence between $\mathbb{C}_{\mathbb{R}}^{2g+1}$ and $L^2_{\mathbb{R}}(F)$ declared above) $\widetilde{Iso}_\delta(u) \cong Iso_{\delta,v}(u)$ for all $\hat{v} \in \overline{B_R(\hat{u}_{0,f})}$. Since by definition $Iso_{\delta,v}(u) \cap Iso_{\delta,v'}(u) = \emptyset$ for $\hat{v}, \hat{v}' \in \overline{B_R(\hat{u}_{0,f})}$ with $\hat{v} \neq \hat{v}'$, this yields a bijective mapping

$$h : \overline{B_R(\hat{u}_{0,f})} \times \widetilde{Iso}_\delta(u) \to \bigcup_{\hat{v} \in \overline{B_R(\hat{u}_{0,f})}} Iso_{\delta,v}(u).$$

We have to prove that both h and its inverse h^{-1} are continuous (with respect to the usual relative topologies of $\mathbb{C}^{2g+1} \times l^2(\Gamma^*_\delta)$ and $L^2_{\mathbb{R}}(F)$). As to the continuity of h, we need to adapt the proof of Theorem 3.3.2. There, we showed that the map (in order to recall the notations, see the proof of Theorem 3.3.2 if needed)

$$\Psi : \widetilde{Iso}_\delta(u_0) \times U \to U, \quad \Psi(\check{u}, a) := \left[-1 + \sqrt{\frac{m_\nu(u_0) - r_\nu(P^{-1}((1+a)\check{u}))}{\widetilde{m}_\nu(u_0)}}\right]_{\nu \in \Gamma^*_\delta}$$

is continuous. Now, we have to show the continuity of the map

$$\Psi : \overline{B_R(\hat{u}_{0,f})} \times \widetilde{Iso}_\delta(u) \times U \to U,$$
$$(\hat{v}, \check{u}, a) \mapsto \left[-1 + \sqrt{\frac{m_\nu(u) - r_\nu(P_{\hat{v}}^{-1}((1+a)\check{u}))}{\widetilde{m}_\nu(u)}}\right]_{\nu \in \Gamma^*_\delta}, \tag{4.46}$$

with the restriction $P_{\hat{v}} := P|_{L^2_{\delta,v}(F)}$ with P as defined in (3.1). As well as the map P^{-1} that we used in Chapter 3 was well-defined as the inverse of $P|_{L^2_{\delta,u_0}(F)}$, the map $P_{\hat{v}}^{-1}$ is well-defined as the inverse of $P|_{L^2_{\delta,v}(F)}$.
In order to prove continuity of the modified map Ψ, we don't have to prove anything new since as in the proof of Theorem 3.3.2, it is ultimately the continuity of the map $u \mapsto r_\nu(u)$ (cf. Lemma 3.2.5) and the decreasing behaviour of r_ν (cf. (3.17)) which yield the desired continuity of Ψ. Now, we can copy the rest of the proof of the continuity of I in the proof of Theorem 3.3.2 and the continuity of h follows.
As to the continuity of h^{-1}, we show that $\overline{B_R(\hat{u}_{0,f})} \times \widetilde{Iso}_\delta(u)$ is compact. Due to an elementary result of calculus (cf. [10, p. 233, p. 713 (Aufgabe 158.6)], for

instance) stating that the inverse of a bijective continuous map from a compact metric space onto some other metric space is continuous, the continuity of h^{-1} then follows.
Since $\overline{B_R(\hat{u}_{0,f})} \subset \mathbb{C}_{\mathbb{R}}^{2g+1}$ is compact, it remains to show that $\widetilde{Iso}_\delta(u)$ is compact. We use the representation of $\widetilde{Iso}_\delta(u)$ from Theorem 3.1.1 in the following. First of all, we remark that $\widetilde{Iso}_\delta(u)$ is a bounded subset of $l^2_{\mathbb{R}}(\Gamma^*_\delta)$ (which immediately follows from Theorem 3.1.1). Let $\epsilon > 0$ and a sequence $(a_n)_{n\in\mathbb{N}} \subset \widetilde{Iso}_\delta(u)$ be given. We show that there exists a convergent subsequence of $(a_n)_n$. Due to the representation in Theorem 3.1.1, there is a $0 < \delta_1 < \delta$ such that $\|(a_n(\nu))_{\nu\in\Gamma^*_{\delta_1}}\|_{l^2} < \frac{\sqrt{\epsilon}}{2\sqrt{2}}$ for all $n \in \mathbb{N}$. We decompose $a_n =: (b_n, c_n)$ with $b_n := (a_n(\nu))_{\nu\in\Gamma^*_\delta\setminus\Gamma^*_{\delta_1}}$ and $c_n := (a_n(\nu))_{\nu\in\Gamma^*_{\delta_1}}$, $n \in \mathbb{N}$. Since $(b_n)_n$ is a bounded sequence in a finite-dimensional vector space (with finite dimension $\#(\Gamma^*_\delta \setminus \Gamma^*_{\delta_1})$), there exists a convergent subsequence $(b_{n_k})_k$. Hence, there exists a $K \in \mathbb{N}$ such that for all $j, k \geq K$

$$\|a_{n_k} - a_{n_j}\|^2_{l^2} = \|b_{n_k} - b_{n_j}\|^2_{l^2} + \|c_{n_k} - c_{n_j}\|^2_{l^2} \leq$$
$$\leq \|b_{n_k} - b_{n_j}\|^2_{l^2} + (\|c_{n_k}\|_{l^2} + \|c_{n_j}\|)^2 < \frac{\epsilon}{2} + \left(2\frac{\sqrt{\epsilon}}{2\sqrt{2}}\right)^2 = \epsilon$$

Thus, $(a_{n_k})_k$ is a Cauchy sequence and converges in $l^2_{\mathbb{R}}(\Gamma^*_\delta)$ because $l^2_{\mathbb{R}}(\Gamma^*_\delta)$ is a Banach space. Since $\widetilde{Iso}_\delta(u)$ is a closed subset of $l^2_{\mathbb{R}}(\Gamma^*_\delta)$ by definition (as a preimage of a closed set under a continuous map), the limit of $(a_{n_k})_k$ lies in $\widetilde{Iso}_\delta(u)$. This shows that $\widetilde{Iso}_\delta(u)$ is compact and thus proves that h is a homeomorphism.
The continuity of the extension of h to $\overline{B_R(\hat{u}_{0,f})} \times \bigcup_{u\in S_R(u_0)} \widetilde{Iso}_\delta(u)$ follows in the same manner as we proved the continuity of h above, where in addition, we also have to use the continuity of $u \mapsto m_\nu(u)$. □

The proof of Lemma 4.2.1 already implies the compactness of $Iso(u_0) \cap \overline{B_R(u_0)}$. We state this important result in the following corollary.

Corollary 4.2.2. *Let $u_0 \in L^2_{\mathbb{R}}(F)$ with smooth Fermi curve and $R > 0$. Then $Iso(u_0) \cap \overline{B_R(u_0)}$ is compact. In particular, if $Iso(u_0)$ is bounded in $L^2_{\mathbb{R}}(F)$, then $Iso(u_0)$ is compact.*

Proof. Let $R > 0$. Then $Iso(u_0) \cap \overline{B_R(u_0)}$ is contained in the image of the map h (4.43) for $u = u_0$. But this image is compact since h is a homeomorphism due to Lemma 4.2.1. Due to the closedness of $Iso(u_0)$ and since closed subsets of compact sets are compact, the compactness of $Iso(u_0) \cap \overline{B_R(u_0)}$ follows.
In particular, if $Iso(u_0)$ is bounded in $L^2_{\mathbb{R}}(F)$, choose $R > 0$ such that $Iso(u_0) \subset \overline{B_R(u_0)}$ and the compactness of $Iso(u_0)$ follows also in this case. □

In the next step, we want to show that for a convergent sequence $(u_n)_{n\in\mathbb{N}} \subset L^2_{\mathbb{R}}(F)$ with $\lim_{n\to\infty} u_n = u_0$, there is an $N \in \mathbb{N}$ such that also the set $\big(Iso(u_0) \cup \bigcup_{n\geq N} Iso(u_n)\big) \cap \overline{B_R(u_0)}$ is compact.

Lemma 4.2.3. *Let $u_0 \in L^2_{\mathbb{R}}(F)$ with smooth Fermi curve, $R > 0$ and $(u_n)_{n\in\mathbb{N}} \subset \overline{B_R(u_0)}$ be a convergent sequence with $\lim_{n\to\infty} u_n = u_0$. Then there exists an $N \in \mathbb{N}$ such that $\left(Iso(u_0) \cup \bigcup_{n\geq N} Iso(u_n)\right) \cap \overline{B_R(u_0)}$ is compact.*

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset \overline{B_R(u_0)}$ with $\lim_{n\to\infty} u_n = u_0$. In the same fashion as the map h (4.43) could be extended to (4.45), we can naturally extend h to a continuous map

$$h : \overline{B_R(\hat{u}_{0,f})} \times \left(\widetilde{Iso}_\delta(u_0) \cup \bigcup_{n\geq N} \widetilde{Iso}_\delta(u_n)\right) \to \bigcup_{\substack{\hat{v}\in\overline{B_R(\hat{u}_{0,f})} \\ n=0 \vee n\geq N}} Iso_{\delta,v}(u_n),$$

with $N \in \mathbb{N}$ sufficiently large. By construction of the map h, the image of this map contains $\left(Iso(u_0) \cup \bigcup_{n\geq N} Iso(u_n)\right) \cap \overline{B_R(u_0)}$. In order to show the compactness of the latter set, we proceed as in the proof of Lemma 4.2.1. At first, we show the compactness of $\overline{B_R(\hat{u}_{0,f})} \times \left(\widetilde{Iso}_\delta(u_0) \cup \bigcup_{n\geq N} \widetilde{Iso}_\delta(u_n)\right)$, where it suffices again to show that $\widetilde{Iso}_\delta(u_0) \cup \bigcup_{n\geq N} \widetilde{Iso}_\delta(u_n)$ is compact. As in the proof of Lemma 4.2.1, let $\epsilon > 0$ and a sequence $(a_k)_{k\in\mathbb{N}} \subset \widetilde{Iso}_\delta(u_0) \cup \bigcup_{n\geq N} \widetilde{Iso}_\delta(u_n)$ be given. If there is an $n \geq N$ or $n = 0$ such that $\widetilde{Iso}_\delta(u_n)$ contains infinitely many elements of $(a_k)_{k\in\mathbb{N}}$, the sequence has a convergent subsequence due to the compactness of $\widetilde{Iso}_\delta(u_n)$. So consider the other case that for $n = 0$ and for all $n \geq N$, $\widetilde{Iso}_\delta(u_n)$ contains at most finitely many elements of the sequence $(a_k)_{k\in\mathbb{N}}$. Then there exist sequences $(k_j)_{j\in\mathbb{N}} \subseteq \mathbb{N}$ and $(n_j)_{j\in\mathbb{N}} \subseteq \mathbb{N}$ such that $a_{k_j} \in \widetilde{Iso}_\delta(u_{n_j})$ for all $j \in \mathbb{N}$. Due to the representation of $\widetilde{Iso}_\delta(u_n)$ in Theorem 3.1.1, there is a sequence $(t_\nu)_\nu$ with $t_\nu \in [0, 2\pi)$ such that $a_{k_j,\nu} = e^{it_\nu}\check{u}_{n_j,\nu}$. Furthermore, there is a $0 < \delta_1 < \delta$ such that $\|\check{u}_0\|_{l^2(\Gamma^*_{\delta_1})} < \frac{\sqrt{\epsilon}}{4\sqrt{2}}$. Due to $\lim_{n\to\infty} u_n = u_0$, we have $\|\check{u}_n - \check{u}_0\|_{l^2(\Gamma^*_\delta)} \to 0$ as $n \to \infty$ and hence, $\|\check{u}_{n_j} - \check{u}_0\|_{l^2(\Gamma^*_{\delta_1})} < \frac{\sqrt{\epsilon}}{4\sqrt{2}}$ for $j \geq J$ with $J \in \mathbb{N}$ sufficiently large. Therefore,

$$\|a_{k_j}\|_{l^2(\Gamma^*_{\delta_1})} \leq \|\check{u}_{n_j} - \check{u}_0\|_{l^2(\Gamma^*_{\delta_1})} + \|\check{u}_0\|_{l^2(\Gamma^*_{\delta_1})} < \frac{\sqrt{\epsilon}}{2\sqrt{2}} \quad \text{for } j \geq J.$$

From this on, we can virtually copy the end of the proof of Lemma 4.2.1 yielding that the sequence $(a_{k_j})_{j\geq J}$ has a convergent subsequence. For sake of completeness, we mention the essential steps one more time: We decompose $a_{k_j} =: (b_{k_j}, c_{k_j})$ with $b_{k_j} := (a_{k_j,\nu})_{\nu\in\Gamma^*_\delta\setminus\Gamma^*_{\delta_1}}$ and $c_{k_j} := (a_{k_j,\nu})_{\nu\in\Gamma^*_{\delta_1}}$, $j \geq J$. Since $(b_{k_j})_j$ is a bounded sequence in a finite-dimensional vector space, there exists a convergent subsequence, without restriction $(b_{k_j})_j$ itself. Hence, there exists a $K \geq J$ such that for all $j, l \geq K$

$$\begin{aligned} \|a_{k_j} - a_{k_l}\|^2_{l^2} &= \|b_{k_j} - b_{k_l}\|^2_{l^2} + \|c_{k_j} - c_{k_l}\|^2_{l^2} \leq \\ &\leq \|b_{k_j} - b_{k_l}\|^2_{l^2} + (\|c_{k_j}\|_{l^2} + \|c_{k_l}\|)^2 < \frac{\epsilon}{2} + \left(2\frac{\sqrt{\epsilon}}{2\sqrt{2}}\right)^2 = \epsilon \end{aligned}$$

yielding that $(a_k)_k$ has a convergent subsequence (compare the end of the proof of Lemma 4.2.1). Hence, both $\overline{B_R(\hat{u}_{0,f})} \times \left(\widetilde{Iso}_\delta(u_0) \cup \bigcup_{n\geq N} \widetilde{Iso}_\delta(u_n)\right)$ and consequently also its image $\bigcup_{\substack{\hat{v}\in\overline{B_R(\hat{u}_{0,f})} \\ n=0\,\vee\, n\geq N}} Iso_{\delta,v}(u_n)$ under the continuous map h are compact. Since closed subsets of compact sets are compact and $\left(Iso(u_0) \cup \bigcup_{n\geq N} Iso(u_n)\right) \cap \overline{B_R(u_0)} \subset \bigcup_{\substack{\hat{v}\in\overline{B_R(\hat{u}_{0,f})} \\ n=0\,\vee\, n\geq N}} Iso_{\delta,v}(u_n)$, the compactness of the set $\left(Iso(u_0) \cup \bigcup_{n\geq N} Iso(u_n)\right) \cap \overline{B_R(u_0)}$ follows by proving that it is closed. To show the closedness, let a convergent sequence $(v_k)_{k\in\mathbb{N}} \subset \left(Iso(u_0) \cup \bigcup_{n\geq N} Iso(u_n)\right) \cap \overline{B_R(u_0)}$ with limit $v \in \overline{B_R(u_0)}$ be given. If there is an $n \geq N$ or $n = 0$ such that infinitely many elements of $(v_k)_k$ lie in $Iso(u_n) \cap \overline{B_R(u_0)}$, this defines a subsequence $(v_{k_j})_j \subset Iso(u_n) \cap \overline{B_R(u_0)}$ which converges to v (due to $\lim_{k\to\infty} v_k = v$). Since $Iso(u_n) \cap \overline{B_R(u_0)}$ is closed (recall that $Iso(u_n)$ is defined as the preimage of the closed set $\{m(u_n)\}$ under the continuous moduli map), $v \in Iso(u_n) \cap \overline{B_R(u_0)}$ follows. If otherwise, each $Iso(u_n)\cap\overline{B_R(u_0)}$ contains at most finitely elements of the sequence $(v_k)_k$, there exist subsequences $(u_{n_j})_{j\in\mathbb{N}}$ and $(v_{k_j})_{j\in\mathbb{N}}$ with $v_{k_j} \in Iso(u_{n_j})\cap\overline{B_R(u_0)}$ for all $j \in \mathbb{N}$. Due to continuity, it follows $m(v) = \lim_{j\to\infty} m(v_{k_j}) = \lim_{j\to\infty} m(u_{n_j}) = m(u_0)$, hence $v \in Iso(u_0) \cap \overline{B_R(u_0)}$. This shows the desired closedness and the lemma is thus proved. □

The next lemma shows that $Iso(u_0) \cap \overline{B_R(u_0)}$ can be uniformly approximated by isopectral sets $Iso(u) \cap \overline{B_R(u_0)}$ provided that $u \in L^2_{\mathbb{R}}(F)$ is in a sufficiently small neighbourhood of u_0.

Lemma 4.2.4. *Let $u_0 \in L^2_{\mathbb{R}}(F)$ with smooth Fermi curve and $R > 0$. Then*

$$\forall_{\epsilon>0}\, \exists_{\eta>0}\, \forall_{u\in B_\eta(u_0)}\, \forall_{v\in Iso(u)\cap\overline{B_R(u_0)}} : \operatorname{dist}(v, Iso(u_0) \cap \overline{B_R(u_0)}) < \epsilon,$$

where $\operatorname{dist}(v, Iso(u_0) \cap \overline{B_R(u_0)}) := \min_{x\in Iso(u_0)\cap\overline{B_R(u_0)}} \|x - v\|_{L^2(F)}$, $v \in L^2_{\mathbb{R}}(F)$.

Proof. Let $\epsilon > 0$ and assume that the assertion to be proved doesn't hold, i.e.

$$\forall_{\eta>0}\, \exists_{u\in B_\eta(u_0)}\, \exists_{v\in Iso(u)\cap\overline{B_R(u_0)}} : \operatorname{dist}(v, Iso(u_0) \cap \overline{B_R(u_0)}) \geq \epsilon.$$

Hence, for all $n \in \mathbb{N}$, there is a $u_n \in B_{1/n}(u_0)$ (which yields a convergent sequence $(u_n)_{n\in\mathbb{N}} \subset \overline{B_R(u_0)}$ converging to u_0) and a $v_n \in Iso(u_n) \cap \overline{B_R(u_0)}$ such that $\operatorname{dist}(v_n, Iso(u_0) \cap \overline{B_R(u_0)}) \geq \epsilon$. Due to Lemma 4.2.3, there is an $N \in \mathbb{N}$ such that $\left(Iso(u_0) \cup \bigcup_{n\geq N} Iso(u_n)\right) \cap \overline{B_R(u_0)}$ is compact. Hence, we may assume without loss of generality that the sequence $(v_n)_{n\geq N}$ converges to some $v \in \overline{B_R(u_0)}$ (otherwise, consider a convergent subsequence). Due to continuity of the moduli, we have $m(v) = \lim_{n\to\infty} m(v_n) = \lim_{n\to\infty} m(u_n) = m(u_0)$. This yields $m(v) = m(u_0)$, and hence $v \in Iso(u_0) \cap \overline{B_R(u_0)}$. This, however, is a contradiction to $\operatorname{dist}(v, Iso(u_0) \cap \overline{B_R(u_0)}) \geq \epsilon$ and the assertion follows. □

With the notation $m_f(u) := (m_\nu(u))_{\nu\in\Gamma_f^*}$, the space $\widetilde{\mathbb{R}}^g$ as in Theorem 4.1.11, the map h (4.45) and $\overline{B_R(\hat{u}_{0,f})} \subset \mathbb{C}_{\mathbb{R}}^{2g+1}$ as in Lemma 4.2.1, we define the map

$$\phi : \overline{B_R(\hat{u}_{0,f})} \times \bigcup_{u\in S_R(u_0)} \widetilde{Iso}_\delta(u) \to \widetilde{\mathbb{R}}^g, \quad (\hat{v}, \widetilde{u}_\delta) \mapsto m_f(h(\hat{v}, \widetilde{u}_\delta)). \tag{4.47}$$

Recall that $m_\nu(h(\hat{v}, \widetilde{u}_\delta)) = m_\nu(u)$ for all $\nu \in \Gamma_\delta^*$, $u \in S_R(u_0)$ by (4.44), i.e. the moduli indexed by $\nu \in \Gamma_\delta^*$ are already correct[16]. In a next step, we would like to achieve the same for the moduli m_f, i.e. the moduli indexed by $\nu \in \Gamma_f^*$.
We use the notation $\phi' := \frac{\partial\phi}{\partial\hat{v}}$ for the partial derivative with respect to the first argument of ϕ. The aim is to apply the Implicit Function Theorem to ϕ. At first, we show that ϕ' has full rank:

Lemma 4.2.5. *Let $u_0 \in L_{\mathbb{R}}^2(F)$ with smooth Fermi curve and $R > 0$. Let further $\widetilde{u}_\delta \in \bigcup_{u\in S_R(u_0)} \widetilde{Iso}_\delta(u)$, $\hat{v} \in \overline{B_R(\hat{u}_{0,f})}$ be given and denote $\phi' := \phi'(\hat{v}, \widetilde{u}_\delta)$. Then the linear map*

$$\phi' : \mathbb{C}_{\mathbb{R}}^{2g+1} \to \widetilde{\mathbb{R}}^g, \quad \hat{w} \mapsto \left(\frac{dm_f(h(\hat{v}, \widetilde{u}_\delta))}{d\hat{v}}\right)\hat{w}$$

has full rank g.

Proof. As several times before, we use again the local isomorphism $L_{\mathbb{R}}^2(F) \to l_{\mathbb{R}}^2(\Gamma^*)$, $u \mapsto (\hat{u}_f, \check{u}_\delta)$ (cf. Theorem 2.4.2) which locally allows us to identify $L_{\mathbb{R}}^2(F)$-potentials with a decomposition into finitely many Fourier coefficients and infinitely many perturbed Fourier coefficients in the asymptotic remainder. If for $u \in S_R(u_0)$, we compose h (4.43) with this map, we get a map denoted by $\widetilde{h}$:

$$\widetilde{h} : \overline{B_R(\hat{u}_{0,f})} \times \widetilde{Iso}_\delta(u) \to l_{\mathbb{R}}^2(\Gamma^*), \quad (\hat{v}, \widetilde{u}_\delta) \mapsto (\hat{h}_f(\hat{v}, \widetilde{u}_\delta), \check{h}_\delta(\hat{v}, \widetilde{u}_\delta)) = (\hat{v}, \check{u}'_\delta), \tag{4.48}$$

where we recall that by definition of h, we have $\hat{h}_f(\hat{v}, \widetilde{u}_\delta) = \hat{v}$ and $\check{u}'_\delta \in l_{\mathbb{R}}^2(\Gamma_\delta^*)$ is defined (as well by definition of h) as

$$\check{u}'_{\delta,\nu} := (1 + a_\nu) \cdot \widetilde{u}_{\delta,\nu} \cdot \sqrt{\frac{\xi(v, \nu)}{\xi(u, \nu)}}, \quad \nu \in \Gamma_\delta^*, \tag{4.49}$$

see (3.55) to recall the definition of a_ν and the mapping between $\widetilde{Iso}_\delta(u)$ and $Iso_\delta(u)$ as well as the construction of h in the proof of Lemma 4.2.1 to recall the terms under the square root. We recall that with the notation

$$L_{\mathbb{R},f}^2(F) := \{v \in L_{\mathbb{R}}^2(F) : \hat{v}(\nu) = 0 \text{ for all } \nu \in \Gamma_\delta^*\} \cong \mathbb{R}^{2g+1},$$

[16]Of course, we are especially interested in the case $u = u_0$.

the map

$$dm_f|_u : L^2_{\mathbb{R},f}(F) \to \widetilde{\mathbb{R}}^g$$

(with slightly different notations compared to Theorem 4.1.11) is onto due to Theorem 4.1.11. By using an analogous matrix representation of $dm_f|_u$ as in the proof of Theorem 4.1.10, we thus get

$$\frac{dm_f(h(\hat{v},\widetilde{u}_\delta))}{d\hat{v}} = \frac{dm_f(u)}{d\hat{u}}|_{\hat{u}=(\hat{v},\hat{u}'_\delta)} \cdot \frac{d(\hat{v},\hat{u}'_\delta)}{d\hat{v}} = \begin{pmatrix} A & B \end{pmatrix} \cdot \begin{pmatrix} \mathbb{1} \\ * \end{pmatrix}, \tag{4.50}$$

where due to Theorem 4.1.11, A is a $g \times (2g+1)$-matrix with full rank g, B is a $g \times \infty$-matrix and $\mathbb{1}$ is the $(2g+1) \times (2g+1)$ unity matrix. By definition, the columns of $(A\ B)$ are the images of the Schauder basis $\mathcal{B} := \{\hat{u}_\kappa\}_{\kappa \in \Gamma^*}$ defined by $\hat{u}_\kappa(\nu) = \delta_{\kappa,\nu}$ for $\kappa, \nu \in \Gamma^*$ under the map $dm_f|_u$. Denote $\mathcal{B}_f := \{\hat{u}_\kappa\}_{\kappa \in \Gamma^*_f}$. The matrix $(A\ B)$ can be transformed into the matrix $(A\ 0)$ by elementary column transformations, more precisely by adding to each column of B a suitable linear combination of the $2g+1$ columns of A such that all entries of the considered column of B are then equal to zero. This is possible due to the full rank of A. The matrix $(A\ 0)$ obtained this way is thus the representation matrix of $dm_f|_u$ with respect to the Schauder basis $\mathcal{B}' := \mathcal{B}_f \cup \mathcal{B}'_\delta$, where the elements of $\mathcal{B}'_\delta$ are exactly the basis vectors $\hat{u}_\kappa$, $\kappa \in \Gamma^*_\delta$ plus a suitable linear combination of elements in $\mathcal{B}_f$ (namely that linear combination we had to add in order to make the columns of B equal to zero). Note that the first $2g+1$ elements of both $\mathcal{B}$ and $\mathcal{B}'$ are identical, namely equal to the elements of $\mathcal{B}_f$. We thus get that $\frac{dm_f(h(\hat{v},\widetilde{u}_\delta))}{d\hat{v}}$ has the same rank as $(A\ 0) \cdot \binom{\mathbb{1}}{*} = A$. Since A has full rank g, the lemma is proved. □

Concerning the smoothness properties of ϕ, we also prove the following statement which we will need later.

Lemma 4.2.6. *The map ϕ (4.47) is smooth. In particular, for given $\widetilde{u}_\delta \in \bigcup_{u \in S_R(u_0)} \widetilde{Iso}_\delta(u)$, the derivative function $\hat{v} \mapsto \phi'(\hat{v},\widetilde{u}_\delta)$ is continuous.*

Proof. By the chain rule (compare also (4.50)), we have to show that both $\frac{dm_f(u)}{d\hat{u}}|_{\hat{u}=(\hat{v},\hat{u}'_\delta)}$ and the term $\frac{d(\hat{v},\hat{u}'_\delta)}{d(\hat{v},\widetilde{u}_\delta)}$ are smooth in $(\hat{v},\widetilde{u}_\delta)$. The smoothness of $\frac{dm_f(u)}{d\hat{u}}|_{\hat{u}=(\hat{v},\hat{u}'_\delta)}$, however, follows from Lemma 4.1.8. It remains to prove the smoothness of $(\hat{v},\widetilde{u}_\delta) \mapsto \frac{d(\hat{v},\hat{u}'_\delta)}{d(\hat{v},\widetilde{u}_\delta)}$. Due to (4.49), it suffices to prove the smoothness of the fixed point a of the fixed point equation $\Psi(\hat{v},\widetilde{u}_\delta, a(\hat{v},\widetilde{u}_\delta)) = a(\hat{v},\widetilde{u}_\delta)$ with Ψ as in (4.46). Deriving this equation yields

$$\frac{da(\hat{v},\widetilde{u}_\delta)}{d(\hat{v},\widetilde{u}_\delta)} = \frac{\partial\Psi}{\partial(\hat{v},\widetilde{u}_\delta)}(\hat{v},\widetilde{u}_\delta, a(\hat{v},\widetilde{u}_\delta)) + \frac{\partial\Psi}{\partial a}(\hat{v},\widetilde{u}_\delta, a(\hat{v},\widetilde{u}_\delta)) \cdot \frac{da(\hat{v},\widetilde{u}_\delta)}{d(\hat{v},\widetilde{u}_\delta)}.$$

$$\Longrightarrow \frac{da(\hat{v},\widetilde{u}_\delta)}{d(\hat{v},\widetilde{u}_\delta)} = \left(\mathbb{1} - \frac{\partial\Psi}{\partial a}(\hat{v},\widetilde{u}_\delta, a(\hat{v},\widetilde{u}_\delta))\right)^{-1} \frac{\partial\Psi}{\partial(\hat{v},\widetilde{u}_\delta)}(\hat{v},\widetilde{u}_\delta, a(\hat{v},\widetilde{u}_\delta)).$$

All operators in this equation are well-defined due to the smoothness property of Ψ (which is an implication of Lemma 3.2.5) and the contraction property of Ψ (see Theorem 3.2.8) yielding that $\mathbb{1} - \frac{\partial \Psi}{\partial a}$ is sufficiently close to $\mathbb{1}$ and hence invertible. The smoothness of $(\hat{v}, \widetilde{u}_\delta) \mapsto \frac{da(\hat{v},\widetilde{u}_\delta)}{d(\hat{v},\widetilde{u}_\delta)}$ now follows from the mentioned smoothness of Ψ. The lemma is proved. □

The property proven in Lemma 4.2.5 that ϕ' has full rank allows us now to prove the existence of potentials $u_1 \in L^2_{\mathbb{R}}(F)$ satisfying (4.41). We prove a slightly more general result in the following lemma.

Lemma 4.2.7. *Let $u_0 \in L^2_{\mathbb{R}}(F)$ with smooth Fermi curve. Then there exists an $N \in \mathbb{N}$ and an associated sequence of finite type potentials $(u_n)_{n \geq N} \subset L^2_{\mathbb{R}}(F)$ with $\lim_{n\to\infty} u_n = u_0$ satisfying the following condition: For all $n \geq N$,*

$$m_\nu(u_n) = \begin{cases} m_\nu(u_0), & \nu \in \Gamma^*, |\nu| \leq n \\ 0, & \nu \in \Gamma^*, |\nu| > n. \end{cases}$$

Proof. Let $u_0 \in L^2_{\mathbb{R}}(F)$ with associated $\delta' > 0$ and $\Gamma^*_{f'} = \Gamma^* \setminus \Gamma^*_{\delta'}$ as defined at the beginning of this section. We decorate δ' with a prime since this δ' is only preliminary because the actual $0 < \delta < \delta'$ will be chosen possibly smaller (for $\delta > 0$, we will then consider $\Gamma^*_f = \Gamma^* \setminus \Gamma^*_\delta$). Due to Theorem 4.1.11, $\frac{dm_{f'}}{d\hat{u}_{f'}}|_{u_0}$ has full rank. Therefore, $\left\| \frac{dm_{f'}}{d\hat{u}_{f'}}|_{u_0} \right\| =: c > 0$. For all $0 < \delta < \delta'$, we have $\Gamma^*_{f'} \subset \Gamma^*_f$ and $\Gamma^*_\delta \subset \Gamma^*_{\delta'}$. Hence for all $0 < \delta < \delta'$, due to

$$\begin{aligned} \left\| \frac{dm_{f'}}{d\hat{u}_{f'}}|_{u_0} \right\| &= \sup \left\{ \left\| \frac{dm_{f'}}{d\hat{u}}|_{u_0}(v) \right\| : \|\hat{v}\|_{l^2(\Gamma^*)} = 1 \wedge \hat{v}(\nu) = 0 \text{ for } \nu \in \Gamma^*_{\delta'} \right\} \leq \\ &\leq \sup \left\{ \left\| \frac{dm_{f'}}{d\hat{u}}|_{u_0}(v) \right\| : \|\hat{v}\|_{l^2(\Gamma^*)} = 1 \wedge \hat{v}(\nu) = 0 \text{ for } \nu \in \Gamma^*_{\delta} \right\} \leq \\ &\leq \sup \left\{ \left\| \frac{dm_f}{d\hat{u}}|_{u_0}(v) \right\| : \|\hat{v}\|_{l^2(\Gamma^*)} = 1 \wedge \hat{v}(\nu) = 0 \text{ for } \nu \in \Gamma^*_{\delta} \right\} = \left\| \frac{dm_f}{d\hat{u}_f}|_{u_0} \right\|, \end{aligned} \tag{4.51}$$

there holds $\left\| \frac{dm_f}{d\hat{u}_f}|_{u_0} \right\| \geq c$. We use again the notation $m_{\delta'} := (m_\nu)_{\nu \in \Gamma^*_{\delta'}}$ and likewise $\widetilde{m}_{\delta'}$ and $r_{\delta'}$ for the model moduli and the error term $r_{\delta'}(\cdot) = m_{\delta'}(\cdot) - \widetilde{m}_{\delta'}(\cdot)$, respectively. Due to Lemma 3.2.6 and (4.39), we have

$$\begin{aligned} \left\| \frac{dm_{\delta'}}{du}|_u \right\| &= \left\| \frac{d\widetilde{m}_{\delta'}}{du}|_u + \frac{dr_{\delta'}}{du}|_u \right\| \leq \left\| \frac{d\widetilde{m}_{\delta'}}{du}|_u \right\| + \left\| \frac{dr_{\delta'}}{du}|_u \right\| = \\ &= \tfrac{1}{|\nu|^2} O(\|(\check{u}_\nu)_\nu\|_{l^2(\Gamma^*_{\delta'})}) + \tfrac{1}{|\nu|^2} o(\|(\check{u}_\nu)_\nu\|_{l^2(\Gamma^*_{\delta'})}) = o(1), \quad \text{as } |\nu| \to \infty, \end{aligned}$$

locally uniform, i.e. uniform for all $u \in U'$, where $U' \subset L^2_{\mathbb{R}}(F)$ is a sufficiently small neighbourhood of u_0. Hence, we may choose $\delta' > 0$ small enough[17] such that for all $u \in U'$, there holds

$$\left\| \frac{dm_{\delta'}}{d\hat{u}}|_u \right\| \leq \frac{c}{8} = \frac{1}{8} \left\| \frac{dm_{f'}}{d\hat{u}_{f'}}|_{u_0} \right\|. \tag{4.52}$$

Now choose $0 < \delta < \delta'$ sufficiently small such that for $w \in L^2_{\mathbb{R}}(F)$ defined by $\hat{w}_f = \hat{u}_{0,f}$ and $\check{w}_\delta = 0$, there holds $w \in U'$ and

$$\left\| \frac{dm_{f'}}{d\hat{u}_f}|_w - \frac{dm_{f'}}{d\hat{u}_f}|_{u_0} \right\| \leq \left\| \frac{dm_{f'}}{d\hat{u}}|_w - \frac{dm_{f'}}{d\hat{u}}|_{u_0} \right\| \leq \frac{c}{4} \overset{(4.51)}{\leq} \frac{1}{4} \left\| \frac{dm_f}{d\hat{u}_f}|_{u_0} \right\|, \tag{4.53}$$

where the second "≤" in (4.53) holds due to continuity for $\delta > 0$ sufficiently small, cf. Lemma 4.1.8. We would like to have an analogous estimate with dm_f instead of $dm_{f'}$ on the left hand side of (4.53). With (4.52) and (4.53), we get

$$\begin{aligned} \left\| \frac{dm_f}{d\hat{u}_f}|_w - \frac{dm_f}{d\hat{u}_f}|_{u_0} \right\| &\leq \left\| \frac{dm_{f'}}{d\hat{u}_f}|_w - \frac{dm_{f'}}{d\hat{u}_f}|_{u_0} \right\| + \left\| \left(\frac{dm_\nu}{d\hat{u}_f}|_w - \frac{dm_\nu}{d\hat{u}_f}|_{u_0} \right)_{\delta'^{-1} < |\nu| \leq \delta^{-1}} \right\| \leq \\ &\leq \frac{1}{4} \left\| \frac{dm_f}{d\hat{u}_f}|_{u_0} \right\| + \left\| \frac{dm_{\delta'}}{d\hat{u}}|_w \right\| + \left\| \frac{dm_{\delta'}}{d\hat{u}}|_{u_0} \right\| \leq \frac{1}{4} \left\| \frac{dm_f}{d\hat{u}_f}|_{u_0} \right\| + \frac{c}{8} + \frac{c}{8} \leq \\ &\leq \frac{1}{4} \left\| \frac{dm_f}{d\hat{u}_f}|_{u_0} \right\| + \frac{1}{4} \left\| \frac{dm_{f'}}{d\hat{u}_{f'}}|_{u_0} \right\| \overset{(4.51)}{\leq} \frac{1}{2} \left\| \frac{dm_f}{d\hat{u}_f}|_{u_0} \right\| \end{aligned} \tag{4.54}$$

as desired. We now apply the Implicit Function Theorem (cf. [23, p. 144], for instance) to the equation

$$\phi(\hat{v}, \widetilde{u}_\delta) = m_f(h(\hat{v}, \widetilde{u}_\delta)) = m_f(u_0). \tag{4.55}$$

This equation is obviously fulfilled for the pair $(\hat{v}, \widetilde{u}_\delta) = (\hat{u}_{0,f}, \check{u}_{0,\delta})$. Due to Lemma 4.2.5, $\phi'(\hat{u}_{0,f}, \check{u}_{0,\delta})$ has full rank. Moreover, by Lemma 4.2.6, ϕ is smooth. Hence, due to the Implicit Function Theorem, there is a neighbourhood U of $\check{u}_{0,\delta}$ in $l^2(\Gamma^*_\delta) \cap \widetilde{Iso}_\delta(u_0)$, a neighbourhood V of $\hat{u}_{0,f}$ in $\overline{B_R(\hat{u}_{0,f})}$, as well as a continuous mapping $U \to V$, $\widetilde{u}_\delta \mapsto \hat{v}(\widetilde{u}_\delta)$ such that the tuple consisting of $\widetilde{u}_\delta \in U$ and $\hat{v} = \hat{v}(\widetilde{u}_\delta)$ satisfies (4.55). Due to (4.54)[18], also $\widetilde{u}_\delta := 0 \in U$ holds. Hence,

[17] More precisely, we choose a priori another $0 < \delta'' < \delta'$ such that with the above definition $c := \left\| \frac{dm_{f'}}{d\hat{u}_{f'}}|_{u_0} \right\|$, we get $\left\| \frac{dm_{\delta''}}{d\hat{u}}|_u \right\| \leq \frac{c}{8}$ in (4.52). Since $\left\| \frac{dm_{f''}}{d\hat{u}_{f''}}|_{u_0} \right\| \geq c$ as explained before, we get $\left\| \frac{dm_{\delta''}}{d\hat{u}}|_u \right\| \leq \frac{1}{8} \left\| \frac{dm_{f''}}{d\hat{u}_{f''}}|_{u_0} \right\|$ so that an additional choice of some $0 < \delta'' < \delta'$ is not necessary if $\delta' > 0$ is chosen small enough.

[18] In the proof of the Inverse Function Theorem (which is used in the proof of the Implicit Function Theorem), cf. [23, p. 142-145], the inequality corresponding to (4.54) shows how large the neighbourhood where invertibility holds can be chosen. In our case, we chose $\delta > 0$ sufficiently small such that the element $(\widehat{v(0)}, 0)$ we are interested in is contained in $V \times U$. By the way, the same argument doesn't only hold for $\widetilde{u}_\delta = 0$ but also for all $\widetilde{u}_\delta \in [0, \check{u}_{0,\delta}]$. We will revisit this fact in the proof of Lemma 4.2.8.

$m_f(h(\hat{v}(0),0)) = m_f(u_0)$ and the potential $v \in L^2_{\mathbb{R}}(F)$ associated to $(\hat{v}(0),0)$ thus fulfills $m_\nu(v) = m_\nu(u_0)$ for $\nu \in \Gamma^*_f$ and $m_\nu(v) = 0$ for $\nu \in \Gamma^*_\delta$. In this fashion we can choose for each $n \in \mathbb{N}$ a $\delta = \delta_n > 0$ which yields neighbourhoods $U' = U'_n$ where (4.54) holds for all $w \in U'_n$. As well, we get neighbourhoods $U = U_n$ and $V = V_n$ as above in the application of the Implicit Function Theorem and a sequence $(u_n)_{n\geq N}$ ($N \in \mathbb{N}$ sufficiently large) satisfying the required properties of the lemma. Since by construction, U'_n can be chosen such that its diameter tends to zero as $n \to \infty$, we have $\lim_{n\to\infty} u_n = u_0$. This proves the lemma. □

For $u_0 \in L^2_{\mathbb{R}}(F)$, $R > 0$, $\widetilde{u}_\delta \in \bigcup_{u\in S_R(u_0)} \widetilde{Iso}_\delta(u)$ and ϕ as in (4.47), we now define the level set

$$\mathcal{L}_R(\widetilde{u}_\delta) := \{\hat{v} \in \overline{B_R(\hat{u}_{0,f})} : \phi(\hat{v}, \widetilde{u}_\delta) = m_f(u_0)\}. \tag{4.56}$$

We are especially interested in the cases $\widetilde{u}_\delta = \check{u}_{0,\delta}$ and $\widetilde{u}_\delta = 0$. Whereas the first case corresponds to the isospectral set $Iso(u_0)$ since $\mathcal{L}_R(\check{u}_{0,\delta})$ encodes the moduli equal to $m(u_0)$, the second case corresponds to the finite type isospectral set $Iso(u_1)$ defined by (4.41) since $\mathcal{L}_R(0)$ encodes those moduli in (4.41). This correspondence will be specified more precisely in our next investigations. We would like to show that these level sets are homeomorphic to one another by constructing a homeomorphism $\mathcal{L}_R(0) \to \mathcal{L}_R(\check{u}_{0,\delta})$. If $Iso(u_0)$ is unbounded, however, there occurs a problem caused by the intersection of the isospectral set with the ball $\overline{B_R(u_0)}$. By constructing the map $\mathcal{L}_R(0) \to \mathcal{L}_R(\check{u}_{0,\delta})$, we will see that there is a natural way to identify two level sets of the form (4.56) with each other. There is, however, no evidence why elements in $\partial B_R(\hat{u}_{0,f}) \cap \mathcal{L}_R(0)$ should be mapped into $\partial B_R(\hat{u}_{0,f})$ since $Iso(u_0)$ has no symmetries with respect to such balls $B_R(\hat{u}_{0,f})$ in general. An element in $\partial B_R(\hat{u}_{0,f}) \cap \mathcal{L}_R(0)$ might also be mapped into $B_R(\hat{u}_{0,f})$ or into $\mathbb{C}^{2g+1}_{\mathbb{R}} \setminus \overline{B_R(\hat{u}_{0,f})}$, where the latter case would be contrary to the well-definition of $\mathcal{L}_R(0) \to \mathcal{L}_R(\check{u}_{0,\delta})$. Furthermore, the mapping behaviour of elements in $\partial B_R(\hat{u}_{0,f}) \cap \mathcal{L}_R(0)$ causes some problems concerning the question whether this map is open. These are the reasons why the following Lemma 4.2.8 requires a slightly more elaborate formulation than just stating that there exists a homeomorphism $\mathcal{L}_R(0) \to \mathcal{L}_R(\check{u}_{0,\delta})$.

Lemma 4.2.8. *Let $u_0 \in L^2_{\mathbb{R}}(F)$ with smooth Fermi curve, $R > 0$ and $\epsilon > 0$ be given. Then there is a $\delta > 0$ (depending on R and ϵ) such that the following holds:*

(i) $\mathcal{L}_{R+2\epsilon}(\widetilde{u}_\delta) \neq \emptyset$ *for all* $\widetilde{u}_\delta \in \bigcup_{u\in S_{R+2\epsilon}(u_0)} \widetilde{Iso}_\delta(u)$ *and*

(ii) for all $\widetilde{u}_\delta \in \widetilde{Iso}_\delta(u_0)$, there exists a continuous and injective map $\mathcal{L}_{R-\epsilon}(0) \to \mathcal{L}_R(\widetilde{u}_\delta)$ whose image contains $\mathcal{L}_{R-2\epsilon}(\widetilde{u}_\delta)$ and whose natural restriction $\mathring{\mathcal{L}}_{R-\epsilon}(0) \to \mathcal{L}_R(\widetilde{u}_\delta)$ is an open map.
Here, $\mathring{\mathcal{L}}_{R-\epsilon}(0) := \{\hat{v} \in B_{R-\epsilon}(\hat{u}_{0,f}) : \phi(\hat{v}, 0) = m_f(u_0)\}$ denotes the interior of $\mathcal{L}_{R-\epsilon}(0)$.

Remark. Clearly, one is interested in small $\epsilon > 0$ as possible. The optimal choice would be $\epsilon = 0$ which, however, won't be feasible in general. If indeed the choice $\epsilon = 0$ is admissible, then the assertion of the lemma simply states that there exists a homeomorphism $\mathcal{L}_R(0) \to \mathcal{L}_R(\widetilde{u}_\delta)$, $\widetilde{u}_\delta \in \widetilde{Iso}_\delta(u_0)$.

Proof. Let $u_0 \in L^2_{\mathbb{R}}(F)$, $R > 0$ and $\epsilon > 0$ be given. To begin with, we want to declare how $\delta > 0$ has to be chosen. This is done in several steps. In the first step, we choose a preliminary $\delta > 0$ which might have to be chosen even smaller in the subsequent steps: Firstly, we choose as before a (preliminary) $\delta > 0$ associated[19] to the radius $R + 2\epsilon$ [20], where we choose δ small enough such that $\|\check{u}_0\|_{l^2(\Gamma^*_\delta)} < \epsilon/16$. Then, we choose in Lemma 4.2.4 an $\eta > 0$ associated to $\epsilon/4$, i.e.

$$\operatorname{dist}(v, Iso(u_0) \cap \overline{B_{R+2\epsilon}(u_0)}) < \epsilon/4 \quad \text{for all } v \in Iso(u) \cap \overline{B_{R+2\epsilon}(u_0)},\ u \in B_\eta(u_0). \tag{4.57}$$

Here, we may choose $\eta > 0$ small enough such that $\|\check{u}\|_{l^2(\Gamma^*_\delta)} \leq 2\|\check{u}_0\|_{l^2(\Gamma^*_\delta)} < \epsilon/8$ for all $u \in B_\eta(u_0)$. This is possible since the map $u \mapsto \check{u}_\delta$ is continuous, cf. Theorem 2.4.2. Due to this Theorem 2.4.2 together with (3.57), we even have $\|\hat{v}\|_{l^2(\Gamma^*_\delta)} \leq 2\|\check{u}\|_{l^2(\Gamma^*_\delta)} < \epsilon/4$ for all $v \in Iso(u) \cap \overline{B_{R+2\epsilon}(u_0)}$ and all $u \in B_\eta(u_0)$, provided that the product of the error term $1 + O(1/|\nu|)$ in (3.57) and the error term $1 + o(1)$ between the $l^2(\Gamma^*_\delta)$-norm of Fourier coefficients and the $l^2(\Gamma^*_\delta)$-norm of perturbed Fourier coefficients is smaller than 2 which can clearly be achieved by choosing δ accordingly. Now choose $\delta > 0$ small enough such that $u \in B_\eta(u_0)$ for all $u \in S_{R+2\epsilon}(u_0) = \{u \in \overline{B_{R+2\epsilon}(u_0)} \ : \ \check{u}_\delta \in [0, \check{u}_{0,\delta}],\ \hat{u}_f = \hat{u}_{0,f}\}$, cf. (4.42). If we define for $u \in S_{R+2\epsilon}(u_0)$ an element $m^* = (m^*_\nu)_\nu \in l^1(\Gamma^*_r)$ by $m^*_\nu := m_\nu(u_0)$ for $\nu \in \Gamma^* \setminus \Gamma^*_\delta$ and $m^*_\nu := m_\nu(u)$ for $\nu \in \Gamma^*_\delta$, we would like to find a $u^* \in B_\eta(u_0)$ such that $m(u^*) = m^*$. We claim that such a potential u^* exists for all $u \in S_{R+2\epsilon}(u_0)$ (provided $\delta > 0$ is sufficiently small yielding that m^* is sufficiently close to $m(u_0)$ in the l^1-norm). Thereto, we have to take a look into the proof of Lemma 4.2.7: If $\widetilde{u}_\delta := 0 \in U$ (with U the neighbourhood in the proof of Lemma 4.2.7), then an arbitrary $\widetilde{u}_\delta \in [0, \check{u}_{0,\delta}]$ is contained in U a fortiori and the existence of $u^* \in B_\eta(u_0)$ follows by the same arguments as in the proof of Lemma 4.2.7.
Finally, the choice of $\delta > 0$ guarantees together with (4.57) and $\|\hat{v}\|_{l^2(\Gamma^*_\delta)} < \epsilon/4$ for all $v \in Iso(u) \cap \overline{B_{R+2\epsilon}(u_0)}$ and all $u \in B_\eta(u_0)$ as explained above, that by definition of the level sets (4.56), we have

$$\operatorname{dist}(x, \mathcal{L}_{R+2\epsilon}(\check{u}_{0,\delta})) < \frac{\epsilon}{2} \quad \text{for all } x \in \mathcal{L}_{R+2\epsilon}(\widetilde{u}_\delta),\ \widetilde{u}_\delta \in \bigcup_{u \in S_{R+2\epsilon}(u_0)} \widetilde{Iso}_\delta(u), \tag{4.58}$$

[19]Cf. the discussion of the choice of $\delta > 0$ for given $R > 0$ on p. 131.

[20]The choice of δ associated to $R + 2\epsilon$ (and not associated to R) has technical reasons and will become clear later in the proof when we will prove the openness property.

where $\operatorname{dist}(x, \mathcal{L}_{R+2\epsilon}(\check{u}_{0,\delta})) := \inf_{y\in\mathcal{L}_{R+2\epsilon}(\check{u}_{0,\delta})} \|x-y\|$ with $\|\cdot\|$ the euclidean norm on $\mathbb{C}_{\mathbb{R}}^{2g+1}$. In short, all level sets relevant for our considerations have a distance less than ϵ to one another and a distance less than $\epsilon/2$ to the level set $\mathcal{L}_{R+2\epsilon}(\check{u}_{0,\delta})$. Moreover, the existence of the above $u^* \in B_\eta(u_0)$ (associated to $u \in S_{R+2\epsilon}(u_0)$) fulfilling $m(u^*) = m^*$ showed that $\mathcal{L}_{R+2\epsilon}(\widetilde{u}_\delta) \neq \emptyset$ for all $\widetilde{u}_\delta \in \bigcup_{u\in S_{R+2\epsilon}(u_0)} \widetilde{Iso}_\delta(u)$. Now that we have chosen $\delta > 0$, let $\widetilde{u}_{1,\delta} \in \bigcup_{u\in S_{R+2\epsilon}(u_0)} \widetilde{Iso}_\delta(u)$ and $\hat{v} \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$ be given. It is $\mathbb{C}_{\mathbb{R}}^{2g+1} = (\ker\phi'(\hat{v},\widetilde{u}_{1,\delta})) \oplus (\ker\phi'(\hat{v},\widetilde{u}_{1,\delta}))^\perp$, where the orthogonal complement is taken with respect to the euclidean standard scalar product on $\mathbb{C}_{\mathbb{R}}^{2g+1}$. Since by Lemma 4.2.5, the rank of $\phi'(\hat{v},\widetilde{u}_{1,\delta})$ equals g, we have

$$\begin{aligned}\dim_{\mathbb{R}}(\ker\phi'(\hat{v},\widetilde{u}_{1,\delta}))^\perp &= 2g+1-\dim_{\mathbb{R}}(\ker\phi'(\hat{v},\widetilde{u}_{1,\delta})) = \\ &= 2g+1-(2g+1-g) = g.\end{aligned}$$

Let $N_1(\hat{v}),\ldots,N_g(\hat{v}) \in \mathbb{C}_{\mathbb{R}}^{2g+1}$ be a basis of $(\ker\phi'(\hat{v},\widetilde{u}_{1,\delta}))^\perp$. Without restriction, we may assume that the $N_i(\hat{v})$ are all normalized by $\|N_i(\hat{v})\| = 1$ $(i = 1,\ldots,g)$. Each $\hat{n} \in (\ker\phi'(\hat{v},\widetilde{u}_{1,\delta}))^\perp$ can thus be represented as $\hat{n} = \sum_{i=1}^g \lambda_i N_i(\hat{v})$ with respective coefficient vector $\lambda := (\lambda_1,\ldots,\lambda_g) \in \mathbb{R}^g$. Now, together with the smoothness of the map ϕ proved in Lemma 4.2.6, the level set $\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$ is a real smooth submanifold (in general with boundary) of $\mathbb{C}_{\mathbb{R}}^{2g+1}$ of dimension $\dim\ker\phi'(\hat{v},\widetilde{u}_{1,\delta})) = g+1$ by the Regular Value Theorem (cf. [23, p. 154], for instance) because for all $\hat{v} \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$, $\phi'(\hat{v},\widetilde{u}_{1,\delta})$ has full rank g due to Lemma 4.2.5 (i.e. $m_f(u_0)$ is a so-called *regular value*). Therefore, the normal spaces spanned by the vectors $N_1(\hat{v}),\ldots,N_g(\hat{v})$ continuously depend on $\hat{v} \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$. We now define the map

$$\Phi : \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta}) \times U \to \widetilde{\mathbb{R}}^g, \quad (\hat{v},\lambda) \mapsto \phi\left(\hat{v} + \sum_{i=1}^g \lambda_i N_i(\hat{v}), \widetilde{u}_{1,\delta}\right), \tag{4.59}$$

where $0 \in U \subseteq \mathbb{R}^g$ is an open set such that $\hat{v} + \sum_{i=1}^g \lambda_i N_i(\hat{v}) \in \overline{B_{R+2\epsilon}(\hat{u}_{0,f})}$ for all $\hat{v} \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$ and all $\lambda \in U$, i.e. such that Φ is well-defined. Note that $\hat{v} \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$, i.e. Φ is also well-defined if $\hat{v} \in \partial B_{R+\epsilon}(\hat{u}_{0,f})$. In particular, the neighbourhood U does not depend on $\hat{v}$. We now consider the partial derivative $\frac{\partial}{\partial\lambda}\Phi$ evaluated at $\lambda = 0$. The linear map $\frac{\partial}{\partial\lambda}\Phi|_{\lambda=0}$ can be identified with a $g\times g$-matrix. By definition of ϕ and respecting the definition of h as well as (4.48), we have

$$\begin{aligned}\tfrac{\partial}{\partial\lambda}\Phi|_{\lambda=0} &= \frac{dm_f(h(\hat{v}+\sum_{i=1}^g \lambda_i N_i(\hat{v}),\widetilde{u}_{1,\delta}))}{d\lambda}|_{\lambda=0} = \\ &= \frac{dm_f(u)}{d\hat{u}}|_{\hat{u}=(\hat{v},\hat{u}'_{1,\delta})} \cdot \frac{d(\hat{v}+\sum_{i=1}^g \lambda_i N_i(\hat{v}),\hat{u}'_{1,\delta})}{d\lambda}|_{\lambda=0} =: \begin{pmatrix} A & B\end{pmatrix}\cdot\begin{pmatrix} N \\ * \end{pmatrix},\end{aligned}$$

where A, B are essentially the same blocks as in the proof of Lemma 4.2.5 and N is the $(2g+1)\times g$-matrix with columns $N_1(\hat{v}),\ldots,N_g(\hat{v})$. Now we proceed

exactly as in the proof of Lemma 4.2.5. By a suitable choice of a Schauder basis of $l^2_{\mathbb{R}}(\Gamma^*)$, the block B can be assumed to be equal to zero so that $\frac{\partial}{\partial\lambda}\Phi|_{\lambda=0}$ has the same rank as $(A\ 0)\binom{N}{*} = (AN_1(\hat{v}), \dots, AN_g(\hat{v}))$. With respect to this basis, we have $\phi'(\hat{v}, \widetilde{u}_{1,\delta}) = A$ (cf. the proof of Lemma 4.2.5). By definition of $N_1(\hat{v}), \dots, N_g(\hat{v})$, the vectors $AN_1(\hat{v}), \dots, AN_g(\hat{v})$ are linearly independent because $0 = \sum_{i=1}^g \lambda_i AN_i(\hat{v}) = A(\sum_{i=1}^g \lambda_i N_i(\hat{v}))$ implies $\sum_{i=1}^g \lambda_i N_i(\hat{v}) \in (\ker A) \cap (\ker A)^\perp = \{0\}$ such that $\lambda = 0$ due to the linear independence of $N_1(\hat{v}), \dots, N_g(\hat{v})$.
Hence, the rank of $\frac{\partial}{\partial\lambda}\Phi|_{\lambda=0}$ is equal to g. This is an important result which will be used in the next step when we apply the Implicit Function Theorem. For $\hat{v} \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$, we want to find local solutions $(\hat{w}, \lambda, \widetilde{u}_\delta)$ of the equation

$$\phi\left(\hat{w} + \sum_{i=1}^g \lambda_i N_i(\hat{w}), \widetilde{u}_\delta\right) = m_f(u_0), \tag{4.60}$$

where $\hat{w} \in W(\hat{v})$ (with $W(\hat{v}) \subset \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$ a sufficiently small neighbourhood of $\hat{v}$), $\lambda \in U$ and $\widetilde{u}_\delta \in \bigcup_{u \in S_{R+2\epsilon}(u_0)} \widetilde{Iso}_\delta(u)$ (in particular, *local* means that λ shall be in a neighbourhood of $0 \in U$ and $\widetilde{u}_\delta$ shall be in a neighbourhood of $\widetilde{u}_{1,\delta}$). Since $\phi(\hat{v}, \widetilde{u}_{1,\delta}) = m_f(u_0)$ (cf. the definition of the level sets (4.56)), the triple consisting of $\hat{w} = \hat{v}$, $\lambda = 0$, $\widetilde{u}_\delta = \widetilde{u}_{1,\delta}$ solves (4.60). Because $\frac{\partial}{\partial\lambda}\Phi|_{\lambda=0}$ is invertible, we may apply the Implicit Function Theorem (cf. [23, p. 144], for instance). This yields that there exists a neighbourhood $V(\hat{v}, \widetilde{u}_{1,\delta}) \subset \bigcup_{u \in S_{R+2\epsilon}(u_0)} \widetilde{Iso}_\delta(u)$ of $\widetilde{u}_{1,\delta}$ and a neighbourhood $W(\hat{v}) \subset \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$ of $\hat{v}$ as well as a unique continuous map

$$W(\hat{v}) \times V(\hat{v}, \widetilde{u}_{1,\delta}) \to U, \quad (\hat{w}, \widetilde{u}_\delta) \mapsto \lambda(\hat{w}, \widetilde{u}_\delta) \tag{4.61}$$

such that

$$\phi\left(\hat{w} + \sum_{i=1}^g \lambda_i(\hat{w}, \widetilde{u}_\delta) N_i(\hat{w}), \widetilde{u}_\delta\right) = m_f(u_0) \quad \text{for all } (\hat{w}, \widetilde{u}_\delta) \in W(\hat{v}) \times V(\hat{v}, \widetilde{u}_{1,\delta}).$$

This defines for each $\hat{v} \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$ and $\widetilde{u}_\delta \in V(\hat{v}, \widetilde{u}_{1,\delta})$ a *local* map

$$W(\hat{v}) \subset \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta}) \to \mathcal{L}_{R+2\epsilon}(\widetilde{u}_\delta), \quad \hat{w} \mapsto \hat{w} + \sum_{i=1}^g \lambda_i(\hat{w}, \widetilde{u}_\delta) N_i(\hat{w}). \tag{4.62}$$

Due to the continuity of $(\hat{w}, \widetilde{u}_\delta) \mapsto \lambda(\hat{w}, \widetilde{u}_\delta)$ and since we can choose continuous normal vectors $\hat{w} \mapsto N_1(\hat{w}), \dots, N_g(\hat{w})$ (due to the continuous dependence of the normal spaces $(\ker \phi'(\hat{v}, \widetilde{u}_{1,\delta}))^\perp$ on $\hat{v} \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$ as explained before), these local maps (4.62) are in particular continuous in their respective domain of definition. We would like to extend these local maps to a *global* map defined on the whole of $\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$.

At first, we argue why the neighbourhoods $V(\hat{v}, \widetilde{u}_{1,\delta})$ can even be chosen independently of $\hat{v}$ by using the compactness of $\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$: Thereto, recall that $\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$ is compact since due to the continuity of ϕ, $\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$ is a closed subset of the compact set $\overline{B_{R+\epsilon}(\hat{u}_{0,f})} \subset \mathbb{C}_{\mathbb{R}}^{2g+1}$ and thus also compact. Now, $\bigcup_{\hat{v} \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})} W(\hat{v})$ is an open covering of $\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$. Since $\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$ is compact, $\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$ can be covered by finitely many of these neighbourhoods, say by $W(\hat{v}_i)$, $i = 1, \ldots, m$. To these, there correspond finitely many neighbourhoods $V(\hat{v}_i, \widetilde{u}_{1,\delta})$, $i = 1, \ldots, m$. Let $d_1, \ldots, d_m$ be the diameters of the $V(\hat{v}_i, \widetilde{u}_{1,\delta})$, $i = 1, \ldots, m$. Choose $\min\{d_1, \ldots, d_m\}$ as diameter of a neighbourhood of $\widetilde{u}_{1,\delta}$ in $\bigcup_{u \in S_{R+2\epsilon}(u_0)} \widetilde{Iso}_\delta(u)$. This defines $V(\widetilde{u}_{1,\delta})$ which is independent of $\hat{v} \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$.
In order to extend the local maps (4.62) to a global map $\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta}) \to \mathcal{L}_{R+2\epsilon}(\widetilde{u}_\delta)$, we have to show that for $\hat{v}, \hat{v}' \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$ with $W(\hat{v}) \cap W(\hat{v}') \neq \emptyset$, the respective maps (4.62) are identical on $W(\hat{v}) \cap W(\hat{v}')$. This, however, follows immediately from the uniqueness of the map $\hat{w} \mapsto \hat{n}(\hat{w}) := \sum_{i=1}^g \lambda_i(\hat{w}, \widetilde{u}_\delta) N_i(\hat{w})$ obtained by the above application of the Implicit Function Theorem. In other words, $\sum_{i=1}^g \lambda_i(\hat{w}, \widetilde{u}_\delta) N_i(\hat{w}) = \sum_{i=1}^g \lambda_i'(\hat{w}, \widetilde{u}_\delta) N_i'(\hat{w})$ for all $\hat{w} \in W(\hat{v}) \cap W(\hat{v}')$, where the $\lambda_i'(\hat{w}, \widetilde{u}_\delta)$ are the coefficients corresponding to the (maybe different) basis vectors $N_1'(\hat{w}), \ldots, N_g'(\hat{w})$ of $(\ker \phi'(\hat{w}, \widetilde{u}_{1,\delta}))^\perp$ in the local map (4.62) defined on $W(\hat{v}')$.
Hence, for all $\widetilde{u}_\delta \in V(\widetilde{u}_{1,\delta})$, the global map

$$\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta}) \to \mathcal{L}_{R+2\epsilon}(\widetilde{u}_\delta), \quad \hat{v} \mapsto \hat{v} + \sum_{i=1}^g \lambda_i(\hat{v}, \widetilde{u}_\delta) N_i(\hat{v}) \tag{4.63}$$

is well-defined. Note that the image of (4.63) is contained in $\overline{B_{R+2\epsilon}(\hat{u}_{0,f})}$ due to (4.58) which explains the index $R + 2\epsilon$ in $\mathcal{L}_{R+2\epsilon}(\widetilde{u}_\delta)$. Moreover, the map (4.63) is continuous since continuity is a local property which has already been justified for the local map (4.62).
Next, we prove that (4.63) is one-to-one. In order to establish this, the diameter of $V(\widetilde{u}_{1,\delta})$ has possibly to be chosen even smaller than we already did. We prove at first that for a sequence $(\widetilde{u}_\delta^n)_{n\in\mathbb{N}} \subset V(\widetilde{u}_{1,\delta})$ with $\lim_{n\to\infty} \widetilde{u}_\delta^n = \widetilde{u}_{1,\delta}$ (in the $l^2(\Gamma_\delta^*)$-norm), there holds

$$\forall_{\epsilon'>0}\, \exists_{k\in\mathbb{N}}\, \forall_{n\geq k}\, \forall_{\hat{v}\in\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})} : \|\hat{n}(\hat{v}, \widetilde{u}_\delta^n)\| < \epsilon', \tag{4.64}$$

with $\|\cdot\|$ the euclidean norm on $\mathbb{C}_{\mathbb{R}}^{2g+1}$ and $\hat{n}(\hat{v}, \widetilde{u}_\delta) := \sum_{i=1}^g \lambda_i(\hat{v}, \widetilde{u}_\delta) N_i(\hat{v})$ as already introduced. As before, we sometimes simply write $\hat{n}(\hat{v})$ if the dependence on $\widetilde{u}_\delta$ is immaterial for the respective consideration. Let $\epsilon' > 0$. We recall that we may assume that all appearing basis vectors $N_i(\hat{v})$ are normalized by $\|N_i(\hat{v})\| = 1$. Choose the neighbourhood $U \subset \mathbb{R}^g$ of $0 \in \mathbb{R}^g$ (cf. (4.61), for instance) small enough such that $\sum_{i=1}^g |\lambda_i| < \epsilon'$ for all $\lambda = (\lambda_1, \ldots, \lambda_g) \in U$. Since $\lambda(\hat{v}, \widetilde{u}_{1,\delta}) = 0$ for all $\hat{v} \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$ and due to the continuity of $(\hat{v}, \widetilde{u}_\delta) \mapsto \lambda(\hat{v}, \widetilde{u}_\delta)$, cf. (4.61), which is even uniform with respect to $\hat{v}$ because $\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$ is compact, it follows $\|\hat{n}(\hat{v}, \widetilde{u}_\delta^n)\| \leq \sum_{i=1}^g |\lambda_i(\hat{v}, \widetilde{u}_\delta^n)| < \epsilon'$ for all $\hat{v} \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$ and for all $n \in \mathbb{N}$

sufficiently large. This proves (4.64).
For $\hat{v} \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$ and $\epsilon' > 0$, we introduce the notation

$$\widetilde{B}_{\epsilon'}(\hat{v}) := B_{\epsilon'}(\hat{v}) \cap \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta}), \tag{4.65}$$

where $B_{\epsilon'}(\hat{v})$ is the ϵ'-ball in $\mathbb{C}^{2g+1}_{\mathbb{R}}$ as usual. By definition, any $\hat{n}(\hat{v}, \widetilde{u}_\delta)$ is perpendicular to any tangent vector of the tangent space $\ker \phi'(\hat{v}, \widetilde{u}_\delta)$ on $\hat{v}$. For $\epsilon' > 0$ sufficiently small and $\hat{w} \in \widetilde{B}_{\epsilon'}(\hat{v})$, $\hat{v} - \hat{w}$ is sufficiently close to such a tangent vector such that the affine subspaces $\hat{v} + (\ker \phi'(\hat{v}, \widetilde{u}_{1,\delta}))^\perp$ and $\hat{w} + (\ker \phi'(\hat{w}, \widetilde{u}_{1,\delta}))^\perp$ are "nearly parallel" to each other. More precisely, we claim that there holds

$$\forall_{\hat{v} \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})} \exists_{\epsilon'>0} \exists_{k \in \mathbb{N}} \forall_{n \geq k} : \hat{w}_1 + \hat{n}(\hat{w}_1, \widetilde{u}^n_\delta) \neq \hat{w}_2 + \hat{n}(\hat{w}_2, \widetilde{u}^n_\delta) \tag{4.66}$$

for all $\hat{w}_1, \hat{w}_2 \in \widetilde{B}_{\epsilon'}(\hat{v})$ with $\hat{w}_1 \neq \hat{w}_2$. In order to prove (4.66), we need to assure that the *curvature* of $\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$ remains locally bounded. What we mean by this becomes clear in the following. Let $\hat{v} \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$ be given. For $\hat{w} \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$ (in a neighbourhood of $\hat{v}$), $0 \neq \hat{t} \in \ker \phi'(\hat{w}, \widetilde{u}_{1,\delta})$, $0 \neq \hat{n} \in (\ker \phi'(\hat{w}, \widetilde{u}_{1,\delta}))^\perp$, we introduce the following notations: Let $E_{\hat{w},\hat{t},\hat{n}} := \hat{w} + span\{\hat{t}, \hat{n}\} \subset \mathbb{C}^{2g+1}_{\mathbb{R}}$ be the two-dimensional (real) affine subspace of $\mathbb{C}^{2g+1}_{\mathbb{R}}$ spanned by $\hat{t}$ and $\hat{n}$ in the point $\hat{w}$. The intersection $\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta}) \cap E_{\hat{w},\hat{t},\hat{n}}$ can thus be locally parameterized by a curve $\gamma = \gamma_{\hat{w},\hat{t},\hat{n}}$ parameterized by $(-\varepsilon, \varepsilon) \to E_{\hat{w},\hat{t},\hat{n}}$, $s \mapsto \gamma(s)$ (with some $\varepsilon > 0$ depending on $\hat{w}, \hat{t}, \hat{n}$) with $\gamma(0) = \hat{w}$, i.e. locally, $\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta}) \cap E_{\hat{w},\hat{t},\hat{n}}$ equals $\mathrm{supp}(\gamma)$. Without restriction, we may consider γ in arc-length parameterization. The curvature $\kappa(s) = \kappa_{\hat{w},\hat{t},\hat{n}}(s)$ (cf. [17, 2.8], for instance) of $\gamma(s)$ is then given by $\kappa(s) := \|\ddot{\gamma}(s)\|$, where we denote derivatives with respect to s with dots. Due to the smoothness of ϕ and $\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$, cf. Lemma 4.2.6 (implying in particular the continuity of the second partial derivatives of ϕ), we get

$$\exists_{\kappa_0>0} \exists_{\epsilon'>0} \forall_{\hat{w} \in \widetilde{B}_{\epsilon'}(\hat{v})} \forall_{0 \neq \hat{t} \in \ker \phi'(\hat{w}, \widetilde{u}_{1,\delta})} \forall_{0 \neq \hat{n} \in (\ker \phi'(\hat{w}, \widetilde{u}_{1,\delta}))^\perp} : \kappa_{\hat{w},\hat{t},\hat{n}}(s) < \kappa_0$$

for all s in the respective domain of definition of $\gamma_{\hat{w},\hat{t},\hat{n}}$. Therefore, there is a radius $r_0 := 1/\kappa_0$ such that for each $\hat{w} \in \widetilde{B}_{\epsilon'}(\hat{v})$ and each $0 \neq \hat{t} \in \ker \phi'(\hat{w}, \widetilde{u}_{1,\delta})$, $0 \neq \hat{n} \in (\ker \phi'(\hat{w}, \widetilde{u}_{1,\delta}))^\perp$, the radius r of the osculating circle $B_r \subset E_{\hat{w},\hat{t},\hat{n}}$ (cf. [17, 2.7], for instance) at $\hat{w} \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta}) \cap E_{\hat{w},\hat{t},\hat{n}}$ satisfies $r \geq r_0$ and $\overline{B_r} \cap \mathrm{supp}(\gamma_{\hat{w},\hat{t},\hat{n}}) = \{\hat{w}\}$. Hence, for all $\hat{w}_1, \hat{w}_2 \in \widetilde{B}_{\epsilon'}(\hat{v})$ and $\hat{n}_1 \in \ker \phi'(\hat{w}_1, \widetilde{u}_{1,\delta})^\perp$, $\hat{n}_2 \in \ker \phi'(\hat{w}_2, \widetilde{u}_{1,\delta})^\perp$, there holds: If $\hat{w}_1 + \hat{n}_1 = \hat{w}_2 + \hat{n}_2$, then necessarily $\|\hat{n}_i\| \geq r_0$ for $i = 1, 2$. This situation is depicted in Figure 4.1, where the point A corresponds to $\hat{w}_1$ and the point B corresponds to $\hat{w}_2$. If we choose $k \in \mathbb{N}$ in (4.64) large enough such that $\|\hat{n}(\hat{v}, \widetilde{u}^n_\delta)\| < r_0/2$ for all $n \geq k$ and all $\hat{v} \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$, the assertion (4.66) follows.
On the other hand, we get as well by (4.64) with the compactness of $\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$

$$\forall_{\hat{v} \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})} \exists_{\epsilon'>0} \exists_{k \in \mathbb{N}} \forall_{n \geq k} : \hat{w}_1 + \hat{n}(\hat{w}_1, \widetilde{u}^n_\delta) \neq \hat{w}_2 + \hat{n}(\hat{w}_2, \widetilde{u}^n_\delta) \tag{4.67}$$

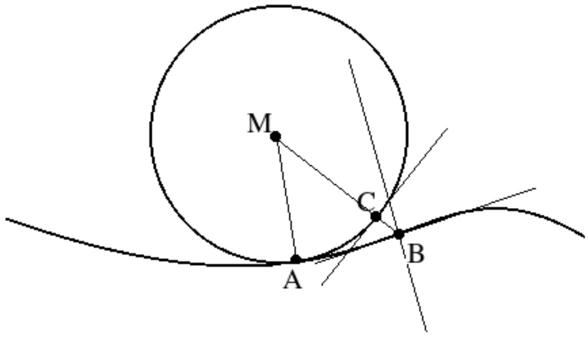

Figure 4.1: Concerning the curvature of $\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$

for all $\hat{w}_1 \in \widetilde{B}_{\epsilon'/2}(\hat{v})$, $\hat{w}_2 \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta}) \setminus \widetilde{B}_{\epsilon'}(\hat{v})$. Summing up (4.66) and (4.67) yields

$$\forall_{\hat{v}\in\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})}\, \exists_{\epsilon'>0}\, \exists_{k\in\mathbb{N}}\, \forall_{n\geq k} : \hat{w}_1 + \hat{n}(\hat{w}_1, \widetilde{u}^n_\delta) \neq \hat{w}_2 + \hat{n}(\hat{w}_2, \widetilde{u}^n_\delta) \tag{4.68}$$

for all $\hat{w}_1, \hat{w}_2 \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$ satisfying either $\hat{w}_1, \hat{w}_2 \in \widetilde{B}_{\epsilon'}(\hat{v})$ with $\hat{w}_1 \neq \hat{w}_2$ or $\hat{w}_1 \in \widetilde{B}_{\epsilon'/2}(\hat{v})$, $\hat{w}_2 \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta}) \setminus \widetilde{B}_{\epsilon'}(\hat{v})$.
In order to point out that the respective ϵ' depends on $\hat{v}$, we write $\epsilon' = \epsilon'(\hat{v})$. Now, $\bigcup_{\hat{v}\in\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})} \widetilde{B}_{\epsilon'(\hat{v})/2}(\hat{v})$ is an open covering of the compact set $\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$. Hence, there exists a finite subcover denoted by $\bigcup_{i=1}^m \widetilde{B}_{\epsilon'_i/2}(\hat{v}_i)$ and integers $k_1, \ldots, k_m$ corresponding to $\hat{v}_1, \ldots, \hat{v}_m$ according to (4.68). Now let $\hat{w}_1, \hat{w}_2 \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$ with $\hat{w}_1 \neq \hat{w}_2$ be given. Then there is an $i \in \{1, \ldots, m\}$ such that $\hat{w}_1 \in \widetilde{B}_{\epsilon'_i/2}(\hat{v}_i)$. Now $\hat{w}_2 \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta}) \setminus \widetilde{B}_{\epsilon'_i}(\hat{v})$ or $\hat{w}_2 \in \widetilde{B}_{\epsilon'_i}(\hat{v})$. For both possibilities, $\hat{w}_1 + \hat{n}(\hat{w}_1, \widetilde{u}^k_\delta) \neq \hat{w}_2 + \hat{n}(\hat{w}_2, \widetilde{u}^k_\delta)$ holds due to (4.68) with $k := \max\{k_1, \ldots, k_m\}$. This proves that (4.63) with $\widetilde{u}_\delta := \widetilde{u}^k_\delta$ is one-to-one. Clearly, this also holds if we restrict (4.63) to $\overline{B_{R-\epsilon}(\hat{u}_{0,f})}$, i.e. if we replace in (4.63) $R+\epsilon$ by $R-\epsilon$ and $R+2\epsilon$ by R [21]. Moreover, due to (4.64), we may assume without restriction that for all $\hat{v} \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$, the coefficient vector $\lambda(\hat{v}, \widetilde{u}^k_\delta) = (\lambda_1(\hat{v}, \widetilde{u}^k_\delta), \ldots, \lambda_g(\hat{v}, \widetilde{u}^k_\delta)) \in \mathbb{R}^g$ appearing in $\hat{n}(\hat{v}, \widetilde{u}^k_\delta) = \sum_{i=1}^g \lambda_i(\hat{v}, \widetilde{u}^k_\delta) N_i(\hat{v})$ satisfies $\lambda(\hat{v}, \widetilde{u}^k_\delta) \in U$ with U as in (4.61) (otherwise, choose k larger once again).
With the above choice $\widetilde{u}_\delta = \widetilde{u}^k_\delta$, we now prove that the restriction of (4.63) to the (open) ball $B_{R-\epsilon}(\hat{u}_{0,f})$, i.e.

$$\mathring{\mathcal{L}}_{R-\epsilon}(\widetilde{u}_{1,\delta}) \to \mathring{\mathcal{L}}_R(\widetilde{u}_\delta), \quad \hat{v} \mapsto \hat{v} + \sum_{i=1}^g \lambda_i(\hat{v}, \widetilde{u}_\delta) N_i(\hat{v}),$$

[21] The reason why we need injectivity on the slightly larger set $\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$ will become clear in the now following proof of openness.

is an open map[22]. Let's temporarily denote the map (4.63) by φ. Let $O \subseteq \mathring{\mathcal{L}}_{R-\epsilon}(\widetilde{u}_{1,\delta})$ be open. We have to prove that $\varphi(O)$ is open in $\mathring{\mathcal{L}}_R(\widetilde{u}_\delta)$. Let $\hat{w} \in \varphi(O)$ and the corresponding[23] $\hat{v} \in O$ with $\varphi(\hat{v}) = \hat{w}$ be given. In order to prove the mentioned openness, we show that there is an $\eta > 0$ such that the ball $\widetilde{B}_\eta(\hat{w}) \subset \mathring{\mathcal{L}}_R(\widetilde{u}_\delta)$ (cf. the notation (4.65)) is contained in $\varphi(O)$.
At first, choose $\eta' > 0$ such that $\overline{B_{\eta'}(\hat{w})} \subset B_R(\hat{u}_{0,f})$. Let $\hat{w}' \in \widetilde{B}_{\eta'}(\hat{w})$ be given. We consider the map

$$\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta}) \to \mathbb{R}_0^+, \quad \hat{x} \mapsto \|\hat{x} - \hat{w}'\|.$$

Due to the continuity of the euclidean norm on $\mathbb{C}_{\mathbb{R}}^{2g+1}$ and the compactness of $\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$, this map has a global minimum $\hat{v}' \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$. Due to (4.58), we even have $\hat{v}' \in \mathring{\mathcal{L}}_{R+\epsilon}(\widetilde{u}_{1,\delta})$. Let $\hat{n} := \hat{w}' - \hat{v}'$. Due to the minimum property of $\hat{v}' \in \mathring{\mathcal{L}}_{R+\epsilon}(\widetilde{u}_{1,\delta})$, we have $\hat{n} \in (\ker \phi'(\hat{v}', \widetilde{u}_{1,\delta}))^\perp$ (since otherwise, there would be an $\hat{x} \in \ker \phi'(\hat{v}', \widetilde{u}_{1,\delta})$ such that the line through $\hat{v}'$ with direction $\hat{x}$ would transversely intersect the ball $B_{|\hat{n}|}(\hat{w}')$ and we would thus find a $\hat{y} \in \mathring{\mathcal{L}}_{R+\epsilon}(\widetilde{u}_{1,\delta})$ with $\|\hat{y} - \hat{w}'\| < \|\hat{v}' - \hat{w}'\|$, a contradiction to the minimum property of $\hat{v}'$). This yields the representation $\hat{w}' = \hat{v}' + \sum_{i=1}^g \lambda_i N_i(\hat{v}')$ with suitable coefficients $\lambda_1, \dots, \lambda_g \in \mathbb{R}$ and the normal vectors $N_1(\hat{v}'), \dots, N_g(\hat{v}')$ as before. Because $\hat{w}' \in \mathring{\mathcal{L}}_R(\widetilde{u}_\delta)$, we thus have the equation

$$\phi\left(\hat{v}' + \sum_{i=1}^g \lambda_i N_i(\hat{v}'), \widetilde{u}_\delta\right) = m_f(u_0).$$

By the choice of $k \in \mathbb{N}$ in the above definition $\widetilde{u}_\delta = \widetilde{u}_\delta^k$, we have $\lambda \in U$ (with U as in (4.61)). Hence, due to the uniqueness of the map $(\hat{v}, \widetilde{u}_\delta) \mapsto \lambda(\hat{v}, \widetilde{u}_\delta)$ obtained by the above application of the Implicit Function Theorem, we get $\lambda_i = \lambda_i(\hat{v}', \widetilde{u}_\delta)$ for all $i = 1, \dots, g$. This shows $\hat{w}' = \hat{v}' + \sum_{i=1}^g \lambda_i(\hat{v}', \widetilde{u}_\delta) N_i(\hat{v}') = \varphi(\hat{v}')$. Hence, the image $\varphi(\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta}))$ contains the ball $\widetilde{B}_{\eta'}(\hat{w})$. In order to prove that also the image $\varphi(O)$ of O contains $\widetilde{B}_\eta(\hat{w})$ with some $0 < \eta \le \eta'$, we show that

$$\exists_{0<\eta\le\eta'} \forall_{\hat{v}' \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})} \forall_{\hat{w}' \in \widetilde{B}_\eta(\hat{w})} : \varphi(\hat{v}') = \hat{w}' \Longrightarrow \hat{v}' \in O. \tag{4.69}$$

The statement (4.69) and $\widetilde{B}_\eta(\hat{w}) \subseteq \varphi(\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta}))$ imply the desired assertion $\widetilde{B}_\eta(\hat{w}) \subseteq \varphi(O)$. So it remains to prove (4.69). Assume that (4.69) doesn't hold, i.e.

$$\forall_{0<\eta\le\eta'} \exists_{\hat{v}' \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})} \exists_{\hat{w}' \in \widetilde{B}_\eta(\hat{w})} : \varphi(\hat{v}') = \hat{w}' \ \wedge \ \hat{v}' \notin O.$$

Hence,

$$\forall_{n \in \mathbb{N}, n \ge 1/\eta'} \exists_{\hat{v}_n \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})} \exists_{\hat{w}_n \in \widetilde{B}_{1/n}(\hat{w})} : \varphi(\hat{v}_n) = \hat{w}_n \ \wedge \ \hat{v}_n \notin O.$$

[22]Note that due to (4.58), φ maps $\mathring{\mathcal{L}}_{R-\epsilon}(\widetilde{u}_{1,\delta})$ into $\mathring{\mathcal{L}}_R(\widetilde{u}_\delta)$.
[23]Note that this $\hat{v}$ is unique due to the just proven injectivity.

This yields sequences $(\hat{w}_n)_n \subset \mathring{\mathcal{L}}_R(\widetilde{u}_\delta)$ and $(\hat{v}_n)_n \subset \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$ with $\lim_{n\to\infty} \hat{w}_n = \hat{w}$. Due to the compactness of $\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$, we may assume that $(\hat{v}_n)_n$ converges to some $\hat{v}' \in \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$ (otherwise, consider a convergent subsequence). Due to the continuity of φ proven above, we get

$$\varphi(\hat{v}') = \lim_{n\to\infty} \varphi(\hat{v}_n) = \lim_{n\to\infty} \hat{w}_n = \hat{w} = \varphi(\hat{v}).$$

Due to the injectivity of φ on $\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$, we get $\hat{v}' = \hat{v}$. In particular, due to the openness of O, we get $\hat{v}_n \in O$ for all but finitely many $n \in \mathbb{N}$, a contradiction to $\hat{v}_n \notin O$ for all $n \geq 1/\eta'$. This shows (4.69) and the desired openness property is proven. By the way, after we've just made use of the injectivity of φ on $\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$, the reader now sees why we considered $\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta})$ in the previous investigations (instead of $\mathcal{L}_{R-\epsilon}(\widetilde{u}_{1,\delta})$ as formulated in the theorem).
In the next step, we show with the help of the previous investigations that for $\widetilde{u}_\delta \in \widetilde{Iso}_\delta(u_0)$, there exists a continuous and injective map $\mathcal{L}_{R-\epsilon}(0) \to \mathcal{L}_R(\widetilde{u}_\delta)$ whose natural restriction $\mathring{\mathcal{L}}_{R-\epsilon}(0) \to \mathcal{L}_R(\widetilde{u}_\delta)$ is an open map. So let $\widetilde{u}_\delta \in \widetilde{Iso}_\delta(u_0)$ be given. We define a path γ in the connected set $[0, \check{u}_{0,\delta}] \cup \widetilde{Iso}_\delta(u_0)$ from 0 to $\widetilde{u}_\delta$ by $\gamma := \gamma_1 + \gamma_2$, where γ_1 is the line from 0 to $\check{u}_{0,\delta}$, i.e. $\mathrm{supp}(\gamma_1) = [0, \check{u}_{0,\delta}]$ and γ_2 is a path in $\widetilde{Iso}_\delta(u_0)$ from $\check{u}_{0,\delta}$ to $\widetilde{u}_\delta$ (note that $\widetilde{Iso}_\delta(u_0)$ is connected, cf. Theorem 3.1.1). In particular, $\mathrm{supp}(\gamma) \subset l^2_{\mathbb{R}}(\Gamma^*_\delta)$ is compact. We have proved that to any $\widetilde{v}_\delta \in \mathrm{supp}(\gamma)$, there is a neighbourhood $V(\widetilde{v}_\delta)$ such that for all $\widetilde{w}_\delta \in V(\widetilde{v}_\delta)$, the map $\mathcal{L}_{R-\epsilon}(\widetilde{v}_\delta) \to \mathcal{L}_R(\widetilde{w}_\delta)$ is continuous and injective. Now, $\bigcup_{\widetilde{v}_\delta \in \mathrm{supp}(\gamma)} V(\widetilde{v}_\delta)$ is an open covering of $\mathrm{supp}(\gamma)$. Due to the compactness of $\mathrm{supp}(\gamma)$, there exists a finite subcover, i.e. finitely many $\widetilde{v}_{1,\delta}, \ldots, \widetilde{v}_{m,\delta} \in \mathrm{supp}(\gamma)$ with $\widetilde{v}_{1,\delta} := 0$ and $\widetilde{v}_{m,\delta} := \widetilde{u}_\delta$ and finitely many continuous and injective maps

$$\varphi_i : \mathcal{L}_{R-\epsilon}(\widetilde{v}_{i,\delta}) \to \mathcal{L}_R(\widetilde{v}_{i+1,\delta}), \quad i = 1, \ldots, m-1.$$

The composition $\varphi := \varphi_{m-1} \circ \cdots \circ \varphi_1 : \mathcal{L}_{R-\epsilon}(0) \to \mathcal{L}_R(\widetilde{u}_\delta)$ is continuous and injective, too. Note that this map is well-defined although at first sight, it might seem unclear whether for $i = 1, \ldots, m-1$, the image $\varphi_i(\mathcal{L}_{R-\epsilon}(\widetilde{v}_{i,\delta})) \subseteq \mathcal{L}_R(\widetilde{v}_{i+1,\delta})$ is contained in the domain of definition $\mathcal{L}_{R-\epsilon}(\widetilde{v}_{i+1,\delta})$ of φ_{i+1}. Even if $\varphi_i(\mathcal{L}_{R-\epsilon}(\widetilde{v}_{i,\delta})) \nsubseteq \mathcal{L}_{R-\epsilon}(\widetilde{v}_{i+1,\delta})$, the composition $\varphi_{i+1} \circ \varphi_i$ is yet well-defined since in the foregoing proof, we even proved continuity and injectivity of maps like φ_{i+1} on larger sets $\mathcal{L}_{R+\epsilon}(\widetilde{v}_{i,\delta})$. With this convention, one might ask if it is possible that the image of φ exceeds the ball $\overline{B_R(\hat{u}_{0,f})}$. This, however is not possible due to (4.58) stating that the level sets in consideration have a distance less than ϵ to one another.
As to the openness property, denote by $\mathring{\varphi}_i$ the restriction of φ_i to $\mathring{\mathcal{L}}_{R-\epsilon}(\widetilde{v}_{i,\delta})$ and set $\mathring{\varphi} := \mathring{\varphi}_{m-1} \circ \cdots \circ \mathring{\varphi}_1 : \mathring{\mathcal{L}}_{R-\epsilon}(0) \to \mathring{\mathcal{L}}_R(\widetilde{u}_\delta)$. Since the composition of finitely many open maps is open, $\mathring{\varphi}$ is open which had to be proved.
In a last step, we show that the image of φ contains $\mathcal{L}_{R-2\epsilon}(\widetilde{u}_\delta)$. The proof is essentially equal to the previous openness proof. For sake of completeness, we give it here anyway: Let $\hat{w} \in \mathcal{L}_{R-2\epsilon}(\widetilde{u}_\delta)$ be given. Hence, $\phi(\hat{w}, \widetilde{u}_\delta) = m_f(u_0)$ by

definition of $\mathcal{L}_{R-2\epsilon}(\widetilde{u}_\delta)$. Consider the map

$$\mathcal{L}_{R-\epsilon}(\widetilde{v}_{m-1,\delta}) \to \mathbb{R}_0^+, \quad \hat{x} \mapsto \|\hat{x} - \hat{w}\|.$$

Due to the continuity of the euclidean norm on $\mathbb{C}_{\mathbb{R}}^{2g+1}$ and the compactness of $\mathcal{L}_{R-\epsilon}(\widetilde{v}_{m-1,\delta})$, this map has a global minimum $\hat{v} \in \mathcal{L}_{R-\epsilon}(\widetilde{v}_{m-1,\delta})$. Due to (4.58), we even have $\hat{v} \in \mathring{\mathcal{L}}_{R-\epsilon}(\widetilde{v}_{m-1,\delta})$. Let $\hat{n} := \hat{w} - \hat{v}$. Due to the minimum property of $\hat{v}$, we have $\hat{n} \in (\ker \phi'(\hat{v}, \widetilde{v}_{m-1,\delta}))^\perp$ (since otherwise, there would be an $\hat{x} \in \ker \phi'(\hat{v}, \widetilde{v}_{m-1,\delta})$ such that the line through $\hat{v}$ with direction $\hat{x}$ would transversely intersect the ball $B_{|\hat{n}|}(\hat{w})$ and we would thus find a $\hat{y} \in \mathring{\mathcal{L}}_{R-\epsilon}(\widetilde{v}_{m-1,\delta})$ with $\|\hat{y} - \hat{w}\| < \|\hat{v} - \hat{w}\|$, a contradiction to the minimum property of $\hat{v}$). This yields the representation $\hat{w} = \hat{v} + \sum_{i=1}^g \lambda_i N_i(\hat{v})$ with suitable coefficients $\lambda_1, \dots, \lambda_g \in \mathbb{R}$ and the normal vectors $N_1(\hat{v}), \dots, N_g(\hat{v})$ as before. We thus have the equation

$$\phi\left(\hat{v} + \sum_{i=1}^{g} \lambda_i N_i(\hat{v}), \widetilde{u}_\delta\right) = m_f(u_0).$$

Due to the uniqueness of the map $V(\widetilde{v}_{m-1,\delta}) \to U$, $\widetilde{v}_\delta \mapsto \lambda(\hat{v}, \widetilde{v}_\delta)$ obtained by the previous application of the Implicit Function Theorem, we get $\lambda_i = \lambda_i(\hat{v}, \widetilde{u}_\delta)$ for all $i = 1, \dots, g$. This shows $\hat{w} = \hat{v} + \sum_{i=1}^g \lambda_i(\hat{v}, \widetilde{u}_\delta) N_i(\hat{v})$ and thus $\varphi_{m-1}(\hat{v}) = \hat{w}$. We carry out this procedure $m-2$ further times, that is, the next step is to find to the just found $\hat{v}$ a $\hat{v}' \in \mathring{\mathcal{L}}_{R-\epsilon}(\widetilde{v}_{m-2,\delta})$ such that $\varphi_{m-2}(\hat{v}') = \hat{v}$, hence $\varphi_{m-1}(\varphi_{m-2}(\hat{v}')) = \hat{w}$. Finally, we arrive at some $\hat{x} \in \mathring{\mathcal{L}}_{R-\epsilon}(0)$ with $\varphi(\hat{x}) = (\varphi_{m-1} \circ \cdots \circ \varphi_1)(\hat{x}) = \hat{w}$. Recall once again that $\hat{x} \in B_{R-\epsilon}(\hat{u}_{0,f})$ holds due to (4.58), i.e. the radius $R+\epsilon$ is not exceeded in each of the $m-1$ steps (for instance, the above $\hat{v}'$ is indeed contained in $B_{R-\epsilon}(\hat{u}_{0,f})$, thanks to (4.58)). Ultimately, the lemma is proved. □

Large parts of the proof of the foregoing lemma were quite technical. We already discussed before that the unnatural intersection of the isospectral set with the ball $\overline{B_R(u_0)}$ and possible isospectral potentials in $\partial B_R(u_0)$ are the reason for these quite extensive technical efforts we've just done. In the following corollary, we want to mention a special case where things turn out to be easier, namely the case where the isospectral set satisfies a certain *boundedness* condition. More precisely:

Corollary 4.2.9. *Let $u_0 \in L^2_{\mathbb{R}}(F)$ with smooth Fermi curve be given. Assume furthermore that the following* boundedness *condition holds:*

$$\exists_{r>0}\, \exists_{R>0}\, \forall_{u \in B_r(u_0)} : Iso(u) \subset B_R(u_0). \tag{4.70}$$

Then for all $\widetilde{u}_\delta \in \widetilde{Iso}_\delta(u_0)$ (with $\delta > 0$ associated to R, cf. p. 131), there exists a homeomorphism $\mathcal{L}_R(0) \to \mathcal{L}_R(\widetilde{u}_\delta)$.

Remark. The reader might ask whether the assertion of the corollary still holds if we replace the condition (4.70) by the weaker assumption that $Iso(u_0)$ is bounded. The answer can't easily be given, at least not without delving into finite type theory. Let's explain what might happen in a worst case scenario if we only require the boundedness of $Iso(u_0)$: Let $(u_n)_{n\in\mathbb{N}} \subset L^2_{\mathbb{R}}(F)$ be a sequence converging to u_0. It might happen that for all $n \in \mathbb{N}$, there is a connected component U_n of $Iso(u_n)$ such that $\lim_{n\to\infty} \inf_{v\in U_n} \|v\|_{L^2(F)} = \infty$. In other words, $Iso(u_n)$ might have unbounded connected components the norm of whose elements uniformly tends to ∞ as $n \to \infty$ such that these components "vanish in $L^2_{\mathbb{R}}(F)$" as $n \to \infty$, at least if we consider the isospectral set as a subset of $L^2_{\mathbb{R}}(F)$. If instead, we assume (4.70), such pathological effects cannot occur.

Proof. Due to (4.70), we may set $\epsilon = 0$ in the proof of Lemma 4.2.8. Now, Lemma 4.2.8 implies that for all $\widetilde{u}_\delta \in \widetilde{Iso}_\delta(u_0)$, there exists a bijective continuous map $\mathcal{L}_R(0) \to \mathcal{L}_R(\widetilde{u}_\delta)$ which is also open due to the fact that the inverse of a bijective continuous map from a compact metric space onto some other metric space is continuous (cf. [10, p. 233, p. 713 (Aufgabe 158.6)], for instance). □

We now prove the main theorem of this section.

Theorem 4.2.10. *Let $u_0 \in L^2_{\mathbb{R}}(F)$ with smooth Fermi curve, $R > 0$ and $\epsilon > 0$ be given. Then there is a $\delta > 0$ and a $u_1 \in L^2_{\mathbb{R}}(F)$ satisfying* (4.41) *as well as a continuous and injective map*

$$\mathcal{I}_{R-\epsilon} : (Iso(u_1) \cap \overline{B_{R-\epsilon}(u_0)}) \times \widetilde{Iso}_\delta(u_0) \to Iso(u_0) \cap \overline{B_R(u_0)}.$$

whose image contains $Iso(u_0) \cap \overline{B_{R-3\epsilon}(u_0)}$ and whose natural restriction to the set $(Iso(u_1) \cap B_{R-\epsilon}(u_0)) \times \widetilde{Iso}_\delta(u_0)$ is an open map.

Proof. Let $u_0 \in L^2_{\mathbb{R}}(F)$, $R > 0$, $\epsilon > 0$ be given. We choose $\delta > 0$ as in the beginning of the proof of Lemma 4.2.8 with the additional requirement that $\delta > 0$ is sufficiently small such that the image $h\left(\overline{B_{R-2\epsilon}(\hat{u}_{0,f})} \times \{0\}\right)$ of the homeomorphic map h (4.43) with respect to the radius $R - \epsilon$ (instead of with respect to R as in Lemma 4.2.1) is contained in $\overline{B_{R-\epsilon}(u_0)}$. With respect to this δ, we choose a $u_1 \in L^2_{\mathbb{R}}(F)$ satisfying (4.41) which is possible by Lemma 4.2.7. Recall that $\widetilde{Iso}_\delta(0)$ contains only the element $0 \in l^2(\Gamma^*_\delta)$. We denote by

$$i : \overline{B_R(\hat{u}_{0,f})} \times \{0\} \to \overline{B_R(\hat{u}_{0,f})}, \quad (\hat{v}, 0) \mapsto \hat{v}$$

the natural isomorphism between $\overline{B_R(\hat{u}_{0,f})} \times \widetilde{Iso}_\delta(0)$ and $\overline{B_R(\hat{u}_{0,f})}$. We show at first that the map

$$\alpha : Iso(u_1) \cap \overline{B_{R-\epsilon}(u_0)} \to \mathcal{L}_{R-\epsilon}(0), \quad u \mapsto i(h^{-1}(u)) \tag{4.71}$$

is continuous, injective, that its image contains $\mathcal{L}_{R-2\epsilon}(0)$ and that its natural restriction to $Iso(u_1) \cap B_{R-\epsilon}(u_0)$ is an open map whose image is contained in $\mathring{\mathcal{L}}_{R-\epsilon}(0)$.
Due to $Iso(u_1) \cap \overline{B_{R-\epsilon}(u_0)} \subseteq \bigcup_{\hat{w} \in \overline{B_{R-\epsilon}(\hat{u}_{0,f})}} Iso_{\delta,w}(v)$, $i(h^{-1}(u))$ is well-defined for $u \in Iso(u_1) \cap \overline{B_{R-\epsilon}(u_0)}$. Moreover, we have $\alpha(u) \in \mathcal{L}_{R-\epsilon}(0)$ by definition of $\mathcal{L}_{R-\epsilon}(0)$, cf. (4.56), and h, cf. in particular (4.44). Hence, the map (4.71) is well-defined. Continuity and injectivity of the map (4.71) immediately follow from the continuity and injectivity of the homeomorphism h^{-1}. Since $h\left(\overline{B_{R-2\epsilon}(\hat{u}_{0,f})} \times \{0\}\right) \subseteq \overline{B_{R-\epsilon}(u_0)}$ as justified above, the image of α contains $\mathcal{L}_{R-2\epsilon}(0)$. Concerning the openness property, we see that $\alpha(u) \in \partial B_{R-\epsilon}(\hat{u}_{0,f})$ implies $\|u\|_{L^2} = \|h(\alpha(u),0)\|_{L^2} = \|\hat{h}(\alpha(u),0)\|_{l^2} \geq \|\alpha(u)\|_{\mathbb{C}^{2g+1}_{\mathbb{R}}} = R-\epsilon$ by definition of α together with the property $\hat{h}_f(\hat{v},0) = \hat{v}$ (cf. Lemma 4.2.1). Hence, $\alpha(u) \in \mathring{\mathcal{L}}_{R-\epsilon}(0)$ for $u \in Iso(u_1) \cap B_{R-\epsilon}(u_0)$. The openness of α now follows since h is homeomorphic. Altogether, the above claims concerning α are proved.
Now, we construct the map $\mathcal{I}_{R-\epsilon}$. Due to Lemma 4.2.8, for each $\widetilde{u}_\delta \in \widetilde{Iso}_\delta(u_0)$, there is a continuous and injective map between $\mathcal{L}_{R-\epsilon}(0)$ and $\mathcal{L}_R(\widetilde{u}_\delta)$ which shall be denoted by $\varphi_{\widetilde{u}_\delta} : \mathcal{L}_{R-\epsilon}(0) \to \mathcal{L}_R(\widetilde{u}_\delta)$. We now define the map $\mathcal{I}_{R-\epsilon}$ as follows:

$$\begin{aligned} \mathcal{I}_{R-\epsilon} : (Iso(u_1) \cap \overline{B_{R-\epsilon}(u_0)}) \times \widetilde{Iso}_\delta(u_0) &\longrightarrow Iso(u_0) \cap \overline{B_R(u_0)} \\ (u, \widetilde{u}_\delta) &\longmapsto h(\varphi_{\widetilde{u}_\delta}(\alpha(u)), \widetilde{u}_\delta). \end{aligned} \tag{4.72}$$

By definition of $h, \alpha, \varphi_{\widetilde{u}_\delta}$, the map $\mathcal{I}_{R-\epsilon}$ maps into $Iso(u_0) \cap \overline{B_R(u_0)}$ such that $\mathcal{I}_{R-\epsilon}$ is well-defined. We now prove that $\mathcal{I}_{R-\epsilon}$ is injective and continuous.
The injectivity of $\mathcal{I}_{R-\epsilon}$ immediately follows from the injectivity of $h, \alpha, \varphi_{\widetilde{u}_\delta}$. We prove the continuity of $\mathcal{I}_{R-\epsilon}$: We note that $h, \alpha, \varphi_{\widetilde{u}_\delta}$ are continuous. In a first step, we want to prove the continuity of $(\hat{v}, \widetilde{u}_\delta) \mapsto \varphi_{\widetilde{u}_\delta}(\hat{v})$. Thereto, we show at first the continuity of $\widetilde{u}_\delta \mapsto \varphi_{\widetilde{u}_\delta}$ with respect to the supremum norm $\|\varphi_{\widetilde{u}_\delta}\|_\infty := \sup_{\hat{v} \in \mathcal{L}_{R-\epsilon}(0)} \|\varphi_{\widetilde{u}_\delta}(\hat{v})\|$ (which is well-defined due to the compactness of $\mathcal{L}_{R-\epsilon}(0)$). Let $\widetilde{u}_\delta \in \widetilde{Iso}_\delta(u_0)$ and $\epsilon' > 0$ be given. Without loss of generality, we may assume that $\widetilde{u_\delta} \in V(0)$ (with $V(0)$ the corresponding neighbourhood of $0 \in l^2(\Gamma^*_\delta)$, compare $V(\widetilde{u}_{1,\delta})$ in (4.63), here with $\widetilde{u}_{1,\delta} := 0$). Otherwise, consider the composition $\varphi_{\widetilde{u}_\delta} := \varphi_{\widetilde{u}_{\delta,m-1}} \circ \dots \circ \varphi_{\widetilde{u}_{\delta,1}}$ of finitely many maps $\varphi_{\widetilde{u}_{\delta,i}}$ of the form (4.63) as in the end of the proof of Lemma 4.2.8, consider each individual $\varphi_{\widetilde{u}_{\delta,i}}$ and then transfer the continuity result for $(\hat{v}, \widetilde{u}_{\delta,i}) \mapsto \varphi_{\widetilde{u}_{\delta,i}}(\hat{v})$ to $(\hat{v}, \widetilde{u}_\delta) \mapsto \varphi_{\widetilde{u}_\delta}(\hat{v})$. Furthermore, we may again assume as before that the basis vectors N_i are normalized by $\|N_i(\hat{w})\| = 1$ for all $i = 1, \dots, g$. Due to the continuity of the map $(\hat{w}, \widetilde{u}'_\delta) \mapsto \lambda(\hat{w}, \widetilde{u}'_\delta)$ (cf. (4.61)) proved in Lemma 4.2.8, for each $\hat{v} \in \mathcal{L}_{R-\epsilon}(0)$, there exists a neighbourhood $U(\hat{v}) \subset \mathcal{L}_{R-\epsilon}(0)$ of $\hat{v}$ and an $\eta_{\hat{v}} > 0$ such that for all $\hat{w} \in U(\hat{v})$ and all $\widetilde{u}'_\delta \in \widetilde{Iso}_\delta(u_0)$ with $\|\widetilde{u}_\delta - \widetilde{u}'_\delta\|_{l^2(\Gamma^*_\delta)} < \eta_{\hat{v}}$, there holds (with

$\|\cdot\|$ denoting the euclidean norm in $\mathbb{C}_{\mathbb{R}}^{2g+1}$), cf. (4.63),

$$\|\varphi_{\widetilde{u}_\delta}(\hat{w}) - \varphi_{\widetilde{u}'_\delta}(\hat{w})\| \le \sum_{i=1}^{g} |\lambda_i(\hat{w}, \widetilde{u}_\delta) - \lambda_i(\hat{w}, \widetilde{u}'_\delta)| \cdot \|N_i(\hat{w})\| < \epsilon'.$$

Since $\bigcup_{\hat{v}\in\mathcal{L}_{R-\epsilon}(0)} U(\hat{v})$ is an open covering of $\mathcal{L}_{R-\epsilon}(0)$, there exists a finite subcover due to the compactness of $\mathcal{L}_{R-\epsilon}(0)$ denoted by $U_1, \ldots, U_m$ as well as corresponding positive numbers $\eta_1, \ldots, \eta_m$. Set $\eta := \min\{\eta_1, \ldots, \eta_m\}$. Again due to the compactness of $\mathcal{L}_{R-\epsilon}(0)$, the supremum $\sup_{\hat{v}\in\mathcal{L}_{R-\epsilon}(0)} \|\varphi_{\widetilde{u}_\delta}(\hat{v}) - \varphi_{\widetilde{u}'_\delta}(\hat{v})\|$ is attained as a maximum at some $\hat{w} = \hat{w}(\widetilde{u}'_\delta) \in \mathcal{L}_{R-\epsilon}(0)$ (depending on $\widetilde{u}'_\delta$), i.e. $\|\varphi_{\widetilde{u}_\delta} - \varphi_{\widetilde{u}'_\delta}\|_\infty = \|\varphi_{\widetilde{u}_\delta}(\hat{w}) - \varphi_{\widetilde{u}'_\delta}(\hat{w})\|$. Hence, for all $\widetilde{u}'_\delta \in \widetilde{Iso}_\delta(u_0)$ with $\|\widetilde{u}_\delta - \widetilde{u}'_\delta\|_{l^2(\Gamma^*_\delta)} < \eta$, there holds

$$\|\varphi_{\widetilde{u}_\delta} - \varphi_{\widetilde{u}'_\delta}\|_\infty \le \sum_{i=1}^{g} |\lambda_i(\hat{w}(\widetilde{u}'_\delta), \widetilde{u}_\delta) - \lambda_i(\hat{w}(\widetilde{u}'_\delta), \widetilde{u}'_\delta)| \cdot \|N_i(\hat{w}(\widetilde{u}'_\delta))\| < \epsilon'.$$

This proves the claimed continuity of $\widetilde{u}_\delta \mapsto \varphi_{\widetilde{u}_\delta}$. Therefore, together with the continuity of $\varphi_{\widetilde{u}_\delta}$ proven in Lemma 4.2.8, for each $(\hat{v}, \widetilde{u}_\delta) \in \mathcal{L}_{R-\epsilon}(0) \times \widetilde{Iso}_\delta(u_0)$ and each $\epsilon' > 0$, there is an $\eta > 0$ such that for all $(\hat{v}', \widetilde{u}'_\delta) \in \mathcal{L}_{R-\epsilon}(0) \times \widetilde{Iso}_\delta(u_0)$ with $\hat{v}' \in B_\eta(\hat{v})$ and $\widetilde{u}'_\delta \in B_\eta(\widetilde{u}_\delta)$ (these balls have to be considered in their respective norms, of course), there holds

$$\begin{aligned}\|\varphi_{\widetilde{u}_\delta}(\hat{v}) - \varphi_{\widetilde{u}'_\delta}(\hat{v}')\| &\le \|\varphi_{\widetilde{u}_\delta}(\hat{v}) - \varphi_{\widetilde{u}_\delta}(\hat{v}')\| + \|\varphi_{\widetilde{u}_\delta}(\hat{v}') - \varphi_{\widetilde{u}'_\delta}(\hat{v}')\| \le \\ &\le \|\varphi_{\widetilde{u}_\delta}(\hat{v}) - \varphi_{\widetilde{u}_\delta}(\hat{v}')\| + \|\varphi_{\widetilde{u}_\delta} - \varphi_{\widetilde{u}'_\delta}\|_\infty < \epsilon'\end{aligned}$$

(where $\|\cdot\|$ denotes the euclidean norm on $\mathbb{C}_{\mathbb{R}}^{2g+1}$ as before). Therefore, $(\hat{v}, \widetilde{u}_\delta) \mapsto \varphi_{\widetilde{u}_\delta}(\hat{v})$ is continuous. Together with the continuity of h and α, the continuity of $\mathcal{I}_{R-\epsilon}$ follows.

Concerning the proof of openness, we have already shown that $\alpha(Iso(u_1) \cap B_{R-\epsilon}(u_0)) \subseteq \mathring{\mathcal{L}}_{R-\epsilon}(0)$. Since the maps α and h are open, this implies together with the openness of $\mathring{\mathcal{L}}_{R-\epsilon}(0) \to \mathcal{L}_R(\widetilde{u}_\delta)$ for all $\widetilde{u}_\delta \in \widetilde{Iso}_\delta(u_0)$ (cf. Lemma 4.2.8) and the well-known fact that the union of open sets is open (in our case: $\bigcup_{\widetilde{u}_\delta\in O_1} \varphi_{\widetilde{u}_\delta}(O_2)$ is open for open sets $O_1 \subseteq \widetilde{Iso}_\delta(u_0)$, $O_2 \subseteq \mathring{\mathcal{L}}_{R-\epsilon}(0)$) that the restriction of $\mathcal{I}_{R-\epsilon}$ to $(Iso(u_1) \cap B_{R-\epsilon}(u_0)) \times \widetilde{Iso}_\delta(u_0)$ is an open map.

At last, we prove that the image of $\mathcal{I}_{R-\epsilon}$ contains $Iso(u_0) \cap \overline{B_{R-3\epsilon}(u_0)}$. So let $u \in Iso(u_0) \cap \overline{B_{R-3\epsilon}(u_0)}$ be given. Since the map $h : \overline{B_{R-\epsilon}(\hat{u}_{0,f})} \times \widetilde{Iso}_\delta(u_0) \to \bigcup_{\hat{v}\in\overline{B_{R-\epsilon}(\hat{u}_{0,f})}} Iso_{\delta,v}(u_0)$ is homeomorphic by Lemma 4.2.1 and $Iso(u_0) \cap \overline{B_{R-\epsilon}(u_0)} \subseteq \bigcup_{\hat{v}\in\overline{B_{R-\epsilon}(\hat{u}_{0,f})}} Iso_{\delta,v}(u_0)$, there is a $(\hat{v}, \widetilde{v}_\delta) \in \overline{B_{R-\epsilon}(\hat{u}_{0,f})} \times \widetilde{Iso}_\delta(u_0)$ with $h(\hat{v}, \widetilde{v}_\delta) = u$. Due to $u \in Iso(u_0) \cap \overline{B_{R-3\epsilon}(u_0)}$, it follows $\hat{v} \in \mathcal{L}_{R-3\epsilon}(\widetilde{v}_\delta)$. By Lemma 4.2.8, there is a $\hat{v}' \in \mathcal{L}_{R-2\epsilon}(0)$ with $\varphi_{\widetilde{v}_\delta}(\hat{v}') = \hat{v}$. By the analogous property of α shown at the beginning of this proof, there is a $w \in Iso(u_1) \cap \overline{B_{R-\epsilon}(u_0)}$ with $\alpha(w) = \hat{v}'$. Hence, $\varphi_{\widetilde{v}_\delta}(\alpha(w)) = \hat{v}$ and $h(\varphi_{\widetilde{v}_\delta}(\alpha(w)), \widetilde{v}_\delta) = u$. Therefore, u is in the image of $\mathcal{I}_{R-\epsilon}$. □

Again, we consider the special case with boundedness condition as in Corollary 4.2.9 yielding a more handsome formulation of Theorem 4.2.10.

Corollary 4.2.11. *Let* $u_0 \in L^2_{\mathbb{R}}(F)$ *with smooth Fermi curve be given and assume that the boundedness condition* (4.70) *holds, i.e.*

$$\exists_{r>0}\, \exists_{R>0}\, \forall_{u \in B_r(u_0)} : Iso(u) \subset B_R(u_0).$$

Then there is a $\delta > 0$ *and a* $u_1 \in L^2_{\mathbb{R}}(F)$ *satisfying* (4.41) *as well as a homeomorphism*

$$\mathcal{I} : Iso(u_1) \times \widetilde{Iso}_\delta(u_0) \to Iso(u_0).$$

Proof. At first, due to (4.70), we may again set $\epsilon = 0$ in Lemma 4.2.8 and we may assume that both $Iso(u_0)$ and $Iso(u_1)$ are contained in $B_R(u_0)$. Hence, (4.72) reduces to

$$\begin{aligned} \mathcal{I} : Iso(u_1) \times \widetilde{Iso}_\delta(u_0) &\longrightarrow Iso(u_0) \\ (u, \widetilde{u}_\delta) &\longmapsto h(\varphi_{\widetilde{u}_\delta}(\alpha(u)), \widetilde{u}_\delta), \end{aligned}$$

where $h, \alpha, \varphi_{\widetilde{u}_\delta}$ are as in the proof of Theorem 4.2.10 (with $\epsilon = 0$), i.e. we have $\alpha : Iso(u_1) \cap \overline{B_R(u_0)} \to \mathcal{L}_R(0)$ and $\varphi_{\widetilde{u}_\delta} : \mathcal{L}_R(0) \to \mathcal{L}_R(\widetilde{u}_\delta)$, $\widetilde{u}_\delta \in \widetilde{Iso}_\delta(u_0)$. This time, also $\varphi_{\widetilde{u}_\delta}$ and α are homeomorphic, cf. also Corollary 4.2.9. Continuity and injectivity of $\mathcal{I}$ are proven as in Theorem 4.2.10. The additional assertion in Theorem 4.2.10 (applied to $\epsilon = 0$) that the image of $\mathcal{I}$ contains $Iso(u_0)$ implies surjectivity of $\mathcal{I}$. The continuity of the inverse $\mathcal{I}^{-1}$ follows once again from the compactness of $Iso(u_1) \times \widetilde{Iso}_\delta(u_0)$ (recall Corollary 4.2.2 and the end of the proof of Lemma 4.2.1 where the compactness of $\widetilde{Iso}_\delta(u_0)$ has been proved) and the bijectivity and continuity of $\mathcal{I}$ (cf. [10, p. 233, p. 713 (Aufgabe 158.6)]). □

4.3 The isospectral set

In this section, we finally want to determine the isospectral set

$$Iso_F(u_0) = \{u \in L^2_{\mathbb{R}}(F) : F(u) = F(u_0)\}$$

we have already introduced at the beginning of Chapter 3. It remains to prove the equivalence $m(u) = m(u_0) \Longleftrightarrow F(u) = F(u_0)$ between moduli and Fermi curves. Before we prove this equivalence, we consider at first the special case that u_0 is a *finite type* potential as defined in Definition 2.5.8. On the other hand, we also go back to the more general setting that the appearing potentials may be arbitrary *complex-valued* potentials which don't need necessarily to be real. By Definition 2.6.1, we see that the moduli only depend on the corresponding Fermi curve $X := F(u)$ (still considered as smooth) of some given potential $u \in L^2(F)$ and not explicitly on u itself. Thus, we can consider the map

$$X \mapsto \left(-16\pi^3 \int_{A_\nu} k_1 dk_2 \right)_{\nu \in \Gamma^* \setminus \{0\}} \tag{4.73}$$

which assigns to each Fermi curve X its moduli, namely the contour integrals $-16\pi^3 \int_{A_\nu} k_1 dk_2$ over the A-cycles of X. The arising question is: Which topological properties does the space the curves X reside in, i.e. the domain of definition of the map (4.73), have? Let's denote this space by $\mathcal{M}$. We can endow $\mathcal{M}$ with a topology such that $\mathcal{M}$ is a complex manifold. In section 3.2 (*Deformations of complex Fermi curves*) of the work [27], this procedure has been carried out. There, the elements of $\mathcal{M}$ had to fulfill certain conditions, in [27] denoted by "Quasi-momenta (i),(ii),(v)" together with the condition that all elements of $\mathcal{M}$ have the same fixed arithmetic genus[24]. In [27, Proposition 3.7] or as well in [19, Lemma C.19], it has been shown that $\mathcal{M}$ is a complex manifold with the property that the tangent space $T_X\mathcal{M}$ for $X \in \mathcal{M}$ is isomorphic to the space of so-called regular 1-forms[25] on X. The latter assertion, namely that we can identify $T_X\mathcal{M}$ with the space of regular 1-forms on X is in fact the crucial statement we will use in the following. For deeper background information, we refer the reader to the mentioned works [19] and [27].
In Theorem 4.1.3, we showed that there exists a set of holomorphic 1-forms dual to the cycles A_ν for $\nu \in \Gamma_N^*/\sigma$. Since we temporarily consider the case of (complex-valued) *finite type* potentials, i.e. all elements of $\mathcal{M}$ have finite genus[26], say g, Theorem 4.1.3 even yields a dual basis of the complete space of holomorphic 1-forms on the curve X (which shall be denoted by $\Omega(X)$) since $\Omega(X)$ is a *finite-dimensional* vector space. Let $A_1, \dots, A_g$ be a suitable numeration of the corresponding A-cycles. The derivative of the map

$$M : \mathcal{M} \to \mathbb{C}^g, \quad X \mapsto \left(-16\pi^3 \int_{A_n} k_1 dk_2 \right)_{n=1,\dots,g}$$

[24]We use the common convention that the infinitely many double points far outside of some compact set that any finite type curve naturally has by definition, don't contribute to the arithmetic genus since otherwise, any finite type Fermi curve would have arithmetic genus equal to ∞.

[25]For the proper definition of a regular 1-form, see for instance [27, p. 34] or [19, Definition 3.4]. We won't deal with this definitions. We only mention that in the case of smooth complex curves, the space of regular 1-forms equals the space of the well-known holomorphic 1-forms. In other words, the concept of regular 1-forms is a generalization to complex curves which might have singularities. Since we consider smooth Fermi curves, the concept of holomorphic 1-forms suffices for us.

[26]Since we consider sufficiently small deformations of a given *smooth* Fermi curve, we may assume that the elements of $\mathcal{M}$ are smooth as well. Hence, we don't need to distinguish between geometric and arithmetic genus (for the definitions, cf. [19, p. 62], for instance) and thus simply speak of the *genus* of the respective curve.

at some (smooth) $X \in \mathcal{M}$ can be represented as

$$dM|_X : \Omega(X) \to \mathbb{C}^g$$

due to $T_X\mathcal{M} \cong \Omega(X)$. We want to show that $dM|_X$ is a vector space isomorphism. In (4.33), we deduced that for $u \in L^2(F)$ (here finite type), $(V, W) := (1, -\frac{u}{4})$ and all $w \in L^2(F)$, there holds

$$\frac{d}{du}\left(\int_{A_\nu} p_1 dp_2\right)|_u(w) = \frac{1}{4}\int_{A_\nu} \omega(V, W, 0, w)$$

for all of those finitely many $\nu \in \Gamma^*$ for which the corresponding handle is not closed. Up to the factor $\frac{1}{4}$ and up to the isomorphism between k- and p-coordinates (compare the end of the proof of Corollary 4.1.7), $dM|_X$ is virtually equal to the map

$$\omega \longmapsto \left(\int_{A_n} \omega\right)_{n=1,\ldots,g} \tag{4.74}$$

since due to Theorem 4.1.3, the space $\Omega(X)$ is generated by elements of the form $\omega(V, W, 0, w)$, $w \in L^2(F)$ (here with $X := F(V, W)/\Gamma^*$) because firstly, the duality relation $\int_{A_i} \omega_k = \delta_{i,k}$ in Theorem 4.1.3 obviously implies linear independence of the ω_k, $k = 1, \ldots, g$, and secondly, the dimension of $\Omega(X)$ (as a vector space over $\mathbb{C}$) on a complex curve X is equal to the genus g of the curve (cf. [6, Remark 17.10]). Now, we show that (4.74) is a vector space isomorphism. We choose a basis of holomorphic 1-forms $(\omega_1, \ldots, \omega_g)$ with the duality relation $\int_{A_i} \omega_j = \delta_{ij}$ whose existence has been shown in Theorem 4.1.3. Let $(a_1, \ldots, a_g) \in \mathbb{C}^g$ be given. Choose $\omega := \sum_{i=1}^g a_i\omega_i$, we get $\left(\int_{A_n} \omega\right)_{n=1,\ldots,g} = (a_1, \ldots, a_g)$ by the duality relation. This shows that (4.74) is onto. Consider two forms $\omega := \sum_{i=1}^g a_i\omega_i$ and $\widetilde{\omega} := \sum_{i=1}^g \widetilde{a}_i\omega_i$ (with coefficients $a_i, \widetilde{a}_i \in \mathbb{C}$) such that

$$\int_{A_n} \omega = \int_{A_n} \widetilde{\omega} \quad \text{for all}\, n = 1, \ldots, g.$$

Then again due to the duality relation, we get

$$\forall_{n=1}^g \quad \sum_{i=1}^g \int_{A_n} (a_i - \widetilde{a}_i)\omega_i = 0 \Longleftrightarrow \forall_{n=1}^g \quad a_n - \widetilde{a}_n = 0. \qquad \Longrightarrow \omega = \widetilde{\omega}.$$

This shows that (4.74) is one-to-one. The linearity of (4.74) is clear. Thus, (4.74) is a vector space isomorphism. Due to the Inverse Function Theorem (cf. [23, p. 142], for example), the map M is locally invertible, in particular locally one-to-one. Hence, we have proved the following lemma.

Lemma 4.3.1. *Let $u_0 \in L^2(F)$ be a finite type potential. Then there exists a tubular neighbourhood $\mathcal{T}$ of $F(u_0)$ in $\mathbb{C}^2$ such that for all $u \in L^2(F)$ with the property that $F(u)$ has the same genus as $F(u_0)$ and $F(u) \subset \mathcal{T}$, there holds: $m(u) = m(u_0)$ implies $F(u) = F(u_0)$.*

Now, we prove the main theorem for generic *real-valued* potentials (not necessarily finite type).

Theorem 4.3.2. *Let $u \in L^2_{\mathbb{R}}(F)$ be a given potential with Fermi curve $F(u)$. Then there is a tubular neighbourhood $\mathcal{T}$ of $F(u)$ in $\mathbb{C}^2$ such that for all $v \in L^2_{\mathbb{R}}(F)$ with $F(v) \subset \mathcal{T}$, there holds the equivalence*

$$m(u) = m(v) \Longleftrightarrow F(u) = F(v).$$

Proof. The implication "$\Leftarrow$" is obvious since the moduli depend, by definition, only on the Fermi curve, not on the potential itself. It remains to show the other direction "$\Rightarrow$", that is, if the moduli of two admissible potentials (in the sense mentioned in the theorem) are equal, then the corresponding Fermi curves are equal. Let $u \in L^2_{\mathbb{R}}(F)$ and consider a suitable tubular neighbourhood $\mathcal{T} \subset \mathbb{C}^2$ of $F(u)$ and the corresponding neighbourhood in the space of potentials $U := \{v \in L^2_{\mathbb{R}}(F) : F(v) \subset \mathcal{T}\}$. This neighbourhood U can be chosen as open: This is clear if we intersect $\mathcal{T}$ with some compact set $K \subset \mathbb{C}^2$. Due to asymptotic freeness (cf. [19, Theorem 2.35]), however, this also holds for the entire Fermi curve. More precisely, by choosing K sufficiently large and for $r > 0$ sufficiently small, there holds $F(v) \subset \mathcal{T}$ for all $v \in B_r(u)$.
Now let $v \in U$ satisfy $m(u) = m(v)$. We construct canonical finite-type sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ converging to u and v, respectively, as $n \to \infty$, as it has been done in Lemma 4.2.7. These sequences fulfill due to $m(u) = m(v)$

$$m_\nu(u_n) = \begin{cases} m_\nu(u), & \nu \in \Gamma^*, |\nu| \leq n \\ 0, & \nu \in \Gamma^*, |\nu| > n. \end{cases}, \quad m_\nu(v_n) = \begin{cases} m_\nu(u), & \nu \in \Gamma^*, |\nu| \leq n \\ 0, & \nu \in \Gamma^*, |\nu| > n \end{cases}$$

for all sufficiently large $n \in \mathbb{N}$. There is an $N \in \mathbb{N}$ such that $u_n, v_n \in U$ for all $n \geq N$. Since $m(u_n) = m(v_n)$ for all $n \geq N$ and $L^2_{\mathbb{R}}(F) \subset L^2(F)$, there holds $F(u_n) = F(v_n)$ for all $n \geq N$ by Lemma 4.3.1. Due to continuity, we get $F(u) = F(v)$ by performing $n \to \infty$. More precisely, in [13, Theorem 4.1.3], it has been proved that Fermi curves are locally described by the zero set of a function holomorphic in k and continuous in u. Continuity of roots (cf. [3, Lemma 3.4.11(1)]) yields continuity of $u \mapsto F(u) \cap K$, where $K \subset \mathbb{C}^2$ is some compact subset. Due to asymptotic freeness (cf. [19, Theorem 2.35]), $F(u_n)$ converges to $F(u)$ (as $n \to \infty$) also in $\mathbb{C}^2 \setminus K$ proving continuity of $u \mapsto F(u)$. The theorem is proved. □

This theorem shows $Iso_F(u_0) = Iso(u_0)$ for $u_0 \in L^2_{\mathbb{R}}(F)$. Hence, we may replace $Iso(u_0)$ by $Iso_F(u_0)$ in Corollary 4.2.11 yielding the desired parameterization of the isopsectral set in the case with boundedness condition. In the general case of unbounded isospectral sets (Theorem 4.2.10), we may replace $Iso(u_0)$ by $Iso_F(u_0)$ as well but we only get a weaker result, namely that of Theorem 4.2.10.

Chapter 5

Outlook

In this chapter, we want to illustrate some perspectives for further research based on the results of this work. These perspectives also include attempts we already tried but unfortunately remained open since we could not find satisfying answers after having spent quite a long time of research on them.

In our work, we finally considered the isospectral problem for *real-valued* potentials with *smooth* Fermi curve. A natural generalization would be to consider arbitrary complex-valued potentials $u_0 \in L^2(F)$ whose Fermi curve $F(u_0)$ might have singularities. A crucial step in which we made use of the smoothness of the Fermi curve was the proof that the map $u \mapsto m_f(u)$ into the first finitely many moduli is a submersion, cf. Theorem 4.1.11. In the proof of Lemma 4.2.8, we used this to apply the Implicit Function Theorem to the equation (4.60),

$$\phi\left(\hat{w} + \sum_{i=1}^{g} \lambda_i N_i(\hat{w}), \widetilde{u}_\delta\right) = m_f(u_0),$$

eventually yielding the map $\mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta}) \to \mathcal{L}_{R+2\epsilon}(\widetilde{u}_\delta)$ (4.63). Thanks to the submersion property of the moduli proved in Theorem 4.1.11, the rank of $\frac{\partial}{\partial\lambda}\Phi|_{\lambda=0}$ (with the map Φ (4.59) as in the proof of Lemma 4.2.8) was equal to g which made the application of the Implicit Function Theorem possible. In the case of singularities, we have $\dim_{\mathbb{R}}(\ker \phi'(\hat{v}, \widetilde{u}_{1,\delta}))^{\perp} < g$ instead of $\dim_{\mathbb{R}}(\ker \phi'(\hat{v}, \widetilde{u}_{1,\delta}))^{\perp} = g$ in the proof of Lemma 4.2.8. So in general, there is a $k > 0$ such that $\dim_{\mathbb{R}}(\ker \phi'(\hat{v}, \widetilde{u}_{1,\delta}))^{\perp} = g - k$, where k can be considered as the number of singularities. We have proved in this work that $m_\nu(u) = 0 \Longleftrightarrow c_\nu = 0$ for all $\nu \in \Gamma^*_\delta$ (cf. (2.85) and Lemma 3.2.2), where the latter condition means that the cycle A_ν is degenerated into one point. If one was able to show the same result for $\nu \in \Gamma^*_f = \Gamma^* \setminus \Gamma^*_\delta$, also the singularities in the compact part could be characterized by $m_\nu = 0$ for the respective $\nu \in \Gamma^*_f$. By neglecting the k moduli satisfying $m_\nu = 0$, one can define -similarly to the map Φ (4.59)- a map

$$\Phi : \mathcal{L}_{R+\epsilon}(\widetilde{u}_{1,\delta}) \times U \to \widetilde{\mathbb{R}}^{g-k}, \quad (\hat{v}, \lambda) \mapsto \phi\left(\hat{v} + \sum_{i=1}^{g-k} \lambda_i N_i(\hat{v}), \widetilde{u}_{1,\delta}\right),$$

where this time $U \subseteq \mathbb{R}^{g-k}$ (instead of $U \subseteq \mathbb{R}^g$ as in the proof of Lemma 4.2.8). The Implicit Function Theorem can be applied again since $\frac{\partial}{\partial\lambda}\Phi|_{\lambda=0}$ has full rank $g-k$. But there is one shortcoming: By neglecting the moduli satisfying $m_\nu = 0$, we have lost the control over them. Whereas in the later part of the proof of Lemma 4.2.8, all other moduli m_ν, $\nu \in \Gamma_f^*$ can be forced to be equal to $m_f(u_0)$, we can't say anything about the just neglected moduli. Will they remain equal to zero (as desired) or not? One needs a condition which enforces those moduli to remain zero. A first step could be to show that $m_\nu(u) = 0$ implies that the derivative $dm_\nu|_u$ vanishes, too. The conjecture that this could be true is motivated by the asymptotic model moduli $\widetilde{m}_\nu$ which fulfill this condition. But here, we are in the compact part of the Fermi curve so that the model moduli are not defined. So one needs to find another way out which might require methods of finite type theory.

Let's discuss the generalization to *complex-valued* potentials, i.e. we have to drop the conditions $\overline{\hat{u}(\nu)} = \hat{u}(-\nu)$ for all $\nu \in \Gamma^*$ as well as $\overline{\check{u}_\nu} = \check{u}_{-\nu}$ for all $\nu \in \Gamma_\delta^*$ for Fourier coefficients and perturbed Fourier coefficients. Already in Theorem 2.5.9, the first problems occur for complex-valued potentials: The estimate for the critical point ζ_ν analogous to (2.82) would be

$$|\zeta_\nu| \leq (|\check{u}_\nu| + |\check{u}_{-\nu}|) \cdot o\left(\frac{1}{|\nu|}\right), \quad \text{as } |\nu| \to \infty. \tag{5.1}$$

In the further course of the proof of Theorem 2.5.9, there appears an estimate for the term $\frac{\zeta_{\nu,1}\cdot\zeta_{\nu,2}}{h_\nu(0)}$. In the complex case, due to (5.1), this term would be equal to

$$\frac{\zeta_{\nu,1}\cdot\zeta_{\nu,2}}{h_\nu(0)} = o\left(\frac{1}{|\nu|^2}\right)\cdot\frac{(|\check{u}_\nu| + |\check{u}_{-\nu}|)^2}{|\check{u}_\nu\check{u}_{-\nu}|} = o\left(\frac{1}{|\nu|^2}\right)\left(\sqrt{\frac{|\check{u}_\nu|}{|\check{u}_{-\nu}|}} + \sqrt{\frac{|\check{u}_{-\nu}|}{|\check{u}_\nu|}}\right)^2,$$

as $|\nu| \to \infty$, which cannot be reasonably estimated. So it's quite doubtful whether the assertion of Theorem 2.5.9 still holds for complex-valued potentials. During our investigations of complex-valued potentials, i.e. before we restricted ourselves to real-valued potentials, quotients as $|\check{u}_\nu|/|\check{u}_{-\nu}|$ appeared several times. Since in the complex case, $\check{u}_\nu$ and $\check{u}_{-\nu}$ have to be considered as independent, there is no evidence why quotients like $|\check{u}_\nu|/|\check{u}_{-\nu}|$ should remain bounded or even become sufficiently small as $|\nu| \to \infty$.

Another problem induced by complex-valued potentials has already been discussed at the end of Section 3.1. For complex-valued $u_0 \in L^2(F)$, the corresponding asymptotic model isospectral set is given by

$$\widetilde{Iso}_\delta(u_0) = \times_{\nu\in\Gamma_\delta^*/\sigma}\{(e^{it}\sqrt{\check{u}_{0,\nu}\cdot\check{u}_{0,-\nu}}, e^{-it}\sqrt{\check{u}_{0,\nu}\cdot\check{u}_{0,-\nu}}) : t \in \mathbb{C}\}.$$

Due to $t \in \mathbb{C}$, the elements of $\widetilde{Iso}_\delta(u_0)$ will eventually leave the domain in which the perturbed Fourier coefficients are well-defined. In fact, this is the same problem we had to deal with in Section 4.2, namely in Theorem 4.2.10, where we

intersected $Iso(u_0)$ with balls $B_R(u_0)$ and had to choose to each $R > 0$ a different $\delta > 0$. An analogous procedure could also be carried out for $\widetilde{Iso}_\delta(u_0)$ if u_0 is complex-valued. The corresponding results are then weaker in the same sense as the assertion of Theorem 4.2.10 is weaker than the homeomorphism assertion of Corollary 4.2.11.
This immediately leads to a next open question: Is it possible to choose a *uniform* $\delta > 0$ for the entire isospectral set? If this question can be answered with "yes", there is hope that a homeomorphism as in Corollary 4.2.11 also exists if $Iso(u_0)$ is unbounded. A first step to solve this problem would be to extend the perturbed Fourier coefficients along $Iso(u_0)$, i.e. one has to show that there exists a $\delta > 0$ such that the Fourier coefficients $(\check{u}_\nu)_{\nu\in\Gamma^*_\delta}$ exist for all $u \in Iso(u_0)$. If Lemma 3.2.2 could be globalized in this sense, we would have $|\check{u}_\nu| = \xi|\nu|^2 m_\nu(u_0)(1+O(1/|\nu|))$, as $|\nu| \to \infty$ for all $u \in Iso(u_0)$, where the estimate by the error term $1+O(1/|\nu|)$ holds uniformly in $u \in Iso(u_0)$. This boundedness property of the perturbed Fourier coefficients along $Iso(u_0)$ would be an important step to extend them beyond their actual domain of definition. The arising question is now: Does the above error term $1 + O(1/|\nu|)$ hold indeed *uniformly* on $Iso(u_0)$? If we take a look into the proof of Lemma 3.2.2, this question can be largely reduced to the question whether the convergence

$$\lim_{|\nu|\to\infty} \left\| \frac{\partial}{\partial k} \mathcal{A}_{\pm,\nu}(k + k^\pm_\nu(\hat{u}_0), u) \right\| = 0$$

in Lemma 2.2.7 holds *uniformly* in $u \in Iso(u_0)$. If one could answer this question, one would have achieved an essential step towards a possible choice of a uniform $\delta > 0$ in $Iso(u_0)$.
Besides the ansatz of determining $Iso(u_0)$ by a Cartesian product $Iso(u_1) \times \widetilde{Iso}_\delta(u_0)$ as in Corollary 4.2.11, there would also be the possibility of imitating the procedure of determining $Iso(u_0)$ if u_0 is a *finite type* potential. In finite type theory, one assigns divisor (cf. [6, 16.1]) $D(u_0)$ to the given potential u_0, where $D(u_0)$ is the pole divisor of the eigenfunction $x \mapsto \psi(k, x)$ of the Schrödinger operator at $k \in X$ normalized by $\psi(k, 0) = 1$, $k \in X$, where X is the compactified normalization of $F(u_0)$, cf. [19, Section 3.2], i.e. the support of the divisor $D(u_0)$ is given by

$$\begin{aligned}\mathrm{supp}D(u_0) = \{k \in X :\ &\text{there is an eigenfunction } 0 \not\equiv \psi(k,\cdot) \text{ of } -\Delta_k + u_0 \\ &\text{with eigenvalue } \lambda = 0 \text{ and } \psi(k, 0) = 0\}.\end{aligned}$$

In [19, Lemma 4.13], it has been shown that a Divisor D on X which is the pole divisor of the normalized eigenfunction associated to some (finite type) potential $u \in Iso(u_0)$ necessarily satisfies the linear equivalence relation

$$D + \sigma(D) \cong K + Q^+ + Q^-, \tag{5.2}$$

where K is the canonical divisor on X, σ the holomorphic involution well-known from Section 2.3 and Q^+ and Q^- are some marked points at infinity which yield the two-point-compactification of X. In finite type theory, one solves the isospectral problem by parameterizing all divisors on X fulfilling this linear equivalence. Since in the finite type case, there exists a 1-1-correspondence between divisors on X and potentials $u \in Iso(u_0)$, the isospectral problem can be solved by parameterizing those set of divisors. So far the finite type case. If we want to transfer this to *infinite type* potentials, one has to find a relation analogous to (5.2). We expect a relation similar to that of [28, Theorem 3.6], namely

$$D + \sigma(D) = b + \left(\frac{1}{\langle \phi, \psi \rangle_{L^2}} \right),$$

where b is the divisor of branch points of the covering $X \to \mathbb{C}$, $k = (k_1, k_2) \mapsto k_1$ and $\langle \phi, \psi \rangle_{L^2}$ is the L^2-scalar product of the eigenfunction ψ with the corresponding dual eigenfunction ϕ (i.e. the eigenfunction corresponding to the transposed Schrödinger operator). Analogously to Chapter 3, one would have to do an asymptotic analysis for the divisor points in the asymptotic part of the Fermi curve. These points are expected to reside within the excluded domains. Again, the model Fermi curve introduced in Section 2.5 should serve as a good approximation where the divisor points of the asymptotic part can be computed easily. The computation of the divisor points of the actual Fermi curve will require perturbation theory once again as it has been done in Chapter 3.
An important step would then be to show that there exists an isomorphism between potentials and divisors. This leads to another interesting question we met during our research: Does there exist a local isomorphism between potentials and divisors? Here, "local" means a local neighbourhood in the L^2-space of potentials (and not restricted to some isospectral set as we've just explained in the finite type case). We tried to answer this question in the following Dirac setting, i.e. instead of the Schrödinger operator, we consider the Dirac operator $\widetilde{D}(V, W, p_1)$ (4.3): Let $(V, W) \in L^2(F) \times L^2(F)$ be given and $(p_{1,n}(V, W), p_{2,n}(V, W))_{n \in \Gamma^*}$ be the corresponding sequence of the divisor points, i.e. $(p_{1,n}(V, W), p_{2,n}(V, W)) \in \mathbb{C}^2$ is the n^{th} pole of the normalized eigenfunction ψ of (4.3) to the eigenvalue $-p_2\pi$. Let $\Omega : (L^2(F) \times L^2(F))^2 \to \mathbb{C}$ be the symplectic form already well-known from (4.19). Consider for $n \in \Gamma^*$ the gradients $\delta p_{1,n} := \frac{\partial p_{1,n}}{\partial(V,W)} \in L^2(F) \times L^2(F)$, $\delta p_{2,n} := \frac{\partial p_{2,n}}{\partial(V,W)} \in L^2(F) \times L^2(F)$ as elements in $L^2(F) \times L^2(F)$ as in [23, p. 127]. The arising question is whether the $\delta p_{1,i}, \delta p_{2,i}$, $i \in \Gamma^*$ can serve as so-called *Darboux coordinates* of $L^2(F) \times L^2(F)$, cf. [14, Theorem 5.1], [28, chapter 6]. More precisely, we are interested in the question whether the following holds:

$$\Omega(\delta p_{1,i}, \delta p_{1,j}) = 0, \quad \Omega(\delta p_{1,i}, \delta p_{2,j}) = \kappa_i \delta_{ij}, \quad \Omega(\delta p_{2,i}, \delta p_{2,j}) = 0, \quad i, j \in \Gamma^*, \tag{5.3}$$

where $\kappa_i \neq 0$ are some constants. The verification of these relations would be an important step in order to prove that $(V, W) \mapsto (p_{1,n}(V, W), p_{2,n}(V, W))_{n \in \Gamma^*}$ is a local isomorphism. In [14, Theorem 5.5], this problem has been solved for the sinh-Gordon equation. In order to gain a first insight into whether such Darboux coordinates also exist in the Dirac case, it makes sense to firstly consider special cases where everything can be computed explicitly. The easiest case is clearly the free case $V = W = 0$. Here, all singularities of the corresponding Fermi curve are ordinary double points and these are exactly the divisor points. The eigenfunctions can be written down by explicit formulas. In this case, we verified (5.3). In a next step, we considered the more general case for $(V, W) \neq (0, 0)$ that all divisor points are contained in the set of points $p = (p_1, p_2) \in \mathbb{C}^2$ where the kernel of $\widetilde{D}(V, W, p_1) + \pi p_2$ is two-dimensional (which, of course, is still not the generic case since divisor points may also be smooth points of the Fermi curve). In particular, this case includes the free case $V = W = 0$. By choosing bases $\{\psi', \psi''\}$ of the respective two-dimensional eigenspaces as well as corresponding bases $\{\phi', \phi''\}$ of the dual eigenspaces (here, the dependence on the index $i \in \Gamma^*$ is suppressed), we could -with a suitable normalization (namely $\psi = (\psi_1, \psi_2) \mapsto \psi_1(0) + \psi_2(0)$) and by making use of the fact that $(p_{1,n}(V, W), p_{2,n}(V, W))_{n \in \Gamma^*}$ are divisor points- compute $\delta p_{1,n}$, $\delta p_{2,n}$ in terms of components of the eigenfunction and the dual eigenfunction. For instance, we computed for $n \in \Gamma^*$

$$\delta p_{1,n} = \frac{-\begin{pmatrix} \frac{1}{c^+}\phi_2\psi_1 \\ \frac{1}{c^-}\phi_1\psi_2 \end{pmatrix}}{\left\langle \phi, \frac{\partial \widetilde{D}}{\partial p_1}\psi \right\rangle_{L^2 \times L^2}} |_{p=(p_{1,n}(V,W), p_{2,n}(V,W))},$$

where $c^{\pm} := \check{\kappa}_2 \pm \check{\kappa}_1$ (with $\check{\kappa}$ as in (4.3)) and $\psi := \psi'' - \psi'$, $\phi := \phi' + \phi''$.
In order to verify the first identity $\Omega(\delta p_{1,m}, \delta p_{1,n}) = 0$ in (5.3), one has to show that for all $m, n \in \Gamma^*$

$$\int_F (\phi_{2,m}\psi_{1,m}\phi_{1,n}\psi_{2,n} - \phi_{2,n}\psi_{1,n}\phi_{1,m}\psi_{2,m}) d^2x = 0. \tag{5.4}$$

Following the idea of the proof of [14, Theorem 5.5], we tried analogously to [14, Lemma 5.2, Lemma 5.3] to deduce differential equations for the components ψ and ϕ in order to express the integrand of the integral in (5.4) as a total variation of some periodic function which would then imply (5.4). In contrast to the sinh-Gordon case, in our setting, not all derivatives of $\psi_1, \psi_2, \phi_1, \phi_2$ can be expressed in terms of these components due to the look of $\widetilde{D}(V, W, p_1)$: One sees directly by (4.3) that $\bar{\partial}\psi_1$ and $\partial\psi_2$ can be expressed by the components of ψ by computing $(\widetilde{D}(V, W, p_1) + \pi p_2)\psi = 0$. For $\bar{\partial}\psi_2$ and $\partial\psi_1$, however, this differential equation is not able to provide analogous expressions. Although we made different kinds of ansatzes, we could not succeed in expressing the mentioned integrand as a total variation. In our eyes, this lack of derivative identities in the Dirac setting (a problem which doesn't occur in the sinh-Gordon setting) is the crucial point why we didn't succeed. On the other hand, we couldn't disprove (5.3) either so that

the question whether there exist Darboux coordinates in the Dirac case is still open. Maybe one needs other tools we didn't see so far.
These were the open questions (we have already been thinking about ourselves) that could be interesting to investigate in the future.
Two other perspectives to continue the isospectral problem (which we, however, haven't thought about ourselves, yet) would be firstly, to generalize this work to higher dimensions (the obvious next generalization would be to consider three-dimensional Schrödinger operators which might also be the most important case considered from a physicist's point of view) and secondly, to consider the isospectral problem for Bloch varieties, cf. (1.4), i.e. to determine the set

$$Iso_B(u_0) := \{u \in L^2(F) : B(u) = B(u_0)\}$$

for a given $u_0 \in L^2(F)$. With these suggestions, we want to conclude this work.

Appendix A

Two lemmata about Dirac operators

In this appendix, we give the proof of [27, Lemma 3.2] (denoted in the following as Lemma A.1) since this lemma turned out to be a very important tool for our considerations in Section 4.1. Its proof is essentially taken from the respective proof in [27]. However, we supplemented the proof by additional computations in order to make it more comprehensible to the reader. Thereafter in the subsequent Lemma A.2, we will show an analogon to Lemma A.1(ii) for *Schrödinger* potentials.

Lemma A.1. *Let $(V, W) \in L^2(F) \times L^2(F)$ and let g be a meromorphic function with finitely many poles on some open neighbourhood $\mathcal{U}$ of $F(V, W)/\Gamma^*$. Then there exists a meromorphic function $A_g^{sing}(p_1)$ from the complex plane $p_1 \in \mathbb{C}$ into the bounded operators from the Banach spaces $L^p(F) \times L^p(F)$ into $L^{p'}(F) \times L^{p'}(F)$ for all $1 < p' < p < \infty$ with the following properties:*

(i) For all $(n_1, n_2) \in \mathbb{Z}^2$ and all $p_1 \in \mathbb{C}$, the following identity is valid:

$$A_g^{sing}(p_1)\psi_{n_1\hat{\kappa}+n_2\check{\kappa}} = \psi_{n_1\hat{\kappa}+n_2\check{\kappa}} A_g^{sing}(p_1 + n_1).$$

(ii) The commutator $[A_g^{sing}(p_1), \widetilde{D}(V, W, p_1)]$ (with $\widetilde{D}(V, W, p_1)$ defined in (4.3)) does not depend on p_1 and there holds

$$[A_g^{sing}(p_1),\ \widetilde{D}(V, W, p_1)] = \begin{pmatrix} 0 & \frac{w_g}{\check{\kappa}_2 - i\check{\kappa}_1} \\ \frac{v_g}{\check{\kappa}_2 + i\check{\kappa}_1} & 0 \end{pmatrix}$$

with some functions $v_g, w_g \in L^2(F)$. In particular, the commutator is equal to $\frac{\partial \widetilde{D}(V,W,p_1)}{\partial (V,W)}(v_g, w_g)$.

(iii) For all $\delta v, \delta w \in L^2(F)$, there holds (4.20), i.e.

$$\Omega((\delta v, \delta w), (v_g, w_g)) = \sum_{\zeta \in \mathcal{U}} res_\zeta(g \cdot \omega(V, W, \delta v, \delta w)),$$

where Ω *denotes the symplectic form defined in* (4.19) *and* $\omega(V, W, \delta v, \delta w)$ *the holomorphic 1-form defined in* (4.18).

Proof. If g is a linear combination of two functions g_1, g_2 with associated functions $A^{sing}_{g_1}$ and $A^{sing}_{g_2}$ fulfilling conditions (i)-(iii), then the linear combination A^{sing}_g of the two associated functions has the required properties. Hence, we may assume that g has a pole at one point $[k'] \in F(V, W)/\Gamma^*$, where $[k] := \{k + \nu \,:\, \nu \in \Gamma^*\}$ for $k \in \mathbb{C}^2$. As in the proof of Lemma 4.1.2, let $\mathsf{F}(\cdot)$ denote the local sum of $g \cdot \mathsf{P}$ (with P as in (4.10)) over all sheets of $F(V, W)/\Gamma^*$ considered as a covering space over $p_1 \in \mathbb{C}$ which contains the element $[k']$ [1], cf. the proof of Lemma 4.1.2 for a more precise explanation of the construction of F. By definition, this is a meromorphic function from some neighbourhood of p_1' into the finite rank operators on $L^2(F) \times L^2(F)$ (recall the relation $k = p_1\hat{\kappa} + p_2\check{\kappa}$ between k- and p-coordinates as in Section 4.1). Therefore, the singular part $p_1 \mapsto \mathsf{F}^{sing}(p_1)$ is a meromorphic function on the whole plane $p_1 \in \mathbb{C}$. We claim that the infinite sum

$$A^{sing}_g(p_1) := \sum_{n \in \mathbb{Z}^2} \psi_{n_1\hat{\kappa}+n_2\check{\kappa}} \mathsf{F}^{sing}(p_1 + n_1)\psi_{-n_1\hat{\kappa}-n_2\check{\kappa}}$$

converges in the strong operator topology. By construction and due to [27, Theorem 2.3], the operator-valued function $\mathsf{F}^{sing}(p_1)$ is a finite sum of operators of the form $\widetilde{\chi} \mapsto \left\langle\left\langle \widetilde{\phi}, \widetilde{\chi} \right\rangle\right\rangle (p_1 - p_1')^{-l}\widetilde{\psi}$ with elements $\widetilde{\phi}$ and $\widetilde{\psi}$ of the Banach spaces $\bigcap_{q<\infty} L^q(F) \times L^q(F) \subset L^2(F) \times L^2(F)$ and with the bilinear form $\langle\langle \cdot, \cdot \rangle\rangle$ as defined in (4.11).
For $l \in \mathbb{N}$, $q_1 \in (0, 1)$ and $p_1 - p_1' \in \mathbb{C} \setminus \mathbb{Z}$, we now consider the series

$$d_l(p_1, q_1) := \sum_{n \in \mathbb{Z}} (p_1 - p_1' + n)^{-l} \exp(2\pi i n q_1).$$

For $l > 1$, the series obviously converges due to the convergence of $\sum_{n \in \mathbb{N}} n^{-l}$ for $l > 1$. So let's consider the case $l = 1$. We claim that

$$d_1(p_1, q_1) = \frac{2\pi i \exp(2\pi i(p_1' - p_1)q_1)}{1 - \exp(2\pi i(p_1' - p_1))} \quad \text{for all } q_1 \in (0, 1),\ p_1 - p_1' \in \mathbb{C} \setminus \mathbb{Z}. \quad \text{(A.1)}$$

Thereto, we have to show that the n^{th} Fourier coefficient of the Fourier decomposition of the right hand side of (A.1) (considered as a periodic function in q_1)

[1] Compare [27, Remark 2.19]: *"In the sequel we shall meet quite often complex spaces, which are locally biholomorphic to finite sheeted coverings over open subsets of* $\mathbb{C}^n$*. If we restrict these coverings to the preimage of small open balls, then different sheets without branch points are not connected with each other. However, in arbitrary small neighbourhoods of branch points several sheets are connected. In the sequel we shall call those sheets, whose restriction to arbitrary small neighbourhoods of a given element contain this element, the local sheets which contain this element."*

equals $1/(p_1 - p_1' + n)$. We compute with $\psi_n(q_1) := \exp(2\pi i n q_1)$

$$\int_0^1 \exp(2\pi i(p_1' - p_1)q_1)\psi_{-n}(q_1)dq_1 = \int_0^1 \exp(2\pi i(p_1' - p_1 - n)q_1)dq_1 =$$
$$= \frac{1}{2\pi i(p_1' - p_1 - n)}(\exp(2\pi i(p_1' - p_1))\underbrace{\exp(-2\pi i n)}_{=1} - 1).$$

This implies

$$\mathcal{F}\left(\frac{2\pi i \exp(2\pi i(p_1' - p_1)\,\cdot)}{1 - \exp 2\pi i(p_1' - p_1)}\right)(n) = \frac{\exp(2\pi i(p_1' - p_1)) - 1}{(1 - \exp(2\pi i(p_1' - p_1)))(p_1' - p_1 - n)} =$$
$$= \frac{1}{p_1 - p_1' + n},$$

which proves the claim. Moreover, we have

$$2\pi i(p_1 - p_1')\sum_{n\in\mathbb{Z}} \frac{\exp(2\pi i n q_1)}{(p_1 - p_1' + n)^{l+1}} + 2\pi i \sum_{n\in\mathbb{Z}} \frac{n \exp(2\pi i n q_1)}{(p_1 - p_1' + n)^{l+1}} =$$
$$= 2\pi i \sum_{n\in\mathbb{Z}} \frac{\exp(2\pi i n q_1)}{(p_1 - p_1' + n)^{l}} = 2\pi i d_l(p_1, q_1),$$

which implies the following recursion equation

$$2\pi i(p_1 - p_1')d_{l+1}(p_1, q_1) + \frac{\partial d_{l+1}(p_1, q_1)}{\partial q_1} = 2\pi i d_l(p_1, q_1). \tag{A.2}$$

We now define $\widetilde{d_l}(p_1, q_1) := \exp(2\pi i(p_1 - p_1')q_1)d_l(p_1, q_1)$. We have the following identitiy of derivatives:

$$\frac{\partial \widetilde{d}_{l+1}(p_1, q_1)}{\partial q_1} = \exp(2\pi i(p_1 - p_1')q_1)\left(\frac{\partial d_{l+1}(p_1, q_1)}{\partial q_1} + 2\pi i(p_1 - p_1')d_{l+1}(p_1, q_1)\right) =$$
$$\overset{(A.2)}{=} 2\pi i \exp(2\pi i(p_1 - p_1')q_1)d_l(p_1, q_1) = 2\pi i \widetilde{d_l}(p_1, q_1) = 2\pi i \frac{\partial}{\partial q_1}\int_0^{q_1} \widetilde{d_l}(p_1, r)dr.$$

Hence, $\widetilde{d}_{l+1}(p_1, q_1) - 2\pi i \int_0^{q_1} \widetilde{d_l}(p_1, r)dr$ is constant with respect to q_1. Therefore,

$$\begin{aligned} &\widetilde{d}_{l+1}(p_1, q_1) - 2\pi i \int_0^{q_1} \widetilde{d_l}(p_1, r)dr \overset{\text{set } q_1 = 0}{=} d_l(p_1, 0) = \\ &\overset{\text{set } q_1 = 1}{=} \exp(2\pi i(p_1 - p_1'))d_l(p_1, 1) - 2\pi i \int_0^1 \widetilde{d_l}(p_1, r)dr. \end{aligned} \tag{A.3}$$

Due to the Riemann-Lebesgue Lemma [26, Theorem IX.7] for $l > 1$, the functions $d_l(p_1, q_1)$ are continuous and periodic with period 1 with respect to q_1. Therefore, $d_l(p_1, 0) = d_l(p_1, 1)$. This yields together with (A.3)

$$d_l(p_1, 0) = \exp(2\pi i(p_1 - p_1'))d_l(p_1, 0) - 2\pi i \int_0^1 \widetilde{d_l}(p_1, r)dr$$

which implies $d_l(p_1,0) = \frac{2\pi i}{\exp(2\pi i(p_1-p_1'))-1}\int_0^1 \widetilde{d}_l(p_1,r)dr$. Finally, again due to (A.3),

$$\widetilde{d}_{l+1}(p_1,q_1) = 2\pi i\left(\int_0^{q_1}\widetilde{d}_l(p_1,r)dr + \frac{1}{\exp(2\pi i(p_1-p_1'))-1}\int_0^1 \widetilde{d}_l(p_1,r)dr\right). \tag{A.4}$$

Since $\widetilde{d}_1(p_1,q_1) = 2\pi i(1-\exp(2\pi i(p_1'-p_1)))^{-1}$, cf. (A.1), the functions $\widetilde{d}_l(p_1,q_1)$ are thus polynomials with respect to q_1 and $(\exp(2\pi i(p_1'-p_1))-1)^{-1}\widetilde{d}_1(p_1,q_1)$. In a next step, we show that the unique solution of (A.4) is given by the generating function

$$\sum_{l\in\mathbb{N}} t^{l-1}\widetilde{d}_l(p_1,q_1) = \frac{2\pi i(1-\exp(2\pi i(p_1'-p_1)))^{-1}\exp(2\pi i q_1 t)}{1-(\exp(2\pi i(p_1-p_1'))-1)^{-1}(\exp(2\pi i t)-1)}. \tag{A.5}$$

The recursion formula (A.4) is an equation of the following form with corresponding parameters $a,b,c\in\mathbb{C}$

$$f_{l+1}(x) = a\left(\int_0^x f_l(r)dr + b\int_0^1 f_l(r)dr\right) \quad \text{with } f_1(x)\equiv c. \tag{A.6}$$

This yields with the ansatz of a generating function $F(t,x) := \sum_{l=0}^{\infty} t^l f_{l+1}(x)$

$$\sum_{l=1}^{\infty} t^l f_{l+1}(x) = a\left(\sum_{l=1}^{\infty}\int_0^x f_l(r)t^l dr + b\sum_{l=1}^{\infty}\int_0^1 f_l(r)t^l dr\right) \Longleftrightarrow$$
$$\Longleftrightarrow F(t,x) - \underbrace{f_1(x)}_{=c} = a\left(t\int_0^x F(t,r)dr + bt\int_0^1 F(t,r)dr\right).$$

A solution of this integral equation is given by $F(t,x) := \frac{c\exp(atx)}{1-b(\exp(at)-1)}$ since

$$a\left(t\cdot\frac{c(\exp(atx)-1)}{at(1-b(\exp(at)-1))} + bt\cdot\frac{c(\exp(at)-1)}{at(1-b(\exp(at)-1))}\right) =$$
$$= \frac{c\exp(atx)-c+cb(\exp(at)-1)}{1-b(\exp(at)-1)} = \frac{c\exp(atx)}{1-b(\exp(at)-1)} - c.$$

Under the restriction that F fulfills the ansatz $F(t,x) = \sum_{l=0}^{\infty} t^l f_{l+1}(x)$, this solution is unique since $f_l(x)$ is for all $l\in\mathbb{N}$ uniquely defined by (A.6) provided that a,b and $c\equiv f_1(x)$ are given. By using the respective parameters a,b,c in (A.6) in our case, cf. (A.4), the representation (A.5) is thus proven. This representation shows that the functions $d_l(p_1,q_1)$ may be considered as meromorphic functions with respect to $p_1\in\mathbb{C}$ whose values are bounded functions of $q_1\in[0,1]$. We will prove the convergence of the infinite sum in A_g^{sing} (in the strong operator topology) by using Fourier decomposition. To this, we recall that due to the definition of Fourier transformation and the inverse Fourier transformation, namely

$$\hat{f}(n) = \int_0^1 e^{-2\pi inx} f(x)dx, \quad f(x) = \sum_{n\in\mathbb{Z}} e^{2\pi inx}\hat{f}(n), \quad \text{with } f \in L^2(\mathbb{R}/\mathbb{Z}),$$

there holds for $f, g \in L^2(\mathbb{R}/\mathbb{Z})$

$$\sum_{n\in\mathbb{Z}} e^{2\pi inx}\hat{f}(n)\hat{g}(n) = \sum_{n\in\mathbb{Z}} \int_0^1 e^{2\pi inx}\hat{f}(n)e^{-2\pi int}g(t)dt = \int_0^1 f(x-t)g(t)dt. \quad \text{(A.7)}$$

Hence, for $q_1 \in (0,1)$ and $q_2 \in \mathbb{R}/\mathbb{Z}$, there holds with $q := (q_1, q_2)$ and the identity $\sum_{n\in\mathbb{Z}} \exp(2\pi in(q_2 - q_2')) = \delta(q_2 - q_2')$ $(*)$

$$\left(\sum_{(n_1,n_2)\in\mathbb{Z}^2} \psi_{n_1\hat{\kappa}+n_2\check{\kappa}}\widetilde{\psi}\frac{\left\langle\left\langle\widetilde{\phi}, \psi_{-n_1\hat{\kappa}-n_2\check{\kappa}}\widetilde{\chi}\right\rangle\right\rangle}{(p_1 - p_1' + n_1)^l}\right)(q_1, q_2) =$$

$$= \widetilde{\psi}(q) \sum_{(n_1,n_2)\in\mathbb{Z}^2} \psi_{n_1\hat{\kappa}+n_2\check{\kappa}}(q)\frac{1}{(p_1 - p_1' + n_1)^l}\sum_{i=1}^2 \int_F \widetilde{\phi}_{3-i}(q')\psi_{-n_1\hat{\kappa}-n_2\check{\kappa}}(q')\widetilde{\chi}_i(q')dq' =$$

$$\overset{(*)}{=} C(\Gamma^*)\widetilde{\psi}(q)\sum_{n\in\mathbb{Z}} e^{2\pi inq_1}\frac{1}{(p_1 - p_1' + n)^l}\sum_{i=1}^2 \int_0^1 \widetilde{\phi}_{3-i}(q_1', q_2)e^{-2\pi inq_1'}\widetilde{\chi}_i(q_1', q_2)dq_1' =$$

$$= C(\Gamma^*)\widetilde{\psi}(q)\sum_{n\in\mathbb{Z}} e^{2\pi inq_1}\mathcal{F}(d_l(p_1, q_1))(n)\sum_{i=1}^2 \mathcal{F}(\widetilde{\phi}_{3-i}(\cdot, q_2)\widetilde{\chi}_i(\cdot, q_2))(n) =$$

$$\overset{\text{(A.7)}}{=} C(\Gamma^*)\int_0^1 d_l(p_1, q_1 - q_1')\begin{pmatrix}\widetilde{\psi}_1(q)\widetilde{\phi}_2(q_1', q_2)\widetilde{\chi}_1(q_1', q_2) + \widetilde{\psi}_1(q)\widetilde{\phi}_1(q_1', q_2)\widetilde{\chi}_2(q_1', q_2)\\ \widetilde{\psi}_2(q)\widetilde{\phi}_2(q_1', q_2)\widetilde{\chi}_1(q_1', q_2) + \widetilde{\psi}_2(q)\widetilde{\phi}_1(q_1', q_2)\widetilde{\chi}_2(q_1', q_2)\end{pmatrix} dq_1',$$

where the constant $C(\Gamma^*)$ only depends on the lattice Γ^* and appears due to a coordinate transformation $\hat{\kappa} \mapsto \binom{1}{0}$, $\check{\kappa} \mapsto \binom{0}{1}$ which we used in the above computation in order to apply the one-dimensional Fourier transform used in (A.7). Due to the properties of $d_l(p_1, q_1)$ discussed before, in particular the boundedness with respect to q_1, the above integral is finite. This proves the convergence of the infinite sum A_g^{sing}. Furthermore, due to Hölder's inequality [25, Theoem III.1(c)], the function A_g^{sing} is a meromorphic function from the complex plane $p_1 \in \mathbb{C}$ into the bounded operators from $L^p(F) \times L^p(F)$ to $L^{p'}(F) \times L^{p'}(F)$ for all $1 < p' < p < \infty$.

The transformation property (i) follows for all $m = (m_1, m_2) \in \mathbb{Z}^2$ from

$$\begin{aligned}\psi_{m_1\hat{\kappa}+m_2\check{\kappa}} A_g^{sing}(p_1 + m_1) &= \sum_{n\in\mathbb{Z}^2} \psi_{(m_1+n_1)\hat{\kappa}+(m_2+n_2)\check{\kappa}} \mathsf{F}^{sing}(p_1 + m_1 + n_1)\psi_{-n_1\hat{\kappa}-n_2\check{\kappa}} = \\ &= \sum_{n\in\mathbb{Z}^2} \psi_{n_1\hat{\kappa}+n_2\check{\kappa}} \mathsf{F}^{sing}(p_1 + n_1)\psi_{(m_1-n_1)\hat{\kappa}+(m_2-n_2)\check{\kappa}} = \\ &= A_g^{sing}(p_1)\psi_{m_1\hat{\kappa}+m_2\check{\kappa}},\end{aligned}$$

where in the second step, we used an index transformation $n = (n_1, n_2) \mapsto n - m$. In order to prove the commutator identity in (ii), we show in a first step the following transformation property of the Dirac operator, namely

$$\widetilde{D}(V, W, p_1)\psi_{n_1\hat{\kappa}+n_2\check{\kappa}} = \psi_{n_1\hat{\kappa}+n_2\check{\kappa}}(\widetilde{D}(V, W, p_1 + n_1) + n_2\pi\mathbb{1}). \tag{A.8}$$

Since the off-diagonal entries of $\widetilde{D}(V, W, p_1)$ are multiplication operators and are independent of p_1, cf. (4.3), the transformation (A.8) obviously holds for the off-diagonal entries of $\widetilde{D}(V, W, p_1)$. So let's consider the diagonal entries. For some test function χ, we have

$$\begin{aligned}-\bar{\partial}(\psi_{n_1\hat{\kappa}+n_2\check{\kappa}}\chi) &= -\frac{1}{2}(\partial_{x_1} + i\partial_{x_2})(e^{2\pi i\langle n_1\hat{\kappa}+n_2\check{\kappa},x\rangle}\chi) = \\ &= e^{2\pi i\langle n_1\hat{\kappa}+n_2\check{\kappa},x\rangle}(-\pi i(n_1\hat{\kappa}_1 + n_2\check{\kappa}_1 + i(n_1\hat{\kappa}_2 + n_2\check{\kappa}_2))\chi - \bar{\partial}\chi) = \\ &= e^{2\pi i\langle n_1\hat{\kappa}+n_2\check{\kappa},x\rangle}(\pi(n_1(\hat{\kappa}_2 - i\hat{\kappa}_1) + n_2(\check{\kappa}_2 - i\check{\kappa}_1))\chi - \bar{\partial}\chi).\end{aligned}$$

Hence,

$$\frac{p_1\pi(\hat{\kappa}_2 - i\hat{\kappa}_1) - \bar{\partial}}{\check{\kappa}_2 - i\check{\kappa}_1}(\psi_{n_1\hat{\kappa}+n_2\check{\kappa}}\chi) = \psi_{n_1\hat{\kappa}+n_2\check{\kappa}}\left(\frac{(p_1 + n_1)\pi(\hat{\kappa}_2 - i\hat{\kappa}_1) - \bar{\partial}}{\check{\kappa}_2 - i\check{\kappa}_1} + n_2\pi\right)\chi.$$

Analogously, we compute for the second diagonal term

$$\begin{aligned}\partial(\psi_{n_1\hat{\kappa}+n_2\check{\kappa}}\chi) &= \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})(e^{2\pi i\langle n_1\hat{\kappa}+n_2\check{\kappa},x\rangle}\chi) = \\ &= e^{2\pi i\langle n_1\hat{\kappa}+n_2\check{\kappa},x\rangle}(\pi i(n_1\hat{\kappa}_1 + n_2\check{\kappa}_1 - i(n_1\hat{\kappa}_2 + n_2\check{\kappa}_2))\chi + \partial\chi) = \\ &= e^{2\pi i\langle n_1\hat{\kappa}+n_2\check{\kappa},x\rangle}(\pi(n_1(\hat{\kappa}_2 + i\hat{\kappa}_1) + n_2(\check{\kappa}_2 + i\check{\kappa}_1))\chi + \partial\chi)\end{aligned}$$

and hence,

$$\frac{p_1\pi(\hat{\kappa}_2 + i\hat{\kappa}_1) + \partial}{\check{\kappa}_2 + i\check{\kappa}_1}(\psi_{n_1\hat{\kappa}+n_2\check{\kappa}}\chi) = \psi_{n_1\hat{\kappa}+n_2\check{\kappa}}\left(\frac{(p_1 + n_1)\pi(\hat{\kappa}_2 + i\hat{\kappa}_1) + \partial}{\check{\kappa}_2 + i\check{\kappa}_1} + n_2\pi\right)\chi.$$

This proves (A.8). Therefore, by writing for simplicity $\widetilde{D}(p_1) = \widetilde{D}(V, W, p_1)$, the commutator is equal to

$$[A_g^{sing}(p_1), \widetilde{D}(p_1)] = \sum_{n\in\mathbb{Z}^2} \left[\psi_{n_1\hat{\kappa}+n_2\check{\kappa}} \mathsf{F}^{sing}(p_1+n_1)\psi_{-n_1\hat{\kappa}-n_2\check{\kappa}}, \widetilde{D}(p_1)\right] =$$

$$\overset{(A.8)}{=} \sum_{n\in\mathbb{Z}^2} \left[\psi_{n_1\hat{\kappa}+n_2\check{\kappa}} \mathsf{F}^{sing}(p_1+n_1)\psi_{-n_1\hat{\kappa}-n_2\check{\kappa}}, \psi_{n_1\hat{\kappa}+n_2\check{\kappa}}\widetilde{D}(p_1+n_1)\psi_{-n_1\hat{\kappa}-n_2\check{\kappa}}\right] =$$

$$= \sum_{n\in\mathbb{Z}^2} \psi_{n_1\hat{\kappa}+n_2\check{\kappa}} \left[\mathsf{F}^{sing}(p_1+n_1), \widetilde{D}(p_1+n_1)\right] \psi_{-n_1\hat{\kappa}-n_2\check{\kappa}}.$$

The operator valued functions $\mathsf{F}(\cdot)$ and $\widetilde{D}(\cdot)$ commute pointwise because the projection P is the spectral projection of $\widetilde{D}(\cdot)$ [2]. Hence,

$$[\mathsf{F}^{sing}(p_1), \widetilde{D}(p_1)] = -[\mathsf{F}^{hol}(p_1), \widetilde{D}(p_1)], \tag{A.9}$$

where $\mathsf{F}^{hol}(\cdot)$ denotes the holomorphic part in the Laurent expansion of $\mathsf{F}(\cdot)$. The Dirac operator

$$\widetilde{D}(p_1) = \begin{pmatrix} \frac{(p_1-p_1')\pi(\hat{\kappa}_2-i\hat{\kappa}_1)-\bar{\partial}}{\check{\kappa}_2-i\check{\kappa}_1} & \frac{W}{\check{\kappa}_2-i\check{\kappa}_1} \\ \frac{V}{\check{\kappa}_2+i\check{\kappa}_1} & \frac{(p_1-p_1')\pi(\hat{\kappa}_2+i\hat{\kappa}_1)+\partial}{\check{\kappa}_2+i\check{\kappa}_1} \end{pmatrix} + \begin{pmatrix} \frac{p_1'\pi(\hat{\kappa}_2-i\hat{\kappa}_1)}{\check{\kappa}_2-i\check{\kappa}_1} & 0 \\ 0 & \frac{p_1'\pi(\hat{\kappa}_2+i\hat{\kappa}_1)}{\check{\kappa}_2+i\check{\kappa}_1} \end{pmatrix}$$

has only linear and constant terms with respect to p_1-p_1'. Therefore, the powers of p_1-p_1' in the left hand side of (A.9) are terms $(p_1-p_1')^j$ for integers $j\le 0$ and the powers of p_1-p_1' in the right hand side of (A.9) are terms $(p_1-p_1')^j$ for integers $j\ge 0$. In order that (A.9) can be satisfied, the commutators corresponding to all integers $j\ne 0$ must necessarily vanish such that $j=0$ is the only remaining integer corresponding to non-trivial commutators. More precisely, denoting by F_{-1} the residue of the operator-valued form $\mathsf{F}(p_1)dp_1$ at the pole $p_1=p_1'$, we get

$$[A_g^{sing}(p_1), \widetilde{D}(p_1)] = \pi \sum_{\kappa\in\Gamma^*} \psi_\kappa \left[\mathsf{F}_{-1}, \begin{pmatrix} \frac{\hat{\kappa}_2-i\hat{\kappa}_1}{\check{\kappa}_2-i\check{\kappa}_1} & 0 \\ 0 & \frac{\hat{\kappa}_2+i\hat{\kappa}_1}{\check{\kappa}_2+i\check{\kappa}_1} \end{pmatrix}\right] \psi_{-\kappa}. \tag{A.10}$$

This shows that the commutator $[A_g^{sing}(p_1), \widetilde{D}(p_1)]$ does not depend on p_1. Set $d_1 := \frac{\hat{\kappa}_2-i\hat{\kappa}_1}{\check{\kappa}_2-i\check{\kappa}_1}$, $d_2 := \frac{\hat{\kappa}_2+i\hat{\kappa}_1}{\check{\kappa}_2+i\check{\kappa}_1}$ and $D := \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$. We decompose $\mathsf{F}_{-1} =: \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$ into its diagonal and off-diagonal part $\mathsf{F}_{-1} = \mathsf{F}_{-1}^{diag} + \mathsf{F}_{-1}^{off} =: \begin{pmatrix} f_{11} & 0 \\ 0 & f_{22} \end{pmatrix} + \begin{pmatrix} 0 & f_{12} \\ f_{21} & 0 \end{pmatrix}$. Clearly, $\left[\mathsf{F}_{-1}^{diag}, D\right] = 0$ since D is diagonal and d_1, d_2 are constant multiplication operators. Hence, we may replace in (A.10) F_{-1} by F_{-1}^{off}. Due to $\begin{pmatrix} 0 & f_{12} \\ f_{21} & 0 \end{pmatrix}\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} = \begin{pmatrix} 0 & f_{12}d_2 \\ f_{21}d_1 & 0 \end{pmatrix}$ and $\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}\begin{pmatrix} 0 & f_{12} \\ f_{21} & 0 \end{pmatrix} = \begin{pmatrix} 0 & f_{12}d_1 \\ f_{21}d_2 & 0 \end{pmatrix}$, the diagonal elements in (A.10) vanish. Moreover, $\left[\mathsf{F}_{-1}^{off}, D\right] = \begin{pmatrix} 0 & f_{12}(d_2-d_1) \\ f_{21}(d_1-d_2) & 0 \end{pmatrix}$. We now compute the

[2]See also the proof of Lemma 4.1.2 where we explicitly showed that $\widetilde{D}(\cdot)$ and $\mathsf{F}(\cdot)$ share the same eigenfunction $\widetilde{\psi}$.

off-diagonal elements of (A.10) explicitly. Thereto, we use $\psi_\kappa[\mathsf{F}_{-1}, D]\psi_{-\kappa} = [\psi_\kappa\mathsf{F}_{-1}\psi_{-\kappa}, D]$ for all $\kappa \in \Gamma^*$ which follows again from the fact that the entries of D are constant multiplication operators. Now recall that, by definition, for $\chi \in L^2(F) \times L^2(F)$ and $\kappa \in \Gamma^*$, $\psi_\kappa\mathsf{F}_{-1}\psi_{-\kappa}(\chi)$ is of the form

$$\psi_\kappa\widetilde{\psi}\left\langle\left\langle\widetilde{\phi}, \psi_{-\kappa}\chi\right\rangle\right\rangle = \psi_\kappa\widetilde{\psi}\int_F(\widetilde{\phi}_2(x')\psi_{-\kappa}(x')\chi_1(x') + \widetilde{\phi}_1(x')\psi_{-\kappa}(x')\chi_2(x'))dx'$$

evaluated at $p_1 = p_1'$ if we assume that the pole p_1' of g is of order one and that P is holomorphic at p_1' [3]. Since $\sum_{\kappa\in\Gamma^*}\psi_\kappa(x)$ is equal to $\mu(F)\delta(x)$ with Dirac's δ-function $\delta(x)$, we have

$$\sum_{\kappa\in\Gamma^*}\psi_\kappa\widetilde{\psi}\left\langle\left\langle\widetilde{\phi}, \psi_{-\kappa}\chi\right\rangle\right\rangle = \mu(F)\widetilde{\psi}(x)\int_F\delta(x-x')(\widetilde{\phi}_2(x')\chi_1(x') + \widetilde{\phi}_1(x')\chi_2(x'))dx' =$$

$$= \mu(F)\widetilde{\psi}(x)(\widetilde{\phi}_2(x)\chi_1(x) + \widetilde{\phi}_1(x)\chi_2(x)) = \mu(F)\begin{pmatrix}\widetilde{\psi}_1\widetilde{\phi}_2 & \widetilde{\psi}_1\widetilde{\phi}_1\\ \widetilde{\psi}_2\widetilde{\phi}_2 & \widetilde{\psi}_2\widetilde{\phi}_1\end{pmatrix}\chi.$$

Hence, for the eigenfunctions $\widetilde{\phi}, \widetilde{\psi}$, the infinite sum $\chi \mapsto \sum_{\kappa\in\Gamma^*}\psi_\kappa\widetilde{\psi}\left\langle\left\langle\widetilde{\phi}, \psi_{-\kappa}\chi\right\rangle\right\rangle$ converges in the strong operator norm topology to the operator $\mu(F)\begin{pmatrix}\widetilde{\psi}_1\widetilde{\phi}_2 & \widetilde{\psi}_1\widetilde{\phi}_1\\ \widetilde{\psi}_2\widetilde{\phi}_2 & \widetilde{\psi}_2\widetilde{\phi}_1\end{pmatrix}$, where the functions in the entries of this matrix are considered as operators of multiplication with these functions. Therefore, there is an $\alpha \in \mathbb{C}\setminus\{0\}$ such that $[A_g^{sing}(p_1), \widetilde{D}(p_1)] = \alpha\mu(F)\pi\begin{pmatrix}0 & (d_2-d_1)\widetilde{\psi}_1\widetilde{\phi}_1\\ (d_1-d_2)\widetilde{\psi}_2\widetilde{\phi}_2 & 0\end{pmatrix}$ with d_1, d_2 the entries of the diagonal matrix in (A.10) as defined before. Since the variation of the Dirac operator $\widetilde{D}(V, W, p_1)$ with respect to (V, W) equals the linear operator $\frac{\partial\widetilde{D}(V,W,p_1)}{\partial(V,W)} : (\delta v, \delta w) \mapsto \begin{pmatrix}0 & \frac{\delta w}{\breve{\kappa}_2 - i\breve{\kappa}_1}\\ \frac{\delta v}{\breve{\kappa}_2+i\breve{\kappa}_1} & 0\end{pmatrix}$, there are unique functions $v_g, w_g \in L^2(F)$ such that

$$[A_g^{sing}(p_1),\ \widetilde{D}(V, W, p_1)] = \frac{\partial\widetilde{D}(V, W, p_1)}{\partial(V, W)}(v_g, w_g) = \begin{pmatrix}0 & \frac{w_g}{\breve{\kappa}_2-i\breve{\kappa}_1}\\ \frac{v_g}{\breve{\kappa}_2+i\breve{\kappa}_1} & 0\end{pmatrix}.$$

More precisely,

$$v_g := -\alpha\mu(F)\pi(d_2-d_1)(\breve{\kappa}_2+i\breve{\kappa}_1)\widetilde{\psi}_2\widetilde{\phi}_2, \quad w_g := \alpha\mu(F)\pi(d_2-d_1)(\breve{\kappa}_2-i\breve{\kappa}_1)\widetilde{\psi}_1\widetilde{\phi}_1 \tag{A.11}$$

[3] This is clearly a special case. For our purposes in Section 4.1, however, the reduction to these assumptions is justified since the meromorphic function g considered there has only poles of first order. Moreover, since possible zeroes of the denominator of P are discrete points on the Fermi curve, we may choose the A-cycles in (4.8) such that possible poles of P are circumnavigated.

with the respective choice of eigenfunction $\widetilde{\psi}$ and dual eigenfunction $\widetilde{\phi}$ as before according to the term F_{-1}. This proves assertion (ii).
In [27, Lemma 3.1], it has been shown that the holomorphic 1-forms (4.18) fulfill the trace formula

$$\omega(V,W,\delta v,\delta w) = \operatorname{tr}\left(\mathsf{P}\begin{pmatrix} 0 & \frac{\delta w}{\breve{\kappa}_2 - i\breve{\kappa}_1} \\ \frac{\delta v}{\breve{\kappa}_2 + i\breve{\kappa}_1} & 0 \end{pmatrix}\right)\frac{dp_1}{\pi}.$$

Here, the trace of an operator $T : L^2(F) \times L^2(F) \to L^2(F) \times L^2(F)$, $\chi \mapsto \left(\begin{smallmatrix} T_{11} & T_{12} \\ T_{21} & T_{22}\end{smallmatrix}\right)\chi$ is defined by (compare [30, Definition VI.5.7] for the definition of the trace operator in functional analysis)

$$\operatorname{tr}(T) := \frac{1}{\mu(F)}\sum_{\kappa\in\Gamma^*}\langle (T_{11}+T_{22})\psi_\kappa, \psi_\kappa\rangle_{L^2(F)} = \frac{1}{\mu(F)}\sum_{\kappa\in\Gamma^*}\int_F \psi_\kappa (T_{11}+T_{22})\psi_{-\kappa},$$

where the property that $\frac{1}{\sqrt{\mu(F)}}\psi_\kappa$, $\kappa \in \Gamma^*$, defines an *orthonormal* (Schauder) basis of $L^2(F)$ is essential.
Now, the total residue of the form $g\cdot\omega(V,W,\delta v,\delta w)$ on $\mathcal{U}$ is equal to

$$\sum_{\zeta\in\mathcal{U}} res_\zeta(g\cdot\omega(V,W,\delta v,\delta w)) = \sum_{\zeta\in\mathcal{U}} res_\zeta\left(\operatorname{tr}\left(g\cdot\mathsf{P}\begin{pmatrix} 0 & \frac{\delta w}{\breve{\kappa}_2 - i\breve{\kappa}_1} \\ \frac{\delta v}{\breve{\kappa}_2 + i\breve{\kappa}_1} & 0 \end{pmatrix}\right)\frac{dp_1}{\pi}\right) =$$
$$= \operatorname{tr}\left(res_{p_1=p_1'}\left(\mathsf{F}(p_1)\begin{pmatrix} 0 & \frac{\delta w}{\breve{\kappa}_2 - i\breve{\kappa}_1} \\ \frac{\delta v}{\breve{\kappa}_2 + i\breve{\kappa}_1} & 0 \end{pmatrix}\frac{dp_1}{\pi}\right)\right) = \frac{1}{\pi}\operatorname{tr}\left(\mathsf{F}_{-1}\begin{pmatrix} 0 & \frac{\delta w}{\breve{\kappa}_2 - i\breve{\kappa}_1} \\ \frac{\delta v}{\breve{\kappa}_2 + i\breve{\kappa}_1} & 0 \end{pmatrix}\right) =$$
$$= \frac{1}{\pi\mu(F)}\sum_{\kappa\in\Gamma^*}\int_F \psi_\kappa\left(f_{12}\frac{\delta v}{\breve{\kappa}_2 + i\breve{\kappa}_1} + f_{21}\frac{\delta w}{\breve{\kappa}_2 - i\breve{\kappa}_1}\right)\psi_{-\kappa}.$$

We already showed in the proof of (ii) that for $\chi \in L^2(F)\times L^2(F)$, there holds for some $\alpha \in \mathbb{C}\setminus\{0\}$

$$(\mathsf{F}_{-1}\chi)(x) = \alpha\widetilde{\psi}(x)\int_F (\widetilde{\phi}_2(x')\chi_1(x') + \widetilde{\phi}_1(x')\chi_2(x'))dx' =$$
$$= \alpha\int_F \begin{pmatrix} \widetilde{\psi}_1(x)\widetilde{\phi}_2(x') & \widetilde{\psi}_1(x)\widetilde{\phi}_1(x') \\ \widetilde{\psi}_2(x)\widetilde{\phi}_2(x') & \widetilde{\psi}_2(x)\widetilde{\phi}_1(x') \end{pmatrix}\begin{pmatrix}\chi_1(x') \\ \chi_2(x')\end{pmatrix}dx'.$$

Therefore, again due to $\sum_{\kappa\in\Gamma^*}\psi_\kappa(x) = \mu(F)\delta(x)$,

$$\frac{1}{\pi\mu(F)}\sum_{\kappa\in\Gamma^*}\int_F \psi_\kappa(x)\left(f_{12}(x)\frac{\delta v(x)}{\breve{\kappa}_2 + i\breve{\kappa}_1} + f_{21}(x)\frac{\delta w(x)}{\breve{\kappa}_2 - i\breve{\kappa}_1}\right)\psi_{-\kappa}(x)dx =$$
$$= \frac{\alpha}{\pi\mu(F)}\sum_{\kappa\in\Gamma^*}\int_F\int_F \psi_\kappa(x)\left(\frac{\widetilde{\psi}_1(x)\widetilde{\phi}_1(x')\delta v(x')}{\breve{\kappa}_2 + i\breve{\kappa}_1} + \frac{\widetilde{\psi}_2(x)\widetilde{\phi}_2(x')\delta w(x')}{\breve{\kappa}_2 - i\breve{\kappa}_1}\right)\psi_{-\kappa}(x')dx'dx =$$
$$= \frac{\alpha}{\pi}\int_F\int_F \delta(x-x')\left(\frac{\widetilde{\psi}_1(x)\widetilde{\phi}_1(x')\delta v(x')}{\breve{\kappa}_2 + i\breve{\kappa}_1} + \frac{\widetilde{\psi}_2(x)\widetilde{\phi}_2(x')\delta w(x')}{\breve{\kappa}_2 - i\breve{\kappa}_1}\right)dx'dx =$$

$$= \frac{\alpha}{\pi}\int_F \left(\frac{\widetilde{\psi}_1(x)\widetilde{\phi}_1(x)\delta v(x)}{\check{\kappa}_2 + i\check{\kappa}_1} + \frac{\widetilde{\psi}_2(x)\widetilde{\phi}_2(x)\delta w(x)}{\check{\kappa}_2 - i\check{\kappa}_1} \right) dx =$$

$$\overset{(A.11)}{=} \frac{1}{\pi^2 \mu(F)(d_2 - d_1)|\check{\kappa}|^2} \int_F (w_g(x)\delta v(x) - v_g(x)\delta w(x))\, dx.$$

In order to finally show assertion (iii), it remains to be proved that $d_2 - d_1 = \frac{2i}{\mu(F)|\check{\kappa}|^2}$. We compute

$$d_2 - d_1 = \frac{\hat{\kappa}_2 + i\hat{\kappa}_1}{\check{\kappa}_2 + i\check{\kappa}_1} - \frac{\hat{\kappa}_2 - i\hat{\kappa}_1}{\check{\kappa}_2 - i\check{\kappa}_1} = \frac{1}{|\check{\kappa}|^2}((\hat{\kappa}_2 + i\hat{\kappa}_1)(\check{\kappa}_2 - i\check{\kappa}_1) - (\hat{\kappa}_2 - i\hat{\kappa}_1)(\check{\kappa}_2 + i\check{\kappa}_1)) =$$
$$= \frac{2i}{|\check{\kappa}|^2}(\hat{\kappa}_1\check{\kappa}_2 - \hat{\kappa}_2\check{\kappa}_1) = \frac{2i}{|\check{\kappa}|^2}\det(\hat{\kappa}, \check{\kappa}) = \frac{2i}{|\check{\kappa}|^2 \mu(F)},$$

where in the last step, we made use of the fact that the generators $\hat{\gamma}, \check{\gamma} \in \mathbb{C}^2$ of the lattice Γ can be chosen such that (written as columns of a 2×2-matrix) $(\hat{\gamma}, \check{\gamma})^T(\hat{\kappa}, \check{\kappa}) = \mathbb{1}$, cf. [27, p. 41] [4]. Hence, $\mu(F) = \det(\hat{\gamma}, \check{\gamma}) = \det((\hat{\kappa}, \check{\kappa})^{-1}) = \frac{1}{\det(\hat{\kappa}, \check{\kappa})}$. Finally, assertion (iii) is proved. □

In the case that (V, W) is a Schrödinger potential, the Fermi curve is symmetric with respect to the holomorphic involution $\sigma : \mathbb{C}^2 \to \mathbb{C}^2$, $k \mapsto -k$, cf. Section 2.3. If we choose the meromorphic function g in Lemma A.1 anti-symmetric with respect to σ as well, we can achieve $v_g = 0$ as the following lemma shows. Moreover, we show another result for symmetric g. These are new results.

Lemma A.2. *Let $u \in L^2(F)$, $(V, W) := \left(1, \frac{-u}{4}\right)$ and let g be a meromorphic function with finitely many poles on some open neighbourhood $\mathcal{U}$ of $F(u)/\Gamma^* = F(V, W)/\Gamma^*$. If g satisfies $g = -g \circ \sigma$, i.e. g is anti-symmetric with respect to the holomorphic involution σ, then there is a function $w_g \in L^2(F)$ such that the operator $A_g^{sing}(p_1)$ obtained in Lemma A.1 satisfies the commutator relation*

$$[A_g^{sing}(p_1),\ \widetilde{D}(V, W, p_1)] = \begin{pmatrix} 0 & \frac{w_g}{\check{\kappa}_2 - i\check{\kappa}_1} \\ 0 & 0 \end{pmatrix},$$

that is, v_g obtained in Lemma A.1(ii) vanishes. If, on the other hand, g is symmetric (i.e. $g = g \circ \sigma$) and $v_g = 0$, then also $w_g = 0$.

Proof. We have to show that under the conditions $(V, W) := \left(1, \frac{-u}{4}\right)$ and $g = -g \circ \sigma$, the function v_g obtained in Lemma A.1(ii) vanishes identically. Due to (A.11) and $g = -g \circ \sigma$, we know that v_g is a linear combination of functions $\left\langle\left\langle \widetilde{\phi}(k), \widetilde{\psi}(k) \right\rangle\right\rangle^{-1} (\widetilde{\phi}_2\widetilde{\psi}_2)(k) + \left\langle\left\langle \widetilde{\phi}(-k), \widetilde{\psi}(-k) \right\rangle\right\rangle^{-1} (\widetilde{\phi}_2\widetilde{\psi}_2)(-k)$ for suitable values $k \in F(u)/\Gamma^*$. Whereas in (A.11), we absorbed the denominator

[4] Actually, it is the other way round: We firstly choose generators $\hat{\gamma}, \check{\gamma}$ of Γ (where we can choose without restriction $\hat{\gamma} = \binom{1}{0}$ as already explained in a footnote after the definition of (4.10)) and afterwards generators $\hat{\kappa}, \check{\kappa}$ of Γ^* satisfying $(\hat{\gamma}, \check{\gamma})^T(\hat{\kappa}, \check{\kappa}) = \mathbb{1}$.

$\langle\langle\phi(k),\psi(k)\rangle\rangle = \left\langle\left\langle\psi_k\widetilde{\phi}(k),\psi_{-k}\widetilde{\psi}(k)\right\rangle\right\rangle = \left\langle\left\langle\widetilde{\phi}(k),\widetilde{\psi}(k)\right\rangle\right\rangle$ of (4.10) into the factor α which is independent on $x\in\mathbb{R}^2$ but dependent on $k\in F(u)/\Gamma^*$, we now have to see how this denominator behaves under the involution $k\mapsto -k$. To this, we need the relation between the components $\widetilde{\psi}_1,\widetilde{\psi}_2$ and $\widetilde{\phi}_1,\widetilde{\phi}_2$ of the eigenfunction $\widetilde{\psi}$ and the dual eigenfunction $\widetilde{\phi}$. We firstly recall that $\widetilde{\psi}$ and $\widetilde{\phi}$ are eigenfunction and dual eigenfunction of the operator $D_k(V,W)$ (4.4). Due to $(V,W)=(1,\frac{-u}{4})$, we have the relation $\widetilde{\psi}_1=-\partial_k\widetilde{\psi}_2$ and $-\bar{\partial}_k\widetilde{\psi}_1-\frac{u}{4}\widetilde{\psi}_2=0$ between the components of the eigenfunction of $D_k(1,-\frac{u}{4})$ which yields the Schrödinger equation $-\Delta_k\widetilde{\psi}_2+u\widetilde{\psi}_2=0$, cf. the proof of Lemma 4.1.1. Since $\partial_k^T=-\partial_{-k}$ and $\bar{\partial}_k^T=-\bar{\partial}_{-k}$, the transposed Dirac operator is given by

$$D_k^T(1,-\tfrac{u}{4}) := \begin{pmatrix} 1 & \bar{\partial}_{-k} \\ -\partial_{-k} & \frac{-u}{4}\end{pmatrix}.$$

Hence, the Dirac equation for the dual eigenfunction $D_k^T(1,-\frac{u}{4})\widetilde{\phi}=0$ is given by the equations

$$\widetilde{\phi}_1+\bar{\partial}_{-k}\widetilde{\phi}_2=0,\quad -\partial_{-k}\widetilde{\phi}_1-\tfrac{u}{4}\widetilde{\phi}_2=0$$

yielding the Schrödinger equation $-\Delta_{-k}\widetilde{\phi}_2+u\widetilde{\phi}_2=0$ for the dual eigenfunction $\widetilde{\phi}_2$. Here, we rediscover $(-\Delta_k+u)^T=-\Delta_{-k}+u$ already well-known from the beginning of Section 2.3. Thus, we have the relation $\widetilde{\phi}_2(k,x)=\widetilde{\psi}_2(-k,x)$ between eigenfunction $\widetilde{\psi}_2$ and dual eigenfunction $\widetilde{\phi}_2$ of the Schrödinger equation. Hence,

$$\widetilde{\phi}_1(k,x)=-\bar{\partial}_{-k}\widetilde{\phi}_2(k,x)=-\bar{\partial}_{-k}\widetilde{\psi}_2(-k,x). \tag{A.12}$$

Now, we can compute the transformation behaviour of the denominator of (4.10) with respect to the involution $\sigma: k\mapsto -k$. We have

$$\begin{aligned}\left\langle\left\langle\widetilde{\phi}(k,x),\widetilde{\psi}(k,x)\right\rangle\right\rangle &= \left\langle\widetilde{\phi}_2(k,x),\widetilde{\psi}_1(k,x)\right\rangle+\left\langle\widetilde{\phi}_1(k,x),\widetilde{\psi}_2(k,x)\right\rangle = \\ &= -\left\langle\widetilde{\psi}_2(-k,x),\partial_k\widetilde{\psi}_2(k,x)\right\rangle-\left\langle\bar{\partial}_{-k}\widetilde{\psi}_2(-k,x),\widetilde{\psi}_2(k,x)\right\rangle\end{aligned}$$

and

$$\begin{aligned}\left\langle\left\langle\widetilde{\phi}(-k,x),\widetilde{\psi}(-k,x)\right\rangle\right\rangle &= -\left\langle\widetilde{\psi}_2(k,x),\partial_{-k}\widetilde{\psi}_2(-k,x)\right\rangle-\left\langle\bar{\partial}_k\widetilde{\psi}_2(k,x),\widetilde{\psi}_2(-k,x)\right\rangle = \\ &= \left\langle\partial_k\widetilde{\psi}_2(k,x),\widetilde{\psi}_2(-k,x)\right\rangle+\left\langle\widetilde{\psi}_2(k,x),\bar{\partial}_{-k}\widetilde{\psi}_2(-k,x)\right\rangle,\end{aligned}$$

where in the last step, we applied integration by parts and used the periodicity of the function $\widetilde{\psi}$. Hence, the denominator of (4.10) $\left\langle\left\langle\widetilde{\phi}(k,x),\widetilde{\psi}(k,x)\right\rangle\right\rangle$ is anti-symmetric with respect to the involution σ. Moreover, due to $g=-g\circ\sigma$, the residue of F_{-1} remains invariant under the involution σ. Therefore, v_g is a linear combination of functions of the form $(\widetilde{\phi}_2\widetilde{\psi}_2)(k,x) - (\widetilde{\phi}_2\widetilde{\psi}_2)(-k,x)$ for suitable values $k \in F(u)/\Gamma^*$. In order to show $v_g = 0$, we have to prove that $(\widetilde{\phi}_2\widetilde{\psi}_2)(k,x)$ is symmetric with respect to σ. This, however, immediately follows from $(\widetilde{\phi}_2\widetilde{\psi}_2)(k,x) = \widetilde{\psi}_2(-k,x)\widetilde{\psi}_2(k,x)$ due to $\widetilde{\phi}_2(k,x) = \widetilde{\psi}_2(-k,x)$ shown above. This proves the first claim of the lemma.

Now, let g be symmetric and $v_g = 0$. We have to show that this implies $w_g = 0$. Due to (A.11) and the symmetry properties of the denominator $\left\langle\left\langle\widetilde{\phi}(k,x),\widetilde{\psi}(k,x)\right\rangle\right\rangle$ shown above, w_g is a linear combination of functions of the form $\widetilde{\phi}_1(k)\widetilde{\psi}_1(k) + \widetilde{\phi}_1(-k)\widetilde{\psi}_1(-k)$. Due to (A.12), we have

$$\widetilde{\phi}_1(k)\widetilde{\psi}_1(k) = \bar{\partial}_{-k}\widetilde{\psi}_2(-k)\partial_k\widetilde{\psi}_2(k).$$

The function v_g is a linear combination (over the same k as in the linear combination of w_g) of functions $\widetilde{\psi}_2(k)\widetilde{\psi}_2(-k) + \widetilde{\psi}_2(-k)\widetilde{\psi}_2(k) = 2\widetilde{\psi}_2(k)\widetilde{\psi}_2(-k)$. If we speak in the following of *the* linear combination, we always mean this same linear combination. We now compute with the notation $k_\pm := \pi i(k_1 \pm ik_2)$ (i.e. the Wirtinger operators read as $\partial_k = \partial + k_-$, $\bar{\partial}_k = \bar{\partial} + k_+$)

$$\begin{aligned}&\partial_k\widetilde{\psi}_2(k)\bar{\partial}_{-k}\widetilde{\psi}_2(-k) + \partial_{-k}\widetilde{\psi}_2(-k)\bar{\partial}_k\widetilde{\psi}_2(k) =\\ &= \partial\widetilde{\psi}_2(k)\bar{\partial}\widetilde{\psi}_2(-k) + k_-\widetilde{\psi}_2(k)\bar{\partial}\widetilde{\psi}_2(-k) - k_+\partial\widetilde{\psi}_2(k)\widetilde{\psi}_2(-k) - k_-k_+\widetilde{\psi}_2(k)\widetilde{\psi}_2(-k) +\\ &+ \partial\widetilde{\psi}_2(-k)\bar{\partial}\widetilde{\psi}_2(k) - k_-\widetilde{\psi}_2(-k)\bar{\partial}\widetilde{\psi}_2(k) + k_+\partial\widetilde{\psi}_2(-k)\widetilde{\psi}_2(k) - k_-k_+\widetilde{\psi}_2(k)\widetilde{\psi}_2(-k).\end{aligned} \tag{A.13}$$

Furthermore, we have the equations

$$\widetilde{\psi}_2(-k)\partial_k\bar{\partial}_k\widetilde{\psi}_2(k) = \widetilde{\psi}_2(-k)\left(\partial\bar{\partial}\widetilde{\psi}_2(k) + k_-\bar{\partial}\widetilde{\psi}_2(k) + k_+\partial\widetilde{\psi}_2(k) + k_-k_+\widetilde{\psi}_2(k)\right), \tag{A.14}$$

$$\begin{aligned}&\widetilde{\psi}_2(k)\partial_{-k}\bar{\partial}_{-k}\widetilde{\psi}_2(-k) =\\ &\qquad \widetilde{\psi}_2(k)\left(\partial\bar{\partial}\widetilde{\psi}_2(-k) - k_-\bar{\partial}\widetilde{\psi}_2(-k) - k_+\partial\widetilde{\psi}_2(-k) + k_-k_+\widetilde{\psi}_2(-k)\right)\end{aligned} \tag{A.15}$$

and

$$\begin{aligned}&\partial\bar{\partial}(\widetilde{\psi}_2(k)\widetilde{\psi}_2(-k)) =\\ &= (\partial\bar{\partial}\widetilde{\psi}_2(k))\widetilde{\psi}_2(-k) + \bar{\partial}\widetilde{\psi}_2(k)\partial\widetilde{\psi}_2(-k) + \partial\widetilde{\psi}_2(k)\bar{\partial}\widetilde{\psi}_2(-k) + \widetilde{\psi}_2(k)\partial\bar{\partial}\widetilde{\psi}_2(-k)\end{aligned} \tag{A.16}$$

Now, the right hand side of (A.13) equals the right hand side of equation (A.16) minus the sum of the right hand sides of equations (A.14) and (A.15). The above mentioned linear combination of the left hand sides of (A.14) and (A.15) equals zero because of $\widetilde{\psi}_2(-k)\partial_k\bar{\partial}_k\widetilde{\psi}_2(k) = \frac{u}{4}\widetilde{\psi}_2(-k)\widetilde{\psi}_2(k)$ and $\widetilde{\psi}_2(k)\partial_{-k}\bar{\partial}_{-k}\widetilde{\psi}_2(-k) = \frac{u}{4}\widetilde{\psi}_2(k)\widetilde{\psi}_2(-k)$ and the condition that the linear combination of $\widetilde{\psi}_2(k)\widetilde{\psi}_2(-k)$ equals zero due to $v_g = 0$. By the same argument together with the fact that the derivative of a constant function equals zero, the linear combination over the left hand side of (A.16) vanishes, too. Therefore, the linear combination over the left hand side of (A.13) vanishes which had to be proved. □

Appendix B

An alternative proof of Theorem 4.1.5

In this appendix, we give an alternative proof of Theorem 4.1.5. Before, we need a lemma which provides us an equivalence between the choice $\delta v = const.$ and a relation between pole divisors of the eigenfunctions of the Dirac operator. Let us recap more precisely how such a divisor is defined: To a given potential $u \in L^2(F)$, one can assign a divisor (cf. [6, 16.1]) $D(u)$, where $D(u)$ is the pole divisor of the eigenfunction $x \mapsto \psi(k,x)$ of the Schrödinger operator at $k \in F(u)/\Gamma^*$ normalized by $\psi(k,0) = 1$, $k \in F(u)/\Gamma^*$, i.e. the support of the divisor $D(u)$ is given by

$$\begin{aligned} \mathrm{supp} D(u) = \{k \in F(u)/\Gamma^* : & \text{ there is an eigenfunction } 0 \not\equiv \psi(k,\cdot) \text{ of } -\Delta_k + u \\ & \text{with eigenvalue } \lambda = 0 \text{ and } \psi(k,0) = 0\}. \end{aligned}$$

We consider in the following Schrödinger potentials $(V,W) := (1, \frac{-u}{4})$. We denote by D_1 the divisor corresponding to (V,W) in the Dirac operator $D_k(V,W)$ (4.4) and by D_2 the divisor corresponding to the transposed Dirac operator. To a variation $(\delta v, \delta w) \in L^2(F) \times L^2(F)$, we assign variations δD_1 and δD_2, respectively. With this notation, we can prove the following lemma.

Lemma B.1. *Let $(V,W) := (1, \frac{-u}{4})$ with variations $(\delta v, \delta w) \in L^2(F) \times L^2(F)$ at (V,W) and corresponding divisor variations δD_1 and δD_2 be given. Then there holds the equivalence*

$$\delta v = const. \Longleftrightarrow \sigma(\delta D_1) = \delta D_2.$$

Proof. Let $(V,W) := (1, \frac{-u}{4})$. We consider the Dirac operator (4.4) and its transposed operator

$$D_k(V,W) = \begin{pmatrix} V & \partial_k \\ -\bar{\partial}_k & W \end{pmatrix}, \quad D_k^T(V,W) = \begin{pmatrix} V & \bar{\partial}_{-k} \\ -\partial_{-k} & W \end{pmatrix}.$$

Denote by $\psi(k,x) = (\psi_1(k,x), \psi_2(k,x))$ the eigenfunction of $D_k(V,W)$ with eigenvalue zero normalized by $\psi_2(k,0) = 1$ and by $\phi(k,x) = (\phi_1(k,x), \phi_2(k,x))$ the corresponding eigenfunction of the transposed operator with normalization $\phi_2(k,0) = 1$. As in the proof of Lemma 4.1.1, ψ_2 fulfills the Schrödinger equation $-\Delta_k\psi_2 + u\psi_2 = 0$ and ϕ_2 fulfills $-\Delta_{-k}\phi_2 + u\phi_2 = 0$ (recall $(\Delta_k + u)^T = \Delta_{-k} + u$, cf. Section 2.3). We therefore get the relation $\phi_2(k,x) = \psi_2(-k,x)$ between eigenfunction and dual eigenfunction of the Schrödinger equation. The relation $\sigma(\delta D_1) = \delta D_2$ is obviously equivalent to $\delta\psi_2(-k,x) = \delta\phi_2(k,x)$. It thus suffices to prove the equivalence $\delta v = const. \iff \delta\psi_2(-k,x) = \delta\phi_2(k,x)$. So let us compute the variations $\delta\psi_2$ and $\delta\phi_2$. The Dirac equation for $(V,W) = (1, \frac{-u}{4})$ is given by

$$\psi_1 + \partial_k\psi_2 = 0, \quad -\bar{\partial}_k\psi_1 + W\psi_2 = 0$$

This yields the variational equations (note $V = 1$)

$$(\delta v)\psi_1 + \delta\psi_1 + (\delta\partial_k)\psi_2 + \partial_k(\delta\psi_2) = 0, \quad (-\delta\bar{\partial}_k)\psi_1 - \bar{\partial}_k\delta\psi_1 + (\delta w)\psi_2 + W\delta\psi_2 = 0.$$

Hence,

$$\delta\psi_2 = -\partial_k^{-1}\left((\delta v)\psi_1 + \delta\psi_1 + (\delta\partial_k)\psi_2\right), \quad \delta\psi_1 = \bar{\partial}_k^{-1}\left((-\delta\bar{\partial}_k)\psi_1 + (\delta w)\psi_2 + W\delta\psi_2\right).$$

$$\begin{aligned}
&\Longrightarrow \delta\psi_2 = -\partial_k^{-1}\left((\delta v)\psi_1 + \bar{\partial}_k^{-1}\left((-\delta\bar{\partial}_k)\psi_1 + (\delta w)\psi_2 + W\delta\psi_2\right) + (\delta\partial_k)\psi_2\right)\\
&\Longrightarrow \underbrace{(\mathbb{1} + \partial_k^{-1}\bar{\partial}_k^{-1}W)}_{=\partial_k^{-1}\bar{\partial}_k^{-1}(\bar{\partial}_k\partial_k + W)}\delta\psi_2 = -\partial_k^{-1}\left((\delta v)\psi_1 + \bar{\partial}_k^{-1}\left((-\delta\bar{\partial}_k)\psi_1 + (\delta w)\psi_2\right) + (\delta\partial_k)\psi_2\right)\\
&\Longrightarrow (\bar{\partial}_k\partial_k + W)\delta\psi_2 = -\bar{\partial}_k((\delta v)\psi_1) + (\delta\bar{\partial}_k)\psi_1 - (\delta w)\psi_2 - \bar{\partial}_k(\delta\partial_k)\psi_2.
\end{aligned}$$

Together with $\psi_1 = -\partial_k\psi_2$, we get

$$\begin{aligned}
(\bar{\partial}_k\partial_k + W)\delta\psi_2 &= \bar{\partial}_k((\delta v)\partial_k\psi_2) - (\delta\bar{\partial}_k)\partial_k\psi_2 - (\delta w)\psi_2 - \bar{\partial}_k(\delta\partial_k)\psi_2 =\\
&= \bar{\partial}_k((\delta v)\partial_k\psi_2) - \delta(\bar{\partial}_k\partial_k)\psi_2 - (\delta w)\psi_2.
\end{aligned}$$

For $\delta\phi_2$, we get completely analogously (by interchanging the operators $\bar{\partial}_k$ and ∂_k by ∂_{-k} and $\bar{\partial}_{-k}$, respectively) due to $\bar{\partial}_k\partial_k = \partial_k\bar{\partial}_k$

$$\begin{aligned}
(\bar{\partial}_{-k}\partial_{-k} + W)\delta\phi_2(k,x) &= \left[\partial_{-k}((\delta v)\bar{\partial}_{-k}\phi_2) - \delta(\bar{\partial}_{-k}\partial_{-k})\phi_2 - (\delta w)\phi_2\right](k,x) =\\
&= \left[\partial_{-k}((\delta v)\bar{\partial}_{-k}\psi_2) - \delta(\bar{\partial}_{-k}\partial_{-k})\psi_2 - (\delta w)\psi_2\right](-k,x).
\end{aligned}$$

If $\delta v = const.$, then $\delta\psi_2(-k,x) = \delta\phi_2(k,x)$ follows. Conversely, if $\delta\psi_2(-k,x) = \delta\phi_2(k,x)$, we have the identity $\bar{\partial}_k((\delta v)\partial_k\psi_2) = \partial_k((\delta v)\bar{\partial}_k\psi_2)$ which can only be fulfilled if $\delta v = const.$ The lemma is proved. □

Theorem B.2 (= Theorem 4.1.5). *Let $u \in L^2(F)$ and $(V,W) := (1, \frac{-u}{4})$ with smooth Fermi curve $F(V,W)/\Gamma^*$. Then for all $N \in \mathbb{N}$, there exist holomorphic 1-forms ω_κ, $\kappa \in \Gamma_N^*/\sigma$, on $F(V,W)/\Gamma^*$ such that for all $\nu \in \Gamma^*$ with $\nu \in \Gamma_N^*/\sigma$, there holds*

$$\int_{A_\nu} \omega_\kappa = \delta_{\kappa,\nu}.$$

Furthermore, these ω_κ can be chosen to be of the form (4.18) *with suitable respective directions* $(0, \delta w) \in L^2(F) \times L^2(F)$. *In particular, the direction δv in* (4.18) *can be chosen to be zero.*

Proof. The first part of the proof is exactly the same as that of Theorem 4.1.5. Only for the proof of (4.31), namely

$$(\{0\} \times L^2(F)) \cap U = U^-,$$

we use an alternative procedure. The inclusion "$\supseteq$" is again trivial and follows from (4.29). We consider at first the special case that u is a finite type potential. In finite type theory, there is a 1-1-correspondence between isospectral potentials u of $F(u)/\Gamma^*$ and divisors $D(u)$, cf. [19, Section II.5]. If we denote again by D_1 the divisor corresponding to the Dirac operator with potential (V,W) and by D_2 the divisor corresponding to the transposed Dirac operator, Schrödinger potentials $(V,W) = (1, \frac{-u}{4})$ can be characterized by $D_2 = \sigma(D_1)$ due to the relation $u^T = u$ already known from the beginning of Section 2.3. As before, let δD_1 be the corresponding variation of D_1 and δD_2 be the variation of $D_2 = \sigma(D_1)$. In [19, Lemma 4.13], the following divisor relation has been shown:

$$D_1 + D_2 \simeq K + Q^+ + Q^-.$$

Here, K denotes the canonical divisor on the Fermi curve and Q^+ and Q^- are some marked points at infinity which yield the two-point-compactification of the (normalized) Fermi curve. In particular, K, Q^+ and Q^- are invariants of the Fermi curve. Therefore, by considering isospectral variations $(\delta v, \delta w) \in L^2(F) \times L^2(F)$ with corresponding (isospectral) variations of the divisors δD_1 and δD_2, we get

$$\delta D_1 + \delta D_2 = 0 \quad \text{modulo linear equivalence.} \tag{B.1}$$

The variation δD_1 can be considered as a set of tangent vectors on the Fermi curve at the points of $\operatorname{supp}(D_1)$. Hence, to each $p \in \operatorname{supp}(D_1)$, we can assign such a tangent vector $\dot{p}$. We would like to represent the relation (B.1) in Jacobi coordinates. More precisely, with $\operatorname{supp}(D_1) =: (p_i)_{i=1,\dots,g}$, with g the genus of the finite type Fermi curve, we consider the Abel map (cf. [6, 21.8])

$$(p_i)_{i=1,\dots,g} \mapsto \left(\sum_{i=1}^{g} \int_{a_i}^{p_i} \omega_k \right)_{k=1,\dots,g} \qquad \text{mod } H_1(X, \mathbb{Z}),$$

where the ω_k are basis vectors of the space of holomorphic differential forms $\Omega(X)$ on the (compactified) Fermi curve $X := F(u)/\Gamma^*$ and the a_i are given points on X. The corresponding tangent map is then given by

$$\delta D_1 = (\dot{p}_i)_{i=1,\dots,g} \mapsto \left(\sum_{i=1}^{g} c_i \omega_k(p_i) \right)_{k=1,\dots,g},$$

where $\omega_k(p_i)$ means here that the 1-form ω_k is evaluated at the point p_i and c_i are the corresponding coefficients of $\dot{p}_i$ with respect to a given basis. For $\omega \in \Omega(X)$, we set $\omega(V, W, \delta D_1) := \sum_{i=1}^{g} c_i \omega(p_i)$. With this notation, (B.1) is equivalent to

$$\omega(V, W, \delta D_1) + \omega(V, W, \delta D_2) = 0 \quad \text{for all } \omega \in \Omega(X). \tag{B.2}$$

Moreover, due to (B.1), the variation δD_2 is uniquely defined by D_1, D_2 and δD_1 (modulo linear equivalence).
We now prove (4.31). Let $(0, w_g) \in (\{0\} \times L^2(F)) \cap U$ be given and let δD_1 be the corresponding variation in terms of divisors and δD_2 the variation corresponding to the transposed Dirac operator as considered before. Since $(0, w_g)$ is an isospectral variation, cf. [27, Lemma 3.2.(ii)], there holds (B.2). Due to Lemma B.1, we have $\sigma(\delta D_1) = \delta D_2$. Hence, (B.2) reads as

$$\omega(V, W, \delta D_1) + \omega(V, W, \sigma(\delta D_1)) = 0 \quad \text{for all } \omega \in \Omega(X). \tag{B.3}$$

In order to prove that $(0, \delta w) \in U^-$, we show that $g + g \circ \sigma = 0$. We may write $g = \frac{1}{2}(g + g \circ \sigma) + \frac{1}{2}(g - g \circ \sigma)$. We denote the symmetric part of g by $g_s := \frac{1}{2}(g + g \circ \sigma)$. By the linearity of the mapping $g \mapsto (v_g, w_g) \mapsto \omega(V, W, v_g, w_g)$, this yields a corresponding decomposition of $\omega = \omega(V, W, 0, w_g)$ into symmetric and anti-symmetric part, namely $\omega = \frac{1}{2}(\omega + \omega \circ \sigma) + \frac{1}{2}(\omega - \omega \circ \sigma)$. We show that the symmetric part vanishes identically. To this, let $(\delta v, \delta w) \in L^2(F) \times L^2(F)$ and $\omega = \omega(V, W, \delta v, \delta w)$ with $\omega = \omega \circ \sigma$. The relation (B.3) yields $\omega(V, W, \delta D_1) = 0$. By definition, $\omega(V, W, \delta D_1)$ is just the right hand side of (4.20) with g_s. Since both ω and g_s are symmetric and ω was chosen arbitrary (of course, with the restriction that it is symmetric), the residue can only be equal to zero if $g_s = 0$. This proves $(0, w_g) \in U^-$. Hence, (4.31) is proven in the finite type case.
For the general infinite type case, it remains to prove the inclusion "$\subseteq$" in (4.31). We use an approximation of finite type potentials. By Theorem 2.4.2, in every neighbourhood in $L^2(F)$ of some potential $u \in L^2(F)$, there are potentials v with the property that all but finitely many of their perturbed Fourier coefficients are equal to zero. In other words, the finite type potentials are dense in $L^2(F)$ and there exists a sequence of finite type potentials $(u_n)_{n \in \mathbb{N}}$ converging to u. Let $(1, W)$ with $W := -\frac{u}{4}$ be given as before. To this, we consider a sequence $(1, W_n)_{n \in \mathbb{N}}$ of finite type potentials converging to $(1, W)$.
Let $(0, w_g) \in (\{0\} \times L^2(F)) \cap U$ be given. Since the Fermi curves $F(1, W_n)$ are all Fermi curves of Schrödinger potentials, they are invariant with respect to

the involution σ. We may associate a sequence g_n of meromorphic functions on suitable open subsets of $F(1, W_n)$ which converge to g as $n \to \infty$. Now, by the foregoing finite type proof, g_n is anti-symmetric with respect to σ for each g_n in $(0, w_{g_n})$. This property carries over to g if we carry out the limit $n \to \infty$. Hence, $(0, w_g) \in U^-$ which had to be proven. This finally proves that the map α (4.28) is surjective and thus an isomorphism. Hence, the identity (4.25) $\Omega_+ = \Omega_0$ follows. The rest of the proof is again exactly the rest of the proof of Theorem 4.1.5. $\square$

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