University of Mannheim
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Bachelor's Thesis

The Closure of Spectral Curves of Constant Mean Curvature Tori of Spectral Genus 2

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Abstract

Constant mean curvature tori are of special interest in the field of surface theory. They can be described through the solution of the elliptic sinh-Gordon equation. Its solutions are defined on the space of potentials. Hence they can be described in terms of spectral curves. We investigate the space of the spectral curves of spectral genus two that describe constant mean curvature tori with some additional conditions. We can show that this special space is a 2-dimensional submanifold of the space of spectral curves of spectral genus two. Furthermore, we use Whitham deformations to get the tangent vector fields on the tangent space.

Contents

1 Introduction

An interesting topic in the field of constant mean curvature surfaces is the construction of constant mean curvature (CMC) tori. In 1984 Wente disproved the Hopf conjecture that any closed, compact surface with constant mean curvature is a sphere by showing that there exists a CMC torus, the so-called Wente torus. Since then, a rich theory has been developed. It is possible to construct many more examples than just the Wente torus. These tori are described by solutions of the so-called sinh-Gordon equation

$$\Delta u + sinh(2u) = 0,$$

which are in return described through potentials, which are polynomials with matrices as coefficients. The determinants of these solutions are called spectral curves and will be of special interest in this thesis. They have the following form

$$y^2 = \lambda a(\lambda) = (-1)^g \lambda \prod_{j=1}^{g} \frac{\overline{\eta}_j}{|\eta_j|}(\lambda - \eta_j)(\lambda - \overline{\eta}_j^{-1}).$$

The number g is called spectral genus. It can be shown that the space $\mathcal{S}_1^2$, which is a space of special spectral curves, is a two-dimensional submanifold of $\mathcal{H}^2$. Ultimately, the goal of this work is to examine this submanifold. To do this we want to complement the elements $a \in \mathcal{S}_1^2$ with a basis (b_1, b_2) of the two-dimensional vector space $\mathcal{B}_a$. Then, a will be in $\mathcal{S}_1^2$ if and only if both polynomials of degree 3 (b_1, b_2) have a root at $\lambda = 1$. Through Whitham deformations we want to obtain two vector fields V_1 and V_2 that map $(a, b_1, b_2) \to (\dot{a}, \dot{b}_1, \dot{b}_2)$. With these vector fields we want to examine a mapping from $\mathcal{S}_1^2 \to \mathbb{S}^1 \times \mathbb{S}^1$.

In *chapter two* we will introduce some preliminaries regarding manifolds, submanifolds and vector fields.

The *third chapter* introduces spectral data, spectral curves, the space of spectral curves and elaborates on the polynomials a, b_1 and b_2. With a better understanding of the special space of spectral curves we can start arguing that it is in fact a submanifold. Furthermore, we refer to [CS16] to argue that we determined the closure of the space of spectral curves of constant mean curvature tori with spectral genus 2.

In *chapter four* we will use Whitham deformations to obtain two vector fields $V_1, V_2 : (a, b_1, b_2) \to (\dot{a}, \dot{b}_1, \dot{b}_2)$. In order to do so we introduce two polynomials of degree 3, which will be called c_1 and c_2, and a third polynomial of degree 2, which will be called Q. All three of these polynomials will satisfy the reality condition. Through the Whitham deformation we will get two partial differential equations and a regular equation depending on a, b_1, b_2, c_1, c_2 and Q. We want to solve them for polynomials $\dot{a}, \dot{b}_1$ and $\dot{b}_2$. This chapter is divided into a theoretical part, in which we derive the sytem of equations and prove that given a, b_1 and b_2 we can uniquely solve it, and into a second part, in which we try to explicitly calculate the vector fields V_1 and V_2.

In the *fifth chapter* we demonstrate that these vector fields commute.

In the *sixth chapter* we use Cayley transforms and try to get simpler results through Whitham deformations than in chapter four.

Finally in *chapter seven* we draw a conclusion.

2 Preliminaries

Since this work covers topics that are not treated equally in every bachelor's program, first of all this chapter will provide some necessary basics of an Analysis III/Differential Topology course based on [For09], [Die73], [Bal15] and [JJ11]. We assume a basic understanding of the notion of a manifold.

Definition 2.1 (Immersion). *Let $T \subset \mathbb{R}^k$ be open. A continuously differentiable function*

$$\phi = (\phi_1, ..., \phi_n) : T \to \mathbb{R}^n$$

is called immersion if

$$rank(d\phi(t)) = k \text{ for all } t \in T.$$

Definition 2.2 (Submanifold). *Let X, Y be two differentiable manifolds and $f : X \to Y$ an immersion. If f is a homeomorphism of X into $f(X) \subset Y$, then $f(X)$ is a submanifold of Y and $f : X \to f(X)$ is a diffeomorphism.*

Since we are interested in vector fields that map points to their tangent vectors the following definition is useful to understand chapter three and four.

Definition 2.3 (Tangents). *Let M be a differentiable manifold and $p \in M$. Two smooth curves γ_0 and γ_1 passing through p. They are called equivalent if for a chart x around p*

$$\left.\frac{d(x \circ \gamma_0)}{dt}\right|_{t=0} = \left.\frac{d(x \circ \gamma_1)}{dt}\right|_{t=0} \text{ holds.}$$

This equivalence is independent from the choice of x and therefore defines a equivalence class on the set of smooth curves passing through p. Hence we use the following definitions:

i) The tangent vector on M in p denotes the corresponding equivalence class.

ii) T_pM denotes the set of all tangent vectors and is called the tangent space.

iii) $TM = \bigcup_{p \in M} T_pM$ is called the tangent bundle.

Theorem 2.4. *Let $f : M \to N$ be a differentiable map, $dimM = n, dimN = n, m \leq n, p \in N$. Let $df(x)$ have rank n for all $x \in M$ with $f(x) = p$. Then $f^{-1}(p)$ is a union of differentiable submanifolds of M of dimension m-n.*

Proof. Let $x \in M$ thus we can write $x = (x_1, x_2, ..., x_n, x_{n+1}, ..., x_m)$. Assume $p = f(x) \in N$ and $rank(df(x)) = n$. By the implicit function theorem there exists an open neighborhood U_x and a differentiable map

$$g(x_{n+1}, ..., x_m) : U_2 \subset \mathbb{R}^{m-n} \to U_1 \subset \mathbb{R}^n$$

with

$$U_x = U_1 \times U_2$$

and

$$f(x) = p \Leftrightarrow (x_1, ..., x_n) = g(x_{n+1}, ..., x_m).$$

With

$$y_k = \begin{cases} x_k - g(x_{n+1}, ..., x_m), & \text{for } k \in \{1, ..., n\} \\ x_k, & \text{for } k \in \{n+1, ..., m\} \end{cases}$$

we then get coordinates for which

$$f(x) = p \Leftrightarrow y_k = 0 fork \in \{1, ..., n\}.$$

Thus $(y_{n+1}, ..., y_m)$ yield local coordinates for $\{f(x) = p\}$ and this implies that in some neighborhood of x $\{f(x) = p\}$ is a submanifold of M of dimension $m - n$. □

We want to show that a special space of spectral curves is a two-dimensional submanifold. We will achieve this with the following corollary.

Corollary 2.5 (Implicit function theorem)**.** *If df of $f : M \to N$ is onto for any chart y of N around $f(p)$ with $y(f(p)) = 0$ there exists a chart x of M around p with $x(p) = 0$ such that*

$$(y \circ f \circ x^{-1})(u_1, ..., u_m) = (u_1, ..., u_n).$$

If additionally q is a regular (smooth point) point of f $L = f^{-1}(q)$ is a submanifold of M of diemsion $m - n$ with $T_pL = ker(df)\forall p \in L$.

Proof. If *df* is onto in $p \in M$ then it is also onto in a neighborhood of p. □

Since we work with polynomials to a large extent the following definition is very useful.

Definition 2.6 (Greatest common divisor of Polynomials). *The greatest common divisor (gcd) of two polynomials is a polynomial of the maximal degree such that it is a factor in both of them.*

Example: $gcd((x+1)(x+2),(x+1)(x+3)) = (x+1)$

We will also make use of the concept of Resultants, which are defined in [GKZ94] as follows.

Definition 2.7 (Resultant). *The Resultant of two polynomials* $f(x) = a_m x^m + a_{m-1}x^{m-1} + ... + a_0$ *and* $g(x) = b_n x^n + b_{n-1}x^{n-1} + ... + b_0$ *is denoted by* $Res(f,g)$ *and is defined as the determinant of the* $Sylvester-Matrix$. *Thus it can be written as*

$$Res(f,g) = \begin{vmatrix} a_m & a_{m-1} & \dots & a_0 & & & \\ & a_m & a_{m-1} & \dots & a_0 & & \\ & & \ddots & & \ddots & & \\ & & a_m & a_{m-1} & \dots & a_0 & \\ b_n & b_{n-1} & \dots & b_0 & & & \\ & b_n & b_{n-1} & \dots & b_0 & & \\ & & \ddots & & \ddots & & \\ & & b_n & b_{n-1} & \dots & b_0 & \end{vmatrix}.$$

3 Spectral curves of CMC tori in $\mathbb{R}^3$

This section will briefly summarize chapter 2 of [CS16] and introduce the concept of spectral curves. Some constant mean curvature immersions of genus one surfaces (surfaces with one hole) in $\mathbb{R}^3$ can be described by so-called spectral data. That is the correspondence to the spectral curve X, which is an algebraic curve, a degree two meromorphic function λ, an anti-holomorphic involution ρ, a point on the unit circle λ_0 and a quarternionic line bundle L on the curve X. The immersions that can be described in such a way are said to be of $finite$ *type*. However, in the case that the spectral data describes a CMC torus certain periodicity conditions need to be satisfied. It is an important fact that all doubly-periodic such immersions correspond to spectral data and hence are of $finite$ *type*. The arithmetic genus of X is called *spectral genus*. In this work it will remain equal two.

We say a polynomial $f(\lambda)$ of degree n satisfies the reality condition if

$$\lambda^n \overline{f(\bar{\lambda}^{-1})} = f(\lambda) \text{ holds.}$$

The space of those polynomials is called $P^n_{\mathbb{R}}$. Now as mentioned in the introduction solutions of the sinh-Gordon equation

$$\Delta u + sinh(2u) = 0,$$

describe CMC tori. The solution of this equation is defined on the space of potentials in [KHS17] the space of potentials for genus two solutions is described as follows:

$$\mathsf{P}^2 := \left\{ \zeta = \begin{pmatrix} \alpha\lambda - \overline{\alpha}\lambda^2 & -\gamma^{-1} + \beta\lambda - \gamma\lambda^2 \\ \gamma\lambda - \overline{\beta}\lambda^2 + \gamma^{-1}\lambda^3 & -\alpha\lambda + \overline{\alpha}\lambda^2 \end{pmatrix} \middle| \, \alpha, \beta \in \mathbb{C}, \gamma \in \mathbb{R}^+ \right\}$$

The determinant of these matrices will now help us to describe the spectral curve X which will from now on be denoted as X_a. It is a Riemann surface and will be described by the equation

$$y^2 = \lambda a(\lambda) = (-1)^2 \lambda \prod_{j=1}^{2} \frac{\overline{\eta}_j}{|\eta_j|} (\lambda - \eta_j)(\lambda - \overline{\eta}_j^{-1}).$$

Then $a \in \mathcal{H}^2 :=$ {space of spectral curves of CMC immersions of $finite$ *type*} are polynomials of degree four that satisfy certain conditions, which are:

1. the reality condition $\lambda^4\overline{a(\bar{\lambda}^{-1})} = a(\lambda)$
2. $\lambda^{-2}a(\lambda) > 0$ for all $\lambda \in \mathbb{S}^1$
3. the highest coefficient is $\frac{\overline{\eta}_1\overline{\eta}_2}{|\eta_1||\eta_2|}$ thus it has absolute value 1
4. the roots of a are pairwise distinct, therefore X_a is smooth

The roots η_1, η_2 of a are in $B_1(0) \setminus \{0\}$. The periodicity conditions can be described through two meromorphic differentials $\Theta_{b_1}, \Theta_{b_2}$ on X_a. To define these differentials we need to define the polynomials b_1 and b_2 first. For any $a \in \mathcal{H}^2$ let $\mathcal{B}_a$ denote the space of polynomials b of degree three that satisfy the reality condition. Also $\Theta_{b_k} := \frac{b(\lambda)d\lambda}{\lambda y}$ has to have purely imaginery periods, that is the periodicity condition. All $b \in \mathcal{B}_a$ are uniquely defined up to adding a holomorphic differential by $b(0) \in \mathbb{C}$. Furthermore, each family of connstant mean curvature immersions of a CMC torus corresponds to a pair $(a, \lambda_0) \in \mathcal{H}^2 \times S^1$ such that there exist linearly independent $b_1, b_2 \in \mathcal{B}_a$ and functions μ_1, μ_2 on X_a. The μ satisfy:

1. $log(\mu_1), log(\mu_2)$ are holomorphic except in
   $x_0 = \lambda^{-1}\{0\}$ and $x_\infty = \lambda^{-1}\{\infty\}$,
   which are simple poles with linearly independent residues
2. $\Theta_{b_1} = dlog(\mu_1), \Theta_{b_2} = dlog(\mu_2)$
3. $\mu_1(\lambda_0) = \mu_2(\lambda_0) = \pm 1$
4. $b_1(\lambda_0) = 0 = b_2(\lambda_0)$

λ_0 is called the *sym point*. We can define the set

$$\mathcal{P}^2_{\lambda_0} := \{a \in \mathcal{H}^2 \mid X_a \text{ is the spectral curve of a CMC torus in } \mathbb{R}^3\},$$

which is contained in the subset

$$\mathcal{S}^2_{\lambda_0} := \{a \in \mathcal{H}^2 \mid \text{all } b \in \mathcal{B}_a \text{ satisfy } b(\lambda_0) = 0\}.$$

The sym point we want to use in this work is one therefore we will choose $\lambda_0 = 1$ later on. Thus we get

$$\mathcal{P}^2 := \{a \in \mathcal{H}^2 \mid X_a \text{ is the spectral curve of a CMC torus in } \mathbb{R}^3\}$$
$$\mathcal{S}^2_1 := \{a \in \mathcal{H}^2 \mid \text{all } b \in \mathcal{B}_a \text{ satisfy } b(1) = 0\}.$$
$$\mathcal{R}^2 := \{a \in \mathcal{H}^2 \mid \text{all } b \in \mathcal{B}_a \text{ have a common root }\}$$
$$\mathcal{S}^2 := \bigcup_{\lambda_0 \in \mathbb{S}^1} \mathcal{S}^2_{\lambda_0}$$

Corollary 3.1. *a is in $\mathcal{S}_1^2$ if and only if $b_1(1) = 0 = b_2(1)$. Thus $\mathcal{S}_1^2$ can be identified with*

$$\mathcal{T} := \{(a, b_1, b_2) \mid a \in \mathcal{S}_1^2, b_1, b_2 \text{ is basis of } \mathcal{B}_a \text{ with } b_1 = 0 = b_2\}.$$

Proof. By Definition of $\mathcal{S}_1^2$. □

This is a subset and a real subvariety of

$$\mathcal{F}^2 := \text{ the frame bundle of } \mathcal{B}^2,$$

where $\mathcal{B}^2 := \mathcal{H}^2 \times (\mathbb{P}^3_{\mathbb{R}})^2$. Its elements are triplets of the form (a, b_1, b_2). For spectral genus two

$$\mathcal{R}^2 = \overline{\mathcal{P}^2} = \mathcal{S}^2$$

holds. The proof is included in chapter (3.2).

3.1 $\mathcal{S}_1^2$ submanifold of $\mathcal{H}^2$

In chapter 3 of [CS16] Carberry and Schmidt introduce an integer invariant of $a \in \mathcal{H}^2$ the so-called winding number. It is defined as

$$n(\tilde{f}) := deg(\tilde{f}).$$

Where $\tilde{f} : \mathbb{S}^1 \to \mathbb{S}^1$ is the mapping $\tilde{f} = \frac{b_0}{b_\infty}$ with $b_0, b_\infty \in P^3_{\mathbb{R}}$. For unique $b_1, b_2 \in \mathcal{B}_a$ with $b_1(0) = 1$ and $b_2(0) = i$ these polynomials become $b_0 = b_2 - ib_1$ and $b_\infty = b_2 + ib_1$. Another usefull mapping is

$$f := \frac{b_1}{b_2} \text{ with } b_1, b_2 \in \mathcal{B}_a.$$

Together $\tilde{f}$ becomes

$$\tilde{f} = \frac{ib_0}{-ib_\infty} = \frac{b_1 + ib_2}{b_1 - ib_2} = \frac{f + i}{f - i}.$$

The polynomials b_1 and b_2 have degree three, therefore

$$deg(f) = deg\Big(\frac{b_1}{b_2}\Big) = 3 - deg(gcd(\mathcal{B}_a)).$$

Chapter 3 also contains a very important Lemma, which is briefely written below. It is Lemma 3.2, which is:

Lemma 3.2.

The following three statements are equivalent:

(i) $g(X_a) = 0$, *where g denotes the arithmetic genus (spectral genus).*

(ii) $deg(f) = 1$.

(iii) $deg(f) = n(\tilde{f})$.

Hence for $g > 0$, *the winding number of f obeys*

$$|n(\tilde{f})| \leq deg(f) - 2.$$

Proof. In [CS16]. □

Corollary 3.3. *b_1 and b_2 have exactly one common root.*

Proof. Let $g = 2$ and $b_1, b_2 \in P^3_{\mathbb{R}}$.
Assumption: Assume b_1, b_2 have at least two common roots. Then

$$deg(f) = deg\left(\frac{b_1}{b_2}\right) = 1$$

therefore equivalence (ii) from the Lemma above is satisfied. Hence also (i) holds which gives $g(X_a) = 0$. This is a contradiction to $g = 2$.
$\Rightarrow b_1, b_2$ have only one common root. □

We observe that

$$|n(\tilde{f})| \geq 0 \text{ and } deg(gcd(\mathcal{B}_a)) \geq 0.$$

With the two equations above we see

$$\begin{aligned} & 0 \leq |n(\tilde{f})| \leq 3 - deg(gcd(\mathcal{B}_a)) - 2 \\ \Leftrightarrow\ & 0 \leq |n(\tilde{f})| + deg(gcd(\mathcal{B}_a)) \leq 1 \\ \Leftrightarrow\ & 0 \leq deg(gcd(\mathcal{B}_a)) \leq 1. \end{aligned}$$

Thus for $a \in \mathcal{R}^2$ we get $deg(gcd(\mathcal{B}_a)) = 1$ and therefore theorem [5.5] from [CS16] holds. This gives us

i) a is a smooth point of $\mathcal{S}^2_1$, of codimension 2 in $\mathcal{H}^2$

ii) a is smooth in $\mathcal{R}^2$, of codimension 1

iii) a belongs to the closure of two different V_j.

Therefore i) gives us with the implicit function theorem (Corollary 2.5) that $\mathcal{S}^2_1$ is a nonempty submanifold of $\mathcal{H}^2$ of dimension two.

3.2 The closure of $\mathcal{P}^2$

We want to determine $\overline{\mathcal{P}^2}$. In fact we will see the following Lemma.

Lemma 3.4.

$$\mathcal{R}^2 = \overline{\mathcal{P}^2} = \mathcal{S}^2$$

The proof resembles some of the key outcomes of [CS16].

Proof. First of all we are going to look at the set $\mathcal{R}^2$. We notice that the condition that all $b \in \mathcal{B}_a$ have a common root is equivalent to $gcd(deg(\mathcal{B}_2)) \geq 1$. Thus

$$\begin{aligned}\mathcal{R}^2 =& \{a \in \mathcal{H}^2 \mid deg(gcd(\mathcal{B}_a)) \geq 1\} \\ \overset{\text{Cor. 3.3}}{=}& \{a \in \mathcal{H}^2 \mid deg(gcd(\mathcal{B}_a)) = 1\} \\ =& \bigcup_{\lambda_0 \in \mathbb{S}^1} \{a \in \mathcal{H}^2 \mid b_1(\lambda_0) = 0 = b_2(\lambda_0)\} \\ =& \mathcal{S}^2\end{aligned}$$

.

Now it remains to show that also $\overline{\mathcal{P}^2}$ is equal to both sets. Therefore we want to show that

i) $\overline{\mathcal{P}^2} \subset \mathcal{R}^2$
ii) $\mathcal{S}^2 \subset \overline{\mathcal{P}^2}$.

The first point i) is a consequence of chapter five of [CS16]. The four conditions [A-D] force $a \in \overline{\mathcal{P}^2}$ to be in $\mathcal{R}^2$. The second point ii) follows directly because any sym point λ_0 is contained in the unit circle $\mathbb{S}^1$. Thus we obtain equality throughout these sets. □

This only holds true for the spectral genus two. For higher spectral genus the conjecture of [CS16] gives that only $\mathcal{S}^g \subset \mathcal{R}^g$ holds and hence equality cannot be proved.

4 Whitham deformations

This chapter is mainly based on chapter 4 from[CS16]. We will now use the so-called Whitham deformations to construct two vector fields with certain conditions from $(a, b_1, b_2) \longmapsto (\dot{a}, \dot{b}_1, \dot{b}_2)$. The vector $(\dot{a}, \dot{b}_1, \dot{b}_2)$ denotes the tangent vector at $t = 0$ and preserves the periods of Θ_{b_1} and Θ_{b_2}. Since the meromorphic differential forms $\frac{d}{dt}\big|_{t=0}\Theta_{b_1}$ and $\frac{d}{dt}\big|_{t=0}\Theta_{b_2}$ have vanishing periods and no residues there exist meromorphic functions $\dot{q}_1$ and $\dot{q}_2$ on the Riemann surface X_a that satisfy

$$d\dot{q}_k = \frac{d}{dt}\bigg|_{t=0}\Theta_{b_k}$$

for $k = 1, 2$. This gives

$$\dot{q}_k = \frac{ic_k(\lambda)}{y}$$

with $c_k \in P^3_{\mathbb{R}}$ and $y = \sqrt{\lambda a(\lambda)}$. Together with the equation above we get the Whitham equation

$$\frac{\partial}{\partial\lambda}\frac{ic_k(\lambda)}{y} = \frac{\partial}{\partial t}\frac{b_k(\lambda)}{y\lambda}\bigg|_{t=0}.$$

Using product and chain rule we get the following expression

$$\frac{ic'_k(\lambda)y - ic_k(\lambda)y'}{y^2} = \frac{\dot{b}_k(\lambda)y\lambda - b_k(\lambda)\dot{y}\lambda}{(y\lambda)^2}$$

here a dot (e.g. $\dot{a}$) denotes the derivative with respect to t, evaluated at $t = 0$ and a prime (e.g. a') denotes the derivative with respect to λ. By the definition of y we get

$$\frac{ic'_k(\lambda)\sqrt{\lambda a(\lambda)} - ic_k(\lambda)(\frac{a(\lambda)+\lambda a'(\lambda)}{2\sqrt{\lambda a(\lambda)}})}{\lambda a(\lambda)} = \frac{\dot{b}_k\lambda\sqrt{\lambda a(\lambda)} - b_k(\lambda)(\frac{\lambda^2\dot{a}(\lambda)}{2\sqrt{\lambda a(\lambda)}})}{\lambda^2\lambda a(\lambda)}.$$

This term can be transformed into

$$\frac{ic'_k(\lambda)}{\sqrt{\lambda a(\lambda)}} - \frac{ic_k(a(\lambda) + \lambda a'(\lambda))}{2\sqrt{\lambda a(\lambda)}^3} = \frac{\dot{b}_k(\lambda)}{\lambda\sqrt{\lambda a(\lambda)}} - \frac{b_k(\lambda)\dot{a}(\lambda)}{2\sqrt{\lambda a(\lambda)}^3}.$$

Multiplying both sides by $2\sqrt{\lambda a(\lambda)}^3$ yields

$$2ic'_k(\lambda)\lambda a(\lambda) - ic_k(\lambda)a(\lambda) - ic_k\lambda a'(\lambda) = 2\dot{b}_k(\lambda)a(\lambda) - b_k\dot{a}(\lambda).$$

Therefore for $k = 1$ we get

$$(2\lambda a c_1' - a c_1 - \lambda a' c_1) i = 2a\dot{b}_1 - \dot{a} b_1 \tag{1}$$

and for $k = 2$

$$(2\lambda a c_2' - a c_2 - \lambda a' c_2) i = 2a\dot{b}_2 - \dot{a} b_2. \tag{2}$$

These two equations are exactly equations [7] and [8] from [CS16]. Since equation (1) and equation (2) are compatible we can calculate c_2·equation (1)-c_1·equation (2), which gives us

$$\begin{aligned} &ic_2 2\lambda a c_1' - ic_2 a c_1 - ic_2 \lambda a' c_1 - ic_1 2\lambda a c_2' + ic_1 a c_2 + ic_1 \lambda a' c_2 \\ &\qquad = c_2 2a\dot{b}_1 - c_2 \dot{a} b_1 - c_1 2a\dot{b}_2 + c_1 \dot{a} b_2. \end{aligned}$$

This expression can be simplified to

$$2a(ic_1' c_2 \lambda - ic_2' c_1 \lambda + c_1 \dot{b}_2 - c_2 \dot{b}_1) = \dot{a}(c_1 b_2 - c_2 b_1).$$

Therefore both sides need to vanish at all roots of a. If $\dot{a}$ does not vanish at all roots of a, $c_1 b_2 - c_2 b_1$ needs to vanish at the remaining ones. Additionally equation (1) and equation (2) yield that c_1 and c_2 also vanish at the roots that a and $\dot{a}$ have in common. Thus we get the following expression

$$c_1 b_2 - c_2 b_1 = Qa, \tag{3}$$

where $Q \in P^2_{\mathbb{R}}$. Q also satisfies the reality condition. We have seen how starting with a given tangent vector $(\dot{a}, \dot{b}_1, \dot{b}_2)$ one can first use equation (1) and equation (2) to get two polynomials $c_1, c_2 \in P^3_{\mathbb{R}}$ and then secondly solve equation (3) for $Q \in P^2_{\mathbb{R}}$. Our goal is now to reverse this process. To do this we will proceed as follows.

1. Solve equation (3) with given $(a, b_1, b_2) \in \mathcal{F}^2$and $Q \in P^2_{\mathbb{R}}$ for $c_1, c_2 \in P^3_{\mathbb{R}}$.
2. Solve equation (1) and equation (2) with given $(a, b_1, b_2, Q, c_1, c_2)$ for the tangent vector $(\dot{a}, \dot{b}_1, \dot{b}_2)$.

4.1 Uniqueness of solutions

First of all we are going to prove that we can indeed solve these equations uniquely with polynomials that satisfy the reality condition. Secondly we want to explicitly solve them given the values of $c_1(1), c_2(1)$. We will be able to derive certain conditions depending on these values such that we can solve the equations (1)-(3) with unique solutions. The first c_1, c_2 we are interested in are such that $c_1(1) = 1$ and $c_2(1) = 0$, while the second c_1, c_2 are such that $c_1(1) = 0$ and $c_2(1) = 1$. Additionally we want $\dot{b}_1, \dot{b}_2$ to have a root at $\lambda = 1$.

Lemma 4.1. *If the polynomials b_1, b_2 have a root at $\lambda = 1$, then Q also has a root at $\lambda = 1$, i.e. $gcd(\mathcal{B}_a)$ divides Q.*

Proof. Assume b_1, b_2 have a root at $\lambda = 1$, then equation (3) evaluated at $\lambda = 1$ gives us $0 = Q(1)a(1)$. So either Q or a needs to have a root at $\lambda = 1$.
Case 1: Q has root of maximal order 2 at $\lambda = 1$.
This case already implies that Q has a root at $\lambda = 1$ and is therefore fairly trivial.
Case 2: a has a root at $\lambda = 1$.
Due to the distinctness of the roots of a the order of the root $\lambda = 1$ is 1. Therefore a' does not have a root at $\lambda = 1$. Evaluating equation (1) at $\lambda = 1$ then yields $-a'(1)c_1(1)i = 0$. Since a' cannot have a root at $\lambda = 1$ c_1 needs to have a root at $\lambda = 1$. The same argumentation applied to equation (2) gives us that also c_2 has to have a root at $\lambda = 1$. Now the right side of equation (3) has a root at $\lambda = 1$ of order 2. Thus Qa also needs to have a root of order 2 at $\lambda = 1$. Since all roots of a only have order 1 Q also needs to have a root at $\lambda = 1$.
In both cases Q has a root at $\lambda = 1$. □

Corollary 4.2. *If either c_1 or c_2 has no root at $\lambda = 1$ a has no root at $\lambda = 1$.*

Proof. Assume a has a root at $\lambda = 1$, then with the same argumentation as in the proof of Lemma (4.1) we get due to equation (1) and (2) that c_1 and c_2 have to have a root at $\lambda = 1$. This is a contradiction to the prequisite that one of both has no root at $\lambda = 1$ (in fact we want to have c_1, c_2 such that either of them is one at $\lambda = 1$). □

Thus a does not have a root at $\lambda = 1$ in the cases we are interested in.

Lemma 4.3. *If a, b_1 and b_2 are such that b_1 and b_2 have only one common root of first order and this root is at $\lambda = 1$ and $a(1) \neq 0$ then there are unique solutions of equation (1), (2) and (3) such that $\dot{b}_1$ and $\dot{b}_2$ vanish at $\lambda = 1$. They are uniquely defined through $c_1(1)$ and $c_2(1)$.*

Proof. If we set $\lambda = 1$ in equation (1) we get:

$$\begin{aligned}
&(2a(1)c_1'(1) - a(1)c_1(1) - a'(1)c_1(1))i = 0 \\
&\qquad\qquad\Leftrightarrow 2a(1)c_1'(1) = a(1)c_1(1) + a'(1)c_1(1) \\
&\qquad\qquad\Leftrightarrow c_1'(1) = \frac{1}{2a(1)}(a(1)c_1(1) + a'(1)c_1(1)) \\
&\qquad\qquad\Leftrightarrow c_1'(1) = \frac{c_1(1)}{2} + \frac{a'(1)c_1(1)}{2a(1)}
\end{aligned}$$

And if we set $\lambda = 1$ in equation (2) we get:

$$
\begin{aligned}
(2a(1)c_2'(1) - a(1)c_2(1) - a'(1)c_2(1))i = 0 \\
\Leftrightarrow 2a(1)c_2'(1) &= a(1)c_2(1) + a'(1)c_2(1) \\
\Leftrightarrow c_2'(1) &= \frac{1}{2a(1)}(a(1)c_2(1) + a'(1)c_2(1)) \\
\Leftrightarrow c_2'(1) &= \frac{c_2(1)}{2} + \frac{a'(1)c_2(1)}{2a(1)}
\end{aligned}
$$

Since we are interested in special c_1, c_2 we can later explicitly calculate these polynomials through these conditions. We will now see that the values of $Q'(1)$ and $Q''(1)$ are defined by $c_1(1)$ and $c_2(1)$. To get this result we will differentiate equation (3) with respect to lambda and set $\lambda = 1$

$$c_1'b_2 + c_1b_2' - c_2b_1 - c_2b_1' = Q'a + Qa'. \tag{3'}$$

Setting $\lambda = 1$ gives us

$$
\begin{aligned}
c_1'(1)b_2(1) + c_1b_2'(1) - c_2(1)b_1(1) - c_2(1)b_1'(1) = Q'(1)a(1) + Q(1)a'(1) \\
\overset{b_1(1)=0, b_2(1)=0, Q(1)=0}{\Leftrightarrow} c_1(1)b_2'(1) - c_2(1)b_1'(1) = Q'(1)a(1) \\
\Leftrightarrow Q'(1) = \frac{c_1(1)b_2'(1) - c_2(1)b_1'(1)}{a(1)}.
\end{aligned}
$$

We also want to obtain the second derivative of Q at $\lambda = 1$ that is $Q''(1)$. Therefore we differentiate (3') with respect to λ to get

$$c_1''b_2 + 2c_1'b_2 + c_1b_2'' - c_2''b_1 - 2c_2'b_1' - c_2b_1'' = Q''a + 2Q'a' + Qa''. \tag{3''}$$

This expression evaluated at $\lambda = 1$ together with $b_1(1) = 0, b_2(1) = 0$ and

Lemma (4.1) ($Q(1) = 0$) gives us:

$$2c_1'b_2'(1) + c_1(1)b_2''(1) - 2c_2'(1)b_1'(1) - c_2(1)b_1''(1) = Q''(1)a(1) + 2Q'(1)a'(1)$$

$$\overset{c_1'(1),c_2'(1)}{\Leftrightarrow}$$

$$\Big[c_1(1) + \frac{a'(1)}{a(1)}c_1(1)\Big]b_2'(1) + c_1(1)b_2''(1) - \Big[c_2(1) + \frac{a'(1)}{a(1)}c_2(1)\Big]b_1'(1)$$
$$- c_2(1)b_1''(1) = Q''(1)a(1) + 2Q'(1)a'(1)$$

$$\overset{Q'(1)}{\Leftrightarrow}$$

$$\Big[c_1(1) + \frac{a'(1)}{a(1)}c_1(1)\Big]b_2'(1) + c_1(1)b_2''(1) - \Big[c_2(1) + \frac{a'(1)}{a(1)}c_2(1)\Big]b_1'(1)$$
$$- c_2(1)b_1''(1) = Q''(1)a(1) + 2\frac{a'(1)}{a(1)}\Big[c_1(1)b_2'(1) - c_2(1)b_1'(1)\Big]$$

$$\Leftrightarrow$$

$$Q''(1)a(1) = \Big[c_1(1) - \frac{a'(1)}{a(1)}c_1(1)\Big]b_2'(1) + c_1(1)b_2''(1)$$
$$- \Big[c_2(1) - \frac{a'(1)}{a(1)}c_2(1)\Big]b_1'(1) - c_2(1)b_1''(1)$$

$$\Leftrightarrow$$

$$Q''(1) =$$
$$\frac{\Big[c_1(1) - \frac{a'(1)}{a(1)}c_1(1)\Big]b_2'(1) + c_1(1)b_2''(1) - \Big[c_2(1) - \frac{a'(1)}{a(1)}c_2(1)\Big]b_1'(1) - c_2(1)b_1''(1)}{a(1)}$$

Since Q is a polynomial of degree two and since we have three conditions on Q it is therefore uniquely defined through the values of $c_1(1)$ and $c_2(1)$. The next step will be to show that equation (3) with the corresponding Q gives us unique c_1 and c_2. To do this we want to use the roots of b_1 and b_2 that both do not have in common. Since they can also have double roots we need to look at all the possible cases, which are:

Case 1: b_1 and b_2 have two other distinct roots
Case 2: b_1 has a double root other than one, b_2 has two other distinct roots
Case 3: b_1 has two other distinct roots, b_2 has a double root other than one
Case 4: b_1 has a double root other than one, b_2 has a double root other than one
Case 5: b_1 has a double root at one, b_2 has two other distinct roots
Case 6: b_1 has two other distinct roots, b_2 hase a double root at one
Case 7: b_1 has a double root at one, b_2 has a double root other than one

Case 8: b_1 has a double root other than one, b_2 has a double root at one

Case 1: Let b_1 have two distinct roots at β_{11} and β_{12} and let b_2 have two distinct roots at β_{21} and β_{22}. Now let us look at equation (3) evaluated at these roots. (3) evaluated at $\lambda = \beta_{11}$ yields

$$\begin{aligned} c_1(\beta_{11})b_2(\beta_{11}) &= Q(\beta_{11})a(\beta_{11}) \\ c_1(\beta_{11}) &= \frac{Q(\beta_{11})a(\beta_{11})}{b_2(\beta_{11})} \end{aligned}$$

for $\lambda = \beta_{12}$ we get

$$\begin{aligned} c_1(\beta_{12})b_2(\beta_{12}) &= Q(\beta_{12})a(\beta_{12}) \\ c_1(\beta_{12}) &= \frac{Q(\beta_{12})a(\beta_{12})}{b_2(\beta_{12})}. \end{aligned}$$

Together with the value of $c_1(1)$ and the derivative $c_1'(1)$ we obtain four conditions on $c_1(\lambda)$:

$$\begin{aligned} I.\ & c_1(1) \\ II.\ & c_1'(1) = \frac{c_1(1)}{2} + \frac{a'(1)}{2a(1)}c_1(1) \\ III.\ & c_1(\beta_{11}) = \frac{Q(\beta_{11})a(\beta_{11})}{b_2(\beta_{11})} \\ IV.\ & c_1(\beta_{12}) = \frac{Q(\beta_{12})a(\beta_{12})}{b_2(\beta_{12})} \end{aligned}$$

Likewise we get the following four conditions for c_2, where β_{21} and β_{22} denote the two distinct roots of b_2:

$$\begin{aligned} I.\ & c_2(1) \\ II.\ & c_2'(1) = \frac{c_2(1)}{2} + \frac{a'(1)}{2a(1)}c_2(1) \\ III.\ & c_2(\beta_{21}) = \frac{Q(\beta_{21})a(\beta_{21})}{b_1(\beta_{21})} \\ IV.\ & c_2(\beta_{22}) = \frac{Q(\beta_{22})a(\beta_{22})}{b_1(\beta_{22})} \end{aligned}$$

Because c_1 and c_2 are both polynomials of degree three they both have a maximum of four different coefficients that need to be determined. Since we found four conditions on each of these polynomials we can now solve them to get the desired coefficients. Since we already showed that Q is uniquley defined by $c_1(1)$ and $c_2(1)$ the obtained solutions will also be unique. Thus we obtain unique c_1 and c_2.

Case 2: Let b_1 have a double root at β_{11} and b_2 have two distinct roots at β_{21} and β_{22}. Then the conditions on c_2 will not change. So we only have to look at c_1. Due to the fact that b_1 has a double root at β_{11} the old conditions III and IV are equal. Therefore we need to replace one of them through a new condition. Following the previous reasoning it is intuitive to look at the derivatives of (3) with respect to λ evaluated at β_{11}, which yields

$$\begin{aligned}&c_1'(\beta_{11})b_2(\beta_{11}) + c_1(\beta_{11})b_2'(\beta_{11}) - c_2'(\beta_{11})b_1(\beta_{11}) - c_2(\beta_{11})b_1'(\beta_{11})\\&= Q'(\beta_{11})a(\beta_{11}) + Q(\beta_{11})a'(\beta_{11}).\end{aligned}$$

Since b_1 has a double root at β_{11} we get $b_1(\beta_{11}) = 0$ and $b_1'(\beta_{11}) = 0$. Thus the expression above simplifies to

$$c_1'(\beta_{11})b_2(\beta_{11}) + c_1(\beta_{11})b_2'(\beta_{11}) = Q'(\beta_{11})a(\beta_{11}) + Q(\beta_{11})a'(\beta_{11}).$$

This equation gives us a fourth equation to uniquely determine c_1. The four conditions for c_1 in this case are:

$$\begin{aligned}I.&\ c_1(1)\\II.&\ c_1'(1) = \frac{c_1(1)}{2} + \frac{a'(1)}{2a(1)}c_1(1)\\III.&\ c_1(\beta_{11}) = \frac{Q(\beta_{11})a(\beta_{11})}{b_2(\beta_{11})}\\IV.&\ c_1'(\beta_{11})b_2(\beta_{11}) + c_1(\beta_{11})b_2'(\beta_{11}) = Q'(\beta_{11})a(\beta_{11}) + Q(\beta_{11})a'(\beta_{11})\end{aligned}$$

The conditions on c_2 remain the same and are therefore:

$$\begin{aligned}I.&\ c_2(1)\\II.&\ c_2'(1) = \frac{c_2(1)}{2} + \frac{a'(1)}{2a(1)}c_2(1)\\III.&\ c_2(\beta_{21}) = \frac{Q(\beta_{21})a(\beta_{21})}{b_1(\beta_{21})}\\IV.&\ c_2(\beta_{22}) = \frac{Q(\beta_{22})a(\beta_{22})}{b_1(\beta_{22})}\end{aligned}$$

Case 3: Let b_1 have two distinct roots at β_{11} and β_{12} and b_2 have a double root at β_{21}. Then we get the converse conditions to case 2. Thus the conditions on c_1 remain the same as in case 1 and the conditions on c_2 can be derived as the conditions for c_1 in case 2. Therefore we obtain the following conditions on c_1:

$$I.\ c_1(1)$$
$$II.\ c_1'(1) = \frac{c_1(1)}{2} + \frac{a'(1)}{2a(1)}c_1(1)$$
$$III.\ c_1(\beta_{11}) = \frac{Q(\beta_{11})a(\beta_{11})}{b_2(\beta_{11})}$$
$$IV.\ c_1(\beta_{12}) = \frac{Q(\beta_{12})a(\beta_{12})}{b_2(\beta_{12})}$$

Taking the derivative of equation (3) with respect to λ evaluated at $\lambda = \beta_{22}$ gives us

$$I.\ c_2(1)$$
$$II.\ c_2'(1) = \frac{c_2(1)}{2} + \frac{a'(1)}{2a(1)}c_2(1)$$
$$III.\ c_2(\beta_{21}) = \frac{Q(\beta_{21})a(\beta_{21})}{b_1(\beta_{21})}$$
$$IV.\ c_2'(\beta_{21})b_1(\beta_{21}) + c_2(\beta_{21})b_1'(\beta_{21}) = Q'(\beta_{21})a(\beta_{21}) + Q(\beta_{21})a'(\beta_{21}).$$

Hence we again get that c_1 and c_2 can be uniquely defined.

Case 4: Let b_1 have a double root at β_{11} and let b_2 have a double root at β_{21}. This is clearly a combination of case 2 and case 3. We can therefore simply take the conditions on c_1 from case 2 and the conditions from case 3 on c_2 to get that both are uniquely defined. Thus our conditions on c_1 are

$$I.\ c_1(1)$$
$$II.\ c_1'(1) = \frac{c_1(1)}{2} + \frac{a'(1)}{2a(1)}c_1(1)$$
$$III.\ c_1(\beta_{11}) = \frac{Q(\beta_{11})a(\beta_{11})}{b_2(\beta_{11})}$$
$$IV.\ c_1'(\beta_{11})b_2(\beta_{11}) + c_1(\beta_{11})b_2'(\beta_{11}) = Q'(\beta_{11})a(\beta_{11}) + Q(\beta_{11})a'(\beta_{11}).$$

While the conditions on c_2 are:

$$I.\ c_2(1)$$
$$II.\ c_2'(1) = \frac{c_2(1)}{2} + \frac{a'(1)}{2a(1)}c_2(1)$$
$$III.\ c_2(\beta_{21}) = \frac{Q(\beta_{21})a(\beta_{21})}{b_1(\beta_{21})}$$
$$IV.\ c_2'(\beta_{21})b_1(\beta_{21}) + c_2(\beta_{21})b_1'(\beta_{21}) = Q'(\beta_{21})a(\beta_{21}) + Q(\beta_{21})a'(\beta_{21})$$

Case 5: Let b_1 have a double root at $\lambda = 1$ and a single root at $\lambda = \beta_{11}$. While b_2 has three distinct roots at $\lambda = 1, \beta_{21}, \beta_{22}$. The conditions on c_2 remain the same, whereas the conditions on c_1 differ slightly. Thus we get

$$\begin{aligned}
I.\ & c_1(1) \\
II.\ & c_1'(1) = \frac{c_1(1)}{2} + \frac{a'(1)}{2a(1)} c_1(1) \\
III.\ & c_1(\beta_{11}) = \frac{Q(\beta_{11})a(\beta_{11})}{b_2(\beta_{11})} \\
IV.\ & c_1'(1)b_2(1) + c_1(1)b_2'(1) = Q'(1)a(1) + Q(1)a'(1)
\end{aligned}$$

and

$$\begin{aligned}
I.\ & c_2(1) \\
II.\ & c_2'(1) = \frac{c_2(1)}{2} + \frac{a'(1)}{2a(1)} c_2(1) \\
III.\ & c_2(\beta_{21}) = \frac{Q(\beta_{21})a(\beta_{21})}{b_1(\beta_{21})} \\
IV.\ & c_2(\beta_{22}) = \frac{Q(\beta_{22})a(\beta_{22})}{b_1(\beta_{22})}.
\end{aligned}$$

Case 6: Let b_2 have a double root at $\lambda = 1$ and a single root at $\lambda = \beta_{21}$. While b_1 has three distinct roots at $\lambda = 1, \beta_{11}, \beta_{12}$. The conditions on c_1 remain the same, whereas the conditions on c_2 differ slightly. Thus we get

$$\begin{aligned}
I.\ & c_1(1) \\
II.\ & c_1'(1) = \frac{c_1(1)}{2} + \frac{a'(1)}{2a(1)} c_1(1) \\
III.\ & c_1(\beta_{11}) = \frac{Q(\beta_{11})a(\beta_{11})}{b_2(\beta_{11})} \\
IV.\ & c_1(\beta_{12}) = \frac{Q(\beta_{12})a(\beta_{12})}{b_2(\beta_{12})}
\end{aligned}$$

and

$$\begin{aligned}
I.\ & c_2(1) \\
II.\ & c_2'(1) = \frac{c_2(1)}{2} + \frac{a'(1)}{2a(1)} c_2(1) \\
III.\ & c_2(\beta_{21}) = \frac{Q(\beta_{21})a(\beta_{21})}{b_1(\beta_{21})} \\
IV.\ & c_2'(1)b_1(1) + c_2(1)b_1'(1) = Q'(1)a(1) + Q(1)a'(1).
\end{aligned}$$

Case 7: Let b_1 have a double root at $\lambda = 1$ and a single root at $\lambda = \beta_{11}$. While b_2 has a single root at $\lambda = 1$ and a double root at $\lambda = \beta_{21}$. This is a combination of case (3) and (5). Thus we get

$$
\begin{aligned}
I.&\ c_1(1)\\
II.&\ c_1'(1) = \frac{c_1(1)}{2} + \frac{a'(1)}{2a(1)}c_1(1)\\
III.&\ c_1(\beta_{11}) = \frac{Q(\beta_{11})a(\beta_{11})}{b_2(\beta_{11})}\\
IV.&\ c_1'(1)b_2(1) + c_1(1)b_2'(1) = Q'(1)a(1) + Q(1)a'(1)
\end{aligned}
$$

and

$$
\begin{aligned}
I.&\ c_2(1)\\
II.&\ c_2'(1) = \frac{c_2(1)}{2} + \frac{a'(1)}{2a(1)}c_2(1)\\
III.&\ c_2(\beta_{21}) = \frac{Q(\beta_{21})a(\beta_{21})}{b_1(\beta_{21})}\\
IV.&\ c_2'(\beta_{21})b_1(\beta_{21}) + c_2(\beta_{21})b_1'(\beta_{21}) = Q'(\beta_{21})a(\beta_{21}) + Q(\beta_{21})a'(\beta_{21}).
\end{aligned}
$$

Case 8: Let b_1 have a double root at β_{11} and b_2 have a double root at $\lambda = 1$ and a single root at β_{21}. Since this combines case 2 and case 6 we get

$$
\begin{aligned}
I.&\ c_1(1)\\
II.&\ c_1'(1) = \frac{c_1(1)}{2} + \frac{a'(1)}{2a(1)}c_1(1)\\
III.&\ c_1(\beta_{11}) = \frac{Q(\beta_{11})a(\beta_{11})}{b_2(\beta_{11})}\\
IV.&\ c_1'(\beta_{11})b_2(\beta_{11}) + c_1(\beta_{11})b_2'(\beta_{11}) = Q'(\beta_{11})a(\beta_{11}) + Q(\beta_{11})a'(\beta_{11})
\end{aligned}
$$

and

$$
\begin{aligned}
I.&\ c_2(1)\\
II.&\ c_2'(1) = \frac{c_2(1)}{2} + \frac{a'(1)}{2a(1)}c_2(1)\\
III.&\ c_2(\beta_{21}) = \frac{Q(\beta_{21})a(\beta_{21})}{b_1(\beta_{21})}\\
IV.&\ c_2'(1)b_1(1) + c_2(1)b_1'(1) = Q'(1)a(1) + Q(1)a'(1).
\end{aligned}
$$

Since b_1 and b_2 can only have exactly one common root due to Corollary (3.3) and we require that this common root is at $\lambda = 1$ we covered all possible cases. And since all cases yield unique solutions for c_1 and c_2 we can always obtain a unique solution. The next and final step to get the result that we can uniquely solve equations (1),(2) and (3) for $\dot{a}, \dot{b}_1$ and $\dot{b}_2$ is to simply solve equations (1) and (2) for these polynomials with the derived polynomials c_1 and c_2. We will start by determining $\dot{a}$. Since $a(\lambda) = \prod_{j=1}^{2} \frac{\overline{\eta}_j}{|\eta_j|}(\lambda-\eta_j)(\lambda-\overline{\eta}_j^{-1})$ all we need to do is to find $\dot{\eta}_1, \dot{\eta}_2$. Since the complex conjugation commutes with differentiation with respect to real variables this will immediately imply $\dot{\overline{\eta}_1^{-1}}, \dot{\overline{\eta}_2^{-1}}$ and therefore $\dot{a}$. We will first calculate the derivatives of a with respect to λ and with respect to t eveluated at $t = 0$.

$$\begin{aligned} a'(\lambda) =& \frac{\overline{\eta}_1\overline{\eta}_2}{|\eta_1||\eta_2|}((\lambda-\overline{\eta}_1^{-1})(\lambda-\eta_2)(\lambda-\overline{\eta}_2^{-1}) + (\lambda-\eta_1)(\lambda-\eta_2)(\lambda-\overline{\eta}_2^{-1}) \\ &+ (\lambda-\eta_1)(\lambda-\overline{\eta}_1^{-1})(\lambda-\overline{\eta}_2^{-1}) + (\lambda-\eta_1)(\lambda-\overline{\eta}_1^{-1})(\lambda-\eta_2)) \end{aligned}$$

and

$$\begin{aligned} \dot{a}(\lambda) =& \frac{(\dot{\overline{\eta}}_1\overline{\eta}_2 + \overline{\eta}_1\dot{\overline{\eta}}_2)|\eta_1||\eta_2| - \overline{\eta}_1\overline{\eta}_2(\dot{|\eta_1|}|\eta_2| + |\eta_1|\dot{|\eta_2|})}{(|\eta_1||\eta_2|)^2}((\lambda-\eta_1)(\lambda-\overline{\eta}_1^{-1}) \\ &(\lambda-\eta_2)(\lambda-\overline{\eta}_2^{-1})) + \frac{\overline{\eta}_1\overline{\eta}_2}{|\eta_1||\eta_2|}(-\dot{\eta}_1(\lambda-\overline{\eta}_1^{-1})(\lambda-\eta_2)(\lambda-\overline{\eta}_2^{-1}) \\ &- \dot{\overline{\eta}_1^{-1}}(\lambda-\eta_1)(\lambda-\eta_2)(\lambda-\overline{\eta}_2^{-1}) - \dot{\eta}_2(\lambda-\eta_1)(\lambda-\overline{\eta}_1^{-1})(\lambda-\overline{\eta}_2^{-1}) \\ &- \dot{\overline{\eta}_2^{-1}}(\lambda-\eta_1)(\lambda-\overline{\eta}_1^{-1})(\lambda-\eta_2)) \end{aligned}$$

Equation (1) and (2) suggest now to evaluate both at the roots of a, which are $\eta_1, \eta_2, \overline{\eta}_1^{-1}, \overline{\eta}_2^{-1}$. From the calculations above we get e.g. for $\lambda = \eta_1$

$$a'(\eta_1) = \frac{\overline{\eta}_1\overline{\eta}_2}{|\eta_1||\eta_2|}(\eta_1-\overline{\eta}_1^{-1})(\eta_1-\eta_2)(\eta_1-\overline{\eta}_2^{-1}).$$

and

$$\begin{aligned} \dot{a}(\eta_1) =& \frac{\overline{\eta}_1\overline{\eta}_2}{|\eta_1||\eta_2|}(-\dot{\eta}_1)(\eta_1-\overline{\eta}_1^{-1})(\eta_1-\eta_2)(\eta_1-\overline{\eta}_2^{-1}) \\ \Rightarrow \dot{a}(\eta_1) =& -\dot{\eta}_1 a'(\eta_1) \\ &\text{likewise for } \eta_2, \overline{\eta}_1^{-1} \text{and } \overline{\eta}_2^{-1} \\ \dot{a}(\eta_2) =& -\dot{\eta}_2 a'(\eta_2) \\ \dot{a}(\overline{\eta}_1^{-1}) =& -\dot{\overline{\eta}_1^{-1}} a'(\overline{\eta}_1^{-1}) \\ \dot{a}(\overline{\eta}_2^{-1}) =& -\dot{\overline{\eta}_2^{-1}} a'(\overline{\eta}_2^{-1}). \end{aligned}$$

By looking at $\dot{a}$ one may realize that it suffices to calculate $\dot{\eta}_1$ and $\dot{\eta}_2$. To obtain these one can use equation (1) or equation (2) both yield similar results. We will evaluate the respective equation at either η_1 or η_2. Without loss of generality we will use equation (1).

$$\begin{aligned}(2\eta_1 a(\eta_1)c_1'(\eta_1) - a(\eta_1)c_1(\eta_1) - \eta_1 a'(\eta_1)c_1(\eta_1))i =& 2a(\eta_1)\dot{b}_1(\eta_1) - \dot{a}(\eta_1)b_1(\eta_1)\\ \Leftrightarrow -\eta_1 a'(\eta_1)c_1(\eta_1)i =& -\dot{a}(\eta_1)b_1(\eta_1)\\ \overset{\dot{a}(\eta_k)=-\dot{\eta}_k a'(\eta_k)}{\Leftrightarrow} -\eta_1 a'(\eta_1)c_1(\eta_1)i =& \dot{\eta}_1 a'(\eta_1)b_1(\eta_1)\\ \Leftrightarrow \dot{\eta}_1 =& \frac{-\eta_1 a'(\eta_1)c_1(\eta_1)i}{a'(\eta_1)b_1(\eta_1)}\\ \Leftrightarrow \dot{\eta}_1 =& \frac{-\eta_1 c_1(\eta_1)i}{b_1(\eta_1)}\end{aligned}$$

With the same calculations one gets

$$\begin{aligned}-\eta_2 a'(\eta_2)c_1(\eta_2)i =& -\dot{a}(\eta_2)b_1(\eta_2)\\ \overset{\dot{a}(\eta_k)=-\dot{\eta}_k a'(\eta_k)}{\Leftrightarrow} -\eta_2 a'(\eta_2)c_1(\eta_2)i =& \dot{\eta}_2 a'(\eta_2)b_1(\eta_2)\\ \Leftrightarrow \dot{\eta}_2 =& \frac{-\eta_2 a'(\eta_2)c_1(\eta_2)i}{a'(\eta_2)b_1(\eta_2)}\\ \Leftrightarrow \dot{\eta}_2 =& \frac{-\eta_2 c_1(\eta_2)i}{b_1(\eta_2)}.\end{aligned}$$

These calculations with equation (2) yield

$$\begin{aligned}\dot{\eta}_1 =& \frac{-\eta_1 c_2(\eta_1)i}{b_2(\eta_1)}\\ &\text{and}\\ \dot{\eta}_2 =& \frac{-\eta_2 c_2(\eta_2)i}{b_2(\eta_2)}.\end{aligned}$$

This acutally provides a nice backcheck with equation (3), which eveluated at η_k yields

$$\begin{aligned}c_1(\eta_k)b_2(\eta_k) - c_2(\eta_k)b_1(\eta_k) =& 0\\ \Leftrightarrow c_1(\eta_k)b_2(\eta_k) =& c_2(\eta_k)b_1(\eta_k)\\ \Leftrightarrow \frac{c_1(\eta_k)}{b_1(\eta_k)} =& \frac{c_2(\eta_k)}{b_2(\eta_k)}.\end{aligned}$$

Since we can now write $\dot{a}(\lambda)$ down we automatically get $\dot{b}_1(\lambda)$ from equation

(1) and $\dot{b}_2(\lambda)$ from equation (2). Which are as follows

$$\begin{aligned}&(2\lambda a c_1' - a c_1 - \lambda a' c_1)i = 2a\dot{b}_1 - \dot{a}b_1\\ \Leftrightarrow &\frac{(2\lambda a c_1' - a c_1 - \lambda a' c_1)i + \dot{a}b_1}{2a} = \dot{b}_1\end{aligned}$$

and

$$\begin{aligned}&(2\lambda a c_2' - a c_2 - \lambda a' c_2)i = 2a\dot{b}_2 - \dot{a}b_2\\ \Leftrightarrow &\frac{(2\lambda a c_2' - a c_2 - \lambda a' c_2)i + \dot{a}b_2}{2a} = \dot{b}_2\end{aligned}$$

Since we uniquely determined c_1, c_2 and $\dot{a}$ above we also get uniquely defined $\dot{b}_1, \dot{b}_2$. Thus we have finally proved Lemma (4.3).

□

Corollary 4.4. *If c_k vanishes at $\lambda = 1$ then so does c_k', whilst $c_l' = \frac{1}{2} + \frac{a'(1)}{a(1)}$ for $k = 1, 2$ and $l \neq k$.*

Proof. Let without loss of generality $c_1(1) = 1$ and $c_2(1) = 0$. We will now evaluate equation (1) and (2) at $\lambda = 1$. Equation (1) yields

$$\begin{aligned}&2a(1)c_1'(1) - a(1)c_1(1) - a'(1)c_1(1) = 0\\ \Leftrightarrow\ &2a(1)c_1' = a(1)c_1(1) + a'(1)c_1(1)\\ \overset{c_1(1)=1}{\Leftrightarrow}\ &c_1'(1) = \frac{1}{2} + \frac{a'(1)}{a(1)}.\end{aligned}$$

Equation (2) yields

$$\begin{aligned}&2a(1)c_2'(1) - a(1)c_2(1) - a'(1)c_2(1) = 0\\ \Leftrightarrow\ &2a(1)c_2'(1) = a(1)c_2(1) + a'(1)c_2(1)\\ \overset{c_2(1)=0}{\Leftrightarrow}\ &c_2'(1) = 0.\end{aligned}$$

□

It remains to show that Q, c_1 and c_2 satisfy the reality condition. To do this we will use the conditions obtained in the proof of Lemma (4.3). First of all we will prove some basic facts about polynomials of degree three that satisfy the reality condition.

Lemma 4.5. *Let $p \in P^3_{\mathbb{R}}$ therefore p is a polynomial of degree three that satisfies the reality condition $\lambda^3\overline{p(\bar{\lambda}^{-1})} = p(\lambda)$. Then p also satisfies:*

i) $p(1) \in \mathbb{R}$

ii) $(p'(1) - \frac{3}{2}p(1)) \in i\mathbb{R}$

iii) $i\Im(p''(1)) = 2p'(1) - 3p(1)$

Proof. Since p satisfies the reality condition we know that it has the form $p_3\lambda^3 + p_2\lambda^2 + \overline{p}_2\lambda + \overline{p}_3$ with $p_3, p_2 \in \mathbb{C}$

i) Thus $p(1) = p_3 + p_2 + \overline{p}_2 + \overline{p}_3 = 2\Re(p_3) + 2\Re(p_{20}) \in \mathbb{R}$.

ii) By differentiating p with respect to λ we get $p'(\lambda) = 3p_3\lambda^2 + 2p_2\lambda + \overline{p}_2$. Thus

$$\begin{aligned} p'(1) - \frac{3}{2}p(1) =& 3p_3 + 2p_2 + \overline{p}_2 - \frac{3}{2}p_3 - \frac{3}{2}p_2 - \frac{3}{2}\overline{p}_2 - \frac{3}{2}\overline{p}_3 \\ =& 3i\Im(p_3) + i\Im(p_2) \in i\mathbb{R} \end{aligned}$$

iii) By differentiating p' with respect to λ we get $p''(\lambda) = 6p_3\lambda + 2p_2$. We then observe that

$$\begin{aligned} p''(1) =& 6p_3 + 2p_2 = 6\Re(p_3) + 6i\Im(p_3) + 2\Re(p_2) + 2i\Im(p_2), \\ \Rightarrow \quad i\Im(p''(1)) =& 6i\Im(p_3) + 2i\Im(p_2) \\ p'(1) =& 3p_3 + 2p_2 + \overline{p}_2 = 3\Re(p_3) + 3i\Im(p_3) + 3\Re(p_2) + i\Im(p_2) \\ p(1) =& 2\Re(p_3) + 2\Re(p_2) \\ \Rightarrow \quad 2p'(1) - 3p(1) =& 6i\Im(p_3) + 2i\Im(p_2) = i\Im(p''(1)). \end{aligned}$$

□

Lemma 4.6. *The polynomial Q satisfies the reality condition.*

Proof. From the conditions on Q we get:

$$\begin{aligned} Q(\lambda) = (1-\lambda)Q'(1) + (1-\lambda)^2\frac{Q''(1)}{2} \\ = \frac{Q''(1)}{2}\lambda^2 + (Q''(1) - Q'(1))\lambda + Q'(1) + \frac{Q''(1)}{2} \end{aligned}$$

with

$$Q'(1) = \frac{c_1(1)b_2'(1) - c_2(1)b_1'(1)}{a(1)}$$

and

$$Q''(1) = \frac{\left[c_1(1) - \frac{a'(1)}{a(1)}c_1(1)\right]b_2'(1) + c_1(1)b_2''(1) - \left[c_2(1) - \frac{a'(1)}{a(1)}c_2(1)\right]b_1'(1) - c_2(1)b_1''(1)}{a(1)}$$

The reality condition $\lambda^2\overline{Q(\bar{\lambda}^{-1})} = Q(\lambda)$ is equivalent to

$$Q'(1) \in i\mathbb{R}$$
$$Q'(1) - Q''(1) \in \mathbb{R}$$

We observe that true to Lemma (4.5) if c_1 and c_2 satisfy the reality condition $c_1(1)$ and $c_2(1)$ both are real and therefore can be neglected. In fact we are only interested in the cases $c_1(1) = 1, c_2(1) = 0$ and $c_1(1) = 0, c_2(1) = 1$, thus we can focus on the polynomials a, b_1 and b_2 to check the reality conditions. We immediately get that $a(1)$ and $a(1)^2$ are real. Thus $a(1), c_1(1)$ and $c_2(1)$ are real. Furthermore, we are in the case that $b_1(1) = 0$ and $b_2(1) = 0$. Since the polynomials b_1 and b_2 satisfy the reality conditions Lemma (4.5) holds. Especially (ii) becomes $b_1'(1) \in i\mathbb{R}$ and $b_2'(1) \in i\mathbb{R}$. Thus $\frac{c_1(1)}{a(1)}$ and $\frac{c_2(1)}{a(1)}$ are real. Since the product of a real and an imaginary number is again imaginary we obtain that $Q'(1) = \frac{c_1(1)}{a(1)}b_2'(1) - \frac{c_2(1)}{a(1)}b_1'(1)$ consists of two imaginary terms and is therefore imaginary.

It remains to show that the imaginary part of $Q'(1) - Q''(1)$ vanishes. First of all we will use $a(1)^2$ as the common denominator of all terms. Thus we get:

$$\frac{c_1(1)a(1)}{a(1)^2} - \frac{c_1(1)}{a(1)^2}a'(1)b_2'(1) + \frac{c_1(1)a(1)}{a(1)}a(1)b_2''(1)$$
$$-\frac{c_2(1)a(1)}{a(1)^2} + \frac{c_2(1)}{a(1)^2}a'(1)b_1'(1) - \frac{c_2(1)a(1)}{a(1)}a(1)b_1''(1)$$

Since $\frac{c_1(1)a(1)}{a(1)^2}, \frac{c_1(1)}{a(1)^2}, \frac{c_1(1)a(1)}{a(1)}, \frac{c_2(1)a(1)}{a(1)^2}, \frac{c_2(1)}{a(1)^2}$ and $\frac{c_2(1)a(1)}{a(1)}$ are real due to the reasoning above, it suffices to look at

$$a'(1)b_1'(1) - a(1)b_1''(1) - a'(1)b_2'(1) + a(1)b_2''(1).$$

Now we observe that for $a(\lambda) = a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + \overline{a_3}\lambda + \overline{a4}$, with $a_4, a_3 \in \mathbb{C}$ and $a_2 \in \mathbb{R}$

$$a(1) = 2\Re(a_4) + 2\Re(a_3) + a_2$$

and

$$a'(1) = 4\Re(a_4) + 4i\Im(a_4) + 4\Re(a_3) + 2i\Im(a_3) + 2a_2$$

holds. Thus we get $\Re(a'(1)) = 2a(1)$.

$$\Rightarrow a'(1) - 2a(1) \in i\mathbb{R}$$

If we substitute $i\Im(b_1''(1))$ and $i\Im(b_2''(1))$ through $2b_1'(1)-3b_1(1)$ and $2b_2'(1)-3b_2(1)$ as described in Lemma (4.5) (iii) we obtain

$$\begin{aligned}&a'(1)b_1'(1)-a(1)(2b_1'(1)-3b_1(1)+\Re(b_1''(1))\\&-a'(1)b_2'(1)+a(1)(2b_2'(1)-3b_2(1)+\Re(b_2''(1))\\=&b_1'(1)(a'(1)-2a(1))-a(1)(-3b_1(1)+\Re(b_1''(1))\\&-b_2'(1)(a'(1)-2a(1))+a(1)(-3b_2(1)+\Re(b_2''(1)).\end{aligned}$$

Since $b_1(1)=0=b_2(1)$ we get

$$b_1'(1)(a'(1)-2a(1))-a(1)\Re(b_1''(1)-b_2'(1)(a'(1)-2a(1))+a(1)\Re(b_2''(1)).$$

Since $b_1'(1),(a'(1)-2a(1)),b_2'(1)$ and $(a'(1)-2a(1))$ are imaginary their product is real and since $a(1)\Re(b_1''(1)$ and $a(1)\Re(b_2''(1))$ are real, the whole term is real. Hence Q satisfies the reality condition. □

It only remains now to show that c_1 and c_2 satisfy the reality conditions as well. Since both are polynomials of degree three, which are uniquely defined through four conditions it suffices to prove that $\lambda^3\overline{c_1(\bar\lambda^{-1})}$ and $\lambda^3\overline{c_2(\bar\lambda^{-1})}$ satisfy the corresponding four conditions as well. If this is the case the polynomials have to be the same due to the unqiueness.

Lemma 4.7. *If $c_k(\lambda)$ satisfies the four conditions from the proof of Lemma (4.3) then so does $\tilde c_k(\lambda):=\lambda^3\overline{c_k(\bar\lambda^{-1})}$ for $k=1,2$.*

Proof. Let c_k for $k=1,2$ be parametrized through

$$\begin{aligned}&c_k(\lambda)=c_{k3}\lambda^3+c_{k2}\lambda^2+c_{k1}\lambda+c_{k0}\\\Rightarrow\ &\tilde c_1(\lambda)=\overline{c_{k0}}\lambda^3+\overline{c_{k1}}\lambda^2+\overline{c_{k2}}\lambda+\overline{c_{k3}}\lambda.\end{aligned}$$

The first condition is $c_k(1)\in\mathbb{R}$. We have to show that this also implies that $\tilde c_k(1)\in\mathbb{R}$. Since $c_k(1)=c_{k3}+c_{k2}+c_{k1}+c_{k0}$ is real we know that $i\Im(c_{k3}+c_{k2}+c_{k1}+c_{k0})=0$. This implies that $-i\Im(c_{k3}+c_{k2}+c_{k1}+c_{k0})=0$, which is $i\Im(\overline{c_{k3}+c_{k2}+c_{k1}+c_{k0}})$. Therefore we see that $c_k(1)=c_{k3}+c_{k2}+c_{k1}+c_{k0}=\overline{c_{k3}+c_{k2}+c_{k1}+c_{k0}}=\tilde c_k(1)$. Hence the first condition satisfies the reality condition.

The second condition refers to the derivative of c_k at $\lambda=1$. We want to show that

$$c_k'(1)=\frac{1}{2}c_k(1)+\frac{a'(1)}{2a(1)}c_k(1)=\tilde{c_k}'(1)$$

with

$$\tilde{c}_k = \lambda^3 \overline{c_k(\bar{\lambda}^{-1})}.$$

To do this we will parametrize c_k again as follows

$$\begin{aligned} & c_k = c_{k3}\lambda^3 + c_{k2}\lambda^2 + c_{k1}\lambda + c_{k0} \\ \Rightarrow\ & c_k'(\lambda) = 3c_{k3}\lambda^2 + 2c_{k2}\lambda^2 + c_{k1} \\ \Rightarrow\ & \tilde{c}_k'(\lambda) = 3\overline{c_{k0}}\lambda^2 + 2\overline{c_{k1}}\lambda + \overline{c_{k2}}. \end{aligned}$$

Thus $c_k'(1) = \frac{1}{2}c_k(1) + \frac{a'(1)}{2a(1)}c_k(1)$ becomes

$$3c_{k3} + 2c_{k2} + c_{k1} = \frac{1}{2}(c_{k3} + c_{k2} + c_{k1} + c_{k0}) + \frac{a'(1)}{2a(1)}(c_{k3} + c_{k2} + c_{k1} + c_{k0}).$$

From the proof of Lemma (4.6) we remember that $\Re(a'(1)) = 2a(1)$. Apllying this we get:

$$\begin{aligned} & 3c_{k3} + 2c_{k2} + c_{k1} = \frac{1}{2}(c_{k3} + c_{k2} + c_{k1} + c_{k0}) \\ & + \frac{2a(1) + i\Im(a'(1))}{2a(1)}(c_{k3} + c_{k2} + c_{k1} + c_{k0}) \\ \Leftrightarrow\ & 3c_{k3} + 2c_{k2} + c_{k1} = \frac{3}{2}(c_{k3} + c_{k2} + c_{k1} + c_{k0}) \\ & + \frac{i\Im(a'(1))}{2a(1)}(c_{k3} + c_{k2} + c_{k1} + c_{k0}) \\ \Leftrightarrow\ & \frac{3}{2}c_{k3} + \frac{1}{2}c_{k2} - \frac{1}{2}c_{k1} - \frac{3}{2}c_{k0} = \frac{i\Im(a'(1))}{2a(1)}c_k(1) \end{aligned}$$

Since $c_k(1), a(1) \in \mathbb{R}$ the right-hand side is imaginary hence we get $\Re(\frac{3}{2}c_{k3} +$

$\frac{1}{2}c_{k2} - \frac{1}{2}c_{k1} - \frac{3}{2}c_{k0}) = 0$. Thus

$$\begin{aligned}
&i\Im(\frac{3}{2}c_{k3} + \frac{1}{2}c_{k2} - \frac{1}{2}c_{k1} - \frac{3}{2}c_{k0}) = \frac{i\Im(a'(1))}{2a(1)}c_1(1)\\
\Leftrightarrow\ & \frac{3}{2}i\Im(c_{k3}) + \frac{1}{2}i\Im(c_{k2}) - \frac{1}{2}i\Im(c_{k1}) - \frac{3}{2}i\Im(c_{k0}) = \frac{i\Im(a'(1))}{2a(1)}c_k(1)\\
\overset{-}{\Leftrightarrow}\ & -\frac{3}{2}i\Im(c_{k3}) - \frac{1}{2}i\Im(c_{k2}) + \frac{1}{2}i\Im(c_{k1}) + \frac{3}{2}i\Im(c_{k0}) = -\frac{i\Im(a'(1))}{2a(1)}c_k(1)\\
\Leftrightarrow\ & \frac{3}{2}\overline{c_{k3}} + \frac{1}{2}\overline{c_{k2}} - \frac{1}{2}\overline{c_{k1}} - \frac{1}{2}\overline{c_{k0}} = -\frac{i\Im(a'(1))}{2a(1)}c_k(1)\\
\Leftrightarrow\ & -3\overline{c_{k0}} - 2\overline{c_{k1}} - \overline{c_{k2}} + \frac{3}{2}(\overline{c_{k3}} + \overline{c_{k2}} + \overline{c_{k1}} + \overline{c_{k0}}) = -\frac{i\Im(a'(1))}{2a(1)}c_k(1)\\
\Leftrightarrow\ & -3\overline{c_{k0}} - 2\overline{c_{k1}} - \overline{c_{k2}} = -\frac{1}{2}\tilde{c}_k(1) - \tilde{c}_k(1) - \frac{i\Im(a'(1))}{2a(1)}c_k(1)\\
\overset{\tilde{c}_k(1)=c_k(1)}{\Leftrightarrow}\ & -3\overline{c_{k0}} - 2\overline{c_{k1}} - \overline{c_{k2}} = -\frac{1}{2}c_k(1) - \frac{2a(1)}{2a(1)}c_k(1) - \frac{i\Im(a'(1))}{2a(1)}c_k(1)\\
\overset{2a(1)=\Im(a'(1))}{\Leftrightarrow}\ & -3\overline{c_{k0}} - 2\overline{c_{k1}} - \overline{c_{k2}} = -\frac{1}{2}c_k(1) - \frac{a'(1)}{2a(1)}c_k(1)\\
\Leftrightarrow\ & -\tilde{c}_k'(1) = -\frac{1}{2}c_k(1) - \frac{a'(1)}{2a(1)}c_k(1)\\
\overset{\cdot(-1)}{\Leftrightarrow}\ & \tilde{c}_k'(1) = \frac{1}{2}c_k(1) + \frac{a'(1)}{2a(1)}c_k(1)\\
\Leftrightarrow\ & \tilde{c}_k'(1) = c_k'(1)
\end{aligned}$$

Thus the second condition satisfies the reality condition.

The third and fourth condition in the case that polynomial b_k has distinct roots can be seen from equation (3). Let β denote any root other than one of the polynomial b_k. First of all we observe that

$$b_k(\beta) = 0 \overset{\text{reality condition}}{\Rightarrow} \beta^3\overline{b_k(\bar{\beta}^{-1})} = 0 \Rightarrow \overline{b_k(\bar{\beta}^{-1})} = 0.$$

We know that:

$$\begin{aligned}
&c_k(\beta) = \frac{Q(\beta)a(\beta)}{b_{3-k}(\beta)}\\
\Leftrightarrow &c_k(\beta) = \frac{\beta^6\overline{Q(\bar{\beta}^{-1})a(\bar{\beta}^{-1})}}{\beta^3\overline{b_{3-k}(\bar{\beta}^{-1})}}\\
\Leftrightarrow &\beta^{-3}c_k(\beta) = \frac{\overline{Q(\bar{\beta}^{-1})a(\bar{\beta}^{-1})}}{\overline{b_{3-k}(\bar{\beta}^{-1})}}
\end{aligned}$$

We will now look at equation (3) and start with $\lambda = \bar{\beta}^{-1}$. Thus b_k vanishes and we obtain

$$\begin{aligned} &\overline{c_k(\bar{\beta}^{-1}) b_{3-k}(\bar{\beta}^{-1})} = \overline{Q(\bar{\beta}^{-1}) a(\bar{\beta}^{-1})} \\ \Leftrightarrow &\overline{c_k(\bar{\beta}^{-1})} = \frac{\overline{Q(\bar{\beta}^{-1}) a(\bar{\beta}^{-1})}}{\overline{b_{3-k}(\bar{\beta}^{-1})}}. \end{aligned}$$

With both calculations together we get

$$\begin{aligned} &\overline{c_k(\bar{\beta}^{-1})} = \beta^{-3} c_k(\beta) \\ \Leftrightarrow &\beta^3 \overline{c_k(\bar{\beta}^{-1})} = c_k(\beta). \end{aligned}$$

Since this holds for both $\beta = \beta_{k1}$ and $\beta = \beta_{k2}$ the third and fourth condition also satisfy the reality condition. Thus in this case the reality condition is satisfied and c_1 and c_2 satisfy the reality condition due to uniqueness. It remains to look at the case when any of the two polynomials b_1 or b_2 has a double root. Since these cases are just limit cases of the case treated above we get a term fully dependend on polynomials that satisfy the reality condition. Thus it satisfies the reality condition. Hence c_1 and c_2 satisfy the reality condition in any of the eight cases from the proof of Lemma (4.3). □

4.2 Vector field V_1

We now want to apply Lemma (4.3) for given $c_1(1), c_2(1)$. Let V_1 denote the vector field corresponding to the conditions $c_1(1) = 1$ and $c_2(1) = 0$. We just proved additional conditions for $(a, b_1, b_2, Q, c_1, c_2)$ such that we can solve equations (1),(2) and (3) to obtain $(\dot{a}, \dot{b}_1, \dot{b}_2)$, which is exactly what we are going to do. First of all a short overview of all conditions:

1. $b_1(1) = 0,\ \dot{b}_1(1) = 0$
2. $b_2(1) = 0,\ \dot{b}_2(1) = 0$
3. $c_1(1) = 1$
4. $c_2(1) = 0$
5. $Q(1) = 0$
6. $a(1) \neq 0$
7. all polynomials satisfy the reality condition
8. c_1, c_2 satisfy the additional conditions from the proof of Lemma (4.3)

From these conditions we can immediately see what the polynomials b_1, b_2, Q look like. That is

$$\begin{aligned}
b_1 &= (\lambda-1)(b_{10}\lambda^2 + b_{11}\lambda + b_{12}), \text{ where } b_{10}, b_{11}, b_{12} \in \mathbb{C} \\
b_2 &= (\lambda-1)(b_{20}\lambda^2 + b_{21}\lambda + b_{22}), \text{ where } b_{20}, b_{21}, b_{22} \in \mathbb{C} \\
Q &= (\lambda-1)(q_0\lambda + q_1), \text{ where } q_0, q_1 \in \mathbb{C}.
\end{aligned}$$

The first step is now to find a Q that satisfies all these conditions from Lemma (4.3) to solve equation (3) for c_1, c_2.

$$\begin{aligned}
Q &= (\lambda-1)(q_0\lambda + q_1) \\
&= q_0\lambda^2 + q_1\lambda - q_0\lambda - q_1
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&Q'(\lambda) = 2q_0\lambda + q_1 - q_0 \\
\Rightarrow &Q'(1) = q_0 + q_1 \\
&Q''(\lambda) = 2q_0 \\
\Rightarrow &Q''(1) = 2q_0.
\end{aligned}$$

The conditions from the Lemma were:

$$I.\ Q'(1) = \frac{c_1(1)b_2'(1) - c_2(1)b_1'(1)}{a(1)}$$

$$II.\ Q''(1) =$$

$$\frac{\left[c_1(1) - \frac{a'(1)}{a(1)}c_1(1)\right]b_2'(1) + c_1(1)b_2''(1) - \left[c_2(1) - \frac{a'(1)}{a(1)}c_2(1)\right]b_1'(1) - c_2(1)b_1''(1)}{a(1)}$$

Given $c_1(1) = 1$ and $c_2(1) = 0$ we get

$$\begin{aligned}
I.\ Q'(1) &= \frac{b_2'(1)}{a(1)} \\
II.\ Q''(1) &= \frac{\left[1 - \frac{a'(1)}{a(1)}\right]b_2'(1) + b_2''(1)}{a(1)}
\end{aligned}$$

Thus we get

$$\begin{aligned}
q_0 + q_1 &= \frac{b_2'(1)}{a(1)} \\
2q_0 &= \frac{\left[1 - \frac{a'(1)}{a(1)}\right]b_2'(1) + b_2''(1)}{a(1)}.
\end{aligned}$$

This yields

$$q_0 = \frac{\left[1 - \frac{a'(1)}{a(1)}\right] b_2'(1) + b_2''(1)}{2a(1)}$$

and therefore

$$\begin{aligned}
q_1 &= \frac{b_2'(1)}{a(1)} - q_0 \\
\Rightarrow q_1 &= \frac{b_2'(1)}{a(1)} - \frac{\left[1 - \frac{a'(1)}{a(1)}\right] b_2'(1) + b_2''(1)}{2a(1)} \\
\Rightarrow q_1 &= \frac{2b_2'(1) - \left[1 - \frac{a'(1)}{a(1)}\right] b_2'(1) - b_2''(1)}{2a(1)} \\
\Rightarrow q_1 &= \frac{\left[1 + \frac{a'(1)}{a(1)}\right] b_2'(1) - b_2''(1)}{2a(1)}.
\end{aligned}$$

Hence we obtain Q as follows

$$\begin{aligned}
Q(\lambda) &= (\lambda - 1)(q_0 \lambda + q_1) \\
\Leftrightarrow Q(\lambda) &= (\lambda - 1)\Big(\frac{\left[1 - \frac{a'(1)}{a(1)}\right] b_2'(1) + b_2''(1)}{2a(1)} \lambda + \frac{\left[1 + \frac{a'(1)}{a(1)}\right] b_2'(1) - b_2''(1)}{2a(1)}\Big)
\end{aligned}$$

$$\Rightarrow Q(\lambda) = (\lambda - 1) \frac{\Big([1 - \frac{a'(1)}{a(1)}] b_2'(1) + b_2''(1)\Big)\lambda + [1 + \frac{a'(1)}{a(1)}] b_2'(1) - b_2''(1)}{2a(1)} \qquad (4)$$

$$\Rightarrow Q(\lambda) = (\lambda - 1) \frac{(\lambda + 1) b_2'(1) + (1 - \lambda)\frac{a'(1)}{a(1)} b_2'(1) - (1 - \lambda) b_2''(1)}{2a(1)}. \qquad (5)$$

Before we continue we want to assure that Q satisfies the reality condition that is basically Lemma (4.6).

$$
\begin{aligned}
\lambda^2\overline{Q(\bar{\lambda}^{-1})} =&\lambda^2\overline{\Big(\frac{b_2'(1)}{2a(1)}\bar{\lambda}^{-2}-\frac{a'(1)b_2'(1)}{2a(1)^2}\bar{\lambda}^{-2}+\frac{b_2''(1)}{2a(1)}\bar{\lambda}^{-2}+\frac{b_2'(1)}{2a(1)}\bar{\lambda}^{-1}}\\
&\overline{+\frac{a'(1)b_2'(1)}{2a(1)^2}\bar{\lambda}^{-1}-\frac{b_2''(1)}{2a(1)}\bar{\lambda}^{-1}-\frac{b_2'(1)}{2a(1)}\bar{\lambda}^{-1}+\frac{a'(1)b_2'(1)}{2a(1)^2}\bar{\lambda}^{-1}}\\
&\overline{-\frac{b_2''(1)}{2a(1)}\bar{\lambda}^{-1}-\frac{b_2'(1)}{2a(1)}-\frac{a'(1)b_2'(1)}{2a(1)^2}+\frac{b_2''(1)}{2a(1)}\Big)}\\
=&\overline{\frac{b_2'(1)}{2a(1)}}-\overline{\frac{a'(1)b_2'(1)}{2a(1)^2}}+\overline{\frac{b_2''(1)}{2a(1)}}+\overline{\frac{a'(1)b_2'(1)}{a(1)^2}}\lambda-\overline{\frac{b_2''(1)}{a(1)}}\lambda\\
&-\overline{\frac{b_2'(1)}{2a(1)}}\lambda^2-\overline{\frac{a'(1)b_2'(1)}{2a(1)^2}}\lambda^2+\overline{\frac{b_2''(1)}{2a(1)}}\lambda^2\\
\overset{(4.6)}{=}&\frac{b_2'(1)}{2a(1)}\lambda^2-\frac{a'(1)b_2'(1)}{2a(1)^2}\lambda^2+\frac{b_2''(1)}{2a(1)}\lambda^2+\frac{a'(1)b_2'(1)}{a(1)^2}\lambda-\frac{b_2''(1)}{a(1)}\lambda\\
&-\frac{b_2'(1)}{2a(1)}-\frac{a'(1)b_2'(1)}{2a(1)^2}+\frac{b_2''(1)}{2a(1)}\\
=&Q(\lambda)
\end{aligned}
$$

With this polyonmial of degree 2 we now want to solve equation (3) for polynomials c_1, c_2. First of all we are going to use $c_1(1) = 1$ on the conditions found in the proof. We limit ourselves to the case where b_1 and b_2 have two distinct roots other than one. We did this because all cases led to rather long terms. Thus these calculations only lead to an exemplary vector field. By applying Corollary (4.4) we get

$$
\begin{aligned}
I.\ c_1(1) &= 1\\
II.\ c_1'(1) &= \frac{1}{2}+\frac{a'(1)}{2a(1)}\\
III.\ c_1(\beta_{11}) &= \frac{Q(\beta_{11})a(\beta_{11})}{b_2(\beta_{11})}\\
IV.\ c_1(\beta_{12}) &= \frac{Q(\beta_{12})a(\beta_{12})}{b_2(\beta_{12})}.
\end{aligned}
$$

The first condition already tells us that:

$$
\begin{aligned}
&c_1(\lambda) = 1 + (\lambda - 1)(c_{10}\lambda^2 + c_{11}\lambda + c_{12}) \\
\Rightarrow &c_1'(\lambda) = c_{10}\lambda^2 + c_{11}\lambda + c_{12} + (\lambda - 1)(2c_{10}\lambda + c_{11}) \\
\Rightarrow &c_1'(1) = c_{10} + c_{11} + c_{12}
\end{aligned}
$$

and

$$
\begin{aligned}
c_1(\beta_{11}) &= 1 + (\beta_{11} - 1)(c_{10}\beta_{11}^2 + c_{11}\beta_{11} + c_{12}) \\
c_1(\beta_{12}) &= 1 + (\beta_{12} - 1)(c_{10}\beta_{12}^2 + c_{11}\beta_{12} + c_{12})
\end{aligned}
$$

Thus it remains to solve the remaining three conditions for the three coefficients. First of all let us replace c_1 in equations II-IV with its definition. This yields

$$
\begin{aligned}
II.\ c_{10} + c_{11} + c_{12} &= \frac{1}{2} + \frac{a'(1)}{2a(1)} \\
III.\ 1 + (\beta_{11} - 1)(c_{10}\beta_{11}^2 + c_{11}\beta_{11} + c_{12}) &= \frac{Q(\beta_{11})a(\beta_{11})}{b_2(\beta_{11})} \\
IV.\ 1 + (\beta_{12} - 1)(c_{10}\beta_{12}^2 + c_{11}\beta_{12} + c_{12}) &= \frac{Q(\beta_{12})a(\beta_{12})}{b_2(\beta_{12})} \\
\Leftrightarrow & \\
II.\ c_{10} + c_{11} + c_{12} &= \frac{1}{2} + \frac{a'(1)}{2a(1)} \\
III.\ c_{10}\beta_{11}^2 + c_{11}\beta_{11} + c_{12} &= \frac{Q(\beta_{11})a(\beta_{11}) - b_2(\beta_{11})}{b_2(\beta_{11})(\beta_{11} - 1)} \\
IV.\ c_{10}\beta_{12}^2 + c_{11}\beta_{12} + c_{12} &= \frac{Q(\beta_{12})a(\beta_{12}) - b_2(\beta_{12})}{b_2(\beta_{12})(\beta_{12} - 1)}.
\end{aligned}
$$

$$III - II := III'.\ c_{10}(\beta_{11}^2 - 1) + c_{11}(\beta_{11} - 1) = \frac{Q(\beta_{11})a(\beta_{11}) - b_2(\beta_{11})}{b_2(\beta_{11})(\beta_{11} - 1)} - \frac{1}{2} - \frac{a'(1)}{2a(1)}$$

$$IV - II := IV'.\ c_{10}(\beta_{12}^2 - 1) + c_{11}(\beta_{12} - 1) = \frac{Q(\beta_{12})a(\beta_{12}) - b_2(\beta_{12})}{b_2(\beta_{12})(\beta_{12} - 1)} - \frac{1}{2} - \frac{a'(1)}{2a(1)}$$

$$\begin{aligned}
-\frac{\beta_{12} - 1}{\beta_{11} - 1} III' + IV' := IV''.\ & c_{10}(\beta_{12}^2 - 1) - \frac{\beta_{12} - 1}{\beta_{11} - 1}(\beta_{12} - 1)c_{10} = \frac{Q(\beta_{12})a(\beta_{12}) - b_2(\beta_{12})}{b_2(\beta_{12})(\beta_{12} - 1)} - \frac{1}{2} - \frac{a'(1)}{2a(1)} \\
& - \frac{(\beta_{12} - 1)(Q(\beta_{11})a(\beta_{11}) - b_2(\beta_{11}))}{(\beta_{11} - 1)b_2(\beta_{11})(\beta_{11} - 1)} + \frac{\beta_{12} - 1}{2(\beta_{11} - 1)} + \frac{(\beta_{12} - 1)a'(1)}{2(\beta_{11} - 1)a(1)} \\
\Leftrightarrow IV''.\ & c_{10}\frac{(\beta_{12}^2 - 1)(\beta_{11} - 1) - (\beta_{12} - 1)(\beta_{11}^2 - 1)}{\beta_{11} - 1} = \frac{Q(\beta_{12})a(\beta_{12}) - b_2(\beta_{12})}{b_2(\beta_{12})(\beta_{12} - 1)} - \frac{1}{2} - \frac{a'(1)}{2a(1)} \\
& - \frac{(\beta_{12} - 1)(Q(\beta_{11})a(\beta_{11}) - b_2(\beta_{11}))}{(\beta_{11} - 1)b_2(\beta_{11})(\beta_{11} - 1)} + \frac{\beta_{12} - 1}{2(\beta_{11} - 1)} + \frac{(\beta_{12} - 1)a'(1)}{2(\beta_{11} - 1)a(1)} \\
\Leftrightarrow c_{10} = & \frac{\beta_{11} - 1}{(\beta_{12}^2 - 1)(\beta_{11} - 1) - (\beta_{12} - 1)(\beta_{11}^2 - 1)} \Big[\frac{Q(\beta_{12})a(\beta_{12}) - b_2(\beta_{12})}{b_2(\beta_{12})(\beta_{12} - 1)} - \frac{1}{2} - \frac{a'(1)}{2a(1)} \\
& - \frac{(\beta_{12} - 1)(Q(\beta_{11})a(\beta_{11}) - b_2(\beta_{11}))}{(\beta_{11} - 1)b_2(\beta_{11})(\beta_{11} - 1)} + \frac{\beta_{12} - 1}{2(\beta_{11} - 1)} + \frac{(\beta_{12} - 1)a'(1)}{2(\beta_{11} - 1)a(1)}\Big]
\end{aligned}$$

$$\begin{aligned}
III'.\ c_{11} &= \frac{1}{(\beta_{11} - 1)}\Big(\frac{Q(\beta_{11})a(\beta_{11}) - b_2(\beta_{11})}{b_2(\beta_{11})(\beta_{11} - 1)} - \frac{1}{2} - \frac{a'(1)}{2a(1)} - (\beta_{11}^2 - 1)c_{10}\Big) \\
II.\ c_{12} &= \frac{1}{2} + \frac{a'(1)}{2a(1)} - c_{10} - c_{11}
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow\ c_{11} = {} & \frac{1}{(\beta_{11}-1)}\Big(\frac{Q(\beta_{11})a(\beta_{11})-b_2(\beta_{11})}{b_2(\beta_{11})(\beta_{11}-1)}-\frac{1}{2}-\frac{a'(1)}{2a(1)} \\
& -\frac{(\beta_{11}-1)(\beta_{11}^2-1)}{(\beta_{12}^2-1)(\beta_{11}-1)-(\beta_{12}-1)(\beta_{11}^2-1)}\Big[\frac{Q(\beta_{12})a(\beta_{12})-b_2(\beta_{12})}{b_2(\beta_{12})(\beta_{12}-1)}-\frac{1}{2}-\frac{a'(1)}{2a(1)} \\
& -\frac{(\beta_{12}-1)(Q(\beta_{11})a(\beta_{11})-b_2(\beta_{11}))}{(\beta_{11}-1)b_2(\beta_{11})(\beta_{11}-1)}+\frac{\beta_{12}-1}{2(\beta_{11}-1)}+\frac{(\beta_{12}-1)a'(1)}{2(\beta_{11}-1)a(1)}\Big]\Big)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\ c_{12} = {} & \frac{1}{2}+\frac{a'(1)}{2a(1)}-\frac{\beta_{11}-1}{(\beta_{12}^2-1)(\beta_{11}-1)-(\beta_{12}-1)(\beta_{11}^2-1)}\Big[\frac{Q(\beta_{12})a(\beta_{12})-b_2(\beta_{12})}{b_2(\beta_{12})(\beta_{12}-1)}-\frac{1}{2}-\frac{a'(1)}{2a(1)} \\
& -\frac{(\beta_{12}-1)(Q(\beta_{11})a(\beta_{11})-b_2(\beta_{11}))}{(\beta_{11}-1)b_2(\beta_{11})(\beta_{11}-1)}+\frac{\beta_{12}-1}{2(\beta_{11}-1)}+\frac{(\beta_{12}-1)a'(1)}{2(\beta_{11}-1)a(1)}\Big] \\
& -\frac{1}{(\beta_{11}-1)}\Big(\frac{Q(\beta_{11})a(\beta_{11})-b_2(\beta_{11})}{b_2(\beta_{11})(\beta_{11}-1)}-\frac{1}{2}-\frac{a'(1)}{2a(1)}-\frac{(\beta_{11}-1)(\beta_{11}^2-1)}{(\beta_{12}^2-1)(\beta_{11}-1)-(\beta_{12}-1)(\beta_{11}^2-1)} \\
& \Big[\frac{Q(\beta_{12})a(\beta_{12})-b_2(\beta_{12})}{b_2(\beta_{12})(\beta_{12}-1)}-\frac{1}{2}-\frac{a'(1)}{2a(1)}-\frac{(\beta_{12}-1)(Q(\beta_{11})a(\beta_{11})-b_2(\beta_{11}))}{(\beta_{11}-1)b_2(\beta_{11})(\beta_{11}-1)}+\frac{\beta_{12}-1}{2(\beta_{11}-1)}+\frac{(\beta_{12}-1)a'(1)}{2(\beta_{11}-1)a(1)}\Big]\Big)
\end{aligned}
$$

$$
\begin{aligned}
c_1(\lambda) = 1 + (\lambda - 1)\Bigg(& \frac{\beta_{11}-1}{(\beta_{12}^2-1)(\beta_{11}-1)-(\beta_{12}-1)(\beta_{11}^2-1)}\Big[\frac{Q(\beta_{12})a(\beta_{12})-b_2(\beta_{12})}{b_2(\beta_{12})(\beta_{12}-1)} - \frac{1}{2} - \frac{a'(1)}{2a(1)} \\
& - \frac{(\beta_{12}-1)(Q(\beta_{11})a(\beta_{11})-b_2(\beta_{11}))}{(\beta_{11}-1)b_2(\beta_{11})(\beta_{11}-1)} + \frac{\beta_{12}-1}{2(\beta_{11}-1)} + \frac{(\beta_{12}-1)a'(1)}{2(\beta_{11}-1)a(1)}\Big]\lambda^2 \\
& + \frac{\lambda}{(\beta_{11}-1)}\Big(\frac{Q(\beta_{11})a(\beta_{11})-b_2(\beta_{11})}{b_2(\beta_{11})(\beta_{11}-1)} - \frac{1}{2} - \frac{a'(1)}{2a(1)} - \frac{(\beta_{11}-1)(\beta_{11}^2-1)}{(\beta_{12}^2-1)(\beta_{11}-1)-(\beta_{12}-1)(\beta_{11}^2-1)} \\
& \Big[\frac{Q(\beta_{12})a(\beta_{12})-b_2(\beta_{12})}{b_2(\beta_{12})(\beta_{12}-1)} - \frac{1}{2} - \frac{a'(1)}{2a(1)} - \frac{(\beta_{12}-1)(Q(\beta_{11})a(\beta_{11})-b_2(\beta_{11}))}{(\beta_{11}-1)b_2(\beta_{11})(\beta_{11}-1)} + \frac{\beta_{12}-1}{2(\beta_{11}-1)} + \frac{(\beta_{12}-1)a'(1)}{2(\beta_{11}-1)a(1)}\Big]\Big) \\
& + \frac{1}{2} + \frac{a'(1)}{2a(1)} - \frac{\beta_{11}-1}{(\beta_{12}^2-1)(\beta_{11}-1)-(\beta_{12}-1)(\beta_{11}^2-1)}\Big[\frac{Q(\beta_{12})a(\beta_{12})-b_2(\beta_{12})}{b_2(\beta_{12})(\beta_{12}-1)} - \frac{1}{2} - \frac{a'(1)}{2a(1)} \\
& - \frac{(\beta_{12}-1)(Q(\beta_{11})a(\beta_{11})-b_2(\beta_{11}))}{(\beta_{11}-1)b_2(\beta_{11})(\beta_{11}-1)} + \frac{\beta_{12}-1}{2(\beta_{11}-1)} + \frac{(\beta_{12}-1)a'(1)}{2(\beta_{11}-1)a(1)}\Big] \\
& - \frac{1}{(\beta_{11}-1)}\Big(\frac{Q(\beta_{11})a(\beta_{11})-b_2(\beta_{11})}{b_2(\beta_{11})(\beta_{11}-1)} - \frac{1}{2} - \frac{a'(1)}{2a(1)} - \frac{(\beta_{11}-1)(\beta_{11}^2-1)}{(\beta_{12}^2-1)(\beta_{11}-1)-(\beta_{12}-1)(\beta_{11}^2-1)}\Big[\\
& \frac{Q(\beta_{12})a(\beta_{12})-b_2(\beta_{12})}{b_2(\beta_{12})(\beta_{12}-1)} - \frac{1}{2} - \frac{a'(1)}{2a(1)} - \frac{(\beta_{12}-1)(Q(\beta_{11})a(\beta_{11})-b_2(\beta_{11}))}{(\beta_{11}-1)b_2(\beta_{11})(\beta_{11}-1)} + \frac{\beta_{12}-1}{2(\beta_{11}-1)} + \frac{(\beta_{12}-1)a'(1)}{2(\beta_{11}-1)a(1)}\Big]\Big)\Bigg)
\end{aligned}
$$

Now we want to calculate c_2 with the following conditions:

$$I.\ c_2(1) = 0$$
$$II.\ c_2'(1) = 0$$
$$III.\ c_2(\beta_{21}) = \frac{Q(\beta_{21})a(\beta_{21})}{b_1(\beta_{21})}$$
$$IV.\ c_2(\beta_{22}) = \frac{Q(\beta_{22})a(\beta_{22})}{b_1(\beta_{22})}$$

The first two conditions yield that c_2 has the form

$$c_2(\lambda) = (\lambda - 1)^2(c_{20}\lambda + c_{21})$$
$$\Rightarrow c_2(\beta_{21}) = (\beta_{21} - 1)^2(c_{20}\beta_{21} + c_{21})$$
$$\Rightarrow c_2(\beta_{22}) = (\beta_{22} - 1)^2(c_{20}\beta_{22} + c_{21}).$$

Therefore it remains to solve condition III and IV for c_{20} and c_{21}.

$$III.\ (\beta_{21} - 1)^2(c_{20}\beta_{21} + c_{21}) = \frac{Q(\beta_{21})a(\beta_{21})}{b_1(\beta_{21})}$$
$$IV.\ (\beta_{22} - 1)^2(c_{20}\beta_{22} + c_{21}) = \frac{Q(\beta_{22})a(\beta_{22})}{b_1(\beta_{22})}$$
$$\Leftrightarrow$$
$$III.\ (c_{20}\beta_{21} + c_{21}) = \frac{Q(\beta_{21})a(\beta_{21})}{b_1(\beta_{21})(\beta_{21} - 1)^2}$$
$$IV.\ (c_{20}\beta_{22} + c_{21}) = \frac{Q(\beta_{22})a(\beta_{22})}{b_1(\beta_{22})(\beta_{22} - 1)^2}$$

$$III - IV := III'.\ c_{21}\beta_{21} - c_{21}\beta_{22} = \frac{Q(\beta_{21})a(\beta_{21})}{b_1(\beta_{21})(\beta_{21} - 1)^2} - \frac{Q(\beta_{22})a(\beta_{22})}{b_1(\beta_{22})(\beta_{22} - 1)^2}$$
$$\Leftrightarrow c_{21}(\beta_{21} - \beta_{22}) = \frac{Q(\beta_{21})a(\beta_{21})}{b_1(\beta_{21})(\beta_{21} - 1)^2} - \frac{Q(\beta_{22})a(\beta_{22})}{b_1(\beta_{22})(\beta_{22} - 1)^2}$$
$$\Leftrightarrow c_{21} = \frac{Q(\beta_{21})a(\beta_{21})}{(\beta_{21} - \beta_{22})b_1(\beta_{21})(\beta_{21} - 1)^2} - \frac{Q(\beta_{22})a(\beta_{22})}{(\beta_{21} - \beta_{22})b_1(\beta_{22})(\beta_{22} - 1)^2}$$

Now IV gives us:

$$c_{22} = \frac{Q(\beta_{21})a(\beta_{21})}{b_1(\beta_{21})(\beta_{21} - 1)^2} - c_{21}\beta_{22}$$
$$\Leftrightarrow c_{22} = \frac{Q(\beta_{21})a(\beta_{21})}{b_1(\beta_{21})(\beta_{21} - 1)^2} - \beta_{22}\Big(\frac{Q(\beta_{21})a(\beta_{21})}{(\beta_{21} - \beta_{22})b_1(\beta_{21})(\beta_{21} - 1)^2} - \frac{Q(\beta_{22})a(\beta_{22})}{(\beta_{21} - \beta_{22})b_1(\beta_{22})(\beta_{22} - 1)^2}\Big)$$

Thus we also obtain c_2:

$$
\begin{aligned}
c_2(\lambda) =&(\lambda-1)^2\Bigg(\Big(\frac{Q(\beta_{21})a(\beta_{21})}{(\beta_{21}-\beta_{22})b_1(\beta_{21})(\beta_{21}-1)^2}-\frac{Q(\beta_{22})a(\beta_{22})}{(\beta_{21}-\beta_{22})b_1(\beta_{22})(\beta_{22}-1)^2}\Big)\lambda+\frac{Q(\beta_{21})a(\beta_{21})}{b_1(\beta_{21})(\beta_{21}-1)^2}\\
&-\beta_{22}\Big(\frac{Q(\beta_{21})a(\beta_{21})}{(\beta_{21}-\beta_{22})b_1(\beta_{21})(\beta_{21}-1)^2}-\frac{Q(\beta_{22})a(\beta_{22})}{(\beta_{21}-\beta_{22})b_1(\beta_{22})(\beta_{22}-1)^2}\Big)\Bigg)
\end{aligned}
$$

Using the definition of Q we get:

$$
\begin{aligned}
c_2(\lambda) =&(\lambda-1)^2\Bigg(\Big(\frac{(\beta_{21}-1)\Big([1-\frac{a'(1)}{a(1)}]b_2'(1)+b_2''(1)\Big)\beta_{21}+[1+\frac{a'(1)}{a(1)}]b_2'(1)-b_2''(1)a(\beta_{21})}{2a(1)(\beta_{21}-\beta_{22})b_1(\beta_{21})(\beta_{21}-1)^2}\\
&-\frac{(\beta_{22}-1)\Big([1-\frac{a'(1)}{a(1)}]b_2'(1)+b_2''(1)\Big)\beta_{22}+[1+\frac{a'(1)}{a(1)}]b_2'(1)-b_2''(1)a(\beta_{22})}{2a(1)(\beta_{21}-\beta_{22})b_1(\beta_{22})(\beta_{22}-1)^2}\Big)\lambda\\
&+\frac{(\beta_{21}-1)\Big([1-\frac{a'(1)}{a(1)}]b_2'(1)+b_2''(1)\Big)\beta_{21}+[1+\frac{a'(1)}{a(1)}]b_2'(1)-b_2''(1)a(\beta_{21})}{2a(1)b_1(\beta_{21})(\beta_{21}-1)^2}\\
&-\beta_{22}\Big(\frac{(\beta_{21}-1)\Big([1-\frac{a'(1)}{a(1)}]b_2'(1)+b_2''(1)\Big)\beta_{21}+[1+\frac{a'(1)}{a(1)}]b_2'(1)-b_2''(1)a(\beta_{21})}{2a(1)(\beta_{21}-\beta_{22})b_1(\beta_{21})(\beta_{21}-1)^2}\\
&-\frac{(\beta_{22}-1)\Big([1-\frac{a'(1)}{a(1)}]b_2'(1)+b_2''(1)\Big)\beta_{22}+[1+\frac{a'(1)}{a(1)}]b_2'(1)-b_2''(1)a(\beta_{22})}{2a(1)(\beta_{21}-\beta_{22})b_1(\beta_{22})(\beta_{22}-1)^2}\Big)\Bigg)
\end{aligned}
$$

The remaining step to obtain V_1 is to solve equation (1) and (2) for $\dot{a}, \dot{b}_1, \dot{b}_2$. From Lemma (4.3) we can take the derivatives of η with respect to t eveluated at $t = 0$. Which were

$$\begin{aligned}
\dot{\eta}_1 &= \frac{-\eta_1 c_1(\eta_1) i}{b_1(\eta_1)} \\
\dot{\overline{\eta}_1^{-1}} &= \frac{-\overline{\eta}_1^{-1} c_1(\overline{\eta}_1^{-1}) i}{b_1(\overline{\eta}_1^{-1})} \\
\dot{\eta}_2 &= \frac{-\eta_2 c_1(\eta_2) i}{b_1(\eta_2)} \\
\dot{\overline{\eta}_2^{-1}} &= \frac{-\overline{\eta}_2^{-1} c_1(\overline{\eta}_2^{-1}) i}{b_1(\overline{\eta}_2^{-1})}
\end{aligned}$$

we can calculate $\dot{|\eta_k|}$ with chain rule

$$\dot{|\eta_k|} = \frac{\eta_k}{|\eta_k|} \dot{\eta}_k$$

for $k = 1, 2$. Since the derivation with respect to t, which is a real variable, and the complex conjugation commute we get

$$\dot{\overline{\eta}}_k = \overline{\dot{\eta}_k} \Rightarrow \dot{\overline{\eta}}_k = \overline{\frac{-\eta_k c_1(\eta_k) i}{b_1(\eta_k)}} \text{ for } k = 1, 2.$$

Now we can use the expression for $\dot{a}$ from the proof of Lemma (4.3), which was

$$\begin{aligned}
\dot{a}(\lambda) =& \frac{(\dot{\overline{\eta}}_1 \overline{\eta}_2 + \overline{\eta}_1 \dot{\overline{\eta}}_2)|\eta_1||\eta_2| - \overline{\eta}_1 \overline{\eta}_2 (\dot{|\eta_1|}|\eta_2| + |\eta_1|\dot{|\eta_2|})}{(|\eta_1||\eta_2|)^2} ((\lambda - \eta_1)(\lambda - \overline{\eta}_1^{-1}) \\
& (\lambda - \eta_2)(\lambda - \overline{\eta}_2^{-1})) + \frac{\overline{\eta}_1 \overline{\eta}_2}{|\eta_1||\eta_2|} (-\dot{\eta}_1 (\lambda - \overline{\eta}_1^{-1})(\lambda - \eta_2)(\lambda - \overline{\eta}_2^{-1}) \\
& - \dot{\overline{\eta}_1^{-1}} (\lambda - \eta_1)(\lambda - \eta_2)(\lambda - \overline{\eta}_2^{-1}) - \dot{\eta}_2 (\lambda - \eta_1)(\lambda - \overline{\eta}_1^{-1})(\lambda - \overline{\eta}_2^{-1}) \\
& - \dot{\overline{\eta}_2^{-1}} (\lambda - \eta_1)(\lambda - \overline{\eta}_1^{-1})(\lambda - \eta_2)).
\end{aligned}$$

Using the derived expressions for the derivatives of the absolute value and the complex conjugation we get

$$\begin{aligned}
&\dot{a}(\lambda) = \\
&\frac{(\overline{\frac{-\eta_1 c_1(\eta_1) i}{b_1(\eta_1)}} \overline{\eta}_2 + \overline{\eta}_1 \overline{\frac{-\eta_2 c_1(\eta_2) i}{b_1(\eta_2)}})|\eta_1||\eta_2| - \overline{\eta}_1 \overline{\eta}_2 (\frac{\eta_1}{|\eta_1|}\dot{\eta}_1 |\eta_2| + |\eta_1| \frac{\eta_2}{|\eta_2|})}{(|\eta_1||\eta_2|)^2} \frac{|\eta_1||\eta_2|}{\overline{\eta}_1 \overline{\eta}_2} a(\lambda) \\
&+ \frac{\overline{\eta}_1 \overline{\eta}_2}{|\eta_1||\eta_2|} (-\dot{\eta}_1 (\lambda - \overline{\eta}_1^{-1})(\lambda - \eta_2)(\lambda - \overline{\eta}_2^{-1}) - \dot{\overline{\eta}_1^{-1}} (\lambda - \eta_1)(\lambda - \eta_2)(\lambda - \overline{\eta}_2^{-1}) \\
&- \dot{\eta}_2 (\lambda - \eta_1)(\lambda - \overline{\eta}_1^{-1})(\lambda - \overline{\eta}_2^{-1}) - \dot{\overline{\eta}_2^{-1}} (\lambda - \eta_1)(\lambda - \overline{\eta}_1^{-1})(\lambda - \eta_2)).
\end{aligned}$$

Finally we plug in the derivatives of the roots of a with respect to t to get $\dot{a}$.

$$\dot{a}(\lambda) =$$

$$\frac{(\overline{\frac{-\eta_1 c_1(\eta_1)i}{b_1(\eta_1)}}\overline{\eta}_2 + \overline{\eta}_1\overline{\frac{-\eta_2 c_1(\eta_2)i}{b_1(\eta_2)}})|\eta_1||\eta_2| - \overline{\eta}_1\overline{\eta}_2(\frac{\eta_1}{|\eta_1|}\frac{-\eta_1 c_1(\eta_1)i}{b_1(\eta_1)}|\eta_2| + |\eta_1|\frac{\eta_2}{|\eta_2|}\frac{-\eta_2 c_1(\eta_2)i}{b_1(\eta_2)})}{(|\eta_1||\eta_2|)^2}$$
$$\frac{|\eta_1||\eta_2|}{\overline{\eta}_1\overline{\eta}_2}a(\lambda) + \frac{\overline{\eta}_1\overline{\eta}_2}{|\eta_1||\eta_2|}(-\frac{-\eta_1 c_1(\eta_1)i}{b_1(\eta_1)}(\lambda-\overline{\eta}_1^{-1})(\lambda-\eta_2)(\lambda-\overline{\eta}_2^{-1})$$
$$-\frac{-\overline{\eta}_1^{-1}c_1(\overline{\eta}_1^{-1})i}{b_1(\overline{\eta}_1^{-1})}(\lambda-\eta_1)(\lambda-\eta_2)(\lambda-\overline{\eta}_2^{-1}) - \frac{-\eta_2 c_1(\eta_2)i}{b_1(\eta_2)}(\lambda-\eta_1)$$
$$(\lambda-\overline{\eta}_1^{-1})(\lambda-\overline{\eta}_2^{-1}) - \frac{-\overline{\eta}_2^{-1}c_1(\overline{\eta}_2^{-1})i}{b_1(\overline{\eta}_2^{-1})}(\lambda-\eta_1)(\lambda-\overline{\eta}_1^{-1})(\lambda-\eta_2)). \tag{6}$$

Since we can now write $\dot{a}(\lambda)$ down we automatically get $\dot{b}_1(\lambda)$ from equation (1) and $\dot{b}_2(\lambda)$ from equation (2). Which are as follows

$$(2\lambda a c_1' - a c_1 - \lambda a' c_1)i = 2a\dot{b}_1 - \dot{a}b_1$$
$$\Leftrightarrow \frac{(2\lambda a c_1' - a c_1 - \lambda a' c_1)i + \dot{a}b_1}{2a} = \dot{b}_1$$
$$\Leftrightarrow \dot{b}_1 = \frac{(2\lambda a c_1' - a c_1 - \lambda a' c_1)i}{2a} + \frac{b_1}{a}\dot{2a} \tag{7}$$

and

$$(2\lambda a c_2' - a c_2 - \lambda a' c_2)i = 2a\dot{b}_2 - \dot{a}b_2$$
$$\Leftrightarrow \frac{(2\lambda a c_2' - a c_2 - \lambda a' c_2)i + \dot{a}b_2}{2a} = \dot{b}_2$$
$$\Leftrightarrow \dot{b}_2 = \frac{(2\lambda a c_2' - a c_2 - \lambda a' c_2)i}{2a} + \frac{b_2}{2a}\dot{a}. \tag{8}$$

$$\begin{aligned}
\dot{b}_1 =& \frac{(2\lambda a c_1' - a c_1 - \lambda a' c_1)i}{2a} + \frac{b_1}{2a}\frac{(\overline{\frac{-\eta_1 c_1(\eta_1)i}{b_1(\eta_1)}}\overline{\eta}_2 + \overline{\eta}_1\overline{\frac{-\eta_2 c_1(\eta_2)i}{b_1(\eta_2)}})|\eta_1||\eta_2| - \overline{\eta}_1\overline{\eta}_2(\frac{\eta_1}{|\eta_1|}\frac{-\eta_1 c_1(\eta_1)i}{b_1(\eta_1)}|\eta_2| + |\eta_1|\frac{\eta_2}{|\eta_2|}\frac{-\eta_2 c_1(\eta_2)i}{b_1(\eta_2)}}{(|\eta_1||\eta_2|)^2})\\
& \frac{|\eta_1||\eta_2|}{\overline{\eta}_1\overline{\eta}_2}a(\lambda) + \frac{\overline{\eta}_1\overline{\eta}_2}{|\eta_1||\eta_2|}(-\frac{-\eta_1 c_1(\eta_1)i}{b_1(\eta_1)}(\lambda - \bar{\eta}_1^{-1})(\lambda - \eta_2)(\lambda - \bar{\eta}_2^{-1}) - \frac{-\bar{\eta}_1^{-1} c_1(\bar{\eta}_1^{-1})i}{b_1(\bar{\eta}_1^{-1})}(\lambda - \eta_1)(\lambda - \eta_2)(\lambda - \bar{\eta}_2^{-1})\\
& - \frac{-\eta_2 c_1(\eta_2)i}{b_1(\eta_2)}(\lambda - \eta_1)(\lambda - \bar{\eta}_1^{-1})(\lambda - \bar{\eta}_2^{-1}) - \frac{-\bar{\eta}_2^{-1} c_1(\bar{\eta}_2^{-1})i}{b_1(\bar{\eta}_2^{-1})}(\lambda - \eta_1)(\lambda - \bar{\eta}_1^{-1})(\lambda - \eta_2))
\end{aligned}$$

$$\begin{aligned}
\dot{b}_2 =& \frac{(2\lambda a c_2' - a c_2 - \lambda a' c_2)i}{2a} + \frac{b_2}{2a}\frac{(\overline{\frac{-\eta_1 c_1(\eta_1)i}{b_1(\eta_1)}}\overline{\eta}_2 + \overline{\eta}_1\overline{\frac{-\eta_2 c_1(\eta_2)i}{b_1(\eta_2)}})|\eta_1||\eta_2| - \overline{\eta}_1\overline{\eta}_2(\frac{\eta_1}{|\eta_1|}\frac{-\eta_1 c_1(\eta_1)i}{b_1(\eta_1)}|\eta_2| + |\eta_1|\frac{\eta_2}{|\eta_2|}\frac{-\eta_2 c_1(\eta_2)i}{b_1(\eta_2)})}{(|\eta_1||\eta_2|)^2}\\
& \frac{|\eta_1||\eta_2|}{\overline{\eta}_1\overline{\eta}_2}a(\lambda) + \frac{\overline{\eta}_1\overline{\eta}_2}{|\eta_1||\eta_2|}(-\frac{-\eta_1 c_1(\eta_1)i}{b_1(\eta_1)}(\lambda - \bar{\eta}_1^{-1})(\lambda - \eta_2)(\lambda - \bar{\eta}_2^{-1}) - \frac{-\bar{\eta}_1^{-1} c_1(\bar{\eta}_1^{-1})i}{b_1(\bar{\eta}_1^{-1})}(\lambda - \eta_1)(\lambda - \eta_2)(\lambda - \bar{\eta}_2^{-1})\\
& - \frac{-\eta_2 c_1(\eta_2)i}{b_1(\eta_2)}(\lambda - \eta_1)(\lambda - \bar{\eta}_1^{-1})(\lambda - \bar{\eta}_2^{-1}) - \frac{-\bar{\eta}_2^{-1} c_1(\bar{\eta}_2^{-1})i}{b_1(\bar{\eta}_2^{-1})}(\lambda - \eta_1)(\lambda - \bar{\eta}_1^{-1})(\lambda - \eta_2)).
\end{aligned}$$

4.3 Vector field V_2

We want to proceed almost like in 4.1 to get another vector field, which we will call V_2. We will again limit ourselves on the case that b_1 and b_2 have two distinct roots other than one. Thus the obtained vector field is again only exemplary. This time we want to have $c_1(1) = 0$ and $c_2(1) = 1$. Thus many of the conditions from 4.1 simply switch to the other corresponding equation. This gives us the following conditions:

1. $b_1(1) = 0,\ \dot{b}_1(1) = 0$
2. $b_2(1) = 0,\ \dot{b}_2(1) = 0$
3. $c_1(1) = 0$
4. $c_2(1) = 1$
5. $Q(1) = 0$
6. $a(1) \neq 0$
7. all polynomials satisfy the reality condition
8. c_1, c_2 satisfy the additional conditions from the proof of Lemma (4.3)

We already now that Q has the form

$$Q(\lambda) = (\lambda - 1)(q_0\lambda + q_0)\ ,\ \text{where } q_0, q1 \in \mathbb{C}.$$

We also know that

$$Q'(1) = q_0 + q_1$$

and

$$Q''(1) = 2q_0.$$

Now we want to repeat the procedure from the proof of Lemma 4.3, where we differentiated equation (3) with respect to λ and then evaluated it at 1. Thus we get

$$q_0 + q_1 = \frac{b_1(1)}{a(1)}$$

and

$$2q_0 = \frac{\left[1 - \frac{a'(1)}{a(1)}\right] b_1'(1) + b_1''(1)}{a(1)}.$$

The latter yields

$$q_0 = \frac{\left[1 - \frac{a'(1)}{a(1)}\right] b_1'(1) + b_1''(1)}{2a(1)}.$$

Which in return gives us

$$\begin{aligned} q_1 &= \frac{b_1(1)}{a(1)} - q0 \\ \Rightarrow q_1 &= \frac{b_1(1)}{a(1)} - \frac{\left[1 - \frac{a'(1)}{a(1)}\right] b_1'(1) + b_1''(1)}{2a(1)} \\ \Rightarrow q_1 &= \frac{\left[1 + \frac{a'(1)}{a(1)}\right] b_1'(1) - b_1''(1)}{2a(1)}. \end{aligned}$$

Therefore we can already write down Q corresponding to V_2.

$$\begin{aligned} Q(\lambda) &= (\lambda - 1)(q_0 \lambda + q_1) \\ \Leftrightarrow Q(\lambda) &= (\lambda - 1)\Big(\frac{\left[1 - \frac{a'(1)}{a(1)}\right] b_1'(1) + b_1''(1)}{2a(1)} \lambda + \frac{\left[1 + \frac{a'(1)}{a(1)}\right] b_1'(1) - b_1''(1)}{2a(1)} \Big) \end{aligned}$$

$$\Rightarrow Q(\lambda) = (\lambda - 1) \frac{\Big([1 - \frac{a'(1)}{a(1)}] b_1'(1) + b_1''(1) \Big) \lambda + [1 + \frac{a'(1)}{a(1)}] b_1'(1) - b_1''(1)}{2a(1)} \quad (9)$$

$$\Rightarrow Q(\lambda) = (\lambda - 1) \frac{(\lambda + 1) b_1'(1) + (1 - \lambda) \frac{a'(1)}{a(1)} b_1'(1) - (1 - \lambda) b_1''(1)}{2a(1)} \quad (10)$$

For simplicity we also assume that b_1 has two additional distinct roots at β_{11} and at β_{12} and that b_2 has two additional distinct roots at β_{21} and at β_{22}. Thus we are in case 1 of the proof and get the following conditions on c_1:

$$\begin{aligned} I.\ c_1(1) &= 0 \\ II.\ c_1'(1) &= 0 \\ III.\ c_1(\beta_{11}) &= \frac{Q(\beta_{11}) a(\beta_{11})}{b_2(\beta_{11})} \\ IV.\ c_1(\beta_{12}) &= \frac{Q(\beta_{12}) a(\beta_{12})}{b_2(\beta_{12})} \end{aligned}$$

By solving this system of equations for the coefficients of c_1 we obtain

$$\begin{aligned}
&c_1(\lambda) = \\
&(\lambda-1)^2\Big(\big(\frac{Q(\beta_{21})a(\beta_{21})}{(\beta_{21}-\beta_{22})(\beta_{21}-1)^2 b_1(\beta_{21})} - \frac{Q(\beta_{22})a(\beta_{22})}{(\beta_{21}-\beta_{22})(\beta_{22}-1)^2 b_1(\beta_{22})}\big)\lambda \\
&+ \frac{Q(\beta_{22})a(\beta_{22})}{(\beta_{22}-1)^2 b_1(\beta_{22}} - \beta_{21}\big(\frac{Q(\beta_{21})a(\beta_{21})}{(\beta_{21}-\beta_{22})(\beta_{21}-1)^2 b_1(\beta_{21})} \\
&- \frac{Q(\beta_{22})a(\beta_{22})}{(\beta_{21}-\beta_{22})(\beta_{22}-1)^2 b_1(\beta_{22})}\big).
\end{aligned} \tag{11}$$

Likewise we get the following four conditions for c_2, where β_{21} and β_{22} denote the two distinct roots of b_2:

$$\begin{aligned}
I.\ & c_2(1) = 1 \\
II.\ & c_2'(1) = \frac{1}{2} + \frac{a'(1)}{2a(1)} \\
III.\ & c_2(\beta_{21}) = \frac{Q(\beta_{21})a(\beta_{21})}{b_1(\beta_{21})} \\
IV.\ & c_2(\beta_{22}) = \frac{Q(\beta_{22})a(\beta_{22})}{b_1(\beta_{22})}
\end{aligned}$$

Solving these four conditions yields the following result for c_2.

$$c_2(\lambda) = 1 + (\lambda - 1)\Bigg(\frac{\beta_{21} - 1}{(\beta_{22}^2 - 1)(\beta_{21} - 1) - (\beta_{22} - 1)(\beta_{21}^2 - 1)}\Big[\frac{Q(\beta_{22})a(\beta_{22}) - b_1(\beta_{22})}{b_1(\beta_{22})(\beta_{22} - 1)} - \frac{1}{2} - \frac{a'(1)}{2a(1)}$$

$$- \frac{(\beta_{22} - 1)(Q(\beta_{21})a(\beta_{21}) - b_2(\beta_{21}))}{(\beta_{21} - 1)b_1(\beta_{21})(\beta_{21} - 1)} + \frac{\beta_{22} - 1}{2(\beta_{21} - 1)} + \frac{(\beta_{22} - 1)a'(1)}{2(\beta_{21} - 1)a(1)}\Big]\lambda^2 + \frac{\lambda}{(\beta_{21} - 1)}\Big(\frac{Q(\beta_{21})a(\beta_{21}) - b_1(\beta_{21})}{b_1(\beta_{21})(\beta_{21} - 1)}$$

$$- \frac{1}{2} - \frac{a'(1)}{2a(1)} - \frac{(\beta_{21} - 1)(\beta_{21}^2 - 1)}{(\beta_{22}^2 - 1)(\beta_{21} - 1) - (\beta_{22} - 1)(\beta_{21}^2 - 1)}\Big[$$

$$\frac{Q(\beta_{22})a(\beta_{22}) - b_1(\beta_{22})}{b_1(\beta_{22})(\beta_{22} - 1)} - \frac{1}{2} - \frac{a'(1)}{2a(1)} - \frac{(\beta_{22} - 1)(Q(\beta_{21})a(\beta_{21}) - b_1(\beta_{21}))}{(\beta_{21} - 1)b_1(\beta_{21})(\beta_{21} - 1)} + \frac{\beta_{22} - 1}{2(\beta_{21} - 1)} + \frac{(\beta_{22} - 1)a'(1)}{2(\beta_{21} - 1)a(1)}\Big]\Big)$$

$$+ \frac{1}{2} + \frac{a'(1)}{2a(1)} - \frac{\beta_{21} - 1}{(\beta_{22}^2 - 1)(\beta_{21} - 1) - (\beta_{22} - 1)(\beta_{21}^2 - 1)}\Big[\frac{Q(\beta_{22})a(\beta_{22}) - b_1(\beta_{22})}{b_2(\beta_{22})(\beta_{22} - 1)} - \frac{1}{2} - \frac{a'(1)}{2a(1)}$$

$$- \frac{(\beta_{22} - 1)(Q(\beta_{21})a(\beta_{21}) - b_1(\beta_{21}))}{(\beta_{21} - 1)b_1(\beta_{21})(\beta_{21} - 1)}$$

$$+ \frac{\beta_{22} - 1}{2(\beta_{21} - 1)} + \frac{(\beta_{22} - 1)a'(1)}{2(\beta_{21} - 1)a(1)}\Big] - \frac{1}{(\beta_{21} - 1)}$$

$$\Big(\frac{Q(\beta_{21})a(\beta_{21}) - b_1(\beta_{21})}{b_1(\beta_{21})(\beta_{21} - 1)} - \frac{1}{2} - \frac{a'(1)}{2a(1)} - \frac{(\beta_{21} - 1)(\beta_{21}^2 - 1)}{(\beta_{22}^2 - 1)(\beta_{21} - 1) - (\beta_{22} - 1)(\beta_{21}^2 - 1)}$$

$$\Big[\frac{Q(\beta_{22})a(\beta_{22}) - b_1(\beta_{22})}{b_1(\beta_{22})(\beta_{22} - 1)} - \frac{1}{2} - \frac{a'(1)}{2a(1)} - \frac{(\beta_{22} - 1)(Q(\beta_{21})a(\beta_{21}) - b_1(\beta_{21}))}{(\beta_{21} - 1)b_1(\beta_{21})(\beta_{21} - 1)} + \frac{\beta_{22} - 1}{2(\beta_{21} - 1)} + \frac{(\beta_{22} - 1)a'(1)}{2(\beta_{21} - 1)a(1)}\Big]\Big)$$

For $\dot{a}$ we get

$$\dot{a}(\lambda) =$$

$$\frac{(\overline{\frac{-\eta_1 c_1(\eta_1)i}{b_1(\eta_1)}}\overline{\eta}_2 + \overline{\eta}_1\overline{\frac{-\eta_2 c_1(\eta_2)i}{b_1(\eta_2)}})|\eta_1||\eta_2| - \overline{\eta}_1\overline{\eta}_2(\frac{\eta_1}{|\eta_1|}\frac{-\eta_1 c_1(\eta_1)i}{b_1(\eta_1)}|\eta_2| + |\eta_1|\frac{\eta_2}{|\eta_2|}\frac{-\eta_2 c_1(\eta_2)i}{b_1(\eta_2)})}{(|\eta_1||\eta_2|)^2}$$
$$\frac{|\eta_1||\eta_2|}{\overline{\eta}_1\overline{\eta}_2}a(\lambda) + \frac{\overline{\eta}_1\overline{\eta}_2}{|\eta_1||\eta_2|}(-\frac{-\eta_1 c_1(\eta_1)i}{b_1(\eta_1)}(\lambda - \overline{\eta}_1^{-1})(\lambda - \eta_2)(\lambda - \overline{\eta}_2^{-1})$$
$$- \frac{-\overline{\eta}_1^{-1}c_1(\overline{\eta}_1^{-1})i}{b_1(\overline{\eta}_1^{-1})}(\lambda - \eta_1)(\lambda - \eta_2)(\lambda - \overline{\eta}_2^{-1}) - \frac{-\eta_2 c_1(\eta_2)i}{b_1(\eta_2)}(\lambda - \eta_1)$$
$$(\lambda - \overline{\eta}_1^{-1})(\lambda - \overline{\eta}_2^{-1}) - \frac{-\overline{\eta}_2^{-1}c_1(\overline{\eta}_2^{-1})i}{b_1(\overline{\eta}_2^{-1})}(\lambda - \eta_1)(\lambda - \overline{\eta}_1^{-1})(\lambda - \eta_2)) \tag{12}$$

with c_1 as listed above. Finally we can also calculate $\dot{b}_1$ and $\dot{b}_2$, which are

$$\dot{b}_1 = \frac{(2\lambda a c_1' - a c_1 - \lambda a' c_1)i}{2a} + \frac{b_1}{2a}\dot{a} \tag{13}$$

and

$$\dot{b}_2 = \frac{(2\lambda a c_2' - a c_2 - \lambda a' c_2)i}{2a} + \frac{b_2}{2a}\dot{a}. \tag{14}$$

5 Commuting vector fields

Unfortunately the vector fields V_1 and V_2 have a rather unwieldy form. To see that they are well defined we have to show that they commute. That means that their Lie bracket needs to vanish. However since these terms are so unhandy the commutation in the most general case is too big of an effort. Therefore we will restrain this chapter on the commutation in the space $\mathcal{T}$.

5.1 Rotation of the spectral parameter

In [5.3] of [CS16] it is shown that the rotation $\lambda \mapsto e^{i*\varphi}\lambda$ acts on the frame bundle $\mathcal{F}^2$ by

$$a_\varphi(\lambda) = a(e^{i\varphi}\lambda),\ b_{1\varphi}(\lambda) = b_1(e^{i\varphi}\lambda) \text{ and } b_{2\varphi} = b_2(e^{i\varphi}\lambda)$$

and preserves the spectral curves. Thus we can rotate λ in a way that we rotate $\frac{\overline{\eta}_1\overline{\eta}_2}{|\eta_1||\eta_2|}$, which is also on $\mathbb{S}^1$, to one.

5.2 Commutation in $\mathcal{T}$

We need to divide the c_1 and c_2 in both vector fields by y to obtain $\dot{q}$ from chapter (4). With this small adaption we see that since a change in b_1 only changes $\dot{b}_1$ and not $\dot{b}_2$ and vice versa these vector fields change μ_1 and μ_2 uniquely at $\lambda = 1$. Hence we get a mapping

$$\varphi : \mathcal{S}_1^2 \to \mathbb{S}^1 \times \mathbb{S}^1.$$

This mapping is in fact a composition that maps $a \in \mathcal{S}_1^2$ to the flows corresponding to V_1 and V_2. That in return map again to $\mathbb{S}^1 \times \mathbb{S}^1$.

In [Hoe15] [6,7] it is shown that any a that has four pairwise distinct roots and additionally has one as highest and also as lowest coefficient induces a period lattice. Due to [CS16] [5.3] or (5.1) any a in $\mathcal{S}_1^2$ can be transformed into an a_ω that has one as highest. Hence for any such $a \in \mathcal{S}_1^2$ there exists a period lattice that can be uniquely defined by two generators. This includes unique b_1 and b_2. Since the Witham deformations are continuous deformations it is uniquely defined how b_1 and b_2 move along when t changes. Furthermore, we notice that $\mathcal{T}$ is locally bijective to $\mathcal{S}_1^2$. Since the derivative of φ is invertible we obtain local (a, b_1, b_2) with the corresponding μ_1, μ_2 that belong to the generators of the lattice at $\lambda = 1$. Thus φ is differentiable and locally invertible. Thus by Schwarz's Theorem the vector fields have to commute in $\mathcal{T}$.

6 Cayley transform and Witham deformations

Cayley transformations are special Möbius transformations. We try to transform the polynomial y to eventually get simpler expressions of the vector fields V_1 and V_2. Let $\tilde{y}$ denote the transformed polynomial. We need to reverse the mapping from definition (2.8), which is

$$\begin{aligned} & \lambda(x) = \frac{x-i}{x+i} \\ \Leftrightarrow\ & \lambda(x+i) = (x-i) \\ \Leftrightarrow\ & x\lambda - x = -i - i\lambda) \\ \Leftrightarrow\ & x(\lambda-1) = -i - i\lambda \\ \Leftrightarrow\ & x = \frac{-i-\lambda}{\lambda-1}. \end{aligned}$$

Then

$$y^2 = \lambda a(\lambda) \overset{\text{Cayley Transform}}{\Rightarrow} \tilde{y}^2 = (1+x^2)a(x), \tag{15}$$

where

i) $x \in \mathbb{R}$
ii) $a(x) \in \mathbb{R}\ \forall x \in \mathbb{R}$
iii) the highest coefficient of a is one.

6.1 Whitham equation under Cayley transform

With the same argumentation as in section (4) we get

$$dq_k = 2\pi i \frac{b_k(x)}{(1+x^2)\tilde{y}} dx,$$

$$\Theta_k = \frac{2\pi i b_k dx}{\tilde{y}(1+x^2)}$$

and

$$\dot{q} = \frac{2\pi i c_k(x)}{\tilde{y}}. \tag{16}$$

Thus we can start with the Whitham equation

$$\begin{aligned} & \frac{\partial \dot{q}}{\partial x} = \frac{\partial}{\partial t} \frac{2\pi i b_k}{\tilde{y}(1+x^2)}\bigg|_{t=0} \\ \Leftrightarrow\ & \frac{\partial}{\partial x}\frac{2\pi i c_k(x)}{\tilde{y}} = \frac{\partial}{\partial t} \frac{2\pi i b_k}{\tilde{y}(1+x^2)}\bigg|_{t=0}. \end{aligned}$$

By differentiating with chain and product rule we get:

$$\Rightarrow \frac{2\pi i c_k' \tilde{y} - \tilde{y}' 2\pi i c_k}{\tilde{y}^2} = \frac{2\pi i \dot{b}_k \tilde{y}(1+x^2) - \dot{\tilde{y}}(1+x^2)2\pi i b_k}{\tilde{y}^2(1+x^2)^2}\Bigg|_{t=0}$$

$$\Leftrightarrow \frac{2\pi i c_k' \sqrt{(1+x^2)a} - \frac{2xa+(1+x^2)a'}{2\sqrt{(1+x^2)a}} 2\pi i c_k}{(1+x^2)a}$$

$$= \frac{2\pi i \dot{b}_k}{\sqrt{a(1+x^2)}} - \frac{(1+x^2)\dot{a} 2\pi i b_k}{2a(1+x^2)^3\sqrt{a(1+x^2)}}$$

$$\Leftrightarrow \frac{2\pi i c_k'}{\sqrt{a(1+x^2)}} - \frac{(2xa+(1+x^2)a')\pi i c_k}{\sqrt{(1+x^2)^3 a^3}}$$

$$= \frac{2\pi i \dot{b}_k}{\sqrt{a(1+x^2)^3}} - \frac{\pi i b_k \dot{a}}{\sqrt{a^3(1+x^2)^3}}$$

$$\overset{\cdot\frac{\sqrt{a^3(1+x^2)^3}}{\pi i}}{\Leftrightarrow} 2(1+x^2)ac_k' - (2xa+(1+x^2)a')c_k = 2\dot{b}_k a - b_k \dot{a}$$

Hence we get equations (1C) and (2C), which are the corresponding equations to (1) and (2) under the Cayley transformation.

$$2(1+x^2)ac_1' - (2xa+(1+x^2)a')c_1 = 2\dot{b}_1 a - b_1\dot{a} \tag{1C}$$

$$2(1+x^2)ac_2' - (2xa+(1+x^2)a')c_2 = 2\dot{b}_2 a - b_2\dot{a} \tag{2C}$$

The operation c_2·(2C)-c_1·(1C) yields:

$$c_2 2(1+x^2)ac_1' + c_2(2xa+(1+x^2)a')c_1 - c_1 2(1+x^2)ac_2'$$
$$- c_1(2xa+(1+x^2)a')c_2 = 2\dot{b}_1 ac_2 - c_2 b_1\dot{a} - 2c_1\dot{b}_2 a + c_1 b_2\dot{a}$$

$\Leftrightarrow$

$$2(1+x^2)ac_1'c_2 - 2(1+x^2)ac_2'c_1 - 2\dot{b}_1 ac_2 + 2c_1\dot{b}_2 a = \dot{a}(c_1b_2 - c_2b_1)$$

With the same reasoning regarding the vanishing at roots of a as in section (4) we get

$$c_1b_2 - c_2b_1 = Qa \tag{3C}$$

with $deg(b_1) = deg(b_2) = 2$, $deg(Q) = 1$ and $deg(a) = 4$. One of the c polynomials has degree three, while the other one has degree two. Hence both sides are of degree five.

Equation (16) yields that for c_k with degree three the highest coefficient c_{k3} is one.

6.2 Vector field $\tilde{V}_1$

By looking at equation (1C) in the case that c_1 has degree three one might be tempted to think that the left-hand side has degree 8 while the right-hand side only has degree 6. We will resolve these issues by simple calculations. In this case we know:

$$\begin{aligned}
&deg(b_1) = deg(b_2) = 2 \text{ and } deg(\dot{b}_1) = deg(\dot{b}_2) = 2\\
&deg(a) = 4 \Rightarrow \; deg(a') = 3 \text{ and } deg(\dot{a}) = 3\\
&deg(c_1) = 3 \Rightarrow \; deg(c_1') = 2\\
&deg(c_2) = 2 \Rightarrow \; deg(c_2') = 1
\end{aligned}$$

Thus $deg(2(1+x^2)ac_1') = 8$ and $deg((2xa + (1+x^2)a')c_1) = 8$ while we only get $deg(\dot{b}_1 a) = 6$. We will use the following parametrizations:

$$\begin{aligned}
&a(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0\\
&c_1(x) = c_{13}x^3 + c_{12}x^2 + c_{11}x + c_0
\end{aligned}$$

We will simply look at the sum of coefficients of x^8 which is

$$\begin{aligned}
&2x^2a_4x^4 3c_{13}x^2 - 2xa_4x^4c_{13}x^3 - x^2 4a_4x^3c_{13}x^3\\
&= 6a_4c_{13}x^8 - 2a_4c_{13}x^8 - 4a_4c_{13}x^8 = 0.
\end{aligned}$$

Thus the right-hand side has maximal degree seven. But it needs to have degree six therefore we will also look at the coefficients of x^7.

$$\begin{aligned}
&2x^2a_4x^4 2c_{12}x + 2x^2a_3x^3 3c_{13}x^2 - 2xa_4x^{14}c_{12}x^2 - 2xa_3x^3c_{13}x^2\\
&- x^2 4a_4x^{13}c_{12}x^2 - 3x^2a_3x^2c_{13}x^3\\
&= 4a_4c_{12}x^7 + 6a_3c_{13}x^7 - 2a_4c_{12}x^7 - 2a_3c_{13}x^7 - 4a_4c_{12}x^7 - 3a_3c_{13}x^7\\
&= a_3c_{13}x^7 - 2a_4c_{12}x^7
\end{aligned}$$

So these coefficients do not vanish on their own. But we still require the left side to have degree six. Thus we need

$$\begin{aligned}
&& a_3c_{13} - 2a_4c_{12} = 0\\
&\Rightarrow & a_3c_{13} = 2a_4c_{12}\\
&\Leftrightarrow & c_{12} = \frac{a_3c_{13}}{2a_4}\\
&\overset{a_4=1}{\Rightarrow} & c_{12} = \frac{a_3c_{13}}{2}\\
&\overset{c_{13}=1}{\Rightarrow} & c_{12} = \frac{a_3}{2}
\end{aligned}$$

In chapter four we mainly collected conditions to solve these three equations. With the condition above we already found one. We will now gain another condition through (2C). We are in the case that $deg(c_1) = 3$ and $deg(c_2) = 2$ and a has degree four. Therefore the left-hand side of (2C) has maximal degree seven whereas the right-hand side only has degree six. Thus we need to look at the coefficients of x^7 on the left side. These are

$$\begin{aligned}&2x^2a_4x^4 2c_{22}x - 2xa_4x^4c_{22}x^2 - x^2 4a_4x^3c_{22}x^{22}\\&= 4a_4c_{22}x^7 - 2a_4c_{22}x^7 - 4a_4c_{22}x^7\\&= -2a_4c_{22}x^7.\end{aligned}$$

Since we require the right-hand side to have degree six and $a_4 = 1$ we can conclude that

$$c_{22} = 0.$$

We will now look at (3C) to already determine the two highest coefficients of Q, which has degree two. Let Q be parametrized through

$$Q(x) = q_1x + q_0.$$

Equation (3c) has degree five on both sides. Thus we want to compare the coeffcients of x^5. By equating the coefficients we gain

$$\begin{aligned}&c_{13}b_{22} = q_1a_4\\ \Leftrightarrow\ &q_1 = \frac{c_{13}b_{22}}{a_4}\\ \overset{c_{13}=1=a_4}{\Leftrightarrow}\ &q_1 = b_{22}.\end{aligned}$$

Equating the coefficients of x^4 yields:

$$\begin{aligned}&c_{13}b_{21} + c_{12}b_{22} + c_{22}b_{12} = q_1a_3 + q_0\\ \overset{c_{12}=\frac{a_3}{2}}{\Rightarrow}\ &c_{13}b_{21} + \frac{a_3b_{22}}{2} + c_{22}b_{12} = q_1a_3 + q_0\\ \overset{c_{22}=0}{\Rightarrow}\ &c_{13}b_{21} + \frac{a_3b_{22}}{2} = q_1a_3 + q_0\\ \overset{c_{13}=1}{\Rightarrow}\ &b_{21} + \frac{a_3b_{22}}{2} = q_1a_3 + q_0\\ \overset{q_1=b_{22}}{\Rightarrow}\ &b_{21} + \frac{a_3b_{22}}{2} = b_{22}a_3 + q_0\\ \Leftrightarrow\ &q_0 = b_{21} - \frac{a_3b_{22}}{2}\end{aligned}$$

Thus we already get an expression for Q, which is

$$Q(x) = b_{22}x + b_{21} - \frac{a_3 b_{22}}{2}. \tag{17}$$

Now we want to solve equation (3C) for the polynomials c_1 and c_2. Since we want to equate the coefficients it is a good idea to recap the parametrization of the polynomials a, Q, b_1, b_2, c_1 and c_2.

$$\begin{aligned}
a(x) &= x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \\
Q(x) &= b_{22}x + b_{21} - \frac{a_3b_{22}}{2} \\
b_1(x) &= b_{12}x^2 + b_{11}x + b_{10} \\
b_2(x) &= b_{22}x^2 + b_{21}x + b_{20} \\
c_1(x) &= x^3 + \frac{a_3}{2}x^2 + c_1x + c_0 \\
c_2(x) &= c_{21}x + c_{20}
\end{aligned}$$

With these parametrizations equation (3C) becomes

$$\begin{aligned}
&b_{22}x^5 + b_{21}x^4 + b_{20}x^3 + \frac{a_3b_{22}}{2}x^4 + \frac{a_3b_{21}}{2}x^3 + \frac{a_3b_{20}}{2}x^2 + c_{11}b_{22}x^3 + c_{11}b_{21}x^2 \\
&+ c_{11}b_{20}x + c_{10}b_{22}x^2 + c_{10}b_{21}x + c_{10}b_{20} - c_{21}b_{12}x^3 - c_{21}b_{11}x^2 - c_{21}b_{10}x \\
&- c_{20}b_{12}x^2 - c_{20}b_{11}x - c_{20}b_{10} = b_{22}x^5 + b_{22}a_3x^4 + b_{22}a_2x^3 + b_{22}a_1x^2 \\
&+ b_{22}a_0x + (b_{21} - \frac{a_3b_{22}}{2})x^4 + (b_{21} - \frac{a_3b_{22}}{2})a_3x^3 + (b_{21} - \frac{a_3b_{22}}{2})a_2x^2 \\
&+ (b_{21} - \frac{a_3b_{22}}{2})a_1x + (b_{21} - \frac{a_3b_{22}}{2})a_0.
\end{aligned}$$

With equating the coefficients of x^3, x^2, x and x^0 we get the following system of equations.

$$\begin{aligned}
I.\ & b_{20} + \frac{a_3b_{21}}{2} + c_{11}b_{22} - c_{21}b_{12} = b_{22}a_2 - (b_{21} - \frac{a_3b_{22}}{2})a_3 \\
II.\ & \frac{a_3b_{10}}{2} + c_{11}b_{21} + c_{10}b_{22} - c_{21}b_{11} - c_{20}b_{12} = b_{22}a_1 + (b_{21} - \frac{a_3b_{22}}{2})a_2 \\
III.\ & c_{11}b_{20} + c_{10}b_{21} - c_{21}b_{10} - c_{20}b_{11} = b_{22}a_0 + (b_{21} - \frac{a_3b_{22}}{2})a_1 \\
IV.\ & b_{20}c_{10} - b_{10}c_{20} = (b_{21} - \frac{a_3b_{22}}{2})a_0
\end{aligned}$$

This is equivalent to:

$$I.\ b_{22}c_{11} - b_{12}c_{21} =_{22} a_2 - (b_{21} - \frac{a_3 b_{22}}{2})a_3 - b_{20} - \frac{a_3 b_{21}}{2}$$
$$II.\ b_{21}c_{11} + b_{22}c_{10} - b_{11}c_{21} - b_{12}c_{20} = b_{22}a_1 + (b_{21} - \frac{a_3 b_{22}}{2})a_2 - \frac{a_3 b_{10}}{2}$$
$$III.\ c_{11}b_{20} + c_{10}b_{21} - c_{21}b_{10} - c_{20}b_{11} = b_{22}a_0 + (b_{21} - \frac{a_3 b_{22}}{2})a_1$$
$$IV.\ b_{20}c_{10} - b_{10}c_{20} = (b_{21} - \frac{a_3 b_{22}}{2})a_0$$

Solutions to it are:

$$c_{11} = (a_3^2 b_{10}^2 b_{21} b_{22} - a_3^2 b_{10} b_{11} b_{20} b_{22} - a_3 b_{10}^3 b_{22} + a_3 b_{10}^2 b_{12} b_{20}$$
$$- a_3 b_{10}^2 b_{21}^2 - a_2 a_3 b_{10}^2 b_{22}^2 + a_3 b_{10} b_{11} b_{20} b_{21} + a_1 a_3 b_{10} b_{11} b_{22}^2 + a_2 a_3 b_{10} b_{12} b_{20} b_{22}$$
$$- a_1 a_3 b_{10} b_{12} b_{21} b_{22} + a_0 a_3 b_{10} b_{12} b_{22}^2 - a_0 a_3 b_{11}^2 b_{22}^2 + a_0 a_3 b_{11} b_{12} b_{21} b_{22}$$
$$- a_0 a_3 b_{12}^2 b_{20} b_{22} + 2 b_{10}^2 b_{20} b_{21} + 2 a_1 b_{10}^2 b_{22}^2 - 2 b_{10} b_{11} b_{20}^2 + 2 a_2 b_{10} b_{11} b_{20} b_{22}$$
$$+ 2 a_1 b_{10} b_{12} b_{21}^2 + 2 a_0 b_{11}^2 b_{21} b_{22} - 2 a_0 b_{11} b_{12} b_{21}^2 + 2 a_0 b_{12}^2 b_{20} b_{21})$$
$$/(2(b_{10}^2 b_{22}^2 - b_{10} b_{11} b_{21} b_{22} - 2 b_{10} b_{12} b_{20} b_{22} + b_{10} b_{12} b_{21}^2$$
$$+ b_{11}^2 b_{20} b_{22} - b_{11} b_{12} b_{20} b_{21} + b_{12}^2 b_{20}^2))$$

$$c_{10} = -(a_3^2 b_{10}^2 b_{22}^2 - a_3^2 b_{10} b_{11} b_{21} b_{22} - a_3^2 b_{10} b_{12} b_{20} b_{22} + a_3^2 b_{11}^2 b_{20} b_{22}$$
$$+ a_3 b_{10}^2 b_{12} b_{21} - a_3 b_{10}^2 b_{21} b_{22} - a_3 b_{10} b_{11} b_{12} b_{20} + a_3 b_{10} b_{11} b_{21}^2 + a_3 b_{10} b_{12} b_{20} b_{21}$$
$$+ a_2 a_3 b_{10} b_{12} b_{21} b_{22} - a_1 a_3 b_{10} b_{12} b_{22}^2 - a_3 b_{11}^2 b_{20} b_{21} - a_2 a_3 b_{11} b_{12} b_{20} b_{22}$$
$$+ a_0 a_3 b_{11} b_{12} b_{22}^2 + a_1 a_3 b_{12}^2 b_{20} b_{22} - a_0 a_3 b_{12}^2 b_{21} b_{22} + 2 b_{10}^2 b_{20} b_{22} - 2 a_2 b_{10}^2 b_{22}^2$$
$$- 2 b_{10} b_{11} b_{20} b_{21} + 2 a_2 b_{10} b_{11} b_{21} b_{22} - 2 b_{10} b_{12} b_{20}^2 + 2 a_2 b_{10} b_{12} b_{20} b_{22}$$
$$- 2 a_2 b_{10} b_{12} b_{21}^2 + 2 a_0 b_{10} b_{12} b_{22}^2 + 2 b_{11}^2 b_{20}^2 - 2 a_2 b_{11}^2 b_{20} b_{22}$$
$$+ 2 a_2 b_{11} b_{12} b_{20} b_{21} + 2 a_1 b_{11} b_{12} b_{20} b_{22} - 2 a_0 b_{11} b_{12} b_{21} b_{22}$$
$$- 2 a_1 b_{12}^2 b_{20} b_{21} - 2 a_0 b_{12}^2 b_{20} b_{22} + 2 a_0 b_{12}^2 b_{21}^2)$$
$$/(2(b_{10}^2 b_{22}^2 - b_{10} b_{11} b_{21} b_{22} - 2 b_{10} b_{12} b_{20} b_{22} + b_{10} b_{12} b_{21}^2$$
$$+ b_{11}^2 b_{20} b_{22} - b_{11} b_{12} b_{20} b_{21} + b_{12}^2 b_{20}^2))$$

$$c_{21} = -(2 b_{11} b_{20}^3 + 2 a_0 b_{12} b_{21}^3 - 2 b_{10} b_{20}^2 b_{21} - a_0 a_3 b_{10} b_{22}^3 - a_3 b_{10} b_{12} b_{20}^2$$
$$+ 2 a_0 b_{10} b_{21} b_{22}^2 + 2 a_0 b_{11} b_{20} b_{22}^2 - 2 a_1 b_{10} b_{20} b_{22}^2 - 2 a_0 b_{11} b_{21}^2 b_{22} - 2 a_1 b_{12} b_{20} b_{21}^2$$
$$+ a_3 b_{10} b_{20} b_{21}^2 + 2 a_1 b_{12} b_{20}^2 b_{22} - 2 a_2 b_{11} b_{20}^2 b_{22} + 2 a_2 b_{12} b_{20}^2 b_{21} - a_3 b_{11} b_{20}^2 b_{21}$$
$$+ a_3 b_{10}^2 b_{20} b_{22} + a_3^2 b_{11} b_{20}^2 b_{22} - 4 a_0 b_{12} b_{20} b_{21} b_{22} + 2 a_1 b_{11} b_{20} b_{21} b_{22}$$
$$+ a_0 a_3 b_{11} b_{21} b_{22}^2 + a_0 a_3 b_{12} b_{20} b_{22}^2 - a_1 a_3 b_{11} b_{20} b_{22}^2 + a_2 a_3 b_{10} b_{20} b_{22}^2$$
$$- a_0 a_3 b_{12} b_{21}^2 b_{22} - a_2 a_3 b_{12} b_{20}^2 b_{22} - a_3^2 b_{10} b_{20} b_{21} b_{22} + a_1 a_3 b_{12} b_{20} b_{21} b_{22})$$
$$/(2(b_{10}^2 b_{22}^2 - b_{10} b_{11} b_{21} b_{22} - 2 b_{10} b_{12} b_{20} b_{22} + b_{10} b_{12} b_{21}^2$$
$$+ b_{11}^2 b_{20} b_{22} - b_{11} b_{12} b_{20} b_{21} + b_{12}^2 b_{20}^2))$$

$$\begin{aligned}
c_{20} = &(2b_{12}b_{20}^3 - 2a_0b_{10}b_{22}^3 - a_3b_{10}b_{21}^3 + 2b_{10}b_{20}b_{21}^2 - 2b_{10}b_{20}^2b_{22} - 2b_{11}b_{20}^2b_{21} \\
&- a_0a_3b_{11}b_{22}^3 + a_1a_3b_{10}b_{22}^3 + 2a_0b_{11}b_{21}b_{22}^2 + 2a_0b_{12}b_{20}b_{22}^2 - 2a_1b_{11}b_{20}b_{22}^2 \\
&+ 2a_2b_{10}b_{20}b_{22}^2 - 2a_0b_{12}b_{21}^2b_{22} + a_3b_{11}b_{20}b_{21}^2 - 2a_2b_{12}b_{20}^2b_{22} - a_3b_{12}b_{20}^2b_{21} \\
&- a_3b_{10}^2b_{21}b_{22} - a_3^2b_{10}b_{20}b_{22}^2 + a_3^2b_{10}b_{21}^2b_{22} + a_3^2b_{12}b_{20}^2b_{22} + a_3b_{10}b_{11}b_{20}b_{22} \\
&+ 2a_1b_{12}b_{20}b_{21}b_{22} + a_3b_{10}b_{20}b_{21}b_{22} + a_0a_3b_{12}b_{21}b_{22}^2 - a_1a_3b_{12}b_{20}b_{22}^2 \\
&- a_2a_3b_{10}b_{21}b_{22}^2 + a_2a_3b_{11}b_{20}b_{22}^2 - a_3^2b_{11}b_{20}b_{21}b_{22}) \\
&/(2(b_{10}^2b_{22}^2 - b_{10}b_{11}b_{21}b_{22} - 2b_{10}b_{12}b_{20}b_{22} + b_{10}b_{12}b_{21}^2 \\
&+ b_{11}^2b_{20}b_{22} - b_{11}b_{12}b_{20}b_{21} + b_{12}^2b_{20}^2))
\end{aligned}$$

We notice that

$$Resultant(b_1, b_2) = \begin{vmatrix} b_{12} & b_{11} & b_{10} & 0 \\ 0 & b_{12} & b_{11} & b_{10} \\ b_{22} & b_{21} & b_{20} & 0 \\ 0 & b_{22} & b_{21} & b_{20} \end{vmatrix} = b_{10}^2b_{22}^2 - b_{10}b_{11}b_{21}b_{22} - 2b_{10}b_{12}b_{20}b_{22}$$
$$+ b_{10}b_{12}b_{21}^2 + b_{11}^2b_{20}b_{22} - b_{11}b_{12}b_{20}b_{21} + b_{12}^2b_{20}^2.$$

That is exactly one half times the denominator of c_{11}, c_{10}, c_{21} and c_{20}. It therefore becomes $2Resultant(b_1, b_2)$.

We also want to show a more sophisticated approach to solving (3C) for c_1 and c_2. Therefore let them be parametrized through

$$c_1(x) = (\gamma_{11}x + \gamma_{10}) + (\gamma_{13}x + \gamma_{12})b_1(x), \text{ with } \gamma_{13}, \gamma_{12}, \gamma_{11} \text{ and } \gamma_{10} \in \mathbb{R}$$

and

$$c_2(x) = (\gamma_{21}x + \gamma_{20}) + (\gamma_{23}x + \gamma_{22})b_2(x), \text{ with } \gamma_{23}, \gamma_{22}, \gamma_{21} \text{ and } \gamma_{20} \in \mathbb{R}.$$

To obtain these polynomials we will first solve for the polynomial of degree one multiplied with b_k for $k = 1, 2$. As above we will pursue the case that c_1 has degree three while c_2 has degree two, which means respectively that it has degree one. We will start with c_1. After multiplying we obtain

$$c_1(x) = \gamma_{13}b_{12}x^3 + (\gamma_{13}b_{11} + \gamma_{12}b_{12})x^2 + (\gamma_{11} + \gamma_{13}b_{10} + \gamma_{12}b_{11})x + \gamma_{10} + \gamma_{12}b_{10}.$$

Since the highest coefficient of c_1 has to be one we get

$$\gamma_{13}b_{12} = 1 \;\Rightarrow\; \gamma_{13} = \frac{1}{b_{12}}.$$

We also know that the second highest coefficient of c_1 is $\frac{a_3}{2}$ thus we obtain

$$\begin{aligned} & \gamma_{13}b_{11} + \gamma_{12}b_{12} = \frac{a_3}{2} \\ \Leftrightarrow\ & \frac{b_{11}}{b_{12}} + \gamma_{12}b_{12} = \frac{a_3}{2} \\ \Leftrightarrow\ & \gamma_{12} = \frac{a_3}{2b_{12}} - \frac{b_{11}}{b_{12}^2}. \end{aligned}$$

Now we also want to obtain γ_{11} and γ_{10}. To do this we will use the roots β_{11} and β_{12} of b_1 and plug these in equation (3C). Since both b_1 and b_2 are real polynomials of degree two we can easily calculate their roots with the quadratic formula. Thus we get

$$\beta_{11} = \frac{-b_{11} + \sqrt{b_{11}^2 - 4b_{12}b_{10}}}{2b_{12}},$$

$$\beta_{12} = \frac{-b_{11} - \sqrt{b_{11}^2 - 4b_{12}b_{10}}}{2b_{12}},$$

$$\beta_{21} = \frac{-b_{21} + \sqrt{b_{21}^2 - 4b_{22}b_{20}}}{2b_{22}},$$

and

$$\beta_{22} = \frac{-b_{21} - \sqrt{b_{21}^2 - 4b_{22}b_{20}}}{2b_{22}}.$$

For $x = \beta_{11}$ we get

$$\begin{aligned} & c_1(\beta_{11})b_2(\beta_{11}) = Q(\beta_{11})a(\beta_{11}) \\ \Leftrightarrow\ & \gamma_{11}\beta_{11} + \gamma_{10} = \frac{Q(\beta_{11})a(\beta_{11})}{b_2(\beta_{11})}. \end{aligned} \tag{18}$$

For $x = \beta_{12}$ we get

$$\begin{aligned} & c_1(\beta_{12})b_2(\beta_{12}) = Q(\beta_{12})a(\beta_{12}) \\ \Leftrightarrow\ & \gamma_{11}\beta_{12} + \gamma_{10} = \frac{Q(\beta_{12})a(\beta_{12})}{b_2(\beta_{12})}. \end{aligned} \tag{19}$$

Thus it only remains to solve the system of equations (18) and (19) for γ_{11} and γ_{10}. By (18)-(19) we get

$$\gamma_{11} = \frac{Q(\beta_{11})a(\beta_{11})b_2(\beta_{12}) - Q(\beta_{12})a(\beta_{12})b_2(\beta_{11})}{(\beta_{11} - \beta_{12})b_2(\beta_{12})b_2(\beta_{11})}. \tag{20}$$

Inserting (20) in (18) gives us

$$\gamma_{10} = \frac{\beta_{11}Q(\beta_{12})a(\beta_{12})b_2(\beta_{11}) - \beta_{12}Q(\beta_{11})a(\beta_{11})b_2(\beta_{12})}{(\beta_{11}-\beta_{12})b_2(\beta_{12})b_2(\beta_{11})}. \tag{21}$$

Thus we received c_1. As we already know the denominator of these coefficients needs to contain the Resultant of b_1 and b_2. And we can easily see that this is true by calculating $b_2(\beta_{12})b_2(\beta_{11})$:

$$\begin{aligned}
&\Big(b_{22}\Big(\frac{-b_{11}-\sqrt{b_{11}^2-4b_{12}b_{10}}}{2b_{12}}\Big)^2 + b_{21}\frac{-b_{11}-\sqrt{b_{11}^2-4b_{12}b_{10}}}{2b_{12}} + b_{20}\Big)\\
&\Big(b_2\Big(\frac{-b_{11}+\sqrt{b_{11}^2-4b_{12}b_{10}}}{2b_{12}}\Big)^2 + b_{21}\frac{-b_{11}+\sqrt{b_{11}^2-4b_{12}b_{10}}}{2b_{12}} + b_{20}\Big)\\
&= b_{20}^2 + \frac{b_{10}b_{21}^2}{b_{12}} + \frac{b_{10}^2b_{22}^2}{b_{12}^2} - \frac{2b_{10}b_{20}b_{22}}{b_{12}} - \frac{b_{11}b_{20}b_{21}}{b_{12}} + \frac{b_{11}^2b_{20}b_{22}}{b_{12}^2} - \frac{b_{10}b_{11}b_{21}b_{22}}{b_{12}^2}\\
&= \frac{Res(b_1,b_2)}{b_{12}^2}
\end{aligned}$$

Thus we have the Resultant in the denominator. We will now use the same procedure again to obtain c_2. We know that the degree of c_2 is one due to the calculations on page 51. Thus the polynomial of degree one that is multiplied by b_2 needs to vanish. Hence $\gamma_{13} = \gamma_{12} = 0$. It remains to solve for γ_{11} and γ_{10}. We will do this by inserting the roots β_{21} and β_{22} of b_2 in equation (3C). For $x = \beta_{21}$ we obtain

$$\begin{aligned}
&c_2(\beta_{21})b_1(\beta_{21}) = Q(\beta_{21})a(\beta_{21})\\
&\Leftrightarrow\ \ \gamma_{21}\beta_{21} + \gamma_{20} = \frac{Q(\beta_{21})a(\beta_{21})}{b_1(\beta_{21})}.
\end{aligned} \tag{22}$$

For $x = \beta_{22}$ we obtain

$$\begin{aligned}
&c_2(\beta_{22})b_1(\beta_{22}) = Q(\beta_{22})a(\beta_{22})\\
&\Leftrightarrow\ \ \gamma_{22}\beta_{22} + \gamma_{20} = \frac{Q(\beta_{22})a(\beta_{22})}{b_1(\beta_{22})}.
\end{aligned} \tag{23}$$

By (22)-(23) we receive

$$\gamma_{21} = \frac{Q(\beta_{21})a(\beta_{21})b_1(\beta_{22}) - Q(\beta_{22})a(\beta_{22})b_1(\beta_{21})}{(\beta_{21}-\beta_{22})b_1(\beta_{22})b_1(\beta_{21})}. \tag{24}$$

By inserting the expression from (24) in (22) we get

$$\gamma_{20} = \frac{\beta_{21}Q(\beta_{22})a(\beta_{22})b_1(\beta_{21}) - \beta_{22}Q(\beta_{21})a(\beta_{21})b_1(\beta_{22})}{(\beta_{21}-\beta_{22})b_1(\beta_{22})b_1(\beta_{21})}. \tag{25}$$

With the same calculations as above we can show that

$$b_1(\beta_{22})b_1(\beta_{21}) = \frac{Res(b_1, b_2)}{b_{22}^2}.$$

Thus we already solved equation (3C) for Q, c_1 and c_2. Now we can commit ourselves to find $\dot{a}, \dot{b}_1$ and $\dot{b}_2$. We will start by creating a new equation (4C) out of (1C) and (2C) by calculating $b_2 \cdot$(1C)$-b_1 \cdot$(2C). This gives us:

$$\begin{aligned}
&2b_2(1+x^2)ac_1' - 2(2xa + (1+x^2)a')(c_1b_2 - c_2b_1) - 2b_1(1+x^2)ac_2' \\
&= 2\dot{b}_1ab_2 - 2\dot{b}_2ab_1 \\
&\overset{(3C)}{\Leftrightarrow} \\
&2b_2(1+x^2)ac_1' - 2(2xa + (1+x^2)a')Qa - 2b_1(1+x^2)ac_2' \\
&= 2\dot{b}_1ab_2 - 2\dot{b}_2ab_1 \\
&\overset{:2a}{\Leftrightarrow}
\end{aligned}$$

$$b_2(1+x^2)c_1' - (2xa + (1+x^2)a')Q - b_1(1+x^2)c_2' = \dot{b}_1b_2 - \dot{b}_2b_1 \qquad \text{(4C)}$$

We will now paramterize $\dot{b}_1$ as follows

$$\dot{b}_1(x) = \dot{b}_{11}x + \dot{b}_{10} + \kappa_{b_1}b_1(x),$$

where $\dot{b}_{11}, \dot{b}_{12}$ and κ_{b_1} are real numbers and $\dot{b}_2$ through

$$\dot{b}_2(x) = \dot{b}_{21}x + \dot{b}_{20} + \kappa_{b_2}b_2(x),$$

where $\dot{b}_{21}, \dot{b}_{22}$ and κ_{b_2} are real numbers. We will undergo the same procedure to determine these coefficients as we did for the polynomials c_1 and c_2. Thus we will start to determine $\dot{b}_{11}$ and $\dot{b}_{12}$ through the roots of b_1. Inserting β_{11} in (4C) gives us

$$\begin{aligned}
&b_2(\beta_{11})(1+\beta_{11}^2)c_1'(\beta_{11}) - 2\beta_{11}c_1(\beta_{11})b_2(\beta_{11}) - (1+\beta_{11}^2)a'(\beta_{11})Q(\beta_{11}) \\
&= \dot{b}_1(\beta_{11})b_2(\beta_{11}) \\
&\Leftrightarrow
\end{aligned}$$

$$\begin{aligned}
&\dot{b}_{11}\beta_{11} + \dot{b}_{10} = \\
&\frac{b_2(\beta_{11})(1+\beta_{11}^2)c_1'(\beta_{11}) - 2\beta_{11}c_1(\beta_{11})b_2(\beta_{11}) - (1+\beta_{11}^2)a'(\beta_{11})Q(\beta_{11})}{b_2(\beta_{11})}.
\end{aligned} \qquad (26)$$

For β_{12} we get

$$\begin{aligned}
&\dot{b}_{11}\beta_{12} + \dot{b}_{10} = \\
&\frac{b_2(\beta_{12})(1+\beta_{12}^2)c_1'(\beta_{12}) - 2\beta_{12}c_1(\beta_{12})b_2(\beta_{12}) - (1+\beta_{12}^2)a'(\beta_{12})Q(\beta_{12})}{b_2(\beta_{12})}.
\end{aligned} \qquad (27)$$

Now (26)-(27) gives us

$$\dot{b}_{11}(\beta_{11}-\beta_{12}) = \frac{b_2(\beta_{11})(1+\beta_{11}^2)c_1'(\beta_{11}) - 2\beta_{11}c_1(\beta_{11})b_2(\beta_{11}) - (1+\beta_{11}^2)a'(\beta_{11})Q(\beta_{11})}{b_2(\beta_{11})} - \frac{b_2(\beta_{12})(1+\beta_{12}^2)c_1'(\beta_{12}) - 2\beta_{12}c_1(\beta_{12})b_2(\beta_{12}) - (1+\beta_{12}^2)a'(\beta_{12})Q(\beta_{12})}{b_2(\beta_{12})}.$$

Hence we get

$$\begin{aligned}\dot{b}_{11} &= [b_2(\beta_{12})(b_2(\beta_{11})(1+\beta_{11}^2)c_1'(\beta_{11}) - 2\beta_{11}c_1(\beta_{11})b_2(\beta_{11}) \\ &- (1+\beta_{11}^2)a'(\beta_{11})Q(\beta_{11})) - b_2(\beta_{11})(b_2(\beta_{12})(1+\beta_{12}^2)c_1'(\beta_{12}) \\ &- 2\beta_{12}c_1(\beta_{12})b_2(\beta_{12}) - (1+\beta_{12}^2)a'(\beta_{12})Q(\beta_{12}))] \\ &/[(\beta_{11}-\beta_{12})b_2(\beta_{11})b_2(\beta_{12})].\end{aligned} \tag{28}$$

Since we know that $b_2(\beta_{11})b_2(\beta_{12})b_{12}^2 = Res(b_1, b_2)$ and that c_1 is a polynomial with $Res(b_1, b_2)$ in its denominator $\dot{b}_{11}$ is an expression with $Res(b_1, b_2)^2$ in its denominator. With $\dot{b}_{11}$ and (26) we will now obtain $\dot{b}_{10}$, which is as follows

$$\begin{aligned}\dot{b}_{10} &= [\beta_{11}b_2(\beta_{11})(b_2(\beta_{12})(1+\beta_{12}^2)c_1'(\beta_{12}) - 2\beta_{12}c_1(\beta_{12})b_2(\beta_{12}) \\ &- (1+\beta_{12}^2)a'(\beta_{12})Q(\beta_{12})) - \beta_{12}b_2(\beta_{12})(b_2(\beta_{11})(1+\beta_{11}^2)c_1'(\beta_{11}) \\ &- 2\beta_{11}c_1(\beta_{11})b_2(\beta_{11}) - (1+\beta_{11}^2)a'(\beta_{11})Q(\beta_{11}))]/[(\beta_{11}-\beta_{12})b_2(\beta_{11})b_2(\beta_{12})].\end{aligned} \tag{29}$$

This expression can also be written with $Res(b_1, b_2)$ in its denominator. It remains now to determine the constant κ_{b_1}. We will do this by using equation (1C) and equating coefficients of x^6. With the latest parametrization of c_1 the left-hand side of (1C) becomes

$$\begin{aligned}&2(1+x^2)(x^4+a_3x^3+a_2x^2+a_1x+a_0)(\gamma_{11}+(\gamma_{13}+\gamma_{12})(2b_{12}x+b_{11}) \\ &+\gamma_{13}(b_{12}x^2+b_{11}x+b_{10})) - 2x(x^4+a_3x^3+a_2x^2+a_1x+a_0)(\gamma_{11}x+\gamma_{10} \\ &+(\gamma_{13}x+\gamma_{12})(b_{12}x^2+b_{11}x+b_{10})) - (1+x^2)(4x^3+3a_3x^2+2a_2x+a_1) \\ &(\gamma_{11}x+\gamma_{10}+(\gamma_{13}x+\gamma_{12})(b_{12}x^2+b_{11}x+b_{10})).\end{aligned}$$

Thus the coefficients of x^6 on the left side are

$$\begin{aligned}&4\gamma_{13}b_{12}+2\gamma_{11}+2\gamma_{12}b_{11}+2\gamma_{13}b_{10}+2\gamma_{13}b_{10}+2a_3\gamma_{13}b_{11} \\ &+4a_3\gamma_{12}b_{12}+2a_3\gamma_{13}b_{11}+4a_2\gamma_{13}b_{12}+2a_2\gamma_{13}b_{12}-2\gamma_{11} \\ &-2\gamma_{13}b_{10}-2\gamma_{12}b_{11}-2a_3\gamma_{13}b_{11}-2a_3\gamma_{12}b_{12}-2a_2\gamma_{13}b_{12} \\ &-4\gamma_{13}b_{12}-4\gamma_{11}-4\gamma_{13}b_{10}-4\gamma_{12}b_{11}-3a_3\gamma_{13}b_{11} \\ &-3a_3\gamma_{12}b_{12}-2a_2\gamma_{13}b_{12} \\ &=-a_3\gamma_{12}b_{12}-4\gamma_{11}-4\gamma_{13}b_{10}-4\gamma_{12}b_{11}-a_3\gamma_{13}b_{11}.\end{aligned}$$

Due to the special parametrization of b_1 the coefficient on the right side is $\kappa_{b_1} b_{12}$. Thus we get

$$\kappa_{b_1} = \frac{a_3\gamma_{12}b_{12} - 4\gamma_{11} - 4\gamma_{13}b_{10} - 4\gamma_{12}b_{11} - a_3\gamma_{13}b_{11}}{b_{12}}. \tag{30}$$

Therefore we calculated all coefficients of $\dot{b}_1$. Now we want to do the same procedure once again to determine all coefficients of $\dot{b}_2$. We will start with $\dot{b}_{21}$ and $\dot{b}_{20}$, which we will get through the roots of b_2. Inserting β_{21} in (4C) gives us

$$\begin{aligned} & b_1(\beta_{21})(1+\beta_{21}^2)c_2'(\beta_{21}) - 2\beta_{21}c_2(\beta_{21})b_1(\beta_{21}) - (1+\beta_{21}^2)a'(\beta_{21})Q(\beta_{21}) \\ & = \dot{b}_2(\beta_{21})b_1(\beta_{21}) \\ & \Leftrightarrow \end{aligned}$$

$$\begin{aligned} & \dot{b}_{21}\beta_{21} + \dot{b}_{20} = \\ & \frac{b_1(\beta_{21})(1+\beta_{21}^2)c_2'(\beta_{21}) - 2\beta_{21}c_2(\beta_{21})b_1(\beta_{21}) - (1+\beta_{21}^2)a'(\beta_{21})Q(\beta_{21})}{b_1(\beta_{21})}. \end{aligned} \tag{31}$$

For β_{22} we get

$$\begin{aligned} & \dot{b}_{21}\beta_{22} + \dot{b}_{20} = \\ & \frac{b_1(\beta_{22})(1+\beta_{22}^2)c_2'(\beta_{22}) - 2\beta_{22}c_2(\beta_{22})b_1(\beta_{22}) - (1+\beta_{22}^2)a'(\beta_{22})Q(\beta_{22})}{b_1(\beta_{22})}. \end{aligned} \tag{32}$$

Through (31)-(32) we obtain

$$\begin{aligned} \dot{b}_{21} = {} & [b_1(\beta_{22})(b_1(\beta_{21})(1+\beta_{21}^2)c_2'(\beta_{21}) - 2\beta_{21}c_2(\beta_{21})b_1(\beta_{21}) \\ & - (1+\beta_{21}^2)a'(\beta_{21})Q(\beta_{21})) - b_1(\beta_{21})(b_1(\beta_{22})(1+\beta_{22}^2)c_2'(\beta_{22}) \\ & - 2\beta_{22}c_2(\beta_{22})b_1(\beta_{22}) - (1+\beta_{22}^2)a'(\beta_{22})Q(\beta_{22}))] \\ & /[(\beta_{21}-\beta_{22})b_1(\beta_{21})b_1(\beta_{22})]. \end{aligned} \tag{33}$$

With $\dot{b}_{21}$ and (31) we will now obtain $\dot{b}_{20}$, which is as follows

$$\begin{aligned} \dot{b}_{20} = {} & [\beta_{21}b_1(\beta_{21})(b_1(\beta_{22})(1+\beta_{22}^2)c_2'(\beta_{22}) - 2\beta_{22}c_1(\beta_{22})b_1(\beta_{22}) \\ & - (1+\beta_{22}^2)a'(\beta_{22})Q(\beta_{22})) - \beta_{22}b_1(\beta_{22})(b_1(\beta_{21})(1+\beta_{21}^2)c_2'(\beta_{21}) \\ & - 2\beta_{21}c_2(\beta_{11})b_1(\beta_{21}) - (1+\beta_{21}^2)a'(\beta_{21})Q(\beta_{21}))]/[(\beta_{21}-\beta_{22})b_1(\beta_{21})b_1(\beta_{22})]. \end{aligned} \tag{34}$$

Now it remains to calculate κ_{b_1}. We will equate the coefficients of x^6 in equation (2C). Due to the parametrization of $c_2 = \gamma_{21}x + \gamma_{20}$ the left-hand side becomes

$$\begin{aligned} & 2(1+x^2)(x^4 + a_3x^3 + a_2x^2 + a_1x + a_0)\gamma_{21} - 2x(x^4 + a_3x^3 + a_2x^2 + a_1x \\ & + a_0)(\gamma_{21}x + \gamma_{20}) - (1+x^2)(4x^3 + 3a_3x^2 + 2a_2x + a_1)(\gamma_{21}x + \gamma_{20}). \end{aligned}$$

Thus the coefficients of x^6 are

$$2\gamma_{21} - 2\gamma_{21} - 4\gamma_{21} = -4\gamma_{21}.$$

The coefficient of x^6 on the right-hand side are $2\kappa_{b_2} b_{22}$. Thus we get

$$\kappa_{b_2} = \frac{-2\gamma_{21}}{b_{22}}. \tag{35}$$

The final step is to obtain $\dot{a}$ from either (1C) or (2C) since the solution of $\dot{b}_2$ is much shorter we will use (2C) to solve for $\dot{a}$. By subtracting $\dot{b}_2 a$ and dividing by $-b_2$ we get

$$\dot{a} = \frac{2(1+x^2)ac_2' - (2xa + (1+x^2)a')c_2 - 2\dot{b}_2 a}{-b_2}. \tag{36}$$

With the parametrizations of the polynomials we get:

$$\dot{a} = \frac{2(1+x^2)a\gamma_{21} - (2xa + (1+x^2)a')(\gamma_{21}x + \gamma 20) - 2(\dot{b}_{21}x + \dot{b}_{20} + \kappa_{b_2} b_2)a}{-b_2}$$

In the way we determined the polynomials $\dot{a}, \dot{b}_1$ and $\dot{b}_2$ under the conditions that $deg(c_2) = 2$ we could now also determine them again in the same way under the condition that $deg(c_1) = 2$ that would lead to another vector field $\tilde{V}_2$. In this case c_1 would actually have degree one as did c_2 in the previous case. Therefore the terms for $\dot{b}_2$ would be longer. All the procedures used in this chapter can be applied to the second case. The next task would be to try simplifying these polynomials.

7 Conclusion

At the end of this work we want to summarize the findings and observations we obtained in this thesis and point to interesting topics that could be investigated in another thesis or even in research papers.

We have seen that the space $\mathcal{S}_1^2$ is a submanifold of the space of spectral curves $\mathcal{H}^2$ that describe constant mean curvature tori. To investigate this submanifold we introduced infinitisemal Whitham deformations, which led to tangent vector fields on the frame bundle $\mathcal{F}$. We were able to prove that given the values of $c_1(1)$ and $c_2(1)$ we can uniquely solve three equations for these tangent vectors.

In a second step we calculated two explicit vector fields V_1 and V_2. Unfortunately the formulas that describe these vector fields are rather long and unwieldy. Thus further calculations were left out. Nevertheless we concluded that the entries of $\dot{b}_1$ and $\dot{b}_2$ were linearly independent. Thus these vector fields commute in $\mathcal{T}$.

With these vector fields we could define a mapping $\varphi : \mathcal{S}_1^2 \to \mathbb{S}^1 \times \mathbb{S}^1$. It would be necessary to extend this mapping to boundary points of $\mathcal{S}_1^2$ to get (a, b_1, b_2) in the boundary of $\mathcal{T}$. It would then be possible to integrate these vector fields.

In [KHS17] the corresponding solutions of the Sinh-Gordon equation are calculated with the restrain that the highest coefficient of a is one. We gave a rotation of λ such that all the coefficients of a can be transformed to an a_ω with highest coefficient one.

An interesting step for future research would be to find reasoning for φ to be a bijection. Thus any point on $\mathbb{S}^1 \times \mathbb{S}^1$ would be in one to one and onto correspondence to a spectral curve in $\mathcal{S}_1^2$.

At the end of this thesis we used Cayley transforms on the spectral parameter to transform it into a real variable. We obtained real polynomials with shorter terms than in chapter four. Hence it was a success. But the polynomials are still very long. Nevertheless one could try to calculate the Lie bracket or use computer programs to simplify the polynomials in another thesis.

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I declare that all statements and information contained herein are true, correct and accurate to the best of my knowledge and belief.

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