Bachelor Thesis

The period lattice of solutions of the sinh-Gordon equation of spectral genus 2

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1 Introduction

The elliptic, non-linear sinh-Gordon equation is given by

$$\Delta u + \sinh(2u) = 0,$$

for twice partially differentiable functions $u : \mathbb{R}^2 \to \mathbb{R}$ where Δ is the Laplacian of $\mathbb{R}^2$ with respect to the Euclidean metric. This differential equation arises in the context of surface theory. There, it is possible to describe constant mean curvature tori using the solutions of the sinh-Gordon equation. Pinkall and Sterling constructed finite-type solutions in terms of two commuting flows on a certain space of matrix-valued polynomials [7]. These polynomials are called potentials and their degree is related to the genus of a naturally assigned algebraic curve. In this thesis, we investigate solutions of spectral genus two. These solutions are doubly periodic and hence, they are associated to a period lattice [4]. The goal of this work is to connect the period lattices and the spectral curves based on the Whithamdeformations from [2] and [5].

In section two we introduce basic definitions which are essential to understand the following results. Furthermore, we summarize important facts about the connection between the set of potentials, polynomial Killing fields and isospectral sets.
In this context, we consider a related system of ordinary differential equations, whose monodromies give insight into the connection between the generators of the period lattice and polynomials associated with the Whitham deformations. These observations in the third part allow us to give a short outline of the underlying theory of constant-mean-curvature surfaces and spectral curves at the beginning of section four.
Once it is clear why it is useful to consider the connection between Whitham deformations, spectral curves, and the period lattice, especially in our case, we use the main part of the thesis to show that the equations of the Whitham deformations can be solved uniquely by introducing reasonable additional constraints. The solution form a vector field which we also calculate explicitly. Furthermore, we obtain a similar vector field by solving a related but different initial value problem. This sheds light on the dependence of the period lattice on the responding spectral curves.
The conclusion summarizes the results of chapter four and give some remarks on how the work could be carried forward. In particular, it seems possible to calculate the dependence of the period lattice from the spectral curve explicitly.

2 Fundamentals

In this section, we introduce basic definitions based on [1]. Furthermore, we summarize the most important results of [4] and [6]. The parametrization of the spectral-genus-two family of solutions of the sinh-Gordon equation is defined on the set of so-called potentials. The solutions of the Lax equations are so-called polynomial Killing fields and the solutions of the sinh-Gordon equation are parametrized by them. Once we introduced the respective definitions, we conclude the main results with the sources for the solutions of the Lax-equations, the corresponding induced group action, and the so-called isospectral sets. Using these, we try to understand the connection between the polynomial $a \in \mathcal{M}_2^1$ and the period lattice.

Definition 2.1. *Let M be a differentiable manifold and p a point in M. Then, two smooth curves c_0 and c_1 through p are called equivalent if*

$$\frac{d(x \circ c_0)}{dt}(0) = \frac{d(x \circ c_1)}{dt}(0)$$

with respect to a chart x around p. This formulation is independent of the particular choice of x and defines an equivalence relation on the set of smooth curves through p. We call $\frac{d(x \circ c_0)}{dt}(0)$ tangent vector of M in the point p, also known as foot point. The set T_pM of all tangent vectors of M in p is called the tangent space of M in p, the set $TM = \bigcup_{p \in M} T_pM$ of all tangent vectors of M is the tangent bundle of M.

Definition 2.2. *A subset $L \subseteq M$ is called l-dimensional diffentiable submanifold of M if, for every $p \in L$, there exists a chart $x : U \to U' \times U''$ of M around p where $U' \subset \mathbb{R}^l$ and $U'' \subset \mathbb{R}^{m-l}$ are open with $0 \in U''$ and $x(U \cap L) = U' \times \{0\}$.*

Definition 2.3. *Let W be a subset of a differentiable manifold M. Then, differential forms of degree k are indexed families ω of alternating k-linear maps $\omega(p) : (T_pM)^k \to \mathbb{R}, p \in W$. They are also called k-forms.*

Definition 2.4.

The set of potentials is the following set of cubic polynomials with matrix-valued coefficients:

$$\mathcal{P}_2 := \left\{ \zeta_\lambda = \begin{pmatrix} 0 & -\gamma^{-1} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \lambda + \begin{pmatrix} -\overline{\alpha} & -\gamma \\ -\overline{\beta} & \bar{\alpha} \end{pmatrix} \lambda^2 \right.$$
$$\left. + \begin{pmatrix} 0 & 0 \\ -\gamma^{-1} & 0 \end{pmatrix} \lambda^3 \mid \alpha, \beta \in \mathbb{C}, \gamma \in \mathbb{R}^+ \right\}$$

where $\lambda \in \mathbb{C}$ is called the *spectral parameter*.
Every $\zeta_\lambda \in \mathcal{P}_2$ can be written as

$$\zeta_\lambda = \begin{pmatrix} \alpha\lambda - \overline{\alpha}\lambda^2 & -\gamma^{-1} + \beta\lambda - \gamma\lambda^2 \\ \gamma\lambda - \overline{\beta}\lambda^2 + \gamma^{-1}\lambda^3 & -\alpha\lambda + \overline{\alpha}\lambda^2 \end{pmatrix} \tag{2.1}$$

and satisfies the reality condition

$$\lambda^3 \overline{\zeta}^t_{1/\overline{\lambda}} = -\zeta_\lambda \tag{2.2}$$

Later, it will be convenient to write ζ_λ abstractly as $\begin{pmatrix} A(\lambda) & B(\lambda) \\ \lambda C(\lambda) & -A(\lambda) \end{pmatrix}$ with complex polynomials $A(\lambda), B(\lambda), C(\lambda)$ of maximal degree two.
Now we can define the polynomial Killing fields on the set of potentials:

Definition 2.5.

Polynomial Killing fields are maps $\zeta_\lambda : \mathbb{R}^2 \to \mathcal{P}_2, (x,y) \mapsto \zeta_\lambda(x,y)$ which solve the *Lax equations*

$$\frac{\partial \zeta_\lambda}{\partial x} = [\zeta_\lambda, U(\zeta_\lambda)] \qquad \frac{\partial \zeta_\lambda}{\partial y} = [\zeta_\lambda, V(\zeta_\lambda)] \tag{2.3}$$

with $\zeta_\lambda(0) = \zeta^0_\lambda \in \mathcal{P}_2$ and

$$U(\zeta_\lambda) := \begin{pmatrix} \frac{\alpha - \overline{\alpha}}{2} & -\gamma^{-1}\lambda^{-1} - \gamma \\ \gamma + \gamma^{-1}\lambda & \frac{\overline{\alpha} - \alpha}{2} \end{pmatrix}$$
$$V(\zeta_\lambda) := i \begin{pmatrix} \frac{\alpha + \overline{\alpha}}{2} & -\gamma^{-1}\lambda^{-1} + \gamma \\ \gamma - \gamma^{-1}\lambda & -\frac{\alpha + \overline{\alpha}}{2} \end{pmatrix}$$

Due to the definition of the set of potentials, we can see that any $\zeta_\lambda \in \mathcal{P}_2$ is composed of a uniquely defined triplet

$$\alpha = (\zeta_\lambda)_\alpha \in \mathbb{C},\ \beta = (\zeta_\lambda)_\beta \in \mathbb{C},\ \gamma = (\zeta_\lambda)_\gamma \in \mathbb{R}^+.$$

Accordingly, the condition to the maps ζ_λ satisfying the differential equations above can be translated in equivalent requirements for the maps α, β, γ, namely that they satisfy some other uniquely defined differential equations. Thus, a polynomial Killing field also induces the following triple of functions

$$\alpha : \mathbb{R}^2 \to \mathbb{C}, (x,y) \mapsto (\zeta_\lambda(x,y))_\alpha$$
$$\beta : \mathbb{R}^2 \to \mathbb{C}, (x,y) \mapsto (\zeta_\lambda(x,y))_\beta$$
$$\gamma : \mathbb{R}^2 \to \mathbb{R}^+, (x,y) \mapsto (\zeta_\lambda(x,y))_\gamma$$

which have to satisfy the following lemma:

Lemma 2.6. *Let* ζ_λ *be a polynomial Killing field. Then, the entries* $\alpha, \beta : \mathbb{R}^2 \to \mathbb{C}$ *and* $\gamma : \mathbb{R}^2 \to \mathbb{R}$ *satisfy the modified Lax equations*

$$\frac{\partial \alpha}{\partial x} = \gamma^2 + \beta\gamma - \overline{\beta}\gamma^{-1} - \gamma^{-2} \qquad \frac{\partial \alpha}{\partial y} = i(\gamma^{-2} + \beta\gamma - \overline{\beta}\gamma^{-1} - \gamma^2)$$
$$\frac{\partial \beta}{\partial x} = -\alpha\beta + \overline{\alpha}\beta - 2\alpha\gamma + 2\overline{\alpha}\gamma^{-1} \qquad \frac{\partial \beta}{\partial y} = i(-\alpha\beta - \overline{\alpha}\beta + 2\alpha\gamma + 2\overline{\alpha}\gamma^{-1})$$
$$\frac{\partial \gamma}{\partial x} = -\alpha\gamma - \overline{\alpha}\gamma \qquad \frac{\partial \gamma}{\partial y} = i(\overline{\alpha}\gamma - \alpha\gamma).$$

Proof. This lemma is a consequence of the commutators structure above and the calculations in [4]. ☐

Remark 2.7.

We conclude for the partial derivatives with respect to x and y of γ:

$$\begin{aligned} \frac{\partial \gamma}{\partial x} &= -\alpha\gamma - \overline{\alpha}\gamma = -2\gamma\Re(\alpha) \in \mathbb{R} \\ \frac{\partial \gamma}{\partial y} &= i(\overline{\alpha}\gamma - \alpha\gamma) = 2\gamma\Im(\alpha) \in \mathbb{R}. \end{aligned} \tag{2.4}$$

Based on [4], we can see that the local flows $\phi_E(x)$ and $\phi_F(x)$ obtained by the Lax equations 2.3 commute and we also get the following equation:

$$[V(\zeta_\lambda), U(\zeta_\lambda)] + \frac{\partial U(\zeta_\lambda)}{\partial y} - \frac{\partial V(\zeta_\lambda)}{\partial x} = 0, \tag{2.5}$$

$$\frac{\partial^2 \zeta_\lambda}{\partial x \partial y} = \frac{\partial^2 \zeta_\lambda}{\partial y \partial x}$$

Here, the commutator equals

$$\begin{aligned} [V(\zeta_\lambda), U(\zeta_\lambda)] &= V(\zeta_\lambda)U(\zeta_\lambda) - U(\zeta_\lambda)V(\zeta_\lambda) \\ &= i\begin{pmatrix} 2(\gamma^2 - \gamma^{-2}) & -2\gamma^{-1}\overline{\alpha}\lambda^{-1} - 2\gamma\alpha \\ -2\gamma\overline{\alpha} - 2\gamma^{-1}\alpha\lambda & 2(\gamma^{-2} - \gamma^2) \end{pmatrix} \\ &= -(\frac{\partial U(\zeta_\lambda)}{\partial y} - \frac{\partial V(\zeta_\lambda)}{\partial x}) \end{aligned} \tag{2.6}$$

The equation 2.5 is called *Maurer-Cartan equation* and can be turned into the sinh-Gordon equation using the following procedure: First, we define the coordinate

$$z := x + iy.$$

and express the x- and y-coordinate in terms of z

$$x = \frac{1}{2}(z + \overline{z}) \qquad y = -\frac{i}{2}(z - \overline{z}).$$

Now, we specify the derivative with respect to z using the chain rule

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}) \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$

Furthermore, we define the variable

$$u := \ln\gamma \;\Leftrightarrow\; e^u = \gamma.$$

In order to calculate the partial derivative with respect to the new coordinate z, we use the results in 2.4:

$$\frac{\partial u}{\partial x} = \frac{1}{\gamma}\frac{\partial \gamma}{\partial x} = -(\alpha + \overline{\alpha}) \qquad \frac{\partial u}{\partial y} = \frac{1}{\gamma}\frac{\partial \gamma}{\partial y} = i(\overline{\alpha} - \alpha)$$

and hence, we obtain

$$\frac{\partial u}{\partial z} = -\alpha \qquad \frac{\partial u}{\partial \overline{z}} = -\overline{\alpha}.$$

In the next step, we express the Maurer-Cartan equation in terms of u and derivatives of u with respect to $z, \overline{z}$. They will be denoted as $u_z, u_{\overline{z}}$. With equation 2.6, we obtain:

$$[V(\zeta_\lambda), U(\zeta_\lambda)] = i\begin{pmatrix} 2(e^{2u} - e^{-2u}) & 2e^{-u}u_{\overline{z}}\lambda^{-1} + 2e^u u_z \\ 2e^u u_{\overline{z}} + 2e^{-u}u_z\lambda & 2(e^{-2u} - e^{2u}) \end{pmatrix} \tag{2.7}$$

To calculate the derivatives of $U(\zeta_\lambda), V(\zeta_\lambda)$, we rewrite the matrices as

$$U(\zeta_\lambda) = \begin{pmatrix} \frac{\alpha - \overline{\alpha}}{2} & -\gamma^{-1}\lambda^{-1} - \gamma \\ \gamma + \gamma^{-1}\lambda & \frac{\alpha - \overline{\alpha}}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(u_{\overline{z}} - u_z) & -e^{-u}\lambda^{-1} - e^u \\ e^u + e^{-u}\lambda & \frac{1}{2}(u_z - u_{\overline{z}}) \end{pmatrix}$$

$$V(\zeta_\lambda) := i\begin{pmatrix} \frac{\alpha + \overline{\alpha}}{2} & -\gamma^{-1}\lambda^{-1} + \gamma \\ \gamma - \gamma^{-1}\lambda & -\frac{\alpha + \overline{\alpha}}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}(u_{\overline{z}} + u_z) & -e^{-u}\lambda^{-1} + e^u \\ e^u - e^{-u}\lambda & \frac{1}{2}(u_z + u_{\overline{z}}) \end{pmatrix}$$

and use the formulas

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}} \qquad = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial y} = -i\left(\frac{\partial}{\partial \overline{z}} - \frac{\partial}{\partial z}\right) \qquad = -\frac{i}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) - \frac{i}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

in order to calculate

$$\frac{\partial U(\zeta_\lambda)}{\partial y} = -i \begin{pmatrix} \frac{1}{2}(u_{\overline{z}\overline{z}} - 2u_{\overline{z}z} + u_{zz}) & -\lambda^{-1}e^{-u}(u_z - u_{\overline{z}}) - e^u(u_{\overline{z}} - u_z) \\ e^u(u_{\overline{z}} - u_z) + \lambda e^{-u}(u_z - u_{\overline{z}}) & \frac{1}{2}(-u_{\overline{z}\overline{z}} + 2u_{\overline{z}z} - u_{zz}) \end{pmatrix}$$

$$\frac{\partial V(\zeta_\lambda)}{\partial x} = i \begin{pmatrix} -\frac{1}{2}(u_{\overline{z}\overline{z}} + 2u_{\overline{z}z} + u_{zz}) & \lambda^{-1}e^{-u}(u_z + u_{\overline{z}}) + e^u(u_{\overline{z}} + u_z) \\ e^u(u_{\overline{z}} + u_z) + \lambda e^{-u}(u_z + u_{\overline{z}}) & \frac{1}{2}(u_{\overline{z}\overline{z}} + 2u_{\overline{z}z} + u_{zz}) \end{pmatrix}.$$

This yields

$$\frac{\partial V(\zeta_\lambda)}{\partial x} - \frac{\partial U(\zeta_\lambda)}{\partial y} = i \begin{pmatrix} -2u_{\overline{z}z} & 2\lambda^{-1}e^{-u}u_{\overline{z}} + 2e^u u_z \\ 2e^u u_{\overline{z}} + 2\lambda e^{-u}u_z & 2u_{\overline{z}z} \end{pmatrix} \tag{2.8}$$

Thus, the Maurer-Cartan equation is satisfied if and only if 2.7 equals 2.8. This observation gives us the condition:

$$\begin{aligned} u_{\overline{z}z} = (e^{-2u} - e^{2u}) \quad &\Leftrightarrow \quad u_{\overline{z}z} + sinh(2u) = 0 \\ &\Leftrightarrow \quad \frac{1}{4}\Delta u + sinh(2u) = 0. \end{aligned}$$

The last equivalence is given by

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (u_{zz} + 2u_{\overline{z}z} + u_{\overline{z}\overline{z}}) - (u_{zz} - 2u_{\overline{z}z} + u_{\overline{z}\overline{z}}) = 4u_{\overline{z}z}.$$

By choosing $z' = \frac{1}{2}z$ and implementing the analogous calculations, we obtain the sinh-Gordon equation from the Maurer-Cartan equation

$$\Delta u + sinh(2u) = 0.$$

This calculation explains the context of analyzing potentials and polynomial killing fields and the sinh-Gordon equation.
In the next step, we investigate the determinant of such $\zeta_\lambda \in \mathcal{P}_2$. Therefore, we define the polynomials $a \in \mathbb{C}^4[\lambda]$ of fourth degree as

$$\det(\zeta_\lambda) = \lambda a(\lambda).$$

Due to the calculation of the determinant of $\zeta_\lambda \in \mathcal{P}_2$, we obtain for a:

$$\begin{aligned} \det(\zeta_\lambda) &= -A(\lambda)^2 + \lambda C(\lambda)B(\lambda) \\ &= -\lambda^2(\alpha - \overline{\alpha}\lambda)^2 - \lambda(-\gamma^{-1} + \beta\lambda - \gamma\lambda^2)(\gamma - \overline{\beta}\lambda + \gamma^{-1}\lambda^2) \\ &= -\lambda^2(\alpha^2 - 2\alpha\overline{\alpha}\lambda + \overline{\alpha}^2\lambda^2) - \lambda(\beta\gamma\lambda - \overline{\beta}\beta\lambda^2 + \beta\gamma^{-1}\lambda^3 \\ &- \gamma^2\lambda^2 + \overline{\beta}\gamma\lambda^3 - \lambda^4 - 1 + \overline{\beta}\gamma^{-1}\lambda - \gamma^{-2}\lambda \\ &= \lambda[\lambda^4 + (-\overline{\alpha}^2 - \beta\gamma^{-1} - \overline{\beta}\gamma)\lambda^3 + (2\alpha\overline{\alpha} + \beta\overline{\beta} + \gamma^2 + \gamma^{-2})\lambda^2 \\ &+ (-\alpha^2 - \overline{\beta}\gamma^{-1} - \beta\gamma)\lambda + 1] \\ &=: \lambda a(\lambda). \end{aligned}$$

a is also determined by the following equations:

$$a(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + \overline{a}_1\lambda + 1$$
$$\text{with } a_1 = -\overline{\alpha}^2 - \beta\gamma^{-1} - \overline{\beta}\gamma \in \mathbb{C} \quad \text{and} \quad a_2 = 2\alpha\overline{\alpha} + \beta\overline{\beta} + \gamma^2 + \gamma^{-2} \in \mathbb{R}.$$

For such determinant-polynomials $a(\lambda)$, we define the set $\mathcal{M}_2^1$ which will come in handy during the following calculations.

Definition 2.8.

$$\begin{aligned} M_2 :&= \{a \in \mathbb{C}^4[\lambda] | \lambda a(\lambda) = \det(\zeta_\lambda) \textit{ for a } \zeta_\lambda \in \mathcal{P}_2\} \\ &= \{a \in \mathbb{C}^4[\lambda] | a(0) = 1, \lambda^4 \overline{a(\overline{\lambda}^{-1})} = a(\lambda), \lambda^{-2} a(\lambda) \geq 0 \textit{ for } \lambda \in \mathbb{S}^1\} \end{aligned}$$

and

$$\mathcal{M}_2^1 := \{a \in \mathcal{M}_2 | a \textit{ has four pairwise distinct roots}\}.$$

Furthermore, we define the map

$$f : \mathcal{P}_2 \to \mathcal{M}_2^1, \quad \zeta_\lambda \mapsto a(\lambda). \tag{2.9}$$

With the above determination of a, due to the one-to-one correspondence between $\mathcal{P}_2$ and $\mathbb{C} \times \mathbb{C} \times \mathbb{R}^+$, we obtain for such a mapping

$$f : \mathbb{C} \times \mathbb{C} \times \mathbb{R}^+ \to \mathbb{C} \times \mathbb{R}^+, \quad \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \mapsto \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{2.10}$$

Definition 2.9. *Level sets of the function f from 2.10 are called isospectral sets. Given any $a \in \mathcal{M}_2^1$ we denote the respective isospectral set by $I(a)$.*

Summarizing the main results about potentials, polynomial Killing fields and isospectral sets of [4] and [6] in terms of the above definitions yields

Lemma 2.10. *Let $\zeta_\lambda \in \mathcal{P}_2$ and $det\zeta_\lambda = \lambda a(\lambda)$ with $a(\lambda) \in \mathcal{M}_2^1$. Then $\zeta_\lambda \in I(a)$ has no roots.*

Proof. According to [4, Theorem 4.5], every root $\tilde{\lambda} \in \mathbb{C}$ of ζ_λ is a double root of $a(\lambda)$. Since $a \in \mathcal{M}_2^1$ has only simple roots, ζ_λ cannot have a root. □

Corollary 2.11. *Given any initial value $(\alpha_0, \beta_0, \gamma_0) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}^+$, the solutions of the modified Lax equations 2.3 are global, i.e. well-defined for all $(x, y) \in \mathbb{R}^2$, and bounded.*
Therefore, given any $\zeta_\lambda \in \mathcal{P}_2$, we obtain a continuous, commutative group action

$$\phi(x, y)(\zeta_\lambda) := \phi_F(y, \phi_E(x, \zeta_\lambda)) \quad \textit{for } (x, y) \in \mathbb{R}^2. \tag{2.11}$$

Furthermore, when $a \in \mathcal{M}_2^1$ and $\zeta_\lambda \in I(a)$, then $\phi(x, y)(\zeta_\lambda) \in I(a)$ for all times $(x, y) \in \mathbb{R}^2$.

Proof. The statement is shown in [4, chapter 5]. □

Corollary 2.12. *The vector fields 2.3 induce the action 2.11 of* $\mathbb{R}^2$ *on* $\mathcal{P}_2$. *For* $a \in \mathcal{M}_2^1$ *the isospectral sets* $I(a) \subset \mathcal{P}_2$ *are compact and two-dimensional compact submanifolds of* $\mathcal{P}_2$ *with transitive group action 2.11, i.e.*

$$I(a) = \{\phi(x, y)\zeta_\lambda | (x, y) \in \mathbb{R}^2\}.$$

Proof. The corollary is proven in [6, chapter 2]. □

3 Period lattice

In this section, we introduce a lattice for $a \in \mathcal{M}_2^1$ and a system of ordinary differential equations and the respective fundamental solution, the so-called frame. We investigate the elements of the lattice and the frame at these elements, i.e. the monodromies. In particular, this leads to a to a connection between the action of the monodromies on the eigenspaces of the initial values ζ_λ^0 and the elements of the lattice. First of all, we define for given $a \in \mathcal{M}_2^1$ and an initial value $\zeta_\lambda \in I(a)$ the set

$$\Gamma_\zeta^a := \{(x,y) \in \mathbb{R}^2 | \phi(x,y)(\zeta_\lambda) = \zeta_\lambda\}. \tag{3.1}$$

3.1 Generators and frame

In [4] and [6], it is proven that the set 3.1 defines a lattice and is independent of the choice of the initial value $\zeta_\lambda \in I(a)$. Moreover, we obtain the following lemma:

Lemma 3.1. *Let $a \in \mathcal{M}_2^1$. Then, the set*

$$\Gamma^a = \{(x,y) \in \mathbb{R}^2 | \phi(x,y)(\zeta_\lambda) = \zeta_\lambda\}$$

does not depend on the choice of $\zeta_\lambda \in I(a)$ and defines a lattice in $\mathbb{R}^2$. In particular, there exist two lineary independent generators $\omega_1, \omega_2 \in \mathbb{R}^2$ such that

$$\Gamma^a = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}.$$

Proof. □

While the choice of such generators is not unique, the idea is to introduce a minimality condition to force them being unique. A pair of generators (ω_1, ω_2) satisfies the minimality condition if $w_1^a \in \Gamma^a \backslash \{0\}$ has minimal length as well as $\omega_2^a \in \Gamma^a \backslash (\omega_1^a \mathbb{Z})$. Based on [4], we define:

Definition 3.2. *Two lattices Γ, Γ' are called isomorphic if there exists an orthogonal linear transformation mapping Γ to Γ', i.e. if there is a linear mapping $\mathbb{C} \to \mathbb{C}$ composite of a rotation and the multiplication with a real number. This mapping is called rotation-dilation.*

In [3], it is proven that each lattice Γ in $\mathbb{C}$ is isomorphic to $\Gamma^\tau := \mathbb{Z} + \mathbb{Z}\tau$ with

$$\tau \in \{\tau \in \mathbb{C} | \Im(\tau) > 0, |\Re(\tau)| \leq \frac{1}{2}, ||\tau|| \geq 1\} \tag{3.2}$$

up to a rotation dilation. We can identify τ uniquely up to the following identifications of τ:

$$-\frac{1}{2}+iy \sim \frac{1}{2}+iy \text{ for } y \in [\frac{\sqrt{3}}{2},\infty)$$
$$-x+i\sqrt{1-x} \sim x+i\sqrt{1-x} \text{ for } x \in [0,\frac{1}{2}].$$

We define the quotient topology of the subset 3.2 of $\mathbb{C}$ divided by the relation $\sim$ of such τ as $\mathcal{F}$. Then, there exists a unique map

$$T:\mathcal{M}_2^1 \to \mathcal{F}, \qquad a \mapsto \tau_a,$$

such that Γ^a is isomorphic to Γ^{τ_a}. In the following, we investigate the dependence of τ_a on $a \in \mathcal{M}_2^1$. To do this, we introduce the following system of ordinary differential equations:

$$\frac{\partial F_\lambda}{\partial x} = F_\lambda U(\zeta_\lambda) \quad \frac{\partial F_\lambda}{\partial y} = F_\lambda V(\zeta_\lambda) \quad F_\lambda(0,0)=\mathbf{1}. \tag{3.3}$$

Using the Maurer-Cartan equation and the Picard-Lindelöf theorem, it is shown in [4] that there exists a unique fundamental solution which solves both equations as well as the initial condition. This fundamental solution is a function of $(x,y)\in\mathbb{R}^2$. Besides, we define

$$M_\lambda^i = F_\lambda(\omega_i), i=1,2 \tag{3.4}$$

for the ω_i which we obtain through the double periodicity of the flows which is reflected in the coefficients $(\alpha,\beta,\gamma)(x,y)$ and hence, also in the respective $U(\zeta_\lambda), V(\zeta_\lambda)$:

$$U((x,y)+\omega_1) = U(x,y) = U((x,y)+\omega_2)$$
$$V((x,y)+\omega_1) = V(x,y) = V((x,y)+\omega_2).$$

We consider

$$\tilde{F}_\lambda^i := (M_\lambda^i)^{-1}F_\lambda((x,y)+\omega_i), i=1,2.$$

Due to the uniqueness of the fundamental solution of 3.3, we obtain $F_\lambda = \tilde{F}_\lambda^i, i=1,2$ and

$$F_\lambda((x,y)+\omega_i) = M_\lambda^i F_\lambda(x,y), i=1,2.$$

Definition 3.3. *The fundamental solution F_λ from 3.3 is called frame and the matrix M_ω for F at any $\omega \in \Gamma^a$ is called monodromy, especially the matrices M_λ^1, M_λ^2 from 3.4.*

In the following, we summarize some lemmata of [4] which investigate the monodromies $M^i_\lambda, i = 1, 2$. Additionally, we analyse how these monodromies act on the eigenspaces of ζ^0_λ for any $\omega \in \Gamma^a$. At the end of this section, we thereby obtain a connection between polynomials b_1, b_2 and the eigenvalues of the monodromies M^1_λ, M^2_λ.

Lemma 3.4. *The monodromies* $M^i_\lambda, i = 1, 2$ *and the initial value of a polynomial Killing field* $\zeta^0_\lambda := \zeta_\lambda(0, 0)$ *commute pairwise.*

Lemma 3.5. *The monodromies 3.4 satisfy* $det(M^i_\lambda) = 1, i = 1, 2$.

Lemma 3.6. *Let* $a \in \mathcal{M}^1_2$ *and let* $\tilde{\lambda}$ *be an arbitrary root of* a. *Then, the eigenvalues of the monodromies satisfy*

$$\mu^i_{\tilde{\lambda}} = \pm 1, \ for \ i = 1, 2.$$

Proof. For the proofs we refer the reader to [4, chapter 7]. □

Remark 3.7.

Since $\det(M^i_\lambda) = 1, i = 1, 2$, we denote the eigenvalues of the monodromies 3.4 as

$$\mu^i_\lambda, \frac{1}{\mu^i_\lambda}.$$

Moreover, we denote the eigenvalues of ζ_λ as $v_\lambda, -v_\lambda$ using the fact that $tr(\zeta_\lambda) = 0$ (this can be easily calculated for ζ_λ in terms of the abstract form). Then, we obtain the following equations:

$$\begin{aligned} 0 &= \det(\zeta_\lambda - v_\lambda \mathbf{1}) \\ &= -\mathrm{A}^2 - \lambda \mathrm{BC} + v^2 \\ &= v^2_\lambda + \det(\zeta_\lambda) \\ &= v^2_\lambda + \lambda a(\lambda). \end{aligned}$$

3.2 Lattice elements and monodromies

The following is based on the work in [6, chapter 3,4]. In the previous subsection, we investigated the set $\Gamma^a = \{(x, y) \in \mathbb{R}^2 | \phi(x, y)(\zeta_\lambda) = \zeta_\lambda\}$ for $a \in \mathcal{M}^1_2$. It defines a lattice in $\mathbb{R}^2$ and is independent of the choice of $\zeta_\lambda \in I(a)$. Especially, it is a lattice

$$\Gamma^a = \omega_1 \mathbb{Z} + \omega_2 \mathbb{Z}$$

with linearly independent generators $\omega_1, \omega_2 \in \mathbb{R}^2$.
In the following, we will obtain a context between the elements of the lattice and polynomials b_ω of degree three which satisfy corresponding reality conditions.
We now consider the monodromy M_ω of the fundamental solution F_λ of the system of ordinary differential equations 3.3 at ω with eigenvalues μ_ω for any $\omega \in \Gamma^a$. Again, we obtain the result: M_ω commutes with ζ_λ^0 for all $\omega \in \Gamma^a$ and maps the eigenspaces of ζ_λ^0 onto themselves. Due to the fact that ζ_λ^0 is traceless and $\zeta_\lambda^0 \in I(a)$ with $a \in \mathcal{M}_2^1$ has no roots (lemma 2.10), the eigenspaces of each ζ_λ^0 are one-dimensional. They are parametrized by

$$\Sigma^* = \{(\lambda, v) \in (\mathbb{C}\backslash\{0\}) \times \mathbb{C} | \det(v\mathbf{1} - \zeta_\lambda^0) = v^2 + \lambda a(\lambda) = 0\}. \tag{3.5}$$

The monodromies M_ω act on these one-dimensional eigenspaces of ζ_λ^0 by multiplication with a function $\mu_\omega : \Sigma^* \to \mathbb{C}\backslash\{0\}$. We observe the involutions

$$\sigma : (\lambda, v) \mapsto (\lambda, -v), \qquad \rho : (\lambda, v) \mapsto (\overline{\lambda^{-1}}, -\overline{\lambda^{-3}}\overline{v}).$$

They act on μ_ω by

$$\sigma^* \mu_\omega = \mu_\omega^{-1}, \qquad \rho^* \mu_\omega = \overline{\mu_\omega^{-1}}. \tag{3.6}$$

We can describe the lattice Γ^a in the following way:

Lemma 3.8. *For all $a \in \mathcal{M}_2^1$ the elements of Γ^a are characterized as those $\omega \in \mathbb{C}$ such that the function $exp(\omega\lambda^{-1}v)$ on Σ^* factorizes into the product of a holomorphic function μ_ω on Σ^* obeying 3.6 with a holomorphic function on Σ^*, which extends holomorphically to $\lambda = 0$ and takes the value 1 there.*

Proof. The lemma is proven in [6, Lemma 3.1]. □

In the proof, it is also shown that the function μ_ω in the lemma is unique and acts on the eigenspaces of ζ_λ^0 equally to the monodromies M_ω. The logarithmic derivative of this function is a meromorphic differential of second kind with second order poles at $\lambda = 0$ and $\lambda = \infty$. Due to 3.6, it takes the form

$$d\ln\mu_\omega = \pi \frac{b_\omega(\lambda)}{v} d\ln\lambda \quad \text{with} \quad b_\omega \in \mathbb{C}^3[\lambda] \quad \text{such that} \quad \overline{\lambda}^3 b_\omega(\overline{\lambda^{-1}}) = \overline{b_\omega(\lambda)}. \tag{3.7}$$

Finally, we obtain the connection between the elements of the lattice Γ^a for $a \in \mathcal{M}_2^1$ and the polynomials b_ω and thus the eigenvalues of the monodromies μ_ω when we consider b_ω at $\lambda = 0$:

Corollary 3.9. *For $a \in \mathcal{M}_2^1$ the elements of Γ^a are the values $\omega = b_\omega(0)$ of those b_ω in 3.7, whose 1-forms $d\ln\mu_\omega$ are the logarithmic derivatives of a holomorphic function μ_ω on Σ^*.*

Proof. The corollary is proven in [6, p.10-12] □

In the following, we introduce the Whitham deformations. For two generators ω_1, ω_2 of the lattice Γ^a for $a \in \mathcal{M}_2^1$, we obtain, using the monodromies $M_\lambda^i, i = 1, 2$ and the above-mentioned logarithmic derivative, two polynomials b_1, b_2. With the minimality condition, the generators are uniquely determined and hence, b_1, b_2 are unique.

4 Whitham deformations

4.1 CMC tori and Whitham equations

In this subsection, we derive the equations of the Whitham deformations. To understand why we can work with them in our case and how they and this work are related to constant mean curvature tori in $\mathbb{R}^3$, we give a short outline. It would exceed the scope of this work to analyze the following in all details. Nevertheless, we will adapt the definitions of the paper [2], accordingly, to investigate the connection between the period lattice and the spectral curves.
Based on [2], constant mean curvature tori can be described by an algebraic curve, called the spectral curve, together with a line bundle on this curve and a point on $\mathbb{S}^1$, called the sym point. Moreover, the spectral data $(X, \lambda, \rho, \lambda_0, L)$ of a CMC torus is introduced. Here, X is an algebraic curve, the spectral curve, with a degree-two meromorphic function λ, anti-holomorphic involution ρ, a point on the unit circle λ_0 and a line bundle L on this curve, which is quaternionic with respect to $\sigma\ \rho$, where σ is the hyperelliptic involution induced by λ. The quadruple $(X, \lambda, \rho, \lambda_0)$ of such spectral data satisfies periodicity conditions for a CMC torus.
We investigate now the spectral curves of spectral genus two. We put some restrictions on the spectral curves $X = X_a$ in $\mathbb{C}^2$ and describe them by an equation of the form

$$v^2 = \lambda a(\lambda) = (-1)^2 \lambda \prod_{i=1}^{2} (\lambda - \alpha_i)(\lambda - \overline{\alpha_i^{-1}})$$

where $a \in \mathcal{M}_2^1$ and thus, it satisfies the following conditions:

1. the reality condition $\overline{a(\lambda)} = \overline{\lambda}^4 a(\overline{\lambda^{-1}})$,
2. $\lambda^{-2} a(\lambda) \geq 0$ for $\lambda \in \mathbb{S}^1$,
3. the lowest and highest coefficient are 1,
4. the roots of a are pairwise distinct, forcing X_a to be smooth.

We want to express the periodicity conditions in terms of a pair of meromorphic differentials on the spectral curve which equal the logarithmic derivatives of the previous section. To do this, we define the space $\mathcal{B}_a$ for each $a \in \mathcal{M}_2^1$, which denotes the 2-dimensional space of polynomials b_ω of degree three satisfying the reality conditions $\overline{b_\omega} = \overline{\lambda}^3 b_\omega(\overline{\lambda^{-1}})$ and such that the meromorphic differential

$$\Theta_b := \pi \frac{b(\lambda)}{v} d\ln\lambda$$

has purely imaginary periods. Every b_ω of $\mathcal{B}_a$ is uniquely determined by $b(0)$ up to adding a holomorphic differential, which is fixed by the condition on the periods. Thus, the elements correspond one-to-one to the numbers $b_\omega(0) \in \mathbb{C}$.
Applying corollary 3.9 to the above-described case, we obtain linearly independent $b_1, b_2 \in \mathcal{B}_a$ with $b_1(0) = \omega_1, b_2(0) = \omega_2$ for the generators ω_1, ω_2 of the lattice Γ^a for $a \in \mathcal{M}_2^1$ and the respective logarithmic derivatives $d\ln\mu_\lambda^i = d\ln\mu_i, i = 1,2$ for functions μ_1, μ_2, as described in lemma 3.8, with

1. the logarithmic differentials $d\ln\mu_i, i = 1,2$ are meromorphic differentials of the second kind with second order poles at $\lambda = 0$ and $\lambda = \infty$,
2. $d\ln\mu_1 = \Theta_{b_1}, \quad d\ln\mu_2 = \Theta_{b_2}$,
3. $d\ln\mu_i, i = 1,2$ takes the value ± 1 at each root of a.

In the next step, we derive the Whitham equations using connection mentioned above. Then, under certain assumptions, we calculate a vector field $(a, b_1, b_2) \to (\dot{a}, \dot{b}_1, \dot{b}_2)$ through these equations. The vector $(\dot{a}, \dot{b}_1, \dot{b}_2)$ denotes the tangent vector at $t = 0$ which infinitesimally preserves the periods of μ_1, μ_2. The periods of the meromorphic differential forms $\frac{d}{dt}\Big|_{t=0} d\ln\mu_i$ for $i = 1,2$ vanish. Also, these forms have no residues and hence, there exist meromorphic functions $\dot{q}_i$ on X_a such that

$$d\dot{q}_i = \frac{d}{dt}\Big|_{t=0} d\ln\mu_i$$

and we may write

$$\dot{q}_i = \pi \frac{ic_i(\lambda)}{v}$$

with polynomials $c_i \in \mathbb{C}^3[\lambda]$ of degree three and $\overline{c_i(\lambda)} = \overline{\lambda^3} c_i(\overline{\lambda^{-1}})$. Thus, we obtain the Whitham equation

$$\frac{\partial}{\partial\lambda} \frac{ic_i(\lambda)}{v} = \frac{\partial}{\partial t} \frac{b_i(\lambda)}{v\lambda}\Big|_{t=0}$$

which yields, using the product and chain rule, to

$$(2\lambda a c_1' - a c_1 - \lambda a' c_1) i = 2a\dot{b}_1 - \dot{a} b_1$$
$$(2\lambda a c_2' - a c_2 - \lambda a' c_2) i = 2a\dot{b}_2 - \dot{a} b_2$$

where the dot and prime denote the derivative with respect to t, evaluated at $t = 0$, and the derivative with respect to λ, respectively. Multiplying the first with c_2 and the second with c_1 and using the compatibility of both, yields

$$2a(c_1' c_2 \lambda - c_2' c_1 \lambda + c_1 \dot{b}_2 - c_2 \dot{b}_1) = \dot{a}(c_1 b_2 - c_2 b_1).$$

This implies that any roots of a at which $\dot{a}$ does not vanish are also roots of $c_1 b_2 - c_2 b_1$. Additionally, when $\dot{a}$ vanishes at a root of a, then c_1 and c_2 vanish at this root, too. Therefore, $c_1 b_2 - c_2 b_1$ vanish at all roots of a and we conclude that

$$c_1 b_2 - c_2 b_1 = Qa$$

with polynomials $Q \in \mathbb{C}^2[\lambda]$ of degree two and $\overline{Q(\lambda)} = \overline{\lambda}^2 Q(\overline{\lambda^{-1}})$.

Remark 4.1.

We obtain the equations of the Whitham deformations for above-mentioned polynomials by summarizing the results with the three equations

$$(2\lambda a c_1' - a c_1 - \lambda a' c_1) i = 2a\dot{b}_1 - \dot{a} b_1 \tag{4.1}$$
$$(2\lambda a c_2' - a c_2 - \lambda a' c_2) i = 2a\dot{b}_2 - \dot{a} b_2 \tag{4.2}$$
$$c_1 b_2 - c_2 b_1 = Qa. \tag{4.3}$$

4.2 Preliminaries

In this subsection, we calculate the above-mentioned vector field $(a, b_1, b_2) \to (\dot{a}, \dot{b}_1, \dot{b}_2)$ under certain conditions . These assumptions and a brief explanation how we compute the vector field are given in the following. Let

$$\begin{aligned}\mathcal{M}_2 &:= \{a \in \mathbb{C}^4[\lambda] | \lambda a(\lambda) = \det(\zeta_\lambda) \textit{ for a } \zeta_\lambda \in \mathcal{P}_2\} \\ &= \{a \in \mathbb{C}^4[\lambda] | a(0) = 1, \lambda^4 \overline{a(\overline{\lambda}^{-1})} = a(\lambda), \lambda^{-2} a(\lambda) \geq 0 \textit{ for } \lambda \in \mathbb{S}^1\}\end{aligned}$$

and

$$\mathcal{M}_2^1 := \{a \in \mathcal{M}_2 | \text{a has four pairwise distinct simple roots}\}$$

as in definition 2.8.
Given $a \in \mathcal{M}_2^1$ and two polynomials b_1 and b_2 obtained through the previous calculations in section 3, 4.1, which satisfy the reality conditions,

$$b_j(\lambda) = \lambda^3 \overline{b_j(\overline{\lambda^{-1}})}, j = 1, 2$$

we obtain a triple (a, b_1, b_2). In this work, such a triple always fulfills the above-mentioned conditions and the polynomials b_1 and b_2 do not have a common root. Under the assumption that $\gcd(b_1, b_2) = 1$, we find, using the equations from the Whitham deformations 4.1,4.2 and 4.3, for every $Q \in \mathbb{C}^2[\lambda]$ satisfying the reality condition

$$\lambda^2 \overline{Q(\overline{\lambda^{-1}})} = Q(\lambda)$$

a unique pair of polynomials (c_1, c_2) of degree 3 which fulfill the reality conditions

$$\lambda^3 \overline{c_1(\overline{\lambda^{-1}})} = c_1(\lambda)$$
$$\text{and } \lambda^3 \overline{c_2(\overline{\lambda^{-1}})} = c_2(\lambda)$$

solving equation 4.3 and satisfying the conditions 4.4,4.5.
Given such (a, b_1, b_2) and uniquely determined (c_1, c_2), the equations 4.1-4.2 have a unique solution and we obtain the triple $(\dot{a}, \dot{b}_1, \dot{b}_2)$.
The pair (c_1, c_2) solving equation 4.3 is uniquely determined by Q through the condition that the highest and lowest coefficient of the polynomial $a \in \mathcal{M}_2^1$ is one and hence, the highest and lowest coefficient of $\dot{a}$ vanishes. Additionally, this requirement can be expressed through the following condition for c_1 and c_2:

$$\sum_{i=1}^{4} \frac{c_1(\alpha_i)}{b_1(\alpha_i)} = 0 \tag{4.4}$$

$$\sum_{i=1}^{4} \frac{c_2(\alpha_i)}{b_2(\alpha_i)} = 0 \tag{4.5}$$

which we obtain, using the equations 4.1 and 4.2, through the calculation

$$a(\lambda) = \sum_{i=1}^{4}(\lambda - \alpha_i) \qquad a'(\lambda) = \sum_{i=1}^{4}\prod_{\substack{j=1\\ j\neq i}}^{4}(\lambda - \alpha_j)$$

$$\dot{a}(\lambda) = -\sum_{i=1}^{4}\dot{\alpha}_i\prod_{\substack{j=1\\ j\neq i}}^{4}(\lambda - \alpha_j)$$

$$\sum_{i=1}^{4}\frac{d}{dt}ln(\alpha_i) = \frac{d}{dt}ln(\prod_{i=1}^{4}\alpha_i) = \frac{d}{dt}ln(1) = 0$$

and

$$\begin{aligned} & i\alpha_i c_1(\alpha_i)a'(\alpha_i) = \dot{a}(\alpha_i)b_1(\alpha_i) \\ \Leftrightarrow \quad & \frac{c_1(\alpha_i)}{b_1(\alpha_i)} = \frac{\dot{a}(\alpha_i)}{i\alpha_i a'(\alpha_i)} = -\frac{\dot{\alpha}_i a'(\alpha_i)}{i\alpha_i a'(\alpha_i)} = -\frac{\dot{\alpha}_i}{i\alpha_i} \\ \Rightarrow \quad & \sum_{i=1}^{4}\frac{c_1(\alpha_i)}{b_1(\alpha_i)} = i\sum_{i=1}^{4}\frac{\dot{\alpha}_i}{\alpha_i} = i\sum_{i=1}^{4}\frac{d}{dt}ln(\alpha_i) = 0. \end{aligned}$$

Analogously for c_2 and b_2, using equation 4.2.

4.3 Solving the Whitham equations

Now we want to calculate the vector field $(a, b_1, b_2) \to (\dot{a}, \dot{b}_1, \dot{b}_2)$. First, we prove that, under the assumptions 4.4 and 4.5, the equation 4.3 has a uniquely determined pair of polynomials (c_1, c_2) of degree three as solution. Second, we construct a c_1 and c_2 and show that they are this exact pair of polynomials. Third, we prove the reality conditions for these (c_1, c_2).
Finally, we calculate the unique triple $(\dot{a}, \dot{b}_1, \dot{b}_2)$ depending on (a, b_1, b_2, c_1, c_2).

Lemma 4.2. *Let a and Q be a complex polynomial of degree 4 and of degree 2, respectively, and b_1 and b_2 two complex polynomials of degree 3 without common roots. Then, there exist unique complex polynomials c_1 and c_2 of degree three, which solve equation 4.3 and the sum of the values of $\frac{c_1}{b_1}$ and $\frac{c_2}{b_2}$ at all roots of a are zero, i.e. they also satisfy the conditions 4.4 and 4.5.*

Proof. Let $c_1 = x_1 b_1 + d_1$ and $c_2 = x_2 b_2 + d_2$, where x_1, x_2 are complex numbers and d_1 and d_2 are complex polynomials of degree at most two.

Thus, it follows that

$$\begin{aligned} & c_1 b_2 - c_2 b_1 = Qa \\ \Leftrightarrow \quad & (x_1 b_1 + d_1) b_2 - (x_2 b_2 + d_2) b_1 = Qa \\ \Leftrightarrow \quad & d_1 b_2 - d_2 b_1 = aQ - (x_1 - x_2) b_1 b_2. \end{aligned} \tag{4.6}$$

Since the left-hand side is a polynomial of degree at most five, $x_1 - x_2$ has to be a unique complex number such that the right-hand side is also a polynomial of degree at most five. We obtain three conditions through equation 4.6 which uniquely determine the polynomials d_1, d_2 of degree at most two. These conditions can be obtained by evaluating the equations at the roots of b_1 and b_2 in the following way:
Case 1: If b_1 or b_2 have three simple roots, the values of the corresponding d at the roots of the respective polynomial b are defined through equation 4.6.
Case 2: If b_1 or b_2 has one root of order two and one root of order one, we need to derive the equation 4.6:

$$d_1' b_2 + d_1 b_2' - d_2' b_1 - d_2 b_1' = a' Q + a Q' - (x_1 - x_2)(b_1' b_2 + b_1 b_2'). \tag{4.7}$$

Thus, we obtain two conditions for the respective d at the root of order two of the according b by the equations 4.6 and 4.7. The third condition is given by the equation 4.6, evaluated at the simple root of the respective b.
Case 3: When b_1 or b_2 have one root of order three we, get the three conditions using

$$\begin{aligned} & d_1'' b_2 + 2 d_1' b_2' + d_1 b_2'' - d_2'' b_1 - 2 d_2' b_1' - d_2 b_1'' \\ = & a'' Q + 2 a' Q' + a Q'' - (x_1 - x_2)(b_1'' b_2 + 2 b_1' b_2' + b_1 b_2'') \end{aligned}$$

and the equations 4.6 and 4.7 at this root of the respective b.
In the following, we assume that d_1 and d_2 are such unique polynomials of degree at most two. Then, $d_1 b_2 - d_2 b_1 - aQ$ vanishes at all roots of b_1 and b_2 and is divisible by b_1 and b_2. Therefore, we obtain the previous equation if $x_1 - x_2$ is this uniquely determined number such that the right-hand side of equation 4.6 is a polynomial of degree at most five. Due to

$$\frac{c_1}{b_1} = \frac{d_1}{b_1} + x_1 \qquad \frac{c_2}{b_2} = \frac{d_2}{b_2} + x_2$$

and the condition that the sum of $\frac{c_1}{b_1}$ and $\frac{c_2}{b_2}$ vanishes at all roots of a, the numbers x_1 and x_2 are clearly determined by:

$$\begin{aligned}\sum_{i=1}^{4}\frac{c_1(\alpha_i)}{b_1(\alpha_i)} &= 0 = \sum_{i=1}^{4}\left(\frac{d_1(\alpha_i)}{b_1(\alpha_i)}+x_1\right)\\ \Leftrightarrow \qquad x_1 &= -\frac{1}{4}\sum_{i=1}^{4}\frac{d_1(\alpha_i)}{b_1(\alpha_i)}\\ \text{and analogous} \qquad x_2 &= -\frac{1}{4}\sum_{i=1}^{4}\frac{d_2(\alpha_i)}{b_2(\alpha_i)}.\end{aligned}$$

Dividing equation 4.6 by b_1b_2 yields

$$\begin{aligned}&(x_1b_1+d_1)b_2-(x_2b_2+d_2)b_1 = Qa\\ \Leftrightarrow \qquad &\frac{d_1}{b_1}+x_1-\left(\frac{d_2}{b_2}+x_2\right) = \frac{aQ}{b_1b_2}.\end{aligned}$$

Because the values at all roots of a of the function on the right-hand side sum up to zero, the obtained x_1 and x_2 solve the equation. Hence, we obtain the lemma. □

Now, we calculate such a pair of polynomials (c_1, c_2) explicitly. We make the same approach as in the proof of the previous lemma with $c_1 = d_1 + x_1b_1$ and $c_2 = d_2 + x_2b_2$:
First, we construct the explicit polynomials d_1 and d_2 and hence, we introduce a case analysis depending on the order of the roots of the respective polynomial b. Second, we use the condition 4.4 and 4.5 to find the complex numbers x_1 and x_2 in each of the cases. Finally, we obtain the uniquely defined pair of polynomials (c_1, c_2).

Lemma 4.3. *Let a, Q, b_1, b_2 be polynomials as in the previous Lemma. Then, we obtain the following unique complex polynomials c_1, c_2 of degree three of lemma 4.2 depending on the order of the roots of b_1, b_2 where b_1 and b_2 can have different orders of roots:*
Case 1: b_r has only roots of order one, b_s is the other polynomial of b_1, b_2. Then,

$$\begin{aligned}c_r(\lambda) &= b_r(\lambda)\left(\sum_{j=1}^{3}\frac{Q(b_{rj})a(b_{rj})}{b_s(b_{rj})(\lambda-b_{rj})b_r'(b_{rj})}-\frac{1}{4}\sum_{i=1}^{4}\sum_{j=1}^{3}\frac{Q(b_{rj})a(b_{rj})}{b_s(b_{rj})(\alpha_i-b_{rj})b_r'(b_{rj})}\right)\\ &= b_r(\lambda)\sum_{j=1}^{3}\frac{Q(b_{rj})a(b_{rj})}{b_s(b_{rj})b_r'(b_{rj})}\left(\frac{1}{(\lambda-b_{rj})}-\frac{1}{4}\sum_{i=1}^{4}\frac{1}{(\alpha_i-b_{rj})}\right) \qquad (4.8)\end{aligned}$$

Case 2: b_r has the root b_{r1} of order two and b_{r2} of order one, b_s is the other polynomial of b_1, b_2. Then,

$$\begin{aligned} c_r(\lambda) = b_r(\lambda)\Bigg(& \left(\frac{2(a'(b_{r1})Q(b_{r1}) + a(b_{r1})Q'(b_{r1}))}{b_r''(b_{r1})b_s(b_{r1})} - y\left(\frac{b_s'(b_{r1})}{b_s(b_{r1})} + \frac{b_r'''(b_{r1})}{3b_r''(b_{r1})}\right)\right)\frac{1}{(\lambda - b_{r1})} \\ &+ \frac{2a(b_{r1})Q(b_{r1})}{b_r''(b_{r1})b_s(b_{r1})}\frac{1}{(\lambda - b_{r1})^2} + \frac{a(b_{r2})Q(b_{r2})}{b_r'(b_{r2})b_s(b_{r2})}\frac{1}{(\lambda - b_{r2})}\Bigg) \\ &- b_r(\lambda)\frac{1}{4}\sum_{i=1}^{4}\Bigg(\left(\frac{2(a'(b_{r1})Q(b_{r1}) + a(b_{r1})Q'(b_{r1}))}{b_r''(b_{r1})b_s(b_{r1})} - y\left(\frac{b_s'(b_{r1})}{b_s(b_{r1})} + \frac{b_r'''(b_{r1})}{3b_r''(b_{r1})}\right)\right)\frac{1}{(\alpha_i - b_{r1})} \\ &+ \frac{2a(b_{r1})Q(b_{r1})}{b_r''(b_{r1})b_s(b_{r1})}\frac{1}{(\alpha_i - b_{r1})^2} + \frac{a(b_{r2})Q(b_{r2})}{b_r'(b_{r2})b_s(b_{r2})}\frac{1}{(\alpha_i - b_{r2})}\Bigg) \end{aligned} \tag{4.9}$$

Case 3: b_r has one root of order three, b_s is the other polynomial of b_1, b_2. Then,

$$\begin{aligned} c_r(\lambda) = \Bigg(&\Bigg(\frac{3(a''(b_{r1})Q(b_{r1}) + 2a'(b_{r1})Q'(b_{r1}) + a(b_{r1})Q''(b_{r1}))}{b_r'''(b_{r1})b_s(b_{r1})} \\ &- \frac{b_s'(b_{r1})}{b_s(b_{r1})}y - \frac{b_s''(b_{r1})}{2b_s(b_{r1})}z\Bigg)\left(\frac{1}{(\lambda - b_{r1})} - \frac{1}{4}\sum_{i=1}^{4}\frac{1}{\alpha_i - b_{r1}}\right) \qquad (4.10) \\ &+ \left(\frac{6(a'(b_{r1})Q(b_{r1}) + a(b_{r1})Q'(b_{r1}))}{b_s(b_{r1})b_r'''(b_{r1})} - \frac{b_s'(b_{r1})}{b_s(b_{r1})}\right)\Bigg(\frac{1}{(\lambda - b_{r1})^2} \qquad (4.11) \\ &- \frac{1}{4}\sum_{i=1}^{4}\frac{1}{\alpha_i - b_{r1}}\Bigg) + \left(\frac{6a(b_{r1})Q(b_{r1})}{b_r'''(b_{r1})b_s(b_{r1})}\right)\left(\frac{1}{(\lambda - b_{r1})^3} - \frac{1}{4}\sum_{i=1}^{4}\frac{1}{\alpha_i - b_{r1}}\right)\Bigg) \qquad (4.12) \end{aligned}$$

Proof. We need to do a case analysis due to the fact that we will get different d_1, d_2 if b_1, b_2 have roots of different order and as a result also different x_1, c_1 and x_2, c_2. Without loss of generality, we consider only for c_1 while c_2 can be constructed similarly depending on b_2 with roots of respective order. It is important to mention that there exists such a pair of polynomials (c_1, c_2) and hence, it is reasonable to calculate c_1 and c_2.
Case 1: b_1 has only roots of order one at $b_{1j}, j = 1, ..., 3$. Then, d_1 is the complex polynomial of degree two

$$d_1(\lambda) = \sum_{j=1}^{3} \frac{Q(b_{1j})a(b_{1j})}{b_2(b_{1j})}\frac{b_1(\lambda)}{(\lambda - b_{1j})b_1'(b_{1j})}$$

Using the rule of l'Hospital we see that d_1 solves equation 4.6 at all roots of b_1:

$$\begin{aligned} d_1(b_{1i}) &= \sum_{j=1}^{3} \lim_{\lambda\to b_{1i}} \frac{Q(b_{1j})a(b_{1j})}{b_2(b_{1j})}\frac{b_1(\lambda)}{(\lambda-b_{1j})b_1'(b_{1j})} \\ &= \lim_{\lambda\to b_{1i}} \frac{Q(b_{1i})a(b_{1i})}{b_2(b_{1i})}\frac{b_1'(\lambda)}{b_1'(b_{1i})} = \frac{Q(b_{1i})a(b_{1i})}{b_2(b_{1i})}. \end{aligned}$$

Using the fact that we obtain the x_1 in the proof of lemma 4.2 by evaluating condition 4.4, we get for the explicit case

$$\begin{aligned} &\sum_{i=1}^{4} \frac{c_1(\alpha_i)}{b_1(\alpha_i)} = \sum_{i=1}^{4} \frac{d_1(\alpha_i)+x_1 b_1(\alpha_i)}{b_1(\alpha_i)} = 0 \\ \Leftrightarrow \quad & x_1 = -\frac{1}{4}\sum_{i=1}^{4}\frac{d_1(\alpha_i)}{b_1(\alpha_i)} = -\frac{1}{4}\sum_{i=1}^{4}\frac{\sum_{j=1}^{3}\frac{Q(b_{1j})a(b_{1j})}{b_2(b_{1j})}\frac{b_1(\alpha_i)}{(\alpha_i-b_{1j})b_1'(b_{1j})}}{b_1(\alpha_i)} \\ \Leftrightarrow \quad & x_1 = -\frac{1}{4}\sum_{i=1}^{4}\sum_{j=1}^{3}\frac{Q(b_{1j})a(b_{1j})}{b_2(b_{1j})b_1'(b_{1j})}\frac{1}{(\alpha_i-b_{1j})}. \end{aligned}$$

The constructed d_1 and x_1 satisfy the conditions in lemma 4.2 and $c_1 = d_1 + x_1 b_1$ is a polynomial as in lemma 4.2 and equals the polynomial 4.8 given in the claim.

Case 2: Let b_{11} be the root of order two and b_{12} the one of order one of b_1. Again, we construct the d_1 and x_1 and we see that the obtained c_1 equals the one in case 2 of the statement. In this instance, the three conditions for d_1 are given through the respective d_1 and the derivative d_1' at b_{11} together with the d_1 at the root b_{12}. d_1 has to solve the equations

$$d_1(b_{11})b_2(b_{11} = a(b_{11})Q(b_{11}) \tag{4.13}$$

$$d_1'(b_{11})b_2(b_{11}) + d_1(b_{11})b_2'(b_{11}) = a'(b_{11})Q(b_{11}) + a(b_{11})Q'(b_{11}) \tag{4.14}$$

$$d_1(b_{12})b_2(b_{12}) = a(b_{12})Q(b_{12}). \tag{4.15}$$

We make the following approach: Let

$$d_1(\lambda) = b_1(\lambda)\left(\frac{x}{(\lambda-b_{11})} + \frac{y}{(\lambda-b_{11})^2} + \frac{z}{(\lambda-b_{12})}\right)$$

with constants $x, y, z \in \mathbb{C}$. Then, we get

$$\begin{aligned} d_1'(\lambda) = b_1'(\lambda)&\left(\frac{x}{(\lambda-b_{11})} + \frac{y}{(\lambda-b_{11})^2} + \frac{z}{(\lambda-b_{12})}\right) \\ - b_1(\lambda)&\left(\frac{x}{(\lambda-b_{11})^2} + \frac{2y}{(\lambda-b_{11})^3} + \frac{z}{(\lambda-b_{12})^2}\right). \end{aligned}$$

Obviously, we obtain z through equation 4.15, if we investigate d_1 at b_{12}. We obtain a similar result as in case 1 for the simple root using the rule of l'Hospital:

$$\begin{aligned}
d_1(b_{12}) &= \frac{xb_1(b_{12})}{(b_{12}-b_{11})} + \frac{yb_1(b_{12})}{(b_{12}-b_{11})^2} + \lim_{\lambda\to b_{12}} \frac{zb_1(\lambda)}{(\lambda-b_{12})}\\
&= \lim_{\lambda\to b_{12}} \frac{zb_1'(\lambda)}{1}\\
\Rightarrow \quad & d_1(b_{12})b_2(b_{12}) = a(b_{12})Q(b_{12})\\
\Leftrightarrow \quad & zb_1'(b_{12})b_2(b_{12}) = a(b_{12})Q(b_{12})\\
\Leftrightarrow \quad & z = \frac{a(b_{12})Q(b_{12})}{b_1'(b_{12})b_2(b_{12})}.
\end{aligned}$$

In the next step, x and y are calculated. First, we obtain y using equation 4.13 and second, compute x depending on this y, employing equation 4.14.

$$\begin{aligned}
d_1(b_{11}) &= \lim_{\lambda\to b_{11}} \frac{b_1(\lambda)}{(\lambda-b_{11})}x + \lim_{\lambda\to b_{11}} \frac{b_1(\lambda)}{(\lambda-b_{11})^2}y + \frac{b_1(b_{11})}{(\lambda-b_{12})}z\\
&= \lim_{\lambda\to b_{11}} \frac{b_1'(\lambda)}{1}x + \lim_{\lambda\to b_{11}} \frac{b_1''(\lambda)}{2}y\\
&= \frac{b_1''(b_{11})}{2}y
\end{aligned}$$

and

$$\begin{aligned}
d_1'(b_{11}) &= \lim_{\lambda\to b_{11}} \frac{b_1'(\lambda)(\lambda-b_{11})-b_1(\lambda)}{(\lambda-b_{11})^2}x + \lim_{\lambda\to b_{11}} \frac{b_1'(\lambda)(\lambda-b_{11})-2b_1(\lambda)}{(\lambda-b_{11})^3}y\\
&+ \lim_{\lambda\to b_{11}} \frac{b_1'(b_{11})(b_{11}-b_{11})-b_1(b_{11})}{(b_{11}-b_{12})^2}z\\
&= \lim_{\lambda\to b_{11}} \frac{b_1''(\lambda)(\lambda-b_{11})+b_1'(\lambda)-b_1'(\lambda)}{2(\lambda-b_{11})}x + \lim_{\lambda\to b_{11}} \frac{b_1''(\lambda)(\lambda-b_{11})+b_1'(\lambda)-2b_1'(\lambda)}{3(\lambda-b_{11})^2}y\\
&= \lim_{\lambda\to b_{11}} \frac{b_1''(\lambda)}{2}x + \lim_{\lambda\to b_{11}} \frac{b_1'''(\lambda)(\lambda-b_{11})+2b_1''(\lambda)-2b_1''(\lambda)}{6(\lambda-b_{11})}y\\
&= \frac{b_1''(b_{11})}{2}x + \frac{b_1'''(b_{11})}{6}y.
\end{aligned}$$

Inserting $d_1(b_{11})$ in 4.13 yields

$$\begin{aligned}
& d_1(b_{11})b_2(b_{11} = a(b_{11})Q(b_{11})\\
\Leftrightarrow \quad & \frac{b_1''(b_{11})}{2}yb_2(b_{11} = a(b_{11})Q(b_{11})\\
\Leftrightarrow \quad & y = \frac{2a(b_{11})Q(b_{11})}{b_1''(b_{11})b_2(b_{11})}.
\end{aligned}$$

Inserting $d_1'(b_{11})$ in 4.14 yields x depending on y:

$$d_1'(b_{11})b_2(b_{11}) + d_1(b_{11})b_2'(b_{11}) = a'(b_{11})Q(b_{11}) + a(b_{11})Q'(b_{11})$$
$$\left(\frac{b_1''(b_{11})}{2}x + \frac{b_1'''(b_{11})}{6}y\right) b_2(b_{11}) + \frac{b_1''(b_{11})b_2'(b_{11})}{2}y = a'(b_{11})Q(b_{11}) + a(b_{11})Q'(b_{11})$$
$$x = \frac{2(a'(b_{11})Q(b_{11}) + a(b_{11})Q'(b_{11}))}{b_1''(b_{11})b_2(b_{11})} - y\left(\frac{b_2'(b_{11})}{b_2(b_{11})} + \frac{b_1'''(b_{11})}{3b_1''(b_{11})}\right).$$

Applying these $x, y, z \in \mathbb{C}$ yields the following d_1:

$$d_1(\lambda) = b_1(\lambda)\left(\frac{\frac{2(a'(b_{11})Q(b_{11})+a(b_{11})Q'(b_{11}))}{b_1''(b_1(b_{11})b_2(b_{11})} - y\left(\frac{b_2'(b_{11})}{b_2(b_{11})} + \frac{b_1'''(b_{11})}{3b_1''(b_{11})}\right)}{(\lambda - b_{11})}\right.$$
$$\left. + \frac{2a(b_{11})Q(b_{11})}{b_1''(b_{11})b_2(b_{11})(\lambda - b_{11})^2} + \frac{a(b_{12})Q(b_{12})}{b_1'(b_{12})b_2(b_{12})(\lambda - b_{12})}\right).$$

Apparently, this d_1 solves 4.13,4.14 and 4.15 at the respective roots of b_1 and equals the d_1 from the proof of lemma 4.2.
Hence, we again construct an x_1 with the result of case 1 and the d_1 of case 2:

$$x_1 = -\frac{1}{4}\sum_{i=1}^{4}\frac{d_1(\alpha_i)}{b_1(\alpha_i)}$$
$$= -\frac{1}{4}\sum_{i=1}^{4}\left(\frac{\frac{2(a'(b_{11})Q(b_{11})+a(b_{11})Q'(b_{11}))}{b_1''(b_{11})b_2(b_{11})} - y\left(\frac{b_2'(b_{11})}{b_2(b_{11})} + \frac{b_1'''(b_{11})}{3b_1''(b_{11})}\right)}{(\alpha_i - b_{11})}\right.$$
$$\left. + \frac{2a(b_{11})Q(b_{11})}{b_1''(b_{11})b_2(b_{11})(\alpha_i - b_{11})^2} + \frac{a(b_{12})Q(b_{12})}{b_1'(b_{12})b_2(b_{12})(\alpha_i - b_{12})}\right).$$

Then, $c_1 = d_1 + x_1b_1$ is the unique complex polynomial c_1 of lemma 4.2 and equals the polynonomial 4.9 in the claim.
Case 3: Let b_{11} be the root of order three of b_1. d_1 must solve the following equations as mentioned in the proof of lemma 4.2:

$$d_1(b_{11})b_2(b_{11}) = a(b_{11})Q(b_{11}) \tag{4.16}$$
$$d_1'(b_{11})b_2(b_{11}) + d_1(b_{11})b_2'(b_{11}) = a'(b_{11})Q(b_{11}) + a(b_{11})Q'(b_{11}) \tag{4.17}$$
$$d_1''(b_{11})b_2(b_{11}) + 2d_1'(b_{11})b_2'(b_{11}) + d_1(b_{11})b_2''(b_{11})$$
$$= a''(b_{11})Q(b_{11}) + 2a'(b_{11})Q'(b_{11}) + a(b_{11})Q''(b_{11}). \tag{4.18}$$

We make the following approach: Let

$$d_1(\lambda) = b_1(\lambda)\left(\frac{x}{(\lambda - b_{11})} + \frac{y}{(\lambda - b_{11})^2} + \frac{z}{(\lambda - b_{11})^3}\right)$$

with constants $x, y, z \in \mathbb{C}$. Calculating the derivatives yields

$$\begin{aligned}
d_1'(\lambda) = b_1'(\lambda)&\left(\frac{x}{(\lambda-b_{11})}+\frac{y}{(\lambda-b_{11})^2}+\frac{z}{(\lambda-b_{11})^3}\right)\\
-b_1(\lambda)&\left(\frac{x}{(\lambda-b_{11})^2}+\frac{2y}{(\lambda-b_{11})^3}+\frac{3z}{(\lambda-b_{12})^4}\right)\\
d_1''(\lambda) = b_1''(\lambda)&\left(\frac{x}{(\lambda-b_{11})}+\frac{y}{(\lambda-b_{11})^2}+\frac{z3}{(\lambda-b_{11})^3}\right)\\
-2b_1'(\lambda)&\left(\frac{x}{(\lambda-b_{11})^2}+\frac{2y}{(\lambda-b_{11})^3}+\frac{3z}{(\lambda-b_{11})^4}\right)\\
+b_1(\lambda)&\left(\frac{2x}{(\lambda-b_{11})^3}+\frac{6y}{(\lambda-b_{11})^4}+\frac{12z}{(\lambda-b_{11})^5}\right).
\end{aligned}$$

In the following, z is derived through equation 4.16, y by use of 4.17 and finally z employing 4.18. Again, we calculate the derivatives of d_1 at the root b_{11}:

$$\begin{aligned}
d_1(b_{11}) &= \lim_{\lambda\to b_{11}}\frac{b_1(\lambda)}{(\lambda-b_{11})}x+\lim_{\lambda\to b_{11}}\frac{b_1(\lambda)}{(\lambda-b_{11})^2}y+\lim_{\lambda\to b_{11}}\frac{b_1(\lambda)}{(\lambda-b_{11})^3}z\\
&= \lim_{\lambda\to b_{11}}\frac{b_1'(\lambda)}{1}x+\lim_{\lambda\to b_{11}}\frac{b_1''(\lambda)}{2}y+\lim_{\lambda\to b_{11}}\frac{b_1'''(\lambda)}{6(\lambda-b_{11})}z\\
&= \frac{b_1'''(b_{11})}{6}z
\end{aligned}$$

$$\begin{aligned}
d_1'(b_{11}) &= \lim_{\lambda\to b_{11}}\frac{b_1'(\lambda)(\lambda-b_{11})-b_1(\lambda)}{(\lambda-b_{11})^2}x+\lim_{\lambda\to b_{11}}\frac{b_1'(\lambda)(\lambda-b_{11})-2b_1(\lambda)}{(\lambda-b_{11})^3}y\\
&+\lim_{\lambda\to b_{11}}\frac{b_1'(b_{11})(b_{11}-b_{11})-3b_1(b_{11})}{(\lambda-b_{11})^4}z\\
&= \lim_{\lambda\to b_{11}}\frac{b_1''(\lambda)(\lambda-b_{11})+b_1'(\lambda)-b_1'(\lambda)}{2(\lambda-b_{11})}x\\
&+\lim_{\lambda\to b_{11}}\frac{b_1'''(\lambda)(\lambda-b_{11})+2b_1''(\lambda)-2b_1''(\lambda)}{6(\lambda-b_{11})}y\\
&+\lim_{\lambda\to b_{11}}\frac{b_1''''(\lambda)(\lambda-b_{11})+3b_1'''(\lambda)-3b_1'''(\lambda)}{24(\lambda-b_{11})}z\\
&= \lim_{\lambda\to b_{11}}\frac{b_1'''(\lambda)}{6}y=\frac{b_1'''(b_{11})}{6}y.
\end{aligned}$$

The last equation sign is valid due to the fact that b_1 is a polynomial of degree three and hence, $b_1''''(\lambda) = 0$.

$$\begin{aligned}
d_1''(b_{11}) = &\lim_{\lambda\to b_{11}} \frac{b_1''(\lambda)(\lambda-b_{11})^2 - 2b_1'(\lambda)(\lambda-b_{11}) + 2b_1(\lambda)}{(\lambda-b_{11})^3}x \\
&+ \lim_{\lambda\to b_{11}} \frac{b_1''(\lambda)(\lambda-b_{11})^2 - 4b_1'(\lambda)(\lambda-b_{11}) + 6b_1(\lambda)}{(\lambda-b_{11})^4}y \\
&+ \lim_{\lambda\to b_{11}} \frac{b_1''(\lambda)(\lambda-b_{11})^2 - 6b_1'(\lambda)(\lambda-b_{11}) + 12b_1(\lambda)}{(\lambda-b_{11})^5}z \\
&= + \lim_{\lambda\to b_{11}} \frac{b_1'''(\lambda)}{3}x = \frac{b_1'''(b_{11})}{3}x
\end{aligned}$$

Equation 4.16 yields

$$\begin{aligned}
& d_1(b_{11})b_2(b_{11}) = a(b_{11})Q(b_{11}) \\
\Leftrightarrow\quad & \frac{b_1'''(b_{11})b_2(b_{11})}{6}z = a(b_{11})Q(b_{11}) \\
\Leftrightarrow\quad & z = \frac{6a(b_{11})Q(b_{11})}{b_1'''(b_{11})b_2(b_{11})}
\end{aligned}$$

Inserting $d_1'(b_{11})$ in equation 4.17 for y depending on the above calculated z:

$$\begin{aligned}
& d_1'(b_{11})b_2(b_{11}) + d_1(b_{11})b_2'(b_{11}) = a'(b_{11})Q(b_{11}) + a(b_{11})Q'(b_{11}) \\
\Leftrightarrow\quad & \frac{b_2(b_{11})b_1'''(b_{11})}{6}y + \frac{b_1'''(b_{11})b_2'(b_{11})}{6}z = a'(b_{11})Q(b_{11}) + a(b_{11})Q'(b_{11}) \\
\Leftrightarrow\quad & y = \frac{6(a'(b_{11})Q(b_{11}) + a(b_{11})Q'(b_{11}))}{b_2(b_{11})b_1'''(b_{11})} - \frac{b_2'(b_{11})}{b_2(b_{11})}
\end{aligned}$$

and finally we get x through 4.18:

$$\begin{aligned}
& d_1''(b_{11})b_2(b_{11}) + 2d_1'(b_{11})b_2'(b_{11}) + d_1(b_{11})b_2''(b_{11}) \\
& = a''(b_{11})Q(b_{11}) + 2a'(b_{11})Q'(b_{11}) + a(b_{11})Q''(b_{11}) \\
\Leftrightarrow\quad & \frac{b_1'''(b_{11})b_2(b_{11})}{3}x + 2\frac{b_2'(b_{11})b_1'''(b_{11})}{6}y + \frac{b_1'''(b_{11})b_2''(b_{11})}{6}z \\
& = a''(b_{11})Q(b_{11}) + 2a'(b_{11})Q'(b_{11}) + a(b_{11})Q''(b_{11}) \\
\Leftrightarrow\quad & x = \frac{3(a''(b_{11})Q(b_{11}) + 2a'(b_{11})Q'(b_{11}) + a(b_{11})Q''(b_{11}))}{b_1'''(b_{11})b_2(b_{11})} \\
& \quad - \frac{b_2'(b_{11})}{b_2(b_{11})}y - \frac{b_2''(b_{11})}{2b_2(b_{11})}z.
\end{aligned}$$

In summary, using this approach for d_1, it is

$$\begin{aligned}
d_1(\lambda) &= b_1(\lambda)\left(\frac{x}{(\lambda-b_{11})}+\frac{y}{(\lambda-b_{11})^2}+\frac{z}{(\lambda-b_{11})^3}\right)\\
&= b_1(\lambda)\Bigg(\Bigg(\frac{3(a''(b_{11})Q(b_{11})+2a'(b_{11})Q'(b_{11})+a(b_{11})Q''(b_{11}))}{b_1'''(b_{11})b_2(b_{11})}\\
&-\frac{b_2'(b_{11})}{b_2(b_{11})}y-\frac{b_2''(b_{11})}{2b_2(b_{11})}z\Bigg)\frac{1}{(\lambda-b_{11})}+\Bigg(\frac{6(a'(b_{11})Q(b_{11})+a(b_{11})Q'(b_{11}))}{b_2(b_{11})b_1'''(b_{11})}\\
&-\frac{b_2'(b_{11})}{b_2(b_{11})}\Bigg)\frac{1}{(\lambda-b_{11})^2}+\left(\frac{6a(b_{11})Q(b_{11})}{b_1'''(b_{11})b_2(b_{11})}\right)\frac{1}{(\lambda-b_{11})^3}\Bigg)
\end{aligned}$$

Obviously, this d_1 solves the three equations 4.16,4.17 and 4.18 at b_{11} and resembles the explicit d_1 mentioned in the proof of lemma 4.2. Similarly to the previous cases, we construct our x_1. The respective

$$\begin{aligned}
x_1 = -\frac{1}{4}\sum_{i=1}^{4}&\Bigg(\Bigg(\frac{3(a''(b_{11})Q(b_{11})+2a'(b_{11})Q'(b_{11})+a(b_{11})Q''(b_{11}))}{b_1'''(b_{11})b_2(b_{11})}\\
&-\frac{b_2'(b_{11})}{b_2(b_{11})}y-\frac{b_2''(b_{11})}{2b_2(b_{11})}z\Bigg)\frac{1}{(\alpha_i-b_{11})}+\Bigg(\frac{6(a'(b_{11})Q(b_{11})+a(b_{11})Q'(b_{11}))}{b_2(b_{11})b_1'''(b_{11})}\\
&-\frac{b_2'(b_{11})}{b_2(b_{11})}\Bigg)\frac{1}{(\alpha_i-b_{11})^2}+\left(\frac{6a(b_{11})Q(b_{11})}{b_1'''(b_{11})b_2(b_{11})}\right)\frac{1}{(\alpha_i-b_{11})^3}\Bigg)
\end{aligned}$$

is a complex number and it equals the x_1 in the proof of lemma 4.2. Thus, we obtain the $c_1 = d_1 + x_1 b_1$ of lemma 4.2. It equals 4.12.
By constructing d_2, x_2 for b_2 and respectively a c_2 analogously, we obtain an explicit pair of unique complex polynomials (c_1, c_2) of degree three which are the polynomials c_1 and c_2 of lemma 4.2 and hence, this completes the proof. □

Lemma 4.4. *Let $a \in \mathcal{M}_2^1, Q \in \mathbb{C}^2[\lambda], b_1, b_2 \in \mathbb{C}^3[\lambda]$ and b_1, b_2 without common roots satisfying the following reality conditions:*

$$\begin{aligned}
a(\lambda) &= \lambda^4\overline{a(\overline{\lambda^{-1}})}\\
Q(\lambda) &= \lambda^2\overline{Q(\overline{\lambda^{-1}})}\\
\overline{b_2(\lambda)} &= \overline{\lambda^3}b_2(\overline{\lambda^{-1}})\\
\overline{b_1(\lambda)} &= \overline{\lambda^3}b_1(\overline{\lambda^{-1}})
\end{aligned}$$

Then, the pair (c_1, c_2) of lemma 4.3 satisfies the reality conditions

$$\lambda^3\overline{c_1(\overline{\lambda^{-1}})} = c_1(\lambda)$$
$$\lambda^3\overline{c_2(\overline{\lambda^{-1}})} = c_2(\lambda).$$

Proof. We know that the c_1, c_2 from lemma 4.3 solve equation 4.3 uniquely under the conditions 4.4 and 4.5. With the involution $\lambda \to \frac{1}{\lambda}$, the roots of a get mapped onto themselves. Therefore, also the second condition in lemma 4.2 is invariant under the involution. We apply the reality conditions of a, Q, b_1 and b_2 as well as the above-mentioned involution to equation 4.3:

$$\begin{aligned}
& Q(\lambda)a(\lambda) = c_1(\lambda)b_2(\lambda) - c_2(\lambda)b_1(\lambda) \\
\Leftrightarrow\quad & \frac{Q(\lambda)a(\lambda)}{b_1(\lambda)b_2(\lambda)} = \frac{c_1(\lambda)}{b_1(\lambda)} - \frac{c_2(\lambda)}{b_2(\lambda)} \\
\Leftrightarrow\quad & \frac{\lambda^2\overline{Q(\overline{\lambda^{-1}})}\lambda^4\overline{a(\overline{\lambda^{-1}})}}{\lambda^3\overline{b_2(\overline{\lambda^{-1}})}\lambda^3\overline{b_1(\overline{\lambda^{-1}})}} = \frac{c_1(\lambda)}{\lambda^3\overline{b_1(\overline{\lambda^{-1}})}} - \frac{c_2(\lambda)}{\lambda^3\overline{b_2(\overline{\lambda^{-1}})}} \\
\Leftrightarrow\quad & \frac{\overline{Q(\overline{\lambda^{-1}})a(\overline{\lambda^{-1}})}}{\overline{b_2(\overline{\lambda^{-1}})}\,\overline{b_1(\overline{\lambda^{-1}})}} = \frac{\lambda^{-3}c_1(\lambda)}{\overline{b_1(\overline{\lambda^{-1}})}} - \frac{\lambda^{-3}c_2(\lambda)}{\overline{b_2(\overline{\lambda^{-1}})}} \\
\Leftrightarrow\quad & \overline{Q(\overline{\lambda^{-1}})a(\overline{\lambda^{-1}})} = \lambda^{-3}c_1(\lambda)\overline{b_2(\overline{\lambda^{-1}})} - \lambda^{-3}c_2(\lambda)\overline{b_1(\overline{\lambda^{-1}})}.
\end{aligned}$$

Since c_1, c_2 solve equation 4.3 for every λ, we can substitute λ by $\frac{1}{\lambda}$ and obtain:

$$\Leftrightarrow\quad \overline{Q(\lambda)a(\lambda)} = \overline{\lambda}^3 c_1(\overline{\lambda^{-1}})\overline{b_2(\lambda)} - \overline{\lambda}^3 c_2(\overline{\lambda^{-1}})\overline{b_1(\lambda)}.$$

Now, we form the complex conjugate of this equation. This yields

$$\Leftrightarrow\quad Q(\lambda)a(\lambda) = \lambda^3\overline{c_1(\overline{\lambda^{-1}})}b_2(\lambda) - \lambda^3\overline{c_2(\overline{\lambda^{-1}})}b_1(\lambda).$$

Due to the fact that c_1, c_2 solve equation 4.3 uniquely, it follows that

$$c_1(\lambda) = \lambda^3\overline{c_1(\overline{\lambda^{-1}})} \qquad c_2(\lambda) = \lambda^3\overline{c_2(\overline{\lambda^{-1}})}$$

Additionally, we can prove this result by calculating $c_1(\lambda), c_2(\lambda)$ and $\lambda^3 c_1(\overline{\lambda^{-1}})$, $\lambda^3 c_2(\overline{\lambda^{-1}})$ using the explicit form of (c_1, c_2) from lemma 4.3. Here, we only compute them for case 1, c_1 of lemma 4.3. First of all, we need the reality condition for b_1' at the roots of b_1 which can be calculated through derivation of the reality condition of b_1 at $b_{1j}, j = 1, ..., 3$:

$$\begin{aligned}
b_1(\lambda) &= \lambda^3 \overline{b_1(\overline{\lambda^{-1}})} \\
\frac{d}{d\lambda} b_1(\lambda)\Big|_{\lambda=b_{1j}} = \frac{d}{d\lambda}\lambda^3 \overline{b_1(\overline{\lambda^{-1}})}\Big|_{\lambda=b_{1j}} &= 3b_{1j}^2 \overline{b_1(\overline{b_{1j}^{-1}})} + b_{1j}^3 \frac{d}{d\lambda}\overline{b_1(\overline{\lambda^{-1}})}\Big|_{\lambda=b_{1j}}
\end{aligned}$$

Let $b_1(\lambda) = \beta_0 + \beta_1\lambda + \overline{\beta}_1\lambda^2 + \overline{\beta}_0\lambda^3$ $\quad \overline{b_1(\overline{\lambda^{-1}})} = \overline{\beta}_0 + \overline{\beta}_1\frac{1}{\lambda} + \beta_1\frac{1}{\lambda^2} + \beta_0\frac{1}{\lambda^3}$.

$$\begin{aligned}
\Rightarrow \quad \frac{d}{d\lambda}\overline{b_1(\overline{\lambda^{-1}})}\Big|_{\lambda=b_{1j}} &= \frac{d}{d\lambda}\left(\overline{\beta}_0 + \overline{\beta}_1\frac{1}{\lambda} + \beta_1\frac{1}{\lambda^2} + \beta_0\frac{1}{\lambda^3}\right)\Big|_{\lambda=b_{1j}} \\
&= -\overline{\beta}_1\frac{1}{b_{1j}^2} - 2\beta_1\frac{1}{b_{1j}^3} - 3\beta_0\frac{1}{b_{1j}^4} \\
\text{We obtain for } b_1' : \quad b_1'(\lambda) &= \beta_1 + 2\overline{\beta}_1\lambda + 3\overline{\beta}_0\lambda^2 \\
\text{and} \quad \overline{b_1'(\overline{\lambda^{-1}})} &= \overline{\beta}_1 + 2\beta_1\frac{1}{\lambda} + 3\beta_0\frac{1}{\lambda^2} \\
\Rightarrow \quad b_1'(b_{1j}) = \frac{d}{d\lambda}b_1(\lambda)\Big|_{\lambda=b_{1j}} &= \frac{d}{d\lambda}\lambda^3\overline{b_1(\overline{\lambda^{-1}})}\Big|_{\lambda=b_{1j}} \\
= 3b_{1j}^2\overline{b_1(\overline{b_{1j}^{-1}})} + b_{1j}^3\frac{d}{d\lambda}\overline{b_1(\overline{\lambda^{-1}})}\Big|_{\lambda=b_{1j}} &= -\overline{\beta}_1 b_{1j} - 2\beta_1 - 3\beta_0\frac{1}{b_{1j}} \\
\Rightarrow \quad b_1'(b_{1j}) &= -b_1(b_{1j})\overline{b_1'(\overline{b_{1j}^{-1}})}
\end{aligned}$$

In the second last line, $b_1(\overline{b_{1j}^{-1}}) = 0$ is valid due to the fact that $\overline{b_{1j}^{-1}} = b_{1j}$. Now, we calculate

$$\lambda^3\overline{c_1(\overline{\lambda^{-1}})} = \lambda^3 \overline{b_1(\overline{\lambda^{-1}}) \sum_{j=1}^{3} \frac{Q(b_{1j})a(b_{1j})}{b_2(b_{1j})b_1'(b_{1j})}\left(\frac{1}{(\overline{\lambda^{-1}} - b_{1j})} - \frac{1}{4}\sum_{i=1}^{4}\frac{1}{(\alpha_i - b_{1j})}\right)}.$$

With the calculation rules for the complex conjugate we obtain (with the reality condition of b_1):

$$\lambda^3\overline{c_1(\overline{\lambda^{-1}})} = b_1(\lambda)\sum_{j=1}^{3}\frac{\overline{Q(b_{1j})a(b_{1j})}}{\overline{b_2(b_{1j})b_1'(b_{1j})}}\left(\frac{1}{(\lambda^{-1} - \overline{b_{1j}})} - \frac{1}{4}\sum_{i=1}^{4}\frac{1}{(\overline{\alpha}_i - \overline{b_{1j}})}\right)$$

and we compute the following equations:

$$\frac{\overline{Q(b_{1j})a(b_{1j})}}{\overline{b_2(b_{1j})b_1'(b_{1j})}} \qquad \left(\frac{1}{(\lambda^{-1} - \overline{b_{1j}})} - \frac{1}{4}\sum_{i=1}^{4}\frac{1}{(\overline{\alpha}_i - \overline{b_{1j}})}\right).$$

Using the reality condition, we conclude for the left one:

$$\frac{\overline{Q(b_{1j})a(b_{1j})}}{\overline{b_2(b_{1j})b_1'(b_{1j})}} = \frac{\overline{b_{1j}^2}Q(\overline{b_{1j}^{-1}})\overline{b_{1j}^4}a(\overline{b_{1j}^{-1}})}{\overline{b_{1j}^3}b_2(\overline{b_{1j}^{-1}})(-\overline{b_{1j}}b_1(\overline{b_{1j}^{-1}}))} = -\frac{Q(\overline{b_{1j}^{-1}})a(\overline{b_{1j}^{-1}})}{\overline{b_{1j}^{-2}}b_2(\overline{b_{1j}^{-1}})b_1(\overline{b_{1j}^{-1}})}.$$

Let $b_{1\tilde{j}} := \overline{b_{1j}^{-1}}$. Then, we get

$$\frac{\overline{Q(b_{1j})a(b_{1j})}}{\overline{b_2(b_{1j})b_1'(b_{1j})}} = -\frac{Q(b_{1\tilde{j}})a(b_{1\tilde{j}})}{b_{1\tilde{j}}^2 b_2(b_{1\tilde{j}})b_1'(b_{1\tilde{j}})}.$$

For the term on the right, we obtain (due to the fact that $\alpha_i = \overline{\alpha_i^{-1}}$ and because the polynomial a has four pairwise distinct roots):

$$\frac{1}{(\lambda^{-1} - \overline{b_{1j}})} = \frac{\lambda\overline{b_{1j}^{-1}}}{(\overline{b_{1j}^{-1}} - \lambda)} = \frac{\lambda b_{1\tilde{j}}}{(b_{1\tilde{j}} - \lambda)}$$

$$-\frac{1}{4}\sum_{i=1}^{4}\frac{1}{(\overline{\alpha}_i - \overline{b_{1j}})} = -\frac{1}{4}\sum_{i=1}^{4}\frac{\overline{\alpha_i^{-1}b_{1j}^{-1}}}{(\overline{b_{1j}^{-1}} - \overline{\alpha_i^{-1}})} = -\frac{1}{4}\sum_{i=1}^{4}\frac{\alpha_i b_{1\tilde{j}}}{(b_{1\tilde{j}} - \alpha_i)}.$$

Inserting this yields

$$\begin{aligned}
\lambda^3\overline{c_1(\overline{\lambda^{-1}})} &= b_1(\lambda)\sum_{j=1}^{3}\frac{\overline{Q(b_{1j})a(b_{1j})}}{\overline{b_2(b_{1j})b_1'(b_{1j})}}\left(\frac{1}{(\lambda^{-1} - \overline{b_{1j}})} - \frac{1}{4}\sum_{i=1}^{4}\frac{1}{(\overline{\alpha}_i - \overline{b_{1j}})}\right)\\
&= -b_1(\lambda)\sum_{j=1}^{3}\frac{Q(b_{1\tilde{j}})a(b_{1\tilde{j}})}{b_{1\tilde{j}}^2 b_2(b_{1\tilde{j}})b_1'(b_{1\tilde{j}})}\left(\frac{\lambda b_{1\tilde{j}}}{(b_{1\tilde{j}} - \lambda)} - \frac{1}{4}\sum_{i=1}^{4}\frac{\alpha_i b_{1\tilde{j}}}{(b_{1\tilde{j}} - \alpha_i)}\right)\\
&= -b_1(\lambda)\sum_{j=1}^{3}\frac{Q(b_{1\tilde{j}})a(b_{1\tilde{j}})}{b_2(b_{1\tilde{j}})b_1'(b_{1\tilde{j}})}\frac{1}{b_{1\tilde{j}}}\left(\frac{\lambda}{(b_{1\tilde{j}} - \lambda)} - \frac{1}{4}\sum_{i=1}^{4}\frac{\alpha_i}{(b_{1\tilde{j}} - \alpha_i)}\right).
\end{aligned}$$

Since $b_{11} = b_{1\tilde{2}}, b_{12} = b_{1\tilde{1}}, b_{13} = b_{1\tilde{3}}$ and, by changing the order of the summands, we have to prove that

$$\begin{aligned}
-\frac{1}{b_{1j}}\left(\frac{\lambda}{(b_{1j} - \lambda)} - \frac{1}{4}\sum_{i=1}^{4}\frac{\alpha_i}{(b_{1j} - \alpha_i)}\right) &= \frac{1}{(\lambda - b_{1j})} - \frac{1}{4}\sum_{i=1}^{4}\frac{1}{(\alpha_i - b_{1j})}\\
\Leftrightarrow \quad \frac{1}{4}\sum_{i=1}^{4}\frac{\alpha_i}{(b_{1j} - \alpha_i)} + \frac{1}{4}\sum_{i=1}^{4}\frac{b_{1j}}{(\alpha_i - b_{1j})} &= \frac{\lambda}{(b_{1j} - \lambda)} + \frac{b_{1j}}{(\lambda - b_{1j})}\\
\Leftrightarrow \quad \frac{1}{4}\sum_{i=1}^{4}\frac{\alpha_i - b_{1j}}{(b_{1j} - \alpha_i)} &= \frac{\lambda - b_{1j}}{(b_{1j} - \lambda)} \quad \Leftrightarrow \quad 1 = 1
\end{aligned}$$

Consequently, $c_1(\lambda) = \lambda^3\overline{c_1(\overline{\lambda^{-1}})}$ and analogously for c_2. Thus, this completes the proof for case 1 of lemma 4.3 explicitly, too. □

The last three lemmata show that the pair (c_1, c_2), constructed in lemma 4.3, is the unique solution of equation 4.3 which also allow 4.1 and 4.2 to be solved. These polynomials c_1 and c_2 are of degree three and satisfy the respective reality conditions.
In the next steps, we prove that 4.1 and 4.2 have a unique solution $(\dot{a}, \dot{b}_1, \dot{b}_2)$ under certain assumptions.
First, we calculate $\dot{a}$ of the equations 4.1 and 4.2 with the uniquely determined pair of polynomials (c_1, c_2) from the lemma 4.3. Furthermore, we consider the equations 4.1 and 4.2 at the roots of a and use the assumption that $\gcd(b_1, b_2) = 1$. For overseeable $\dot{\alpha}_i$, we do not insert the explicit polynomials c_1 and c_2 from lemma 4.3. Then, we obtain the following case differentiation depending on the polynomials b_1, b_2 at the roots of a:
Case 1: a,b_1 have a root at α_i. In this case, b_2 cannot have a root at α_i and we obtain, using 4.2,

$$\begin{aligned} & i\alpha_i c_2(\alpha_i) a'(\alpha_i) = \dot{a}(\alpha_i) b_2(\alpha_i) \\ \Leftrightarrow \quad & i\alpha_i \frac{c_2(\alpha_i)}{b_2(\alpha_i)} = \frac{\dot{a}(\alpha_i)}{a'(\alpha_i)} = -\frac{\dot{\alpha}_i \prod_{r=1, r\neq i}^4 (\alpha_i - \alpha_r)}{\prod_{r=1, r\neq i}^4 (\alpha_i - \alpha_r)} \\ \Leftrightarrow \quad & \dot{\alpha}_i = -i\alpha_i \frac{c_2(\alpha_i)}{b_2(\alpha_i)}. \end{aligned}$$

Case 2: We get a similar result if a and b_2 have a root at α_i. Then, b_1 cannot have a root at α_i and we obtain, using 4.1,

$$\dot{\alpha}_i = -i\alpha_i \frac{c_1(\alpha_i)}{b_1(\alpha_i)}.$$

Case 3: α_i is a root of a but neither for b_1 nor b_2. We obtain, using 4.1 and 4.2,

$$\dot{\alpha}_i = -i\alpha_i \frac{c_2(\alpha_i)}{b_2(\alpha_i)} = -i\alpha_i \frac{c_1(\alpha_i)}{b_1(\alpha_i)}$$

Since (c_1, c_2) solve equation 4.3, i.e.

$$\begin{aligned} & c_1 b_2 - c_2 b_1 = aQ \\ \Leftrightarrow \quad & \frac{c_1(\alpha_i)}{b_1(\alpha_i)} = \frac{c_2(\alpha_i)}{b_2(\alpha_i)}, \end{aligned}$$

the last equation is satisfied.
Due to the fact that the pair of polynomials (c_1, c_2) is uniquely determined,

we obtain the uniquely defined

$$\dot{a}(\lambda) = -\sum_{i=1}^{4} \dot{\alpha}_i \prod_{\substack{j=1 \\ j\neq i}}^{4} (\lambda - \alpha_j)$$

for given $(a, b_1, b_2), Q$ and the respective (c_1, c_2).
In the next step, we calculate $(\dot{b}_1, \dot{b}_2)$ of the equations 4.1 and 4.2.

$$\begin{aligned} & (2\lambda a c_1' - a c_1 - \lambda a' c_1) i = 2a\dot{b}_1 - \dot{a} b_1 \\ \Leftrightarrow \quad & \dot{b}_1 = i\left(\frac{2\lambda a c_1' - a c_1 - \lambda a' c_1}{2a}\right) + \frac{b_1}{2a}\dot{a} \end{aligned}$$

and

$$\begin{aligned} & (2\lambda a c_2' - a c_2 - \lambda a' c_2) i = 2a\dot{b}_2 - \dot{a} b_2 \\ \Leftrightarrow \quad & \dot{b}_2 = i\left(\frac{2\lambda a c_2' - a c_2 - \lambda a' c_2}{2a}\right) + \frac{b_2}{2a}\dot{a}. \end{aligned}$$

Again, we do not insert c_1, c_2 and $\dot{a}$ for neatly arranged $(\dot{b}_1, \dot{b}_2)$.

Corollary 4.5. *Given a polynomial $a \in \mathcal{M}_2^1$, two polynomials $b_1, b_2 \in \mathbb{C}^3[\lambda]$ without common roots and a pair of polynomials (c_1, c_2) of lemma 4.3, the system 4.1-4.2 has a unique solution for (a, b_1, b_2, c_1, c_2).*

Proof. Let α_i be a root of a. Through equation 4.1 and 4.2 $\dot{a}$ is uniquely determined at the roots of a due to the fact that only b_1 or b_2 can have a root at each α_i. If b_1 or b_2 have a root at α_i, the other one cannot have one due to the fact that $\gcd(b_1, b_2) = 1$ and the respective equation 4.1 or 4.2 determines $\dot{a}$ at α_i uniquely as mentioned in the calculations above. If neither b_1 nor b_2 have a root at α_i the respective $\dot{\alpha}_i$ obtained by 4.1 and 4.2 is the same because (c_1, c_2) solve 4.3 at α_i, i.e.

$$\frac{c_1(\alpha_i)}{b_1(\alpha_i)} = \frac{c_2(\alpha_i)}{b_2(\alpha_i)}.$$

This $\dot{a}$ is computed in the case differentiation above depending on b_1, b_2 at the roots of $\alpha_i, i = 1, ..., 4$.
For such $(a, b_1, b_2, c_1, c_2, \dot{a})$, the system 4.1-4.2 has a unique solution for $(\dot{b}_1, \dot{b}_2)$ given by

$$\begin{aligned} \dot{b}_1 &= i\left(\frac{2\lambda a c_1' - a c_1 - \lambda a' c_1}{2a}\right) + \frac{b_1}{2a}\dot{a} \\ \dot{b}_2 &= i\left(\frac{2\lambda a c_2' - a c_2 - \lambda a' c_2}{2a}\right) + \frac{b_2}{2a}\dot{a}. \end{aligned}$$

□

Finally, we write the polynomials a, Q, c_1, c_2 in the following way:

$$\begin{aligned} a(\lambda) &= \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + \overline{a}_1\lambda + 1 \\ Q(\lambda) &= q_0 + q_1\lambda + \overline{q}_0\lambda^2 \\ c_1(\lambda) &= \gamma_0 + \gamma_1\lambda + \overline{\gamma}_1\lambda^2 + \overline{\gamma}_0\lambda^3 \\ c_2(\lambda) &= \tilde{\gamma}_0 + \tilde{\gamma}_1\lambda + \overline{\tilde{\gamma}}_1\lambda^2 + \overline{\tilde{\gamma}}_0\lambda^3 \end{aligned}$$

with $a_1, q_0, \gamma_0, \gamma_1, \tilde{\gamma}_0, \tilde{\gamma}_1 \in \mathbb{C}$ and $q_1 \in \mathbb{R}, a_2 \in \mathbb{R}^+$. Denoting the complex parameters in terms of the real and imaginary part, we obtain three real degrees of freedom for $\dot{a}$ and Q and four real degrees of freedom for c_1 and c_2. Since Q and $\dot{a}$ both have three real degrees of freedom, we can investigate the map $f : \mathbb{R}^3 \to \mathbb{R}^3, Q \mapsto \dot{a}$ and prove that it is an invertible linear map:

Lemma 4.6. *Let $a \in \mathcal{M}_2^1$, b_1 and b_2 as above defined and without common root, and let $\dot{a}$ be the partial derivative of a with respect to t evaluated at $t = 0$ and $Q \in \mathbb{C}^2[\lambda]$ satisfying the reality condition $Q(\lambda) = \lambda^2\overline{Q(\overline{\lambda^{-1}})}$. Then, $f : \mathbb{R}^3 \to \mathbb{R}^3, Q \mapsto \dot{a}$ is an invertible linear map.*

Proof. Using the previous calculations, we see that the function $g : \mathbb{R}^3 \to \mathbb{R}^4 \times \mathbb{R}^4, Q \mapsto (c_1, c_2)$ is a linear map. With the computations before corollary 4.5 also the function $h : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^3, (c_1, c_2) \to \dot{a}$ is a linear map. Therefore, the composition of these functions is a linear map, i.e. $f = h \circ g : \mathbb{R}^3 \to \mathbb{R}^3, Q \mapsto \dot{a}$. Now we can show that f is bijective. In order to do this, we show that $ker(f) = \{0\}$. Thus, we know that f is injective and since f is a linear mapping it is also bijective.
Let $\dot{a} = 0$. We consider the following equations 4.1 and 4.2:

$$\begin{aligned} (2\lambda a c_1' - a c_1 - \lambda a' c_1)i &= 2a\dot{b}_1 - \dot{a}b_1, \\ (2\lambda a c_2' - a c_2 - \lambda a' c_2)i &= 2a\dot{b}_2 - \dot{a}b_2. \end{aligned}$$

The terms $2\lambda a c_i' - a c_i, i = 1, 2$ and the right-hand sides of the equations vanish at every root of a. Since $a \in \mathcal{M}_2^1$, a has four simple roots $\alpha_i, i = 1, ...4$ and the derivative a' with respect to λ cannot have a root in common with a. Thus, c_1 and c_2 equal zero at every α_i. Due to the fact that a has four roots but c_1 and c_2 are only polynomials of degree three, it follows that $c_1 = 0$ and $c_2 = 0$. Then, we obtain $\dot{a} = 0 \Rightarrow Q = 0$ by 4.3 and $ker(f) = \{0\}$. With the above-mentioned argumentation, f is an invertible linear map. □

5 Conclusion and outlook

In this section, we briefly summarize the results of the thesis and give some remarks on how the work could be carried forward.
First of all, we introduced fundamental definitions and explained how $a \in \mathcal{M}_2^1$ and the respective period lattice are connected. For the two generators of such a lattice we obtained polynomials b_1 and b_2. Then, we described the spectral curves of CMC tori in $\mathbb{R}^3$ by an equation of the form $v^2 = \lambda a(\lambda)$ equally to a polynomial a in $\mathcal{M}_2^1$. This explained the context between the underlying theory of CMC tori and the calculations in section four. Additionally, we described the Whitham deformations and introduced the vector field $(a, b_1, b_2) \to (\dot{a}, \dot{b}_1, \dot{b}_2)$. Afterwards, we calculated such a vector field explicitly using the equations 4.1,4.2 and 4.3 for given $(a, b_1, b_2), Q$. We obtained a unique pair of polynomials (c_1, c_2) of degree three satisfying $c_i(\lambda) = \lambda^3 \overline{c_i(\overline{\lambda^{-1}})}, i = 1, 2$ for given $a \in \mathcal{M}_2^1$, two polynomials $b_1, b_2 \in \mathbb{C}^3[\lambda]$ which have no common root and satisfy the reality conditions $b_i(\lambda) = \lambda^3 \overline{b_i(\overline{\lambda^{-1}})}, i = 1, 2$ and $Q \in \mathbb{C}^2[\lambda]$ with $Q(\lambda) = \lambda^2 \overline{Q(\overline{\lambda^{-1}})}$. For these uniquely determined (c_1, c_2) and (a, b_1, b_2), the equations 4.1 and 4.2 have a unique solution $(\dot{a}, \dot{b}_1, \dot{b}_2)$. Finally, the mapping of such a polynomial Q and $\dot{a}$ was investigated and we proved that this mapping is linear and invertible.

The goal of these calculations is to obtain the dependence of this period lattice from the spectral curve. In particular, we are interested in how a change of the polynomial a affects the period lattice. In the following we describe how, based on our work, this connection can be made explicit. This means, we need to calculate along a continuous differentiable path parametrized by t in the set $\mathcal{M}_2^1$, which starts at the given (a, b_1, b_2), the polynomials b_1, and b_2. Since we proved that the mapping of Q and $\dot{a}$ is invertible, we can also compute a unique Q and $\dot{b}_1, \dot{b}_2$ for given $(a, b_1, b_2), \dot{a}$ under the same assumptions as before. In this case, we know the polynomials b_1, b_2 at the beginning of the path and also $\dot{a}$ along the whole path. Thus, we can calculate Q for such an $\dot{a}$ and hence, the respective $\dot{b}_1, \dot{b}_2$ along the whole path. By integration along the whole path, it is possible to obtain the polynomials b_1, b_2 from $\dot{b}_1, \dot{b}_2$.

Consequently, we receive the respective b_1, b_2 for any $a \in \mathcal{M}_2^1$. The generators of the associated period lattice are determined at $\lambda = 0$ as we described in section three. Finally, this explains the dependence of the period lattice on the spectral curve and facilitates the investigation on how the period lattice changes when a changes.

6 References

[1] Werner Ballmann. *Einführung in die Geometrie und Topologie*. Springer-Verlag, 2015.

[2] Emma Carberry and Martin Ulrich Schmidt. The prevalence of tori amongst constant mean curvature planes in r3. *Journal of Geometry and Physics*, 106:352–366, 2016.

[3] Eberhard Freitag and Rolf Busam. *Complex analysis*. Springer Science & Business Media, 2006.

[4] Ricardo Pena Hoepner. Solutions of the sinh-gordon equation of spectral genus two. *Master thesis*, 2015.

[5] Martin Kilian, Martin U Schmidt, and Nicholas Schmitt. Flows of constant mean curvature tori in the 3-sphere: The equivariant case. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2015(707):45–86, 2015.

[6] Markus Knopf, Ricardo Pena Hoepner, and Martin U Schmidt. Solutions of the sinh-gordon equation of spectral genus two and constrained willmore tori i. *arXiv preprint arXiv:1708.00887*, 2017.

[7] Ulrich Pinkall and Ivan Sterling. On the classification of constant mean curvature tori. *Annals of Mathematics*, 130(2):407–451, 1989.

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I declare that all statements and information contained herein are true, correct and accurate to the best of my knowledge and belief.

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