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Master's Thesis

Solutions of the Sinh-Gordon Equation of Spectral Genus Two

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Abstract

The elliptic sinh-Gordon equation arises in the context of surface theory. We investigate solutions of spectral genus two. These solutions are parametrized by a certain class of Polynomial Killing fields, which can be regarded as periodic flows on complex matrix-valued polynomials of degree three. The determinant is an integral of motion with respect to these flows and the main part of this elaboration examines the corresponding isospectral sets in dependence on the position of the determinant's roots. Thereby, solutions of spectral genus one and zero are investigated as well. The determinants with pairwise distinct zeroes generate lattices of periods. Finally, the mapping from the set of these determinants to the set of equivalent classes of isomorphic lattices is continuously extended to those with several roots.

Zusammenfassung

Die elliptische sinh-Gordon Gleichung entsteht im Kontext von Flächentheorie. Wir untersuchen Lösungen vom sogenannten Spektralgeschlecht zwei. Diese Lösungen werden durch eine Klasse polynomialer Killingfelder parametrisiert, welche man als periodische Flüsse auf komplexen Polynomen von Grad drei mit Matrixkoeffizienten auffassen kann. Deren Determinante ist bezüglich dieser Flüsse ein Integral der Bewegung und das Hauptaugenmerk meiner Ausarbeitung liegt auf der Untersuchung der resultierenden Isospektralmengen in Abhängigkeit von der Lage der Determinantennullstellen. Dabei werden auch Lösungen von Spektralgeschlecht eins und null untersucht. Die Determinanten mit paarweise verschiedenen Nullstellen induzieren Periodengitter. Zuletzt wird die Abbildung, welche diese Determinanten auf die Äquivalenzklassenmenge isomorpher Gitter abbildet, auf solche mit mehrfachen Nullstellen stetig fortgesetzt.

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Contents

1 Introduction

The elliptic, nonlinear sinh-Gordon equation

$$\Delta u + \sinh(2u) = 0$$

for twice partially differentiable functions $u : \mathbb{R}^2 \to \mathbb{R}$ has applications in the context of surfaces of constant mean curvature. Due to its algebraic structure, there are infinite-type as well as finite-type solutions. The finite-type class is parametrized in the following way: For each nonnegative integer $g \in \mathbb{N}_0$ - the so-called spectral genus - there exists a family of solutions whose complexity increases with g. The simplest case $g = 0$ corresponds only to the constant zero solution (vacuum solution). The case $g = 1$ can be expressed in terms of elliptic functions and is well-known (as the Delaunay solution). In this thesis, solutions of spectral genus $g = 2$ are investigated. Almost all of these solutions are doubly periodic and the examination of the corresponding lattices of periods forms a crucial element of this work.
Now a short overview of the following chapters is given.

The *second chapter* puts emphasis on various fundamental findings regarding both submanifolds and orbits of ordinary differential equations. This will matter tremendously during the analysis of solutions of the Lax equations on compact isospectral sets.

The *third chapter* deals with commutators of vector fields. We will figure out how a commutator is defined and that it basically is a vector field itself. Furthermore, we will prove a relationship between the commutator of two vector fields and the commutativity of the respective local flows.

In the *fourth chapter* the set of potentials is introduced as the set on which the parametrization of solutions will be defined. Moreover, we investigate the moduli space being the space of corresponding determinant polynomials.

Chapter five examines Polynomial Killing fields which are defined on the potential set from chapter four and parametrize solutions of the sinh-Gordon equation of spectral genus $g = 2$. By means of chapters two and three, we are able to make clear statements on the behavior of flows induced by the Lax equations.

In *chapter six* the isospectral sets are structurally analyzed in dependence on the roots' position of the determinant polynomials from the moduli space. Most essentially, we will investigate whether the flows induced by the Lax equations act transitively on the isospectral sets.

The *seventh chapter* deals with the lattice of periods induced by particular determinant polynomials. We understand what isomorphy of lattices means and start to analyze the mapping which transfers such determinant polynomials on the set of equivalent classes of isomorphic lattices by giving an intuition of the mapping's behavior on the boundary of its domain.

Chapter eight summarizes the most important results and gives a short outlook on possible future research topics.

2 Preliminaries

In this chapter, the main tools of the upcoming analysis are presented. One of the most important theorems of this thesis is the Implicit Function Theorem A.2, which forms a fundamental part of every *Advanced Calculus* lecture. Closely connected is the study of submanifolds as spaces where - heuristically spoken - the Implicit Function Theorem is applicable. The second part of this section quickly repeats basic results from the theory of ordinary differential equations.

2.1 Submanifolds

The main parts of this chapter originate Königsberger [1]. We denote by $\mathbb{R}^d_0 \subset \mathbb{R}^n$ the d-dimensional subspace

$$\mathbb{R}^d_0 := \{x \in \mathbb{R}^n \mid x_{d+1} = \cdots = x_n = 0\}.$$

Furthermore, for the entire chapter we denote by V, W finite-dimensional, normed $\mathbb{K}$-vector spaces.

Definition 2.1 (Submanifold)**.**
A nonempty set $M \subset V$ is called *d-dimensional differentiable submanifold of V* if, given any point $a \in M$, there is a neighbourhood $O \subset V$ of a, an open set $O' \subset \mathbb{R}^n$ and a diffeomorphism $\phi : O \to O'$ such that

$$\phi(M \cap O) = \mathbb{R}^d_0 \cap O'. \tag{2.1}$$

Such a diffeomorphism ϕ is called a *chart for M* and the set $\{\phi_i\}_{i\in I}$ of charts is called an *atlas for M* if $\{O_i\}_{i\in I}$ is a cover of M.

Obviously, the dimension d is uniquely determined since any diffeomorphism maintains dimensions. For the rest of this work, *differentiable submanifold* will be abbreviated *submanifold*.

Example 2.2.
The sphere $\mathbb{S}^{n-1}$ is a $(n-1)$-dimensional submanifold of $\mathbb{R}^n$:
Let $N = (0, \dots, 0, 1)$ be the north pole and $S = (0, \dots, 0, -1)$ the south pole. Consider the stereographic projections $\phi_K : \mathbb{R}^n \setminus K \to \mathbb{R}^{n-1}_0$ with $K \in \{S, N\}$. Stereographic projections are diffeomorphisms and thus, ϕ_N and ϕ_S are charts. Furthermore, $\mathbb{S}^{n-1} \setminus N$ and $\mathbb{S}^{n-1} \setminus S$ form a cover of $\mathbb{S}^{n-1}$, so $\{\phi_S, \phi_N\}$ is an atlas for $\mathbb{S}^{n-1}$.

The next theorem characterizes submanifolds locally as solutions of certain equation systems.

Theorem 2.3.
A nonempty subset M of a n-dimensional normed vector space V is a d-dimensional submanifold if and only if given any point $a \in M$ there is a neighbourhood $O \subset V$ of a as well as $(n-d)$ continuously differentiable functions $f_1, \dots, f_{n-d} : O \to \mathbb{R}$ such that

(i) $M \cap O = \{x \in O \mid f_1(x) = \cdots = f_{n-d}(x) = 0\}$ *and*

(ii) the differentials $df_1(a), \dots, df_{n-d}(a)$ in a are linearly independent.

Proof.
First, let M be a submanifold and $a \in M$ arbitrary. Consider the Cartesian components of the mapping $\phi = (\phi_1, \dots, \phi_n)$ and define

$$f_k := \phi_{d+k} \text{ with } k = 1, \dots, n-d.$$

Then (i) follows by (2.1).
Since ϕ is a diffeomorphism, $d\phi(a) = (d\phi_1(a), \dots, d\phi_n(a)) : V \to \mathbb{R}^n$ is an isomorphism and (ii) directly follows.
Conversely, assume the condition above is satisfied and select an arbitrary $a \in M$. Notice that $df_1(a), \dots, df_{n-d}(a) \in \mathcal{L}(V, \mathbb{R})$ and $\dim(\mathcal{L}(V, \mathbb{R})) = \dim(V)\dim(\mathbb{R}) = n$. By (ii) and Basis Extension Theorem, there exist d linear maps $l_1, \dots, l_d \in \mathcal{L}(V, \mathbb{R})$ such that

$$l_1, \dots, l_d, df_1(a), \dots, df_{n-d}(a)$$

form a basis of $\mathcal{L}(V, \mathbb{R})$. Consider the map

$$g : O \to \mathbb{R}^n, x \mapsto (l_1(x), \dots, l_d(x), f_1(x), \dots, f_{n-d}(x)).$$

By construction,

$$dg(a) : V \to \mathbb{R}, x \mapsto (l_1, \dots, l_d, df_1(a), \dots, df_{n-d}(a))(x)$$

is an isomorphism and we can apply the Inverse Function Theorem A.1 on g at the point a. Thus, there is an open set $U' \subset O$ containing a such that the restriction

$$\phi := g\big|_{U'} : U' \to O'$$

with $O' := g(U')$ is a diffeomorphism. Finally, (i) yields

$$\phi(M \cap U') = \mathbb{R}^d_0 \cap O'$$

and (2.1) is satisfied. **q.e.d.**

Definition 2.4.
$x \in O \subset V$ is called *regular point* of a differentiable map $f : O \to W$ when the differential $df(x) : V \to W$ is onto.
Furthermore, $y \in W$ is called *regular value* of f when all $x \in f^{-1}(y)$ are regular points or when the pre-image is empty.

Corollary 2.5.
Let $O \subset V$ be an open set, $f : O \to W$ a continuously differentiable map, $c \in W$ a regular value and $M := f^{-1}(c)$ its level set. If M is nonempty, then M is a submanifold of V with dimension

$$\dim M = \dim V - \dim W.$$

Proof.
We need to check the conditions of Theorem 2.3. Let $a \in M$. Since a is a regular point, the differential $df(a) : V \to W$ is a surjective and linear mapping which implies

$$\dim(\text{im}(df(a))) = \dim(W)$$

and by the Rank Theorem we have

$$\begin{aligned}\dim(V) &= \dim(\ker(df(a))) + \dim(\operatorname{im}(df(a)))\\ &= \dim(\ker(df(a))) + \dim(W)\end{aligned}$$

or simply $\dim(V) \geq \dim(W)$. Let $n := \dim V$ and $d := \dim V - \dim W \geq 0$. Notice the fact that $\dim V/\ker(df(a)) = n - d$ and with the Fundamental Theorem on Homomorphisms we obtain that there exists an isomorphism $i : W \to \mathbb{R}^{n-d}$. Consider the mapping

$$F : V \to \mathbb{R}^{n-d},\ x \mapsto i \circ f(x) - C$$

with $C := i(c)$. We define $O := V$ and focus on the Cartesian components

$$F = (F_1, \dots, F_{n-d})$$

which are continuously differentiable. Furthermore,

$$M \cap V = M = F^{-1}(0) = \{x \in V \mid F_1(x) = \cdots = F_{n-d}(x) = 0\}$$

by construction. Last but not least, given any $a \in M$ the differential $dF(a) = i \circ df(a)$ is surjective, so the components $dF_1, \dots, dF_{n-d}$ are linearly independent. **q.e.d.**

Definition 2.6.
Let $M \subset V$ be a nonempty subset. A vector $v \in V$ is called *tangent vector of M in the point $a \in M$* if there exists a continuously differentiable curve

$$\alpha : (-\varepsilon, \varepsilon) \to M$$

with $\varepsilon > 0$, $\alpha(0) = a$ and $\alpha'(0) = v$.
The set of tangent vectors of M in the point $a \in M$ is called *tangent cone of M in a* and will be denoted as T_aM.
In addition, if T_aM is a vector space, we call it *tangent space*.

Theorem 2.7.
Let M be a d-dimensional differentiable submanifold of V. Then given any arbitrary point $a \in M$ the following statements hold:

(i) T_aM is a $\mathbb{R}$-vector space of dimension d.

(ii) If there is a continuously differentiable function $f : O \to W$ on an open set $O \subset V$ and a regular value $c \in W$ with pre-image $M = f^{-1}(c)$, then

$$T_aM = \ker df(a).$$

In particular, when $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$,

$$T_aM = \{v \in \mathbb{R}^n \mid f'(a)v = 0\}.$$

Proof.
For (i), consider first the simplest d-dimensional submanifold $\tilde{M} = \mathbb{R}^d_0 \cap O'$ for $O' \subset \mathbb{R}^n$ open and $\tilde{a} \in \tilde{M}$. Obviously,

$$T_{\tilde{a}}(\mathbb{R}^d_0 \cap O') = \mathbb{R}^d_0 \tag{2.2}$$

and therefore, (i) holds. Regarding the general case, take a chart $\phi : O \to O'$ with the notation from Definition 2.1 and notice that ϕ establishes a one-to-one relationship between the curve $\alpha : (-\varepsilon, \varepsilon) \to M \cap O$ of M and the curve $\tilde{\alpha} := \phi \circ \alpha$ of $\tilde{M}$. Making use of the chain rule gives

$$\alpha'(0) = (d\phi(a))^{-1}\ \tilde{\alpha}'(0)$$

and consequently with (2.2)

$$T_a M = T_a(M \cap O) = (d\phi(a))^{-1}\ T_{\phi(a)}(\mathbb{R}^d_0 \cap O') = (d\phi(a))^{-1}\ \mathbb{R}^d_0.$$

This proves (i).
For (ii), consider a curve $\alpha : (-\varepsilon, \varepsilon) \to M$ and recognize the fact that by assumption, $f \circ \alpha = c$ and therefore, $df(a)\alpha'(0) = 0$. Hence,

$$T_a M \subset \ker df(a).$$

Since c is a regular value, $df(a) : V \to W$ maps surjectively and therefore with the help of the Rank-Nullity Theorem as well as Corollary 2.5 and (i) we obtain

$$\begin{aligned} \dim \ker df(a) &= \dim V - \dim(\operatorname{im}(df(a))) \\ &= \dim V - \dim W \\ &= \dim M \\ &= d \\ &= \dim T_a M \end{aligned}$$

and (ii) is proven. **q.e.d.**

2.2 Orbits of Ordinary Differential Equations

This subsection presents some useful facts which will be essential during our analysis of the Lax equations. The proof of the Picard-Lindelöf Theorem is presented in its full length since some arguments work analogously in the ensuing proposition. This proposition provides information about the solution's behaviour close to the maximal interval's boundaries. Main parts of the argumentation originate from Barreira et al. [6].

Theorem 2.8 (Picard-Lindelöf**).**
Let $f : D \to \mathbb{R}^n$ be a continuous function on an open set $D \subset \mathbb{R} \times \mathbb{R}^n$ which is locally Lipschitz continuous with respect to the second variable. Then for any point $(t_0, x_0) \in D$, there exists a unique solution of the initial value problem

$$\begin{aligned} x'(t) &= f(t, x), \\ x(t_0) &= x_0 \end{aligned}$$

in some open interval containing t_0.

Proof.
Let $C(a, b)$ be the set of all bounded continuous functions $y : (a, b) \to \mathbb{R}^n$. We need to find

a uniquely defined function $x \in C(a,b)$ in an appropriate open interval (a,b) containing t_0 such that

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds \tag{2.3}$$

for every $t \in (a,b)$. As this will be done with Banach's Fixed Point Theorem, we need to construct a metric space (X,d) such that the transformation of the right hand side of (2.3) maps X to itself and is a contraction. Choose constants $a < t_0 < b$ and $\beta > 0$ such that the compact set satisfies

$$K := [a,b] \times \overline{B(x_0,\beta)} \subset D.$$

Moreover, we define

$$X := \{x \in C(a,b) \mid x(t) \in \overline{B(x_0,\beta)} \text{ for } t \in (a,b)\} \subset C(a,b),$$

and show that X is a complete metric space with distance

$$d(x,y) = \sup\{\|x(t) - y(t)\| : t \in (a,b)\}.$$

Take any Cauchy sequence $(x_k)_{k\in\mathbb{N}}$ in X. Since $(C(a,b),d)$ is a complete metric space, it converges to a function $x \in C(a,b)$ and even $x \in X$ holds because

$$\|x(t) - x_0\| = \lim_{p\to\infty} \|x_k(t) - x_0\| \leq \beta.$$

As mentioned before, we want to apply Banach Fixed-Point Theorem on X. Consider the continuous transformation

$$T(x)(t) := x_0 + \int_{t_0}^{t} f(s, x(s))ds$$

and notice

$$\|T(x)(t) - x_0\| \leq \| \int_{t_0}^{t} f(s, x(s))ds\| \leq (b-a)M$$

with

$$M := \max\{\|f(t,x) \mid (t,x) \in K\|\} < \infty$$

by compactness of the set K and continuity of the function f. Thus, for $b - a$ sufficiently small we have $(b-a)M \leq \beta$ and $T(X) \subset X$. Furthermore, given $x, y \in X$, locally Lipschitz continuity yields

$$\|T(x)(t) - T(y)(t)\| \leq \int_{t_0}^{t} L\|x(s) - y(s)\|ds \leq (b-a)Ld(x,y),$$

where L is the Lipschitz constant for the compact set K. Therefore,

$$d(T(x), T(y)) \leq (b-a)Ld(x,y)$$

for all $x, y \in X$ and with $b - a$ sufficiently small we have

$$(b-a)L < 1$$

in addition to $(b-a)M \leq \beta$ and T is a contraction in the complete metric space X. By Banach Theorem, we conclude that T has a unique fixed point $x \in X$. **q.e.d.**

Proposition 2.9.
Let $f : D \to \mathbb{R}^n$ be a continuous function on an open set $D \subset \mathbb{R} \times \mathbb{R}^n$ which is locally Lipschitz continuous with respect to the second variable. If a solution $x(t)$ of the equation $x'(t) = f(t,x)$ has maximal interval $(a,b) \subset \mathbb{R}$, then for each compact set $K \subset D$ there exists $\varepsilon > 0$ such that

$$(t, x(t)) \in D \setminus K \quad \text{for every } t \in (a, a+\varepsilon) \cup (b-\varepsilon, b)$$

(when $a = -\infty$ the first interval is empty and when $b = \infty$ so is the second).

Proof (Sketch).
We consider only the endpoint b since the argument for a is entirely analogous. We proceed by contradiction, so we assume that for some compact set $K \subset D$ there exists a sequence $(t_p)_{p\in\mathbb{N}}$ in $\mathbb{R}$ with $t_p \uparrow b$ for $p \to \infty$ and

$$(t_p, x(t_p)) \in K \quad \text{for every } p \in \mathbb{N}.$$

Since K is a compact subset of $\mathbb{R} \times \mathbb{R}^n$, any sequence in K has a convergent subsequence with limit in K. In particular, there exists a subsequence $(t_{p_k})_{k\in\mathbb{N}}$ and a point $(b, x_0) \in K$ such that

$$\lim_{k\to\infty} (t_{p_k}, x(t_{p_k})) = (b, x_0).$$

Take $\alpha, \beta > 0$ such that the following compact set satisfies

$$K_{\alpha\beta} := [b-\alpha, b+\alpha] \times \overline{B(x_0, \beta)} \subset D$$

and define the finite real number

$$M := \sup\{\|f(t,x)\| : (t,x) \in K_{\alpha\beta}\} < \infty.$$

As in the proof of Picard-Lindelöf, there exist $\alpha, \beta > 0$ with

$$2M\alpha \leq \beta.$$

Moreover, for each $k \in \mathbb{N}$ we consider the compact set

$$L_k := \left[t_{p_k} - \frac{\alpha}{2}, t_{p_k} + \frac{\alpha}{2}\right] \times \overline{B\left(x(t_{p_k}), \frac{\beta}{2}\right)}$$

and notice the fact that by construction $L_k \subset K_{\alpha\beta}$ for sufficiently small p holds true. Therefore,

$$2\sup\{\|f(t,x)\| : x \in L_k\}\frac{\alpha}{2} \leq 2M\frac{\alpha}{2} \leq \frac{\beta}{2}.$$

In complete analogy to the proof of Picard-Lindelöf, one can show that there exists a unique solution

$$y : (t_{p_k} - \frac{\alpha}{2}, t_{p_k} + \frac{\alpha}{2}) \to \mathbb{R}^n$$

of the initial value problem

$$y'(t) = f(t,y)$$
$$y(t_{p_k}) = x(t_{p_k}).$$

Since $t_{p_k} + \frac{\alpha}{2} > b$ for p sufficiently large, this means we have found an extension of the solution x to the interval $(a, t_{p_k} + \frac{\alpha}{2})$. This contradicts the fact that b is the right bound of the maximal interval. **q.e.d.**

Theorem 2.10 (Criterion for Global Solutions).
Let $f : D \to \mathbb{R}^n$ be a continuous function on an open set $D \subset \mathbb{R}^n$ which is locally Lipschitz continuous and consider the autonomous equation

$$x'(t) = f(x(t)). \tag{2.4}$$

Then any solution of this equation whose orbit is contained in a compact subset of D is global.

Proof.
Let $F : \mathbb{R} \times D \to \mathbb{R}^n$ be the function $F(t,x) = f(x)$. Then we can rewrite (2.4) in the form

$$x'(t) = F(t, x(t)).$$

By Picard-Lindelöf Theorem 2.8, there exists a unique solution of the respective initial value problem in some open intervall $(a,b) \subset \mathbb{R}$ comprising t_0. We want to figure out how this maximal interval looks like. Given an orbit $\{x(t) \,|\, t \in (a,b)\}$ of a solution $x(t)$ which is contained in some compact set $K \subset D$ and $m \in \mathbb{R}$ consider the compact set

$$K_m := [-m, m] \times K \subset \mathbb{R} \times D.$$

From Proposition 2.9 follows that there exists $\varepsilon_m > 0$ such that

$$(t, x(t)) \notin K_m \quad \text{for every } t \in (a, a+\varepsilon_m) \cap (b - \varepsilon_m, b).$$

Hence, for any given $m \in \mathbb{R}$ we have found some $t \in (a,b)$ so that one of the following scenarios holds true:

(i) $t \notin [-m, m]$ and $x(t) \in K$

(ii) $t \in [-m, m]$ and $x(t) \notin K$

(iii) $t \notin [-m, m]$ and $x(t) \notin K$.

But the cases (ii) and (iii) are impossible due to

$$\{x(t) \,|\, t \in (a,b)\} \subset K.$$

Consequently, for each $m \in \mathbb{R}$ there exists $\varepsilon_m > 0$ such that

$$t \notin [-m, m] \quad \text{for all } t \in (a, a+\varepsilon_m) \cap (b - \varepsilon_m, b).$$

Regarding $m \to \infty$ we conclude that $a = -\infty$ and $b = \infty$. **q.e.d.**

3 Commutators of Vector Fields

In the upcoming chapters we will be confronted with some ordinary differential equations of the form

$$\frac{\partial u}{\partial x} = [E, F](x).$$

Therefore, we need to understand what the right hand side means and how a commutator of vector fields is defined - is it a vector field itself and what properties does it have? All of these questions will be answered in this section without using manifold terminology. The main results are to a large extent extracted from Schmidt [8].

Definition 3.1.
A *vector field* on an open subset $\Omega \subset \mathbb{R}^n$ is a vector-valued map $E : \Omega \to \mathbb{R}^n$. Given a vector field E, a point $x_0 \in \Omega$ and an open interval $I_{x_0} \subset \mathbb{R}$ containing zero, a parametric curve

$$x : I_{x_0} \to \Omega,\ t \mapsto x(t) = (x_1(t), \dots, x_n(t))$$

with Cartesian coordinates is called *integral curve of E* passing through $x_0 \in \Omega$ if it satisfies $x(0) = x_0$ and if it is a solution of the autonomous system of differential equations

$$x'(t) = E(x(t)). \tag{3.1}$$

A continuously differentiable vector field $E : \Omega \to \mathbb{R}^n$ is Lipschitz continuous and therefore induces for $x_0 \in \Omega$ a uniquely defined maximal integral curve by Picard Lindelöf 2.8 (in the sense of I_{x_0} being maximal). We can assign two objects to such a vector field:

i) The *local flow* ϕ_E *of* E is defined on

$$W_E := \bigcup_{x_0 \in \Omega} I_{x_0} \times \{x_0\} \subset \mathbb{R} \times \Omega$$

and describes maximal integral curves

$$\phi_E : W_E \to \mathbb{R}^n,\ \phi_E(t, x_0) := x(t).$$

In particular, a local flow satisfies

$$\frac{\partial \phi_E}{\partial t}(t, x_0) = E(\phi_E(t, x_0)). \tag{3.2}$$

When $I_{x_0} = \mathbb{R}$ for all $x_0 \in \Omega$, thus $W_E = \mathbb{R} \times \Omega$, the local flow is global.

ii) The first-order linear *differential operator* $L_E : C^1(\Omega) \to C(\Omega)$ computes the directional derivative along E:

$$(L_E f)(x) = E(x) \cdot \nabla f(x) = \sum_{i=1}^{n} E_i(x) \frac{\partial f(x)}{\partial x_i}.$$

A quick calculation shows that this operator satisfies the product rule

$$L_E(fg) = (L_E f)g + f(L_E g).$$

Two continuously differentiable vector fields E, F on $\Omega \subset \mathbb{R}^n$ have commuting local flows ϕ_E, ϕ_F if

$$\phi_E(s, \phi_F(t, x)) = \phi_F(t, \phi_E(s, x))$$

for all $s, t \in \mathbb{R}$, $x \in \Omega$ such that $(t, x) \in W_F$, $(s, \phi_F(t, x)) \in W_E$ and $(s, x) \in W_E$, $(t, \phi_E(s, x)) \in W_F$. Notice the fact that this set is always a neighbourhood of $\{0\} \times \{0\} \times \Omega$.
In order to measure the degree of non-commutativity between the local flows ϕ_E and ϕ_F we look at a smooth function f on Ω and for appropriate s, t, x the difference is

$$\triangle_f(s, t, x) := f(\phi_E(s, \phi_F(t, x)) - f(\phi_F(t, \phi_E(s, x)).$$

Obviously, $\triangle_f$ is a differentiable function which equals zero whenever $s = 0$ or $t = 0$ and consequently, at $(s, t) = (0, 0)$ the first partial derivatives with respect to s or t disappear. This justifies having a closer look at second partial derivatives.

Lemma 3.2.
The mixed partial derivative of $\triangle_f$ with respect to s and t at $(s, t) = (0, 0)$ equals the commutator of the corresponding differential operators applied on f:

$$\frac{\partial^2}{\partial t \partial s}\bigg|_{s=t=0} \triangle_f(s, t, x) = (L_F L_E f - L_E L_F f)(x).$$

Proof.
Calculation yields

$$\begin{aligned}
\frac{\partial}{\partial s}\bigg|_{s=0} f(\phi_E(s, \phi_F(t, x)) &= \left(\nabla f(\phi_E(s, \phi_F(t, x))) \cdot \frac{\partial \phi_E}{\partial s}(s, \phi_F(t, x))\right)\bigg|_{s=0} \\
&= (\nabla f(\phi_E(s, \phi_F(t, x))) \cdot E(\phi_E(s, \phi_F(t, x))))\bigg|_{s=0} \\
&= \nabla f(\phi_F(t, x)) \cdot E(\phi_F(t, x)) \\
&= (L_E f)(\phi_F(t, x)).
\end{aligned}$$

The third equation holds due to the fact that integral curves solve the system of ordinary differential equations (3.2). For reasons of simplicity, we substitute the function $L_E f$ by g and obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\bigg|_{t=0} (L_E f)(\phi_F(t, x)) &= \frac{\partial}{\partial t}\bigg|_{t=0} g(\phi_F(t, x)) \\
&= \left(\nabla g(\phi_F(t, x)) \cdot \frac{\partial \phi_F}{\partial t}(t, x)\right)\bigg|_{t=0} \\
&= (\nabla g(\phi_F(t, x)) \cdot F(\phi_F(t, x)))\bigg|_{t=0} \\
&= (L_F g)(\phi_F(t, x))\bigg|_{t=0} \\
&= (L_F L_E f)(x).
\end{aligned}$$

In total, we obtain

$$\frac{\partial^2}{\partial t \partial s}\bigg|_{s=t=0} f(\phi_E(s, \phi_F(t, x)) = (L_F L_E f)(x).$$

By Schwarz' Theorem, the mixed partial derivative of the other term can be calculated in reverse order such that the calculation works accordingly. **q.e.d.**

With other words, we established a relationship between the (non-)commutativity of local flows and above mentioned differential operators. In the following, we want to specify this relationship.
Inspired by Lemma 3.2, we define the commutator of operators

$$[L_F, L_E] := L_F L_E - L_E L_F$$

which, at first sight, wrongly seems to be a second-order operator.

Lemma 3.3.
$L_F L_E - L_E L_F$ is a first-order linear differential operator and satisfies the product rule.

Proof.
Let $E = (E_1, \dots, E_n)$ and $F = (F_1, \dots, F_n)$ the Cartesian components of the vector fields E, F on the open set $\Omega \subset \mathbb{R}^n$. Product rule yields

$$L_F L_E f = \sum_{i=1}^{n} F_i \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{n} E_j \frac{\partial}{\partial x_j} f \right) = \sum_{i,j=1}^{n} F_i \frac{\partial E_j}{\partial x_i} \frac{\partial}{\partial x_j} f + \sum_{i,j=1}^{n} F_i E_j \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

When subtracting $L_E L_F$ from this expression the last sum disappears by Schwarz' Theorem and we obtain

$$(L_F L_E - L_E L_F) f = \sum_{i,j=1}^{n} \left(F_i \frac{\partial E_j}{\partial x_i} - E_i \frac{\partial F_j}{\partial x_i} \right) \frac{\partial f}{\partial x_j}.$$

Thus, $[L_F, L_E]$ equals a first-order differential operator L_G with $G(x) = E'(x)F(x) - F'(x)E(x)$. In particular, the commutator is linear and the product rule holds. **q.e.d.**

Remark 3.4.
More generally, one can quickly show by calculation that if any two differential operators satisfy the product rule, the corresponding commutator also satisfies the product rule.

Definition 3.5.
The vector field

$$G : \Omega \to \mathbb{R}^n,\ x \mapsto G(x) = E'(x)F(x) - F'(x)E(x)$$

with $L_G = [L_F, L_E]$ is called *commutator* of the vector fields F and E and will be denoted $[E, F]$. The operation $[\cdot, \cdot]$ is called *Lie Bracket*.

Lemma 3.6 (Properties of the Lie Bracket)**.**
Let E, F and D be sufficiently differentiable vector fields defined on an open subset $\Omega \subset \mathbb{R}^n$. The Lie Bracket satisfies the following identities:

(a) Bilinearity: For $a, b \in \mathbb{R}$,

$$[aE + bF, D] = a[E, D] + b[F, D]$$
$$[D, aE + bF] = a[D, E] + b[D, F]$$

(b) *Antisymmetry:*

$$[E,F] = -[F,E]$$

(c) *Jacobi-Identitiy:*

$$[E,[F,D]] + [F,[D,E]] + [D,[E,F]] = 0.$$

Proof.
(a) and (b) are obvious consequences of the definition and can be proven quickly using linearity of differentiation.
For Jacobi-Identity, the corresponding differential operators need to be taken into consideration. Linearity of differentiation yields

$$L_{E+F+D} = L_E + L_F + L_D,$$

so we are allowed to look at the single differential operators and add them afterwards. The definition of the Lie Bracket results in

$$\begin{aligned} L_{[E,[F,D]]} &= [L_E, L_{[F,D]}] = [L_E, L_F L_D - L_D L_F] \\ &= L_E L_F L_D - L_E L_D L_F - L_F L_D L_E + L_D L_F L_E \end{aligned}$$

and analogously,

$$\begin{aligned} L_{[F,[D,E]]} &= L_F L_D L_E - L_F L_E L_D - L_D L_E L_F + L_E L_D L_F \\ L_{[D,[E,F]]} &= L_D L_E L_F - L_D L_F L_E - L_E L_F L_D + L_F L_E L_D. \end{aligned}$$

The sum of these three terms equals zero. Consequently, the differential operator and, respectively, the vector field disappear. **q.e.d.**

Remark 3.7.
Lemma 3.6 demonstrates that the space of first order linear vector fields on an open subset $\Omega \subset \mathbb{R}$ together with the commutator $[\cdot,\cdot]$ form a *Lie Algebra*.

Theorem 3.8.
Let E and F be continuously differentiable vector fields defined on an open subset $\Omega \subset \mathbb{R}^n$. The local flows ϕ_E and ϕ_F commute if and only if the commutator $[E,F]$ disappears.

Proof.
When the flows commute, $\triangle_f$ equals zero for all (s,t,x) such that $\triangle_f$ is well-defined. In particular, the second mixed partial derivative with respect to s and t in $(s,t) = (0,0)$ equals zero as well as $[L_F, L_E]$ by Lemma 3.2. By definition of the vector field commutator,

$$[E,F] = 0$$

holds.
Conversely, assume that $[E,F]$ disappears. We define for $(s,x) \in W_E$ and sufficiently small $t \in \mathbb{R}$ the differentiable local flow

$$\phi_s : (t,x) \mapsto \phi_E(-s, \phi_F(t, \phi_E(s,x))).$$

Notice that t needs to be chosen small enough in order to obtain

$$(t, \phi_E(s,x)) \in W_F \text{ and } (-s, \phi_F(t, \phi_E(s,x))) \in W_E.$$

This is possible since

$$(0, \phi_E(s,x)) \in W_F \text{ and } (-s, \phi_F(0, \phi_E(s,x))) \in W_E.$$

Any continuously differentiable local flow of this form belongs to a vector field, so denote by B_s the vector field associated to ϕ_s. We want to determine what B_s looks like by using the fact that the flow ϕ_s solves (3.2). In our case, this means

$$\frac{\partial \phi_s}{\partial t}(t,x) = B_s(\phi_s(t,x)).$$

By setting $t = 0$ we obtain the explicit form using the chain rule

$$\begin{aligned} B_s(x) &= \frac{\partial \phi_s}{\partial t}(t,x)\Big|_{t=0} \\ &= \frac{\partial}{\partial t}\Big|_{t=0} \phi_E(-s, \phi_F(t, \phi_E(s,x))) \\ &= \phi_E'(-s, \phi_F(t, \phi_E(s,x)))\Big|_{t=0} \frac{\partial}{\partial t}\Big|_{t=0} \phi_F(t, \phi_E(s,x)) \\ &= \phi_E'(-s, \phi_E(s,x))\, F(\phi_E(s,x)). \end{aligned}$$

Here - and for the remaining part of the proof - ϕ_E' denotes the derivative with respect to the space coordinate.
Our next goal is to differentiate for a given $x \in \Omega$ the map $s \mapsto B_s(x)$. In doing so, it is quite challenging to differentiate the part $\phi_E'(-s, \phi_E(s,x))$ directly. An elegant bypass of this problem is to employ the convenient property of

$$\mathbb{R}^n \to \mathbb{R}^n,\ w \mapsto \phi_E'(-s, \phi_E(s,x))w$$

being a linear map. More precisely, we avoid the difficulties in calculation by differentiating its inverse mapping and then use differentiation properties of linear mappings in order to deduce the desired derivative. To find the inverse function, we look at $\phi_E(-s, \phi_E(s,x)) = x$, differentiate both sides with respect to x and obtain using chain rule

$$\phi_E'(-s, \phi_E(s,x))\phi_E'(s,x) = 1. \tag{3.3}$$

Hence, the inverse function of $\phi_E'(-s, \phi_E(s,x))$ must be $\phi_E'(s,x)$. Applying Schwarz' Theorem, differentiation with respect to s yields

$$\frac{\partial \phi_E'}{\partial s}(s,x) = \frac{\partial^2 \phi_E}{\partial s \partial x}(s,x) = \frac{\partial^2 \phi_E}{\partial x \partial s}(s,x) = \frac{\partial E(\phi_E(s,x))}{\partial x} = E'(\phi_E(s,x))\phi_E'(s,x).$$

Again, the third equality holds by (3.2) and the fourth by the chain rule.

As a quick excursion, we consider a linear mapping $A = A(s)$ depending on s and its inverse A^{-1}. Differentiation on both sides of $AA^{-1} = 1$ leads to

$$(\frac{\partial}{\partial s}A)A^{-1} + A(\frac{\partial}{\partial s}A^{-1}) = 0$$

and thus,

$$(\frac{\partial}{\partial s}A) = -A(\frac{\partial}{\partial s}A^{-1})A.$$

We apply this relationship on our case (3.3) and obtain

$$\begin{aligned}\frac{d}{ds}\phi_E'(-s,\phi_E(s,x)) &= -\phi_E'(-s,\phi_E(s,x))E'(\phi_E(s,x))\phi_E'(s,x)\phi_E'(-s,\phi_E(s,x))\\ &= -\phi_E'(-s,\phi_E(s,x))E'(\phi_E(s,x)).\end{aligned}$$

Then we can calculate the derivative of B_s with respect to s

$$\begin{aligned}\frac{d}{ds}B_s(x) &= -\phi_E'(-s,\phi_E(s,x))E'(\phi_E(s,x))F(\phi_E(s,x))\\ &\quad \phi_E'(-s,\phi_E(s,x))F'(\phi_E(s,x))E(\phi_E(s,x))\\ &= -\phi_E'(-s,\phi_E(s,x))[E,F](\phi_E(s,x))\\ &= 0.\end{aligned}$$

The last equality is valid since by assumption the commutator $[E,F]$ disappears. Thus, the mapping $s \mapsto B_s(x)$ remains constant for all $x \in \Omega$. Setting $s = 0$ yields $\phi_0 = \phi_F$, so $B_0 = F$ and consequently, B_s equals F for all s. In particular, both local flows coincide for sufficiently small s:

$$\phi_E(-s,\cdot) \circ \phi_F(t,\cdot) \circ \phi_E(s,\cdot) = \phi_s(t,\cdot) = \phi_F(t,\cdot).$$

We conclude by linking $\phi_E(s,\cdot)$ on both sides

$$\phi_F(t,\cdot) \circ \phi_E(s,\cdot) = \phi_E(s,\cdot) \circ \phi_F(t,\cdot).$$

q.e.d.

4 Potentials

During the next two chapters we will become familiar with the mentioned spectral genus $g = 2$ family of solutions of the sinh-Gordon equation.
Before we turn towards the parametrization of solutions, we first analyze properties of the set it will be defined on by introducing algebraic data. This set will be the set of so-called potentials. Potentials have interesting symmetry-features which will be a subject of discussion throughout the following pages.

Definition 4.1 (Potentials).
The set of potentials is the following set of cubic polynomials with matrix-valued coefficients:

$$\mathcal{P}_2 := \left\{ \zeta_\lambda = \begin{pmatrix} 0 & -\gamma^{-1} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \lambda + \begin{pmatrix} -\bar{\alpha} & -\gamma \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \lambda^2 \right.$$
$$\left. + \begin{pmatrix} 0 & 0 \\ \gamma^{-1} & 0 \end{pmatrix} \lambda^3 \,\middle|\, \alpha, \beta \in \mathbb{C}, \gamma \in \mathbb{R}^+ \right\}$$

where $\lambda \in \mathbb{C}$ is called the *spectral parameter*.

Every $\zeta_\lambda \in \mathcal{P}_2$ can be compactly written as

$$\zeta_\lambda = \begin{pmatrix} \alpha\lambda - \bar{\alpha}\lambda^2 & -\gamma^{-1} + \beta\lambda - \gamma\lambda^2 \\ \gamma\lambda - \bar{\beta}\lambda^2 + \gamma^{-1}\lambda^3 & -\alpha\lambda + \bar{\alpha}\lambda^2 \end{pmatrix} \tag{4.1}$$

and satisfies the *reality condition*

$$\begin{aligned} \lambda^3 \,\overline{\zeta}^{\,t}_{1/\bar{\lambda}} &= \lambda^3 \begin{pmatrix} \bar{\alpha}\lambda^{-1} - \alpha\lambda^{-2} & \gamma\lambda^{-1} - \beta\lambda^{-2} + \gamma^{-1}\lambda^{-3} \\ -\gamma^{-1} + \bar{\beta}\lambda^{-1} - \gamma\lambda^{-2} & -\bar{\alpha}\lambda^{-1} + \alpha\lambda^{-2} \end{pmatrix} \\ &= -\zeta_\lambda. \end{aligned} \tag{4.2}$$

Sometimes it will be useful to write ζ_λ abstractly as $\begin{pmatrix} A(\lambda) & B(\lambda) \\ \lambda C(\lambda) & -A(\lambda) \end{pmatrix}$ with complex polynomials $A(\lambda), B(\lambda), C(\lambda)$ of maximal degree two. Then it follows due to the reality condition

$$\begin{aligned} \lambda^3\, \overline{A(\bar{\lambda}^{-1})} &= -A(\lambda) \\ \lambda^2\, \overline{B(\bar{\lambda}^{-1})} &= -C(\lambda) \\ \lambda^2\, \overline{C(\bar{\lambda}^{-1})} &= -B(\lambda). \end{aligned}$$

Now we are interested in the determinant of $\zeta_\lambda \in \mathcal{P}_2$:

$$\begin{aligned} \det \zeta_\lambda &= -A^2(\lambda) - \lambda B(\lambda) C(\lambda) \\ &= -\lambda^2(\alpha - \bar{\alpha}\lambda)^2 - \lambda(-\gamma^{-1} + \beta\lambda - \gamma\lambda^2)(\gamma - \bar{\beta}\lambda + \gamma^{-1}\lambda^2) \\ &= -\lambda^2(\alpha^2 - 2\alpha\bar{\alpha}\lambda + \bar{\alpha}^2\lambda^2) - \lambda(\beta\gamma\lambda - \bar{\beta}\beta\lambda^2 + \beta\gamma^{-1}\lambda^3 \\ &\quad - \gamma^2\lambda^2 + \bar{\beta}\gamma\lambda^3 - \lambda^4 - 1 + \bar{\beta}\gamma^{-1}\lambda - \gamma^{-2}\lambda^2) \\ &= \lambda\big[\lambda^4 + (-\bar{\alpha}^2 - \beta\gamma^{-1} - \bar{\beta}\gamma)\lambda^3 + (2\alpha\bar{\alpha} + \beta\bar{\beta} + \gamma^2 + \gamma^{-2})\lambda^2 \\ &\quad + (-\alpha^2 - \bar{\beta}\gamma^{-1} - \beta\gamma)\lambda + 1\big] \\ &=: \lambda a(\lambda). \end{aligned}$$

By the Fundamental Theorem of Algebra, $a(\lambda)$ has four (possibly multiple) roots in $\mathbb{C} \setminus \{0\}$. Moreover, the highest and lowest coefficient equal one and therefore, the roots' product equals one. This can quickly be verified if we consider $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C}$ to be the roots of $a(\lambda)$, compute

$$a(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)$$

and verify after explicitly calculating the products that the lowest coefficient equals

$$\lambda_1 \lambda_2 \lambda_3 \lambda_4 \overset{!}{=} 1.$$

Notice that the polynomial $a(\lambda)$ takes the following form

$$a(\lambda) = \lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + \bar{a}_1 \lambda + 1 \tag{4.3}$$

with

$$\begin{aligned} a_1 &= -\bar{\alpha}^2 - \beta\gamma^{-1} - \bar{\beta}\gamma && \in \mathbb{C} \\ a_2 &= 2\alpha\bar{\alpha} + \beta\bar{\beta} + \gamma^2 + \gamma^{-2} && \in \mathbb{R}. \end{aligned}$$

Apart from this structural symmetry, the polynomial $a(\lambda)$ bears even more nice features due to the properties of ζ_λ. Consider the following calculation using reality condition of ζ_λ:

$$\lambda a(\lambda) = \det \zeta_\lambda = \det(-\lambda^3 \, \overline{\zeta^t_{1/\bar{\lambda}}}) = \lambda^6 \, \overline{\det \zeta_{1/\bar{\lambda}}} = \lambda^6 \, \overline{\bar{\lambda}^{-1} a(\bar{\lambda}^{-1})} = \lambda^5 \, \overline{a(\bar{\lambda}^{-1})}.$$

So the $a(\lambda)$ inherits the *reality condition*

$$a(\lambda) = \lambda^4 \, \overline{a(\bar{\lambda}^{-1})}. \tag{4.4}$$

In particular, if λ_0 is a root of $a(\lambda)$ another root is given by $\bar{\lambda}_0^{-1}$.
It is obvious that, in general, $a(\lambda)$ assumes values in the complex plane. However, again by taking advantage of the suitably structured ζ_λ, we will demonstrate that

$$\lambda^{-2} a(\lambda) \geq 0 \quad \text{for } \lambda \in \mathcal{S}^1$$

holds. To do so, we point out some facts from Linear algebra.

Definition 4.2.
A square matrix A is called *skew-Hermitian* if its conjugate transpose equals its negative:

$$\overline{A}^t = -A.$$

Consider A to be skew-Hermitian, a an eigenvector and λ the respective eigenvalue. Without loss of generality, we assume $\bar{a}^t a = 1$ (otherwise we could take $\frac{a}{\|a\|}$ instead of a). Now we conduct a simple calculation

$$\bar{\lambda} = \bar{\lambda}\bar{a}^t a = \bar{a}^t \bar{\lambda} a = \overline{\bar{a}^t \lambda a}^t = \bar{a}^t \bar{A}^t a = \bar{a}^t(-A)a = -\bar{a}^t \lambda a = -\lambda$$

and infer that skew-Hermitian matrices have purely imaginary eigenvalues.
Now consider the very specific case of A being 2×2, skew-Hermitian and traceless. Since

the trace equals the sum of its eigenvalues, there exists a real number $k \in \mathbb{R}$ such that the eigenvalues are

$$\lambda_1 = ik,\ \lambda_2 = -ik.$$

Consequently, the determinant is real and non-negative, because

$$\det A = \lambda_1 \lambda_2 = ik\,(-ik) = k^2 \geq 0.$$

Now we are prepared to demonstrate the assertion above.
Obviously,

$$\lambda^{-\frac{3}{2}}\zeta_\lambda = \begin{pmatrix} \alpha\lambda^{-\frac{1}{2}} - \bar{\alpha}\lambda^{\frac{1}{2}} & -\gamma^{-1}\lambda^{-\frac{3}{2}} + \beta\lambda^{-\frac{1}{2}} - \gamma\lambda^{\frac{1}{2}} \\ \gamma\lambda^{-\frac{1}{2}} - \bar{\beta}\lambda^{\frac{1}{2}} + \gamma^{-1}\lambda^{\frac{3}{2}} & -\alpha\lambda^{-\frac{1}{2}} + \bar{\alpha}\lambda^{\frac{1}{2}} \end{pmatrix}$$

is two-by-two, traceless and, for $\lambda \in \mathcal{S}^1$, also skew-Hermitian. Due to the previous preconsiderations, $\det(\lambda^{-\frac{3}{2}}\zeta_\lambda) \geq 0$ for $\lambda \in \mathcal{S}^1$ follows directly. As a result, for $\lambda \in \mathcal{S}^1$

$$0 \leq \det(\lambda^{-\frac{3}{2}}\zeta_\lambda) = \lambda^{-3}\det(\zeta_\lambda) = \lambda^{-2}a(\lambda)$$

holds. These properties are not only necessary for $\zeta_\lambda \in \mathcal{P}_2$, but even sufficient and fully characterize $a(\lambda)$, as shown in the upcoming

Theorem 4.3.
The above features fully characterize the determinant-polynomials $a(\lambda)$. This means, the following sets are actually the same:

$$\begin{aligned} \mathcal{M}_2 :&= \{a \in \mathbb{C}^4[\lambda] \mid \lambda a(\lambda) = \det(\zeta_\lambda) \text{ for a } \zeta_\lambda \in \mathcal{P}_2\} \\ &= \{a \in \mathbb{C}^4[\lambda] \mid a(0) = 1,\ \lambda^4\overline{a(\bar{\lambda}^{-1})} = a(\lambda),\ \lambda^{-2}a(\lambda) \geq 0 \text{ for } \lambda \in \mathcal{S}^1\}. \end{aligned}$$

Proof.
After the above findings, it remains to show $\supset$. To do so, given any $a(\lambda)$ from the lower set, we need to find appropriate $(\alpha, \beta, \gamma) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}^+$, such that the respective $\zeta_\lambda \in \mathcal{P}_2$ satisfies $\det(\zeta_\lambda) = \lambda a(\lambda)$. Since $a(0) = 1$ and $\lambda^4\overline{a(\bar{\lambda}^{-1})} = a(\lambda)$ the polynomial $a(\lambda)$ is uniquely defined by the two roots $\lambda_1, \lambda_2 \in \mathbb{C}$, namely

$$a(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \bar{\lambda}_1^{-1})(\lambda - \bar{\lambda}_2^{-1}). \tag{4.5}$$

Notice that $a(0) = 1$ implies $\lambda_1\lambda_2\bar{\lambda}_1^{-1}\bar{\lambda}_2^{-1} = 1$ and therefore

$$(\lambda_1\bar{\lambda}_2^{-1})^{-1} = \lambda_2\bar{\lambda}_1^{-1} \tag{4.6}$$

$$\bar{\lambda}_1\lambda_1^{-1} = \lambda_2\bar{\lambda}_2^{-1} \tag{4.7}$$

holds. Choose $\alpha = 0$. Then, given β and γ, the ζ_λ should finally look like

$$\zeta_\lambda = \begin{pmatrix} 0 & B(\lambda) \\ \lambda C(\lambda) & 0 \end{pmatrix}$$

with

$$B(\lambda) = -\gamma\lambda^2 + \beta\lambda - \gamma^{-1} \tag{4.8}$$

$$C(\lambda) = \gamma^{-1}\lambda^2 - \bar{\beta}\lambda + \gamma. \tag{4.9}$$

Knowing the fact that in the end

$$\lambda a(\lambda) = \det(\zeta_\lambda) = -\lambda B(\lambda) C(\lambda)$$

must pertain, the idea is to define B and C by distributing the roots of $a(\lambda)$ in a suitable way and by constructing γ (and implicitly β) such that the polynomials take the forms (4.8) and (4.9). Accordingly, we define

$$\begin{aligned} B(\lambda) &:= -\gamma(\lambda - \lambda_1)(\lambda - \bar{\lambda}_2^{-1}) && (4.10)\\ &= -\gamma\lambda^2 + \gamma(\lambda_1 + \bar{\lambda}_2^{-1})\lambda - \gamma\lambda_1\bar{\lambda}_2^{-1}, && (4.11)\\ C(\lambda) &:= \gamma^{-1}(\lambda - \lambda_2)(\lambda - \bar{\lambda}_1^{-1}) && (4.12)\\ &= \gamma^{-1}\lambda^2 - \gamma^{-1}(\lambda_2 + \bar{\lambda}_1^{-1})\lambda + \gamma^{-1}\lambda_2\bar{\lambda}_1^{-1}. && (4.13) \end{aligned}$$

(4.11) is of the form (4.8) if and only if

$$\begin{aligned} \gamma &= (\lambda_1\bar{\lambda}_2^{-1})^{-\frac{1}{2}} && (4.14)\\ \beta &= \gamma(\lambda_1 + \bar{\lambda}_2^{-1}) = (\lambda_1\bar{\lambda}_2)^{\frac{1}{2}} + (\lambda_1\bar{\lambda}_2)^{-\frac{1}{2}}. && (4.15) \end{aligned}$$

Thus, it remains to examine first, whether these β, γ also make (4.13) look like (4.9) and second, whether $\gamma \in \mathbb{R}^+$.
The first part is easily verified using (4.6), since

$$\gamma^{-1}\lambda_2\bar{\lambda}_1^{-1} = \gamma^{-1}(\lambda_1\bar{\lambda}_2^{-1})^{-1} = (\lambda_1\bar{\lambda}_2^{-1})^{\frac{1}{2}}(\lambda_1\bar{\lambda}_2^{-1})^{-1} = \gamma$$

and

$$\begin{aligned} \gamma^{-1}(\lambda_2 + \bar{\lambda}_1^{-1}) &= (\lambda_1\bar{\lambda}_2^{-1})^{\frac{1}{2}}(\lambda_2 + \bar{\lambda}_1^{-1})\\ &= (\lambda_2\bar{\lambda}_1^{-1})^{-\frac{1}{2}}(\lambda_2 + \bar{\lambda}_1^{-1})\\ &= (\lambda_2\bar{\lambda}_1)^{\frac{1}{2}} + (\lambda_2\bar{\lambda}_1)^{-\frac{1}{2}}\\ &= \bar{\beta} \end{aligned}$$

hold. For the second part we need to apply the last property of $a(\lambda)$. It is favourable to regard it in the explicit form

$$\lambda^{-2}a(\lambda) = \lambda^2 - a_1\lambda + a_2 - \bar{a}_1\lambda^{-1} + \lambda^{-2} \geq 0 \ \text{ for } \lambda \in \mathcal{S}^1$$

with

$$\begin{aligned} &a_1 = \lambda_1 + \lambda_2 + \bar{\lambda}_1^{-1} + \bar{\lambda}_2^{-1}\\ &a_2 = (\bar{\lambda}_1\bar{\lambda}_2)^{-1} + \lambda_1\bar{\lambda}_1^{-1} + \lambda_1\bar{\lambda}_2^{-1} + \lambda_2\bar{\lambda}_2^{-1} + \lambda_2\bar{\lambda}_1^{-1} + \lambda_1\lambda_2. \end{aligned}$$

Define

$$\tilde{\lambda} := i\sqrt{\lambda_1\bar{\lambda}_1^{-1}} = i\frac{\lambda_1}{\sqrt{\lambda_1\bar{\lambda}_1}} = i\frac{\lambda_1}{|\lambda_1|} \in \mathcal{S}^1$$

and apply the last property using (4.7):

$$\begin{aligned} \tilde{\lambda}^{-2}a(\tilde{\lambda}) &= -\lambda_1\bar{\lambda}_1^{-1} - a_1\tilde{\lambda} + a_2 - \bar{a}_1\tilde{\lambda}^{-1} - \bar{\lambda}_1\lambda_1^{-1}\\ &= -\lambda_1\bar{\lambda}_1^{-1} - a_1\tilde{\lambda} + a_2 - \bar{a}_1\tilde{\lambda}^{-1} - \lambda_2\bar{\lambda}_2^{-1} \geq 0\\ \tilde{\lambda}^{-2}a(-\tilde{\lambda}) &= -\lambda_1\bar{\lambda}_1^{-1} + a_1\tilde{\lambda} + a_2 + \bar{a}_1\tilde{\lambda}^{-1} - \lambda_2\bar{\lambda}_2^{-1} \geq 0. \end{aligned}$$

Adding $\tilde{\lambda}^{-2}a(\tilde{\lambda})$ and $\tilde{\lambda}^{-2}a(-\tilde{\lambda})$ yields

$$(\bar{\lambda}_1\bar{\lambda}_2)^{-1} + \lambda_1\bar{\lambda}_2^{-1} + \lambda_2\bar{\lambda}_1^{-1} + \lambda_1\lambda_2 \geq 0$$

which is equivalent to

$$\lambda_1\lambda_2\left(\frac{1}{|\lambda_1\lambda_2|^2} + \frac{1}{|\lambda_1|^2} + \frac{1}{|\lambda_2|^2} + 1\right) \geq 0.$$

Since the term in brackets is real and positive, we infer $\lambda_1\lambda_2 \geq 0$ and due to $a(0) = 1$ both roots are nonzero. As a result,

$$\lambda_1\lambda_2 > 0$$

is valid and in particular

$$\gamma = \left(\frac{\lambda_1}{\bar{\lambda}_2}\right)^{-\frac{1}{2}} = \left(\frac{\lambda_1\lambda_2}{|\lambda_2|^2}\right)^{-\frac{1}{2}} > 0.$$

q.e.d.

Definition 4.4.
The set $\mathcal{M}_2$ is called *moduli space*.

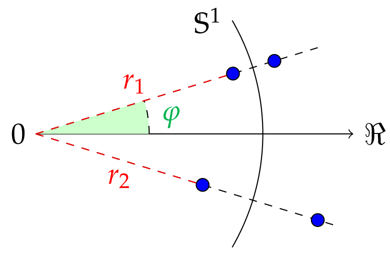


Figure 1: Example of an element $a \in \mathcal{M}_2$. Roots $r_1e^{i\varphi}, r_1^{-1}e^{i\varphi}, r_2e^{-i\varphi}, r_2^{-1}e^{-i\varphi}$ marked as blue bullet points.

Lastly, a relationship between the roots of ζ_λ and the corresponding polynomial $a(\lambda)$ is essential. Therefore, we state

Theorem 4.5.
Let $\zeta_\lambda \in \mathcal{P}_2$ and $\det\zeta_\lambda = \lambda a(\lambda)$ with $a(\lambda) \in \mathcal{M}_2$.
If $\tilde{\lambda} \in \mathbb{C}$ is a root of ζ_λ, then $\tilde{\lambda}$ is a double root of $a(\lambda)$.
Conversely, if $\tilde{\lambda} \in \mathcal{S}^1$ is a root of $a(\lambda)$, then $\tilde{\lambda}$ is a root of ζ_λ.

Proof.
Consider

$$\zeta_\lambda = \begin{pmatrix} A(\lambda) & B(\lambda) \\ \lambda C(\lambda) & -A(\lambda) \end{pmatrix}$$

with a root $\tilde{\lambda} \in \mathbb{C} \setminus \{0\}$. In particular, $\tilde{\lambda}$ is a root of each of its entries, so there are polynomials $\tilde{A}, \tilde{B}, \tilde{C}$ of maximal degree one such that

$$\begin{aligned} A(\lambda) &= (\lambda - \tilde{\lambda})\tilde{A}(\lambda) \\ B(\lambda) &= (\lambda - \tilde{\lambda})\tilde{B}(\lambda) \\ C(\lambda) &= (\lambda - \tilde{\lambda})\tilde{C}(\lambda). \end{aligned}$$

Now the assertion immediately follows since

$$\lambda a(\lambda) = \det \zeta_\lambda = -(\lambda - \tilde{\lambda})^2 \left[\tilde{A}(\lambda) - \lambda \tilde{B}(\lambda)\tilde{C}(\lambda)\right].$$

For the reverse direction, we first conduct certain preconsiderations. We consider for $A, B \in \mathbb{C}^{m \times n}$ the so-called *Frobenius product* (details in Horn et al. [11])

$$\langle A, B \rangle_F := \operatorname{tr}(\bar{A}^t B)$$

which is an inner product of matrices. Obviously, the modified Frobenius product

$$\langle A, B \rangle := \frac{1}{2}\langle A, B \rangle_F = \frac{1}{2}\operatorname{tr}(\bar{A}^t B)$$

stays an inner product and induces a norm

$$\|A\| = \sqrt{\langle A, A \rangle}.$$

Furthermore, remember the fact that the determinant of any 2×2 matrix A can be expressed by its trace in the following way

$$\det(A) = \frac{1}{2}\left[\operatorname{tr}(A)^2 - \operatorname{tr}(A^2)\right].$$

Consequently, when A is 2×2, traceless and skew-Hermitian, the modified Frobenius norm becomes

$$\begin{aligned} \|A\| &= \sqrt{\frac{1}{2}\operatorname{tr}(\bar{A}^t A)} = \sqrt{\frac{1}{2}\operatorname{tr}(-A^2)} = \sqrt{-\frac{1}{2}\operatorname{tr}(A^2)} \\ &= \sqrt{\frac{1}{2}\left[\operatorname{tr}(A)^2 - \operatorname{tr}(A^2)\right]} = \sqrt{\det A} \end{aligned}$$

whereby the second identity comes from the skew-Hermitian property and the fourth from the assumption of A being traceless. Thus, we have shown that the determinant of 2×2, traceless, skew-Hermitian matrices defines a norm (squared). We have already seen that $\lambda^{-\frac{3}{2}}\zeta_\lambda$ is such a matrix when $\lambda \in \mathcal{S}^1$ so

$$\|\lambda^{-\frac{3}{2}}\zeta_\lambda\|^2 = \det(\lambda^{-\frac{3}{2}}\zeta_\lambda) = \lambda^{-2}a(\lambda) \quad \text{for } \lambda \in \mathcal{S}^1.$$

Now, let $\tilde{\lambda} \in \mathcal{S}^1$ be a root of $a(\lambda)$. Then the above formula yields

$$\|\tilde{\lambda}^{-\frac{3}{2}}\zeta_{\tilde{\lambda}}\|^2 = 0$$

and therefore, by separating points property of norms, ζ_λ also has the root $\tilde{\lambda}$. **q.e.d.**

5 Polynomial Killing Fields

Now we want to introduce solutions of the sinh-Gordon equation of spectral genus $g = 2$. They are parametrized by flows on the set of potentials which are induced by a particular system of ordinary differential equations, the Lax equations. These so-called Polynomial Killing fields hold symmetry features with which we acquainted ourselves in the last chapter and we will see that the determinant function remains constant on the flows.

Definition 5.1.
Polynomial Killing fields are maps $\zeta_\lambda : \mathbb{R}^2 \to \mathcal{P}_2$, $(x, y) \mapsto \zeta_\lambda(x, y)$ which solve the *Lax equations*

$$\frac{\partial \zeta_\lambda}{\partial x} = [\zeta_\lambda, U(\zeta_\lambda)] \qquad \frac{\partial \zeta_\lambda}{\partial y} = [\zeta_\lambda, V(\zeta_\lambda)] \tag{5.1}$$

with $\zeta_\lambda(0) = \zeta_\lambda^0 \in \mathcal{P}_2$ and

$$\begin{aligned} U(\zeta_\lambda) &:= \begin{pmatrix} 0 & -\gamma^{-1} \\ 0 & 0 \end{pmatrix} \lambda^{-1} + \begin{pmatrix} \frac{\alpha-\bar{\alpha}}{2} & -\gamma \\ \gamma & \frac{\bar{\alpha}-\alpha}{2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \gamma^{-1} & 0 \end{pmatrix} \lambda \\ &= \begin{pmatrix} \frac{\alpha-\bar{\alpha}}{2} & -\gamma^{-1}\lambda^{-1} - \gamma \\ \gamma + \gamma^{-1}\lambda & \frac{\bar{\alpha}-\alpha}{2} \end{pmatrix} \\ V(\zeta_\lambda) &:= \begin{pmatrix} 0 & -\gamma^{-1} \\ 0 & 0 \end{pmatrix} i\lambda^{-1} + \begin{pmatrix} \frac{\alpha+\bar{\alpha}}{2} & \gamma \\ \gamma & -\frac{\alpha+\bar{\alpha}}{2} \end{pmatrix} i + \begin{pmatrix} 0 & 0 \\ -\gamma^{-1} & 0 \end{pmatrix} i\lambda \\ &= i \begin{pmatrix} \frac{\alpha+\bar{\alpha}}{2} & -\gamma^{-1}\lambda^{-1} + \gamma \\ \gamma - \gamma^{-1}\lambda & -\frac{\alpha+\bar{\alpha}}{2} \end{pmatrix}. \end{aligned}$$

Obviously, the Lax equations (5.1) are a system of autonomous ordinary differential equations. Intuitively, these equations are well-defined since the product of the λ^{-1} coefficient of $U(\zeta_\lambda)$ and the λ^0 coefficient of ζ_λ equals zero regardless of the multiplication order. Therefore, the right hand side defines a polynomial of degree four at most. As the product of the λ coefficient of $U(\zeta_\lambda)$ and the λ^3 coefficient of ζ_λ equals zero as well, we obtain third degree polynomials. The same arguments hold for $V(\zeta_\lambda)$.
We have already seen that any $\zeta_\lambda \in \mathcal{P}_2$ consists of a uniquely defined triplet

$$\alpha = (\zeta_\lambda)_\alpha \in \mathbb{C},\ \beta = (\zeta_\lambda)_\beta \in \mathbb{C}\ ,\ \gamma = (\zeta_\lambda)_\gamma \in \mathbb{R}^+.$$

Consequently, the prerequisite that ζ_λ is a map satisfying some differential equations can be boiled down to some other uniquely defined differential equations on the maps α, β, γ. More precisely, a Polynomial Killing induces a triple of complex functions

$$\begin{aligned} \alpha &: \mathbb{R}^2 \to \mathbb{C}, (x, y) \mapsto (\zeta_\lambda(x, y))_\alpha \\ \beta &: \mathbb{R}^2 \to \mathbb{C}, (x, y) \mapsto (\zeta_\lambda(x, y))_\beta \\ \gamma &: \mathbb{R}^2 \to \mathbb{R}^+, (x, y) \mapsto (\zeta_\lambda(x, y))_\gamma \end{aligned}$$

that need to satisfy a preferably manageable system of autonomous ordinary differential equations. As in their standard form the Lax equations are rather unwieldy, this approach is often beneficial. In our next step we will discern, what this system of equations in α, β, γ

concretely looks like. To achieve this, we need to calculate the commutators on the right hand side of (5.1) and compare the entries with the structure of ζ_λ. We obtain

$$\begin{aligned}\zeta_\lambda U(\zeta_\lambda) &= \begin{pmatrix} \alpha\lambda - \bar{\alpha}\lambda^2 & -\gamma^{-1} + \beta\lambda - \gamma\lambda^2 \\ \gamma\lambda - \bar{\beta}\lambda^2 + \gamma^{-1}\lambda^3 & -\alpha\lambda + \bar{\alpha}\lambda^2 \end{pmatrix} \begin{pmatrix} \frac{\alpha-\bar{\alpha}}{2} & -\gamma^{-1}\lambda^{-1} - \gamma \\ \gamma + \gamma^{-1}\lambda & \frac{\bar{\alpha}-\alpha}{2} \end{pmatrix} \\ &= \begin{pmatrix} -1 & -\frac{1}{2}\bar{\alpha}\gamma^{-1} - \frac{1}{2}\alpha\gamma^{-1} \\ 0 & -1 \end{pmatrix} \\ &\quad + \begin{pmatrix} -\gamma^{-2} + \frac{1}{2}\alpha^2 - \frac{1}{2}\alpha\bar{\alpha} + \beta\gamma & \frac{1}{2}\bar{\alpha}\beta - \frac{1}{2}\alpha\beta - \alpha\gamma + \bar{\alpha}\gamma^{-1} \\ -\frac{1}{2}\alpha\gamma - \frac{1}{2}\bar{\alpha}\gamma & \frac{1}{2}\alpha^2 - \frac{1}{2}\alpha\bar{\alpha} + \bar{\beta}\gamma^{-1} - \gamma^2 \end{pmatrix}\lambda \\ &\quad + \begin{pmatrix} -\frac{1}{2}\alpha\bar{\alpha} + \frac{1}{2}\bar{\alpha}^2 + \beta\gamma^{-1} - \gamma^2 & \frac{1}{2}\alpha\gamma + \frac{1}{2}\bar{\alpha}\gamma \\ -\frac{1}{2}\alpha\bar{\beta} + \frac{1}{2}\bar{\alpha}\bar{\beta} - \alpha\gamma^{-1} + \bar{\alpha}\gamma & -\frac{1}{2}\alpha\bar{\alpha} + \frac{1}{2}\bar{\alpha}^2 + \bar{\beta}\gamma - \gamma^{-2} \end{pmatrix}\lambda^2 \\ &\quad + \begin{pmatrix} -1 & 0 \\ \frac{1}{2}\alpha\gamma^{-1} + \frac{1}{2}\bar{\alpha}\gamma^{-1} & -1 \end{pmatrix}\lambda^3\end{aligned}$$

as well as

$$\begin{aligned}U(\zeta_\lambda)\zeta_\lambda &= \begin{pmatrix} \frac{\alpha-\bar{\alpha}}{2} & -\gamma^{-1}\lambda^{-1} - \gamma \\ \gamma + \gamma^{-1}\lambda & \frac{\bar{\alpha}-\alpha}{2} \end{pmatrix} \begin{pmatrix} \alpha\lambda - \bar{\alpha}\lambda^2 & -\gamma^{-1} + \beta\lambda - \gamma\lambda^2 \\ \gamma\lambda - \bar{\beta}\lambda^2 + \gamma^{-1}\lambda^3 & -\alpha\lambda + \bar{\alpha}\lambda^2 \end{pmatrix} \\ &= \begin{pmatrix} -1 & \frac{1}{2}\bar{\alpha}\gamma^{-1} + \frac{1}{2}\alpha\gamma^{-1} \\ 0 & -1 \end{pmatrix} \\ &\quad + \begin{pmatrix} -\gamma^2 + \frac{1}{2}\alpha^2 - \frac{1}{2}\alpha\bar{\alpha} + \bar{\beta}\gamma^{-1} & -\frac{1}{2}\bar{\alpha}\beta + \frac{1}{2}\alpha\beta + \alpha\gamma - \bar{\alpha}\gamma^{-1} \\ \frac{1}{2}\alpha\gamma + \frac{1}{2}\bar{\alpha}\gamma & \frac{1}{2}\alpha^2 - \frac{1}{2}\alpha\bar{\alpha} + \beta\gamma - \gamma^{-2} \end{pmatrix}\lambda \\ &\quad + \begin{pmatrix} -\frac{1}{2}\alpha\bar{\alpha} + \frac{1}{2}\bar{\alpha}^2 + \bar{\beta}\gamma - \gamma^{-2} & -\frac{1}{2}\alpha\gamma - \frac{1}{2}\bar{\alpha}\gamma \\ \frac{1}{2}\alpha\bar{\beta} - \frac{1}{2}\bar{\alpha}\bar{\beta} + \alpha\gamma^{-1} - \bar{\alpha}\gamma & -\frac{1}{2}\alpha\bar{\alpha} + \frac{1}{2}\bar{\alpha}^2 + \beta\gamma^{-1} - \gamma^2 \end{pmatrix}\lambda^2 \\ &\quad + \begin{pmatrix} -1 & 0 \\ -\frac{1}{2}\alpha\gamma^{-1} - \frac{1}{2}\bar{\alpha}\gamma^{-1} & -1 \end{pmatrix}\lambda^3.\end{aligned}$$

For the commutator's computation, we can regard ζ_λ and $U(\zeta_\lambda)$ as linear mappings due to their two-by-two matrix form. The derivative of a linear mapping equals the mapping itself, so Definition 3.5 boils down to a subtraction of the latter matrix from the first one:

$$\begin{aligned}[\zeta_\lambda, U(\zeta_\lambda)] &= \zeta_\lambda U(\zeta_\lambda) - U(\zeta_\lambda)\zeta_\lambda \\ &= \begin{pmatrix} 0 & -\bar{\alpha}\gamma^{-1} - \alpha\gamma^{-1} \\ 0 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} \gamma^2 + \beta\gamma - \bar{\beta}\gamma^{-1} - \gamma^{-2} & -\alpha\beta + \bar{\alpha}\beta - 2\alpha\gamma + 2\bar{\alpha}\gamma^{-1} \\ -\alpha\gamma - \bar{\alpha}\gamma & -\gamma^2 - \beta\gamma + \bar{\beta}\gamma^{-1} + \gamma^{-2} \end{pmatrix}\lambda \\ &\quad + \begin{pmatrix} -\gamma^2 + \beta\gamma^{-1} - \bar{\beta}\gamma + \gamma^{-2} & \alpha\gamma + \bar{\alpha}\gamma \\ -\alpha\bar{\beta} + \bar{\alpha}\bar{\beta} - 2\alpha\gamma^{-1} + 2\bar{\alpha}\gamma & \gamma^2 - \beta\gamma^{-1} + \bar{\beta}\gamma - \gamma^{-2} \end{pmatrix}\lambda^2 \\ &\quad + \begin{pmatrix} 0 & 0 \\ \alpha\gamma^{-1} + \bar{\alpha}\gamma^{-1} & 0 \end{pmatrix}\lambda^3.\end{aligned}$$

In complete analogy, we conduct the same calculation in order to obtain the right hand side commutator of (5.1):

$$\begin{aligned}
\zeta_\lambda V(\zeta_\lambda) &= i\begin{pmatrix} \alpha\lambda - \bar{\alpha}\lambda^2 & -\gamma^{-1} + \beta\lambda - \gamma\lambda^2 \\ \gamma\lambda - \bar{\beta}\lambda^2 + \gamma^{-1}\lambda^3 & -\alpha\lambda + \bar{\alpha}\lambda^2 \end{pmatrix}\begin{pmatrix} \frac{\alpha+\bar{\alpha}}{2} & -\gamma^{-1}\lambda^{-1} + \gamma \\ \gamma - \gamma^{-1}\lambda & -\frac{\alpha+\bar{\alpha}}{2} \end{pmatrix} \\
&= \begin{pmatrix} -1 & \frac{1}{2}\bar{\alpha}\gamma^{-1} - \frac{1}{2}\alpha\gamma^{-1} \\ 0 & -1 \end{pmatrix} i \\
&+ \begin{pmatrix} \gamma^{-2} + \frac{1}{2}\alpha^2 + \frac{1}{2}\alpha\bar{\alpha} + \beta\gamma & -\frac{1}{2}\bar{\alpha}\beta - \frac{1}{2}\alpha\beta + \alpha\gamma + \bar{\alpha}\gamma^{-1} \\ -\frac{1}{2}\alpha\gamma + \frac{1}{2}\bar{\alpha}\gamma & \frac{1}{2}\alpha^2 + \frac{1}{2}\alpha\bar{\alpha} + \bar{\beta}\gamma^{-1} + \gamma^2 \end{pmatrix} i\lambda \\
&+ \begin{pmatrix} -\frac{1}{2}\alpha\bar{\alpha} - \frac{1}{2}\bar{\alpha}^2 - \beta\gamma^{-1} - \gamma^2 & \frac{1}{2}\alpha\gamma - \frac{1}{2}\bar{\alpha}\gamma \\ -\frac{1}{2}\alpha\bar{\beta} - \frac{1}{2}\bar{\alpha}\bar{\beta} + \alpha\gamma^{-1} + \bar{\alpha}\gamma & -\frac{1}{2}\alpha\bar{\alpha} - \frac{1}{2}\bar{\alpha}^2 - \bar{\beta}\gamma - \gamma^{-2} \end{pmatrix} i\lambda^2 \\
&+ \begin{pmatrix} 1 & 0 \\ \frac{1}{2}\alpha\gamma^{-1} - \frac{1}{2}\bar{\alpha}\gamma^{-1} & 1 \end{pmatrix} i\lambda^3
\end{aligned}$$

and

$$\begin{aligned}
V(\zeta_\lambda)\zeta_\lambda &= i\begin{pmatrix} \frac{\alpha+\bar{\alpha}}{2} & -\gamma^{-1}\lambda^{-1} + \gamma \\ \gamma - \gamma^{-1}\lambda & -\frac{\alpha+\bar{\alpha}}{2} \end{pmatrix}\begin{pmatrix} \alpha\lambda - \bar{\alpha}\lambda^2 & -\gamma^{-1} + \beta\lambda - \gamma\lambda^2 \\ \gamma\lambda - \bar{\beta}\lambda^2 + \gamma^{-1}\lambda^3 & -\alpha\lambda + \bar{\alpha}\lambda^2 \end{pmatrix} \\
&= \begin{pmatrix} -1 & -\frac{1}{2}\bar{\alpha}\gamma^{-1} + \frac{1}{2}\alpha\gamma^{-1} \\ 0 & -1 \end{pmatrix} i \\
&+ \begin{pmatrix} \gamma^2 + \frac{1}{2}\alpha^2 + \frac{1}{2}\alpha\bar{\alpha} + \bar{\beta}\gamma^{-1} & \frac{1}{2}\bar{\alpha}\beta + \frac{1}{2}\alpha\beta - \alpha\gamma - \bar{\alpha}\gamma^{-1} \\ \frac{1}{2}\alpha\gamma - \frac{1}{2}\bar{\alpha}\gamma & \frac{1}{2}\alpha^2 + \frac{1}{2}\alpha\bar{\alpha} + \beta\gamma + \gamma^{-2} \end{pmatrix} i\lambda \\
&+ \begin{pmatrix} -\frac{1}{2}\alpha\bar{\alpha} - \frac{1}{2}\bar{\alpha}^2 - \bar{\beta}\gamma - \gamma^{-2} & -\frac{1}{2}\alpha\gamma + \frac{1}{2}\bar{\alpha}\gamma \\ \frac{1}{2}\alpha\bar{\beta} + \frac{1}{2}\bar{\alpha}\bar{\beta} - \alpha\gamma^{-1} - \bar{\alpha}\gamma & -\frac{1}{2}\alpha\bar{\alpha} - \frac{1}{2}\bar{\alpha}^2 - \beta\gamma^{-1} - \gamma^2 \end{pmatrix} i\lambda^2 \\
&+ \begin{pmatrix} 1 & 0 \\ -\frac{1}{2}\alpha\gamma^{-1} + \frac{1}{2}\bar{\alpha}\gamma^{-1} & 1 \end{pmatrix} i\lambda^3.
\end{aligned}$$

In total, we obtain

$$\begin{aligned}
[\zeta_\lambda, V(\zeta_\lambda)] &= \zeta_\lambda V(\zeta_\lambda) - V(\zeta_\lambda)\zeta_\lambda \\
&= \begin{pmatrix} 0 & \bar{\alpha}\gamma^{-1} - \alpha\gamma^{-1} \\ 0 & 0 \end{pmatrix} i \\
&+ \begin{pmatrix} -\gamma^2 + \beta\gamma - \bar{\beta}\gamma^{-1} + \gamma^{-2} & -\alpha\beta - \bar{\alpha}\beta + 2\alpha\gamma + 2\bar{\alpha}\gamma^{-1} \\ -\alpha\gamma + \bar{\alpha}\gamma & \gamma^2 - \beta\gamma + \bar{\beta}\gamma^{-1} - \gamma^{-2} \end{pmatrix} i\lambda \\
&+ \begin{pmatrix} -\gamma^2 - \beta\gamma^{-1} + \bar{\beta}\gamma + \gamma^{-2} & \alpha\gamma - \bar{\alpha}\gamma \\ -\alpha\bar{\beta} - \bar{\alpha}\bar{\beta} + 2\alpha\gamma^{-1} + 2\bar{\alpha}\gamma & \gamma^2 + \beta\gamma^{-1} - \bar{\beta}\gamma - \gamma^{-2} \end{pmatrix} i\lambda^2 \\
&+ \begin{pmatrix} 0 & 0 \\ \alpha\gamma^{-1} - \bar{\alpha}\gamma^{-1} & 0 \end{pmatrix} i\lambda^3.
\end{aligned}$$

By comparing the entries of the commutators with those from ζ_λ we have justified the following

Lemma 5.2.
Let ζ_λ be a Polynomial Killing field. Then the entries $\alpha, \beta : \mathbb{R}^2 \to \mathbb{C}$ and $\gamma : \mathbb{R}^2 \to \mathbb{R}^+$ satisfy the

modified Lax equations:

$$\frac{\partial\alpha}{\partial x} = \gamma^2 + \beta\gamma - \bar{\beta}\gamma^{-1} - \gamma^{-2} \qquad \frac{\partial\alpha}{\partial y} = i\left(\gamma^{-2} + \beta\gamma - \bar{\beta}\gamma^{-1} - \gamma^2\right)$$
$$\frac{\partial\beta}{\partial x} = -\alpha\beta + \bar{\alpha}\beta - 2\alpha\gamma + 2\bar{\alpha}\gamma^{-1} \qquad \frac{\partial\beta}{\partial y} = i\left(-\alpha\beta - \bar{\alpha}\beta + 2\alpha\gamma + 2\bar{\alpha}\gamma^{-1}\right)$$
$$\frac{\partial\gamma}{\partial x} = -\alpha\gamma - \bar{\alpha}\gamma \qquad \frac{\partial\gamma}{\partial y} = i\left(\bar{\alpha}\gamma - \alpha\gamma\right).$$

Proof.
The assertion is a direct consequence of the above commutators' structure. **q.e.d.**

Remark 5.3.
Notice that

$$\frac{\partial\gamma}{\partial x} = -\alpha\gamma - \bar{\alpha}\gamma = -2\gamma\,\Re(\alpha) \in \mathbb{R}$$

and

$$\frac{\partial\gamma}{\partial y} = i\left(\bar{\alpha}\gamma - \alpha\gamma\right) = 2\gamma\,\Im(\alpha) \in \mathbb{R}.$$

A first application of the modified Lax equations comes across when analysing the role of the determinant polynomial $a(\lambda)$ with reference to the Lax equations. We will demonstrate that $a(\lambda)$ is a so-called *integral of motion*, which means a constant quantity along the trajectories of the Lax equations.

Lemma 5.4.
The determinant polynomial $a(\lambda)$ from (4.3) is an integral of motion with respect to the Lax equations.

Proof.
From (4.3), we recall that the coefficients of $a(\lambda)$ are

$$\begin{aligned} a_1 &= -\bar{\alpha}^2 - \beta\gamma^{-1} - \bar{\beta}\gamma &&\in \mathbb{C} \\ a_2 &= 2\alpha\bar{\alpha} + \beta\bar{\beta} + \gamma^2 + \gamma^{-2} &&\in \mathbb{R}. \end{aligned}$$

We calculate the derivatives by inserting the Lax equations and conclude that the coefficients remain constant.

$$\begin{aligned}\frac{\partial a_1}{\partial x} &= -2\bar{\alpha}\frac{\partial \bar{\alpha}}{\partial x} - \frac{\partial \beta}{\partial x}\gamma^{-1} + \beta\gamma^{-2}\frac{\partial \gamma}{\partial x} - \bar{\beta}\frac{\partial \gamma}{\partial x} - \frac{\partial \bar{\beta}}{\partial x}\gamma \\
&= -2\bar{\alpha}(\gamma\bar{\beta} - \gamma^{-2} + \gamma^2 - \beta\gamma^{-1}) - (-\alpha\beta + \bar{\alpha}\beta - 2\alpha\gamma + 2\bar{\alpha}\gamma^{-1})\gamma^{-1} \\
&\quad + \beta\gamma^{-2}(-\alpha\gamma - \bar{\alpha}\gamma) - \bar{\beta}(-\alpha\gamma - \bar{\alpha}\gamma) - (-\bar{\alpha}\bar{\beta} + \alpha\bar{\beta} - 2\bar{\alpha}\gamma + 2\alpha\gamma^{-1})\gamma \\
&= -2\bar{\alpha}\bar{\beta}\gamma + 2\bar{\alpha}\gamma^{-2} - 2\bar{\alpha}\gamma^2 + 2\bar{\alpha}\beta\gamma^{-1} + \alpha\beta\gamma^{-1} - \bar{\alpha}\beta\gamma^{-1} + 2\alpha - 2\bar{\alpha}\gamma^{-2} \\
&\quad - \alpha\beta\gamma^{-1} - \bar{\alpha}\beta\gamma^{-1} + \alpha\bar{\beta}\gamma + \bar{\alpha}\bar{\beta}\gamma + \bar{\alpha}\bar{\beta}\gamma - \alpha\bar{\beta}\gamma + 2\bar{\alpha}\gamma^2 - 2\alpha \\
&= 0 \\
\frac{\partial a_1}{\partial y} &= -2\bar{\alpha}\frac{\partial \bar{\alpha}}{\partial y} - \frac{\partial \beta}{\partial y}\gamma^{-1} + \beta\gamma^{-2}\frac{\partial \gamma}{\partial y} - \bar{\beta}\frac{\partial \gamma}{\partial y} - \frac{\partial \bar{\beta}}{\partial y}\gamma \\
&= -2\bar{\alpha}(i(-\gamma^{-2} - \bar{\beta}\gamma + \beta\gamma^{-1} + \gamma^2)) - (i(-\alpha\beta - \bar{\alpha}\beta + 2\alpha\gamma + 2\bar{\alpha}\gamma^{-1}))\gamma^{-1} \\
&\quad + \beta\gamma^{-2}(i(\bar{\alpha}\gamma - \alpha\gamma)) - \bar{\beta}(i(\bar{\alpha}\gamma - \alpha\gamma)) - (i(\bar{\alpha}\bar{\beta} + \alpha\bar{\beta} - 2\bar{\alpha}\gamma - 2\alpha\gamma^{-1}))\gamma \\
&= 2i\bar{\alpha}\bar{\beta}\gamma + 2i\bar{\alpha}\gamma^{-2} - 2i\bar{\alpha}\gamma^2 - 2i\bar{\alpha}\beta\gamma^{-1} - 2i\alpha + i\alpha\beta\gamma^{-1} + i\bar{\alpha}\beta\gamma^{-1} - 2i\bar{\alpha}\gamma^{-2} \\
&\quad - i\alpha\beta\gamma^{-1} + i\bar{\alpha}\beta\gamma^{-1} - i\bar{\alpha}\bar{\beta}\gamma + i\alpha\bar{\beta}\gamma + 2i\alpha - i\alpha\bar{\beta}\gamma - i\bar{\alpha}\bar{\beta}\gamma + 2i\bar{\alpha}\gamma^2 \\
&= 0\end{aligned}$$

Therefore, the coefficient $\bar{a}_1$ of course remains constant as well.

$$\begin{aligned}\frac{\partial a_2}{\partial x} &= 2\frac{\partial \alpha}{\partial x}\bar{\alpha} + 2\alpha\frac{\partial \bar{\alpha}}{\partial x} + \frac{\partial \bar{\beta}}{\partial x}\beta + \bar{\beta}\frac{\partial \beta}{\partial x} + 2\gamma\frac{\partial \gamma}{\partial x} - 2\gamma^{-3}\frac{\partial \gamma}{\partial x} \\
&= 2\bar{\alpha}(\gamma^2 + \beta\gamma - \bar{\beta}\gamma^{-1} - \gamma^{-2}) + 2\alpha(\gamma^2 + \bar{\beta}\gamma - \beta\gamma^{-1} - \gamma^{-2}) \\
&\quad + \beta(-\bar{\alpha}\bar{\beta} + \alpha\bar{\beta} - 2\bar{\alpha}\gamma + 2\alpha\gamma^{-1}) + \bar{\beta}(-\alpha\beta + \bar{\alpha}\beta - 2\alpha\gamma + 2\bar{\alpha}\gamma^{-1}) \\
&\quad + 2\gamma(-\alpha\gamma - \bar{\alpha}\gamma) - 2\gamma^{-3}(-\alpha\gamma - \bar{\alpha}\gamma) \\
&= 2\bar{\alpha}\gamma^2 + 2\bar{\alpha}\beta\gamma - 2\bar{\alpha}\bar{\beta}\gamma^{-1} - 2\bar{\alpha}\gamma^{-2} + 2\alpha\gamma^2 + 2\alpha\bar{\beta}\gamma - 2\alpha\beta\gamma^{-1} \\
&\quad - 2\alpha\gamma^{-2} - \bar{\alpha}\beta\bar{\beta} + \alpha\beta\bar{\beta} - 2\bar{\alpha}\beta\gamma + 2\alpha\beta\gamma^{-1} - \alpha\beta\bar{\beta} + \bar{\alpha}\beta\bar{\beta} \\
&\quad - 2\alpha\bar{\beta}\gamma + 2\bar{\alpha}\bar{\beta}\gamma^{-1} - 2\alpha\gamma^2 - 2\bar{\alpha}\gamma^2 + 2\alpha\gamma^{-2} + 2\bar{\alpha}\gamma^{-2} \\
&= 0\end{aligned}$$

$$\begin{aligned}\frac{\partial a_2}{\partial y} &= 2\frac{\partial \alpha}{\partial y}\bar{\alpha} + 2\alpha\frac{\partial \bar{\alpha}}{\partial y} + \frac{\partial \bar{\beta}}{\partial y}\beta + \bar{\beta}\frac{\partial \beta}{\partial y} + 2\gamma\frac{\partial \gamma}{\partial y} - 2\gamma^{-3}\frac{\partial \gamma}{\partial y} \\
&= 2\bar{\alpha}(i(\gamma^{-2} + \beta\gamma - \bar{\beta}\gamma^{-1} - \gamma^2)) + 2\alpha(i(-\gamma^{-2} - \bar{\beta}\gamma + \beta\gamma^{-1} + \gamma^2)) \\
&\quad + \beta(i(\bar{\alpha}\bar{\beta} + \alpha\bar{\beta} - 2\bar{\alpha}\gamma - 2\alpha\gamma^{-1})) + \bar{\beta}(i(-\alpha\beta - \bar{\alpha}\beta + 2\alpha\gamma + 2\bar{\alpha}\gamma^{-1})) \\
&\quad + 2\gamma(i(\bar{\alpha}\gamma - \alpha\gamma)) - 2\gamma^{-3}(i(\bar{\alpha}\gamma - \alpha\gamma)) \\
&= 2i\bar{\alpha}\gamma^{-2} + 2i\bar{\alpha}\beta\gamma - 2i\bar{\alpha}\bar{\beta}\gamma^{-1} - 2i\bar{\alpha}\gamma^2 - 2i\alpha\gamma^{-2} - 2i\alpha\bar{\beta}\gamma \\
&\quad + 2i\alpha\beta\gamma^{-1} + 2i\alpha\gamma^2 + i\bar{\alpha}\beta\bar{\beta} + i\alpha\beta\bar{\beta} - 2i\bar{\alpha}\beta\gamma - 2i\alpha\beta\gamma^{-1} - i\alpha\beta\bar{\beta} \\
&\quad - i\bar{\alpha}\beta\bar{\beta} + 2i\alpha\bar{\beta}\gamma + 2\bar{\alpha}\bar{\beta}\gamma^{-1} + 2i\bar{\alpha}\gamma^2 - 2i\alpha\gamma^2 - 2i\bar{\alpha}\gamma^{-2} + 2i\alpha\gamma^{-2} \\
&= 0.\end{aligned}$$

q.e.d.

The next issue to be solved is whether such a Polynomial Killing field exists at all. Inspired by the Lax equations, we have a look at the continuously differentiable vector fields

$$E(\zeta_\lambda) := [\zeta_\lambda, U(\zeta_\lambda)]$$
$$F(\zeta_\lambda) := [\zeta_\lambda, V(\zeta_\lambda)].$$

As discussed in section 3, they induce local flows $\phi_E(x, \zeta_\lambda), \phi_F(y, \zeta_\lambda)$ in $\mathcal{P}_2$, which satisfy the respective parts of the Lax equations. Applying previous results, we obtain the following

Lemma 5.5.
The local flows $\phi_E(x)$ and $\phi_F(y)$ obtained by the Lax equations commute.

Proof.
According to Theorem 3.8 it suffices to show that the commutator of the vector fields E, F disappears.

$$\begin{aligned}
[E,F](\zeta_\lambda) &= E'F(\zeta_\lambda) - F'E(\zeta_\lambda) \\
&= [F(\zeta_\lambda), U(\zeta_\lambda)] + [\zeta_\lambda, U'(\zeta_\lambda)(F(\zeta_\lambda))] \\
&\quad - [E(\zeta_\lambda), V(\zeta_\lambda)] - [\zeta_\lambda, V'(\zeta_\lambda)(E(\zeta_\lambda))] \\
&= [[\zeta_\lambda, V(\zeta_\lambda)], U(\zeta_\lambda)] + \left[\zeta_\lambda, U'(\zeta_\lambda)\left(\frac{\partial \zeta_\lambda}{\partial y}\right)\right] \\
&\quad - [[\zeta_\lambda, U(\zeta_\lambda)], V(\zeta_\lambda)] - \left[\zeta_\lambda, V'(\zeta_\lambda)\left(\frac{\partial \zeta_\lambda}{\partial x}\right)\right] \\
&= [[\zeta_\lambda, V(\zeta_\lambda)], U(\zeta_\lambda)] - [[\zeta_\lambda, U(\zeta_\lambda)], V(\zeta_\lambda)] \\
&\quad + \left[\zeta_\lambda, \frac{\partial U(\zeta_\lambda)}{\partial y}\right] - \left[\zeta_\lambda, \frac{\partial V(\zeta_\lambda)}{\partial x}\right] \\
&= \left[\zeta_\lambda, [V(\zeta_\lambda), U(\zeta_\lambda)] + \frac{\partial U(\zeta_\lambda)}{\partial y} - \frac{\partial V(\zeta_\lambda)}{\partial x}\right]
\end{aligned}$$

The last equality is a quick calculation using the properties of the Lie bracket from Lemma 3.6. Jacobi-Identity and Antisymmetry yield

$$\begin{aligned}
0 &= [\zeta_\lambda, [U(\zeta_\lambda), V(\zeta_\lambda)]] + [U(\zeta_\lambda), [V(\zeta_\lambda), \zeta_\lambda]] + [V(\zeta_\lambda), [\zeta_\lambda, U(\zeta_\lambda)]] \\
&= [\zeta_\lambda, [U(\zeta_\lambda), V(\zeta_\lambda)]] - [[V(\zeta_\lambda), \zeta_\lambda], U(\zeta_\lambda)] - [[\zeta_\lambda, U(\zeta_\lambda)], V(\zeta_\lambda)] \\
&= -[\zeta_\lambda, [V(\zeta_\lambda), U(\zeta_\lambda)]] + [[\zeta_\lambda, V(\zeta_\lambda)], U(\zeta_\lambda)] - [[\zeta_\lambda, U(\zeta_\lambda)], V(\zeta_\lambda)].
\end{aligned}$$

Thus, it remains to prove the fact that the last expression equals zero. With the Lax equations from Lemma 5.2 we can explicitly state the differentiation terms

$$\begin{aligned}
\frac{\partial U(\zeta_\lambda)}{\partial y} &= \begin{pmatrix} 0 & \frac{\partial \gamma}{\partial y}\gamma^{-2} \\ 0 & 0 \end{pmatrix} \lambda^{-1} + \begin{pmatrix} \frac{1}{2}\left(\frac{\partial \alpha}{\partial y} - \frac{\partial \bar{\alpha}}{\partial y}\right) & -\frac{\partial \gamma}{\partial y} \\ \frac{\partial \gamma}{\partial y} & \frac{1}{2}\left(-\frac{\partial \alpha}{\partial y} + \frac{\partial \bar{\alpha}}{\partial y}\right) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\frac{\partial \gamma}{\partial y}\gamma^{-2} & 0 \end{pmatrix} \lambda \\
&= \begin{pmatrix} 0 & i\gamma^{-1}(\bar{\alpha} - \alpha \\ 0 & 0 \end{pmatrix} \lambda^{-1} \\
&\quad + \begin{pmatrix} i(\gamma^{-2} - \gamma^2 + \frac{1}{2}(\beta + \bar{\beta})(\gamma - \gamma^{-1})) & -i\gamma(\bar{\alpha} - \alpha) \\ i\gamma(\bar{\alpha} - \alpha) & i(\gamma^2 - \gamma^{-2} + \frac{1}{2}(\beta + \bar{\beta})(\gamma^{-1} - \gamma)) \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & 0 \\ i\gamma^{-1}(\alpha - \bar{\alpha} & 0 \end{pmatrix} \lambda
\end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial V(\zeta_\lambda)}{\partial x} &= i\begin{pmatrix} 0 & i\frac{\partial \gamma}{\partial x}\gamma^{-2} \\ 0 & 0 \end{pmatrix}\lambda^{-1} + i\begin{pmatrix} \frac{1}{2}\left(\frac{\partial \alpha}{\partial x} + \frac{\partial \bar{\alpha}}{\partial x}\right) & \frac{\partial \gamma}{\partial x} \\ \frac{\partial \gamma}{\partial x} & \frac{1}{2}\left(-\frac{\partial \alpha}{\partial x} - \frac{\partial \bar{\alpha}}{\partial x}\right) \end{pmatrix} + i\begin{pmatrix} 0 & 0 \\ \frac{\partial \gamma}{\partial x}\gamma^{-2} & 0 \end{pmatrix}\lambda \\
&= i\begin{pmatrix} 0 & -\gamma^{-1}(\alpha+\bar{\alpha}) \\ 0 & 0 \end{pmatrix}\lambda^{-1} \\
&\quad + i\begin{pmatrix} \gamma^2 - \gamma^{-2} + \frac{1}{2}(\beta+\bar{\beta})(\gamma-\gamma^{-1}) & -\gamma(\alpha+\bar{\alpha}) \\ -\gamma(\alpha+\bar{\alpha}) & \gamma^{-2} - \gamma^2 + \frac{1}{2}(\beta+\bar{\beta})(\gamma^{-1}-\gamma) \end{pmatrix} \\
&\quad + i\begin{pmatrix} 0 & 0 \\ -\gamma^{-1}(\alpha+\bar{\alpha} & 0 \end{pmatrix}\lambda.
\end{aligned}
$$

Combining both results in

$$
\begin{aligned}
\frac{\partial U(\zeta_\lambda)}{\partial y} - \frac{\partial V(\zeta_\lambda)}{\partial x} &= i\begin{pmatrix} 0 & 2\gamma^{-1}\bar{\alpha} \\ 0 & 0 \end{pmatrix}\lambda^{-1} \\
&\quad + i\begin{pmatrix} 2(\gamma^{-2}-\gamma^2) & 2\gamma\alpha \\ 2\gamma\bar{\alpha} & 2(\gamma^2-\gamma^{-2}) \end{pmatrix} \\
&\quad + i\begin{pmatrix} 0 & 0 \\ 2\gamma^{-1}\alpha & 0 \end{pmatrix}\lambda.
\end{aligned}
$$

Furthermore, $[V(\zeta_\lambda), U(\zeta_\lambda)]$ needs to be calculated.

$$
\begin{aligned}
V(\zeta_\lambda)U(\zeta_\lambda) &= \begin{pmatrix} \frac{\alpha+\bar{\alpha}}{2} & -\gamma^{-1}\lambda^{-1}+\gamma \\ \gamma-\gamma^{-1}\lambda & -\frac{\alpha+\bar{\alpha}}{2} \end{pmatrix} i\begin{pmatrix} \frac{\alpha-\bar{\alpha}}{2} & -\gamma^{-1}\lambda^{-1}-\gamma \\ \gamma+\gamma^{-1}\lambda & \frac{\bar{\alpha}-\alpha}{2} \end{pmatrix} \\
&= i\begin{pmatrix} -1 & -\gamma^{-1}\bar{\alpha} \\ 0 & -1 \end{pmatrix}\lambda^{-1} \\
&\quad + i\begin{pmatrix} \gamma^2-\gamma^{-2}+\frac{1}{4}((\alpha+\bar{\alpha})(\alpha-\bar{\alpha})) & -\gamma\alpha \\ -\gamma\bar{\alpha} & \gamma^{-2}-\gamma^2-\frac{1}{4}((\bar{\alpha}+\alpha)(\bar{\alpha}-\alpha)) \end{pmatrix} \\
&\quad + i\begin{pmatrix} 1 & 0 \\ -\gamma^{-1}\alpha & 1 \end{pmatrix}\lambda \\
U(\zeta_\lambda)V(\zeta_\lambda) &= \begin{pmatrix} \frac{\alpha-\bar{\alpha}}{2} & -\gamma^{-1}\lambda^{-1}-\gamma \\ \gamma+\gamma^{-1}\lambda & \frac{\bar{\alpha}-\alpha}{2} \end{pmatrix} i\begin{pmatrix} \frac{\alpha+\bar{\alpha}}{2} & -\gamma^{-1}\lambda^{-1}+\gamma \\ \gamma-\gamma^{-1}\lambda & -\frac{\alpha+\bar{\alpha}}{2} \end{pmatrix} \\
&= i\begin{pmatrix} -1 & \gamma^{-1}\bar{\alpha} \\ 0 & -1 \end{pmatrix}\lambda^{-1} \\
&\quad + i\begin{pmatrix} -\gamma^2+\gamma^{-2}+\frac{1}{4}((\alpha+\bar{\alpha})(\alpha-\bar{\alpha})) & \gamma\alpha \\ \gamma\bar{\alpha} & -\gamma^{-2}+\gamma^2-\frac{1}{4}((\bar{\alpha}+\alpha)(\bar{\alpha}-\alpha)) \end{pmatrix} \\
&\quad + i\begin{pmatrix} 1 & 0 \\ \gamma^{-1}\alpha & 1 \end{pmatrix}\lambda
\end{aligned}
$$

So the commutator equals

$$\begin{aligned}[V(\zeta_\lambda), U(\zeta_\lambda)] &= V(\zeta_\lambda)U(\zeta_\lambda) - U(\zeta_\lambda)V(\zeta_\lambda) \\ &= i\begin{pmatrix} 0 & -2\gamma^{-1}\bar{\alpha} \\ 0 & 0 \end{pmatrix}\lambda^{-1} \\ &\quad + i\begin{pmatrix} 2(\gamma^2-\gamma^{-2}) & -2\gamma\alpha \\ -2\gamma\bar{\alpha} & 2(\gamma^{-2}-\gamma^2) \end{pmatrix} \\ &\quad + i\begin{pmatrix} 0 & 0 \\ -2\gamma^{-1}\alpha & 0 \end{pmatrix}\lambda \\ &= -\left(\frac{\partial U(\zeta_\lambda)}{\partial y} - \frac{\partial V(\zeta_\lambda)}{\partial x}\right).\end{aligned} \tag{5.2}$$

All in all, we have demonstrated that

$$[V(\zeta_\lambda), U(\zeta_\lambda)] + \frac{\partial U(\zeta_\lambda)}{\partial y} - \frac{\partial V(\zeta_\lambda)}{\partial x} = 0 \tag{5.3}$$

and therefore,

$$[E, F] = 0.$$

q.e.d.

Remark 5.6.
With other words, we have implicitly shown that the second partial derivatives commute since

$$F'E(\zeta_\lambda) = [\frac{\partial\zeta_\lambda}{\partial x}, V(\zeta_\lambda)] + [\frac{\partial, V(\zeta_\lambda)}{\partial x}, \zeta_\lambda] = \frac{\partial}{\partial x}[\zeta_\lambda, V(\zeta_\lambda)] = \frac{\partial^2\zeta_\lambda}{\partial x\partial y}$$

and analogously,

$$E'F(\zeta_\lambda) = \frac{\partial^2\zeta_\lambda}{\partial y\partial x}.$$

However this notation is rather symbolic because the expressions on the right hand side are not well-defined yet (at this point we do not know if there exists a $\zeta_\lambda(x,y)$ that satisfies both Lax equations).

Remark 5.7 (Link to sinh-Gordon equation)**.**
Equation (5.3) is called *Maurer-Cartan equation* and we will quickly demonstrate that its validity is deduced from the sinh-Gordon equation. This motivates the analysis of potentials and polynomial killing fields. We introduce a new coordinate

$$z := x + iy$$

and express the x, y-coordinates in terms of z

$$x = \frac{1}{2}(z+\bar{z}) \qquad y = -\frac{i}{2}(z-\bar{z}).$$

The derivative with respect to the new coordinate z can then be specified using the chain rule

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{5.4}$$

It is essential to define the following variable

$$u := \ln\gamma \qquad \Leftrightarrow \qquad e^u = \gamma$$

and to calculate the derivative of u with respect to the new coordinate z using

$$\frac{\partial u}{\partial x} = \frac{1}{\gamma}\frac{\partial \gamma}{\partial x} = -(\alpha + \bar{\alpha}) \qquad \frac{\partial u}{\partial y} = \frac{1}{\gamma}\frac{\partial \gamma}{\partial y} = i(\bar{\alpha} - \alpha).$$

Now we obtain with (5.4)

$$\frac{\partial u}{\partial z} = -\alpha \qquad \frac{\partial u}{\partial \bar{z}} = -\bar{\alpha}.$$

Subsequently, we express the Maurer-Cartan equation in terms of u and derivatives of u with respect to $z, \bar{z}$ (which will be denoted $u_z, u_{\bar{z}}$ for reasons of simplicity). First, the commutator can be computed with the help of (5.2):

$$[V(\zeta_\lambda), U(\zeta_\lambda)] = i\begin{pmatrix} 2(e^{2u} - e^{-2u}) & 2e^{-u}u_{\bar{z}}\lambda^{-1} + 2e^u u_z \\ 2e^u u_{\bar{z}} + 2e^{-u}u_z\lambda & 2(e^{-2u} - e^{2u}) \end{pmatrix}. \tag{5.5}$$

For the derivative terms, we rewrite the matrices $U(\zeta_\lambda), V(\zeta_\lambda)$

$$\begin{aligned} U(\zeta_\lambda) = & \begin{pmatrix} \frac{\alpha-\bar{\alpha}}{2} & -\gamma^{-1}\lambda^{-1} - \gamma \\ \gamma + \gamma^{-1}\lambda & \frac{\bar{\alpha}-\alpha}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(u_{\bar{z}} - u_z) & -e^{-u}\lambda^{-1} - e^u \\ e^u + e^{-u}\lambda & \frac{1}{2}(u_z - u_{\bar{z}}) \end{pmatrix} \\ V(\zeta_\lambda) = i & \begin{pmatrix} \frac{\alpha+\bar{\alpha}}{2} & -\gamma^{-1}\lambda^{-1} + \gamma \\ \gamma - \gamma^{-1}\lambda & -\frac{\alpha+\bar{\alpha}}{2} \end{pmatrix} = i\begin{pmatrix} -\frac{1}{2}(u_z + u_{\bar{z}}) & -e^{-u}\lambda^{-1} + e^u \\ e^u - e^{-u}\lambda & \frac{1}{2}(u_z + u_{\bar{z}}) \end{pmatrix} \end{aligned}$$

and use the formulas

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \qquad \frac{\partial}{\partial y} = -i\left(\frac{\partial}{\partial \bar{z}} - \frac{\partial}{\partial z}\right)$$

in order to calculate

$$\begin{aligned} \frac{\partial U(\zeta_\lambda)}{\partial y} &= -i\begin{pmatrix} \frac{1}{2}(u_{\bar{z}\bar{z}} - 2u_{\bar{z}z} + u_{zz}) & -\lambda^{-1}e^{-u}(u_z - u_{\bar{z}}) - e^u(u_{\bar{z}} - u_z) \\ e^u(u_{\bar{z}} - u_z) + \lambda e^{-u}(u_z - u_{\bar{z}}) & \frac{1}{2}(-u_{\bar{z}\bar{z}} + 2u_{\bar{z}z} - u_{zz}) \end{pmatrix} \\ \frac{\partial V(\zeta_\lambda)}{\partial x} &= \quad i\begin{pmatrix} -\frac{1}{2}(u_{\bar{z}\bar{z}} + 2u_{\bar{z}z} + u_{zz}) & \lambda^{-1}e^{-u}(u_z + u_{\bar{z}}) + e^u(u_{\bar{z}} + u_z) \\ e^u(u_{\bar{z}} + u_z) + \lambda e^{-u}(u_z + u_{\bar{z}}) & \frac{1}{2}(u_{\bar{z}\bar{z}} + 2u_{\bar{z}z} + u_{zz}) \end{pmatrix}. \end{aligned}$$

This yields

$$\frac{\partial V(\zeta_\lambda)}{\partial x} - \frac{\partial U(\zeta_\lambda)}{\partial y} = i\begin{pmatrix} -2u_{\bar{z}z} & 2\lambda^{-1}e^{-u}u_{\bar{z}} + 2e^u u_z \\ 2e^u u_{\bar{z}} + 2\lambda e^{-u}u_z & 2u_{\bar{z}z} \end{pmatrix}. \tag{5.6}$$

Hence, the Maurer-Cartan equation is satisfied if and only if (5.5) equals (5.6) which reduces to the condition

$$\begin{aligned} u_{\bar{z}z} = (e^{-2u} - e^{2u}) \quad &\Leftrightarrow \quad u_{\bar{z}z} + \sinh(2u) = 0 \\ &\Leftrightarrow \quad \frac{1}{4}\Delta u + \sinh(2u) = 0 \end{aligned}$$

where the last equivalence is due to

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (u_{zz} + 2u_{\bar{z}z} + u_{\bar{z}\bar{z}}) - (u_{zz} - 2u_{\bar{z}z} + u_{\bar{z}\bar{z}}) = 4u_{\bar{z}z}.$$

If we choose $z' = \frac{1}{2}z$ (which implies $u_{\bar{z}'z'} = 4u_{\bar{z}z}$) and conduct the same computations, the Maurer-Cartan equation finally turns into the sinh-Gordon equation

$$\Delta u + \sinh(2u) = 0.$$

In the next step we show that the local flows ϕ_E, ϕ_F are global. We already know that both flows keep the determinant constant or, equivalently, that the flows stay in the starting point's levels set of the map

$$f : \mathcal{P}_2 \to \mathcal{M}_2, \quad \zeta_\lambda \mapsto a(\lambda). \tag{5.7}$$

Instead of looking at the quite unmanageable mapping f itself, we will again benefit from the one-to-one correspondence between $\mathcal{P}_2$ and $\mathbb{C} \times \mathbb{C} \times \mathbb{R}^+$. Thus, we can interpret f as a mapping

$$f : \mathbb{C} \times \mathbb{C} \times \mathbb{R}^+ \to \mathbb{C} \times \mathbb{R}^+, \quad \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \mapsto \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{5.8}$$

with a_1, a_2 from (4.3). Due to this correspondence, we will not distinguish between these two interpretations anymore as it will be clear from the context, which mapping is meant.

Definition 5.8.
Level sets of the function f from (5.8) are called *isospectral sets*.
Given any $a \in \mathcal{M}_2$ we denote the respective isospectral set by $I(a)$.

Proposition 5.9.
Fix $a = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + \bar{a}_1\lambda + 1 \in \mathcal{M}_2$. Then $I(a)$ is compact.

Proof.
By Heine-Borel, it is sufficient to show closedness and boundedness.
Let π_1, π_2 be the projections on the first and second coefficient. The functions

$$f_1 := \pi_1 \circ f, \quad (\alpha, \beta, \gamma) \mapsto a_1 = -\bar{\alpha}^2 - \beta\gamma^{-1} - \bar{\beta}\gamma$$
$$f_2 := \pi_2 \circ f \quad (\alpha, \beta, \gamma) \mapsto a_2 = 2\alpha\bar{\alpha} + \beta\bar{\beta} + \gamma^2 + \gamma^{-2}$$

are continuous, so the pre-images of closed sets are closed and consequently,

$$I(a) = f_1^{-1}(a_1) \cap f_2^{-1}(a_2)$$

is closed. For boundedness, it suffices to prove that $f_2^{-1}(a_2)$ is bounded since the intersection of a bounded set with any other set is bounded. By definition of f_2,

$$f_2^{-1}(a_2) \subset B\left(0, \sqrt{\frac{\bar{a}_2}{2}}\right) \times B\left(0, \sqrt{\bar{a}_2}\right) \times \left(0, \sqrt{\bar{a}_2}\right)$$

holds true and as the set on the right hand side is bounded, so is the one on the left hand side. **q.e.d.**

Corollary 5.10.
Given any initial value $(\alpha_0, \beta_0, \gamma_0) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}^+$ *the solutions of the modified Lax equations (5.2) are global, i.e. well-defined for all* $(x, y) \in \mathbb{R}^2$*, and bounded.*
Therefore, given any $\zeta_\lambda \in \mathcal{P}_2$ *we obtain a continuous, commutative group action*

$$\phi(x,y)(\zeta_\lambda) := \phi_F(y, \phi_E(x, \zeta_\lambda)) \quad \textit{for } (x,y) \in \mathbb{R}^2.$$

Furthermore, when $a \in \mathcal{M}_2$ *and* $\zeta_\lambda \in I(a)$ *then* $\phi(x,y)(\zeta_\lambda) \in I(a)$ *for all times* $(x,y) \in \mathbb{R}^2$*.*

Proof.
Let ζ_λ^0 be the element in $\mathcal{P}_2$ that belongs to $(\alpha_0, \beta_0, \gamma_0) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}^+$ and $a_0(\lambda) \in \mathcal{M}_2$ its determinant polynomial. Obiously, the modified Lax equations (5.2) are of type

$$\frac{\partial}{\partial x}(\alpha, \beta, \gamma) = g(\alpha, \beta, \gamma) \qquad \frac{\partial}{\partial y}(\alpha, \beta, \gamma) = h(\alpha, \beta, \gamma)$$

with functions

$$g, h : \mathbb{C} \times \mathbb{C} \times \mathbb{R}^+ \to \mathbb{C} \times \mathbb{C} \times \mathbb{R}$$

continuously differentiable. In particular, these mappings are locally Lipschitz continuous. Furthermore, any solution's orbit is contained in $I(a_0)$, which is compact by Proposition 5.9. All in all, we can apply Theorem 2.10 and the first part of the assertion is provided.
For the second part, we consider the Lax equations in its original form (5.1). With the first part, the local flows from Lemma 5.5 become global and commute. Clearly,

$$\phi(0,0)(\zeta_\lambda) = \zeta_\lambda.$$

The (two-dimensional) flow property (or compatibility condition) is a simple calculation using commutativity in the third line:

$$\begin{aligned}
\phi(x_2, y_2)(\phi(x_1, y_1)(\zeta_\lambda)) &= \phi_F(y_2, \phi_E(x_2, \phi(x_1, y_1)(\zeta_\lambda))) \\
&= \phi_F\big(y_2, \phi_E\big(x_2, \phi_F\big(y_1, \phi_E(x_1, \zeta_\lambda)\big)\big)\big) \\
&= \phi_F\big(y_2, \phi_F\big(y_1, \phi_E\big(x_2, \phi_E(x_1, \zeta_\lambda)\big)\big)\big) \\
&= \phi_F(y_1 + y_2, \phi_E(x_1 + x_2, \zeta_\lambda)) \\
&= \phi(x_1 + x_2, y_1 + y_2)(\zeta_\lambda).
\end{aligned}$$

It remains to show continuity of the map $(x, y, \zeta_\lambda) \mapsto \phi(x,y)(\zeta_\lambda)$. We can rewrite this mapping by splitting it up into several maps

$$(x, y, \zeta_\lambda) \mapsto (y, x, \zeta_\lambda) \overset{id \times \phi_E}{\mapsto} (y, \phi_E(x, \zeta_\lambda)) \overset{\phi_F}{\mapsto} \phi_F(y, \phi_E(x, \zeta_\lambda)) = \phi(x,y)(\zeta_\lambda).$$

All of these mappings are continuous, so the combination is continuous as well.
The last statement is a direct consequence of Lemma 5.4. **q.e.d.**

Remark 5.11.
We will often call the group action $\phi(x,y)(\zeta_\lambda)$ (two dimensional) *flow* although, strictly speaking, a flow must have one-dimensional time paramter domain by definition - this is here clearly not the case. Analogously, we will call the compatibility condition flow property as already done above.

6 Isospectral Sets

In the previous section we have already detected that the level sets of the function f from (5.8) are compact. In this part, we will extend our analysis of the isospectral sets with respect to structural features. First, we define the following subsets of $\mathcal{M}_2$

$$\mathcal{M}_2^0 := \{a \in \mathcal{M}_2 \,|\, \lambda^{-2}a(\lambda) > 0 \text{ for } \lambda \in \mathcal{S}^1\} \tag{6.1}$$

$$\mathcal{M}_2^1 := \{a \in \mathcal{M}_2 \,|\, a \text{ has four pairwise distinct roots}\}. \tag{6.2}$$

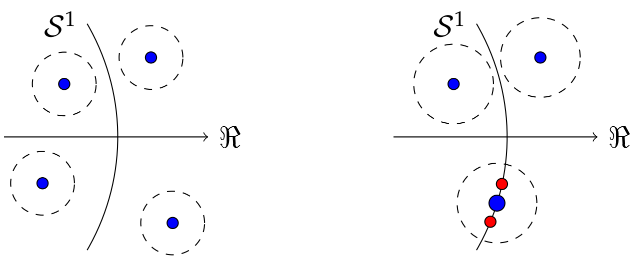
We will first confine ourselves to isospectral sets which originate from determinant polynomials $a \in \mathcal{M}_2^1$. This very restricting scenario will be the basic starting point from which we intend to gain similar results for determinant polynomials with differently positioned roots.

6.1 The classical case: $a(\lambda)$ has four pairwise distinct roots

Take $a \in \mathcal{M}_2^1$. If there was a $\lambda \in \mathcal{S}^1$ such that $\lambda^{-2}a(\lambda) = 0$, then this λ would be a root of a. By reality condition (4.4), this root must be double, which contradicts the definition of $\mathcal{M}_2^1$. Therefore,

$$\mathcal{M}_2^1 \subset \mathcal{M}_2^0 \subset \mathcal{M}_2 \tag{6.3}$$

holds. Obviously, these subsets are open because the roots of the elements of both subsets are not located on the unit circle. Consequently, we can always move the roots within a certain ε-neighbourhood ensuring that the resulting a remains in the respective subset (Figure 2a).



(a) $\mathcal{M}_2^1$ is open as well as $\mathcal{M}_2^0$ (completely analogous).

(b) $\mathcal{M}_2$ is not open. Red bullet points indicate the roots of the counter example.

Figure 2: ε-neighbourhoods (dashed circles) of given roots (blue points).

Crucially, the set $\mathcal{M}_2$ itself is not open: Consider an $a \in \mathcal{M}_2 \setminus \mathcal{M}_2^0$, i.e. a determinant polynomial having at least one double root on $\mathcal{S}^1$. Then any ε-environment contains elements absent $\mathcal{M}_2$, namely the ones which result when pushing the roots away from the double root towards opposite directions along $\mathcal{S}^1$ (Figure 2b).
A fairly useful result is provided in the following

Lemma 6.1.
Each of the three sets (6.3) is path-connected and the set $\mathcal{M}_2^0 \setminus \mathcal{M}_2^1$ is one-dimensional.

Proof.
The set $\mathcal{M}_2$ is convex (so in particular path-connected) since all conditions on its elements a are linear. This finding and analogous argumentation yield path-connectedness of the set $\mathcal{M}_2^0$. Nevertheless, it is not obvious to infer path- connectedness of $\mathcal{M}_2^1$ from path-connectedness of $\mathcal{M}_2^0$ only because of (6.3). In general, subsets of path-connected sets do not necessarily need to be path-connected since subsets may consist of two disjunct subsets. However, our setting allows this conclusion, because the complement $\mathcal{M}_2^0 \setminus \mathcal{M}_2^1$ has co-dimension of at least two, i.e. is one-dimensional. Therefore, it is impossible that $\mathcal{M}_2^1$ looks like described above and thus, it inherits path-connectedness from $\mathcal{M}_2^0$.
In order to complete the proof, we demonstrate the second part of the assertion, namely one-dimensionality of the set $\mathcal{M}_2^0 \setminus \mathcal{M}_2^1$. Take an arbitrary element $a \in \mathcal{M}_2^0 \setminus \mathcal{M}_2^1$. The polynomial a has at least one double root $\lambda_0 \in \mathbb{C}$. If $\lambda_0 \in \mathcal{S}^1$, then

$$\lambda_0^{-2} a(\lambda_0) = 0$$

which contradicts $a \in \mathcal{M}_2^0$. Hence, $\lambda_0 \notin \mathcal{S}^1$. But then, since λ_0 is a double root, the reality condition (4.4) demands $\bar{\lambda}_0^{-1}$ to be a double root of a as well, so

$$a(\lambda) = (\lambda - \lambda_0)^2 (\lambda - \bar{\lambda}_0^{-1})^2.$$

Moreover, the condition $a(0) = 1$ is reflected in

$$\begin{aligned}(\lambda_0 \bar{\lambda}_0^{-1})^2 = 1 \;&\Leftrightarrow\; \left(\frac{\lambda_0}{|\lambda_0|}\right)^2 = \frac{\lambda_0}{\bar{\lambda}_0} \in \{-1, 1\} \\ &\Leftrightarrow\; \frac{\lambda_0}{|\lambda_0|} \in \{-1, 1, i, -i\} \\ &\Leftrightarrow\; \lambda_0 \in (\mathbb{R} \cup i\mathbb{R}) \setminus \{0\}.\end{aligned}$$

Since $\lambda_0 \notin \mathcal{S}^1$ we obtain

$$\lambda_0 \in (\mathbb{R} \cup i\mathbb{R}) \setminus \{-1, 1, -i, i, 0\}.$$

Assume, $\lambda_0 = ir$ with $r \in \mathbb{R} \setminus \{-1, 0, 1\}$ and $\bar{\lambda}_0^{-1} = r^{-1} i$ respectively. Then

$$\begin{aligned}\lambda^{-2} a(\lambda) &= \lambda^{-2} (\lambda - ri)^2 (\lambda - r^{-1} i)^2 \\ &= \lambda^{-2} (\lambda^2 - 2ri\lambda - r^2)(\lambda^2 - 2r^{-1} i\lambda - r^{-2}) \\ &= \lambda^2 - (r + r^{-1}) 2i\lambda - (4 + r^2 + r^{-2}) + (r + r^{-1}) 2i\lambda^{-1} + \lambda^{-2}.\end{aligned}$$

If we insert $\lambda = 1 \in \mathcal{S}^1$ into this formula, we obtain

$$-(2 + r^2 + r^{-2}) < 0$$

which contradicts the condition that $a \in \mathcal{M}_2^0$. Since r was arbitrary, we directly obtain

$$\lambda_0 \in \mathbb{R} \setminus \{-1, 0, 1\}$$

and therefore, $\mathcal{M}_2^0 \setminus \mathcal{M}_2^1$ has dimension of at most one. **q.e.d.**

Proposition 6.2.
Let $a \in \mathcal{M}_2^1$.
Then the isospectral set $I(a)$ is a compact, two-dimensional submanifold of $\mathbb{C} \times \mathbb{C} \times \mathbb{R}^+$.

Proof.
Compactness has already been proven. For the main assertion, we want to apply Corollary 2.5, thus, we need to show that any $a \in \mathcal{M}_2^1$ is a regular value of the function f from (5.8). This is the case if and only if given any $a \in \mathcal{M}_2^1$ all $\zeta_\lambda \in I(a)$ are regular points, i.e. the derivatives

$$df((\alpha, \beta, \gamma)^t) : \mathbb{C} \times \mathbb{C} \times \mathbb{R}^+ \to \mathbb{C} \times \mathbb{R}$$

map surjectively - which implies that they possess rank three.
Let $a \in \mathcal{M}_2^1$ and consider

$$\zeta_\lambda = \begin{pmatrix} \alpha\lambda - \bar{\alpha}\lambda^2 & -\gamma^{-1} + \beta\lambda - \gamma\lambda^2 \\ \gamma\lambda - \bar{\beta}\lambda^2 + \gamma^{-1}\lambda^3 & -\alpha\lambda + \bar{\alpha}\lambda^2 \end{pmatrix} = \begin{pmatrix} A(\lambda) & B(\lambda) \\ \lambda\, C(\lambda) & -A(\lambda) \end{pmatrix} \in I(a). \tag{6.4}$$

By Theorem 4.5 we know that ζ_λ has no roots (a root in ζ_λ leads to a double root in a which is impossible by assumption). This fact will be the fundament of our argumentation. Instead of looking at f in its original form, we rather consider the complexification

$$f_c : \mathbb{C}^4 \times \mathbb{R}^+ \to \mathbb{C}^2 \times \mathbb{R}^+, \begin{pmatrix} \alpha \\ \bar{\alpha} \\ \beta \\ \bar{\beta} \\ \gamma \end{pmatrix} \mapsto \begin{pmatrix} a_1 \\ \bar{a_1} \\ a_2 \end{pmatrix} = \begin{pmatrix} -\bar{\alpha}^2 - \beta\gamma^{-1} - \bar{\beta}\gamma \\ -\alpha^2 - \bar{\beta}\gamma^{-1} - \beta\gamma \\ 2\alpha\bar{\alpha} + \beta\bar{\beta} + \gamma^2 + \gamma^{-2} \end{pmatrix} \tag{6.5}$$

and calculate the Jacobian matrix

$$J_{f_c} = \begin{pmatrix} 0 & -2\bar{\alpha} & -\gamma^{-1} & -\gamma & (\beta\gamma^{-2} - \bar{\beta}) \\ -2\alpha & 0 & -\gamma & -\gamma^{-1} & (\bar{\beta}\gamma^{-2} - \beta) \\ 2\bar{\alpha} & 2\alpha & \bar{\beta} & \beta & 2(\gamma - \gamma^{-3}) \end{pmatrix}. \tag{6.6}$$

We need to demonstrate that J_{f_c} has full rank. First assume $\alpha \neq 0$. Then the Jacobian can be transformed via Gaussian elimination into

$$J_{f_c} = \begin{pmatrix} -2\alpha & 0 & -\gamma & -\gamma^{-1} & (\bar{\beta}\gamma^{-2} - \beta) \\ 0 & -2\bar{\alpha} & -\gamma^{-1} & -\gamma & (\beta\gamma^{-2} - \bar{\beta}) \\ 0 & 0 & U & V & W \end{pmatrix}$$

where

$$\begin{aligned} U &= \bar{\beta} - \gamma\alpha^{-1}\bar{\alpha} - \gamma^{-1}\bar{\alpha}^{-1}\alpha \\ &= -\alpha^{-1}\bar{\alpha}(\gamma - \bar{\alpha}^{-1}\alpha\bar{\beta} + (\bar{\alpha}^{-1}\alpha)^2\gamma^{-1}) \\ &= -\alpha^{-1}\bar{\alpha}\; C(\bar{\alpha}^{-1}\alpha) \\ V &= \beta - \gamma^{-1}\alpha^{-1}\bar{\alpha} - \gamma\bar{\alpha}^{-1}\alpha \\ &= \alpha^{-1}\bar{\alpha}(-\gamma^{-1} + \bar{\alpha}^{-1}\alpha\beta - (\bar{\alpha}^{-1}\alpha)^2\gamma) \\ &= \alpha^{-1}\bar{\alpha}\; B(\bar{\alpha}^{-1}\alpha) \\ W &= 2(\gamma - \gamma^{-3}) + \bar{\alpha}^{-1}\alpha(\beta\gamma^{-2} - \bar{\beta}) + \alpha^{-1}\bar{\alpha}(\bar{\beta}\gamma^{-2} - \beta). \end{aligned}$$

If the Jacobian J_{f_c} had no full rank, then $U = V = W = 0$. But then ζ_λ had a root in $\bar{\alpha}^{-1}\alpha$ which contradicts the assumption $a \in \mathcal{M}_2^1$. Therefore, J_{f_c} has full rank when $\alpha \neq 0$.

If $\alpha = 0$, $A(\lambda) = 0$ and consequently, ζ_λ is off-diagonal and has a root if and only if the polynomials $B(\lambda)$ and $C(\lambda)$ have a common root. This is the case if and only if the resultant $\operatorname{res}(B, C)$ equals zero. This argumentation and the insight that ζ_λ has no root leads to the conclusion that the resultant must be nonzero

$$\operatorname{res}(B, C) \neq 0.$$

Nevertheless, let us explicitly calculate the resultant as the determinant of the Sylvester matrix (details can be found in Brieskorn [9]):

$$\begin{aligned}
\operatorname{res}(B, C) &= \det \begin{pmatrix} -\gamma & \beta & -\gamma^{-1} & 0 \\ 0 & -\gamma & \beta & -\gamma^{-1} \\ \gamma^{-1} & -\bar{\beta} & \gamma & 0 \\ 0 & \gamma^{-1} & -\bar{\beta} & \gamma \end{pmatrix} \\
&= -\gamma(-\gamma^3 - \gamma^{-1}\bar{\beta}^2 + \gamma^{-1} + \gamma\beta\bar{\beta}) + \gamma^{-1}(\gamma\beta^2 + \gamma^{-3} - \gamma^{-1}\bar{\beta}\beta - \gamma) \\
&= \beta^2 + \bar{\beta}^2 - \beta\bar{\beta}(\gamma^2 + \gamma^{-2}) + \gamma^4 + \gamma^{-4} - 2 \qquad (6.7) \\
&\neq 0.
\end{aligned}$$

The second equation is due to Laplace expansion with respect to the first column. Now we focus again on the complexified Jacobian J_{f_c}. Since $\alpha = 0$, the first two columns of (6.6) disappear and we complete by showing that the determinant of the remaining 3×3 matrix is nonzero.

$$\begin{aligned}
\det J_{f_c} &= \begin{pmatrix} -\gamma^{-1} & -\gamma & (\beta\gamma^{-2} - \bar{\beta}) \\ -\gamma & -\gamma^{-1} & (\bar{\beta}\gamma^{-2} - \beta) \\ \bar{\beta} & \beta & 2(\gamma - \gamma^{-3}) \end{pmatrix} \\
&= 2\gamma^{-2}(\gamma - \gamma^{-3}) - \beta\gamma(\beta\gamma^{-2} - \bar{\beta}) - \bar{\beta}\gamma(\bar{\beta}\gamma^{-2} - \beta) + \bar{\beta}\gamma^{-1}(\beta\gamma^{-2} - \bar{\beta}) \\
&\quad + \beta\gamma^{-1}(\bar{\beta}\gamma^{-2} - \beta) - 2\gamma^2(\gamma - \gamma^{-3}) \\
&= 2\gamma^{-1} - 2\gamma^{-5} - \beta^2\gamma^{-1} + \beta\bar{\beta}\gamma - \bar{\beta}^2\gamma^{-1} + \beta\bar{\beta}\gamma + \beta\bar{\beta}\gamma^{-3} - \bar{\beta}^2\gamma^{-1} \\
&\quad + \beta\bar{\beta}\gamma^{-3} - \beta^2\gamma^{-1} - 2\gamma^3 + 2\gamma^{-1} \\
&= -2\gamma^{-1}\left(\beta^2 + \bar{\beta}^2 - \beta\bar{\beta}(\gamma^2 + \gamma^{-2}) + \gamma^4 + \gamma^{-4} - 2\right) \\
&= -2\gamma^{-1} \operatorname{res}(B, C).
\end{aligned}$$

Since the resultant is nonzero, so is the determinant of the Jacobian matrix and the assertion is proved. With the second part of Corollary 2.5 and the original map f, we obtain that the dimension of the submanifold equals

$$\dim I(a) = 5 - 3 = 2.$$

q.e.d.

Lemma 6.3.
Consider $a \in \mathcal{M}_2^1$ and let $I(a)$ be its isospectral set. Fix a triplet $(\alpha, \beta, \gamma) \in \mathbb{C} \times \mathbb{C} \times \mathbb{R}^+$ such that $\zeta_\lambda \in I(a)$ is satisfied.
Then the right hand side terms of the Lax equations (5.1)

$$([\zeta_\lambda, U(\zeta_\lambda)], [\zeta_\lambda, V(\zeta_\lambda)])$$

form a basis of the tangent space of $I(a)$ in ζ_λ.

Proof.
With Theorem 2.7 (i) and Proposition 6.2 we immediately obtain that $T_{\zeta_\lambda}(I(a))$ is a two-dimensional $\mathbb{R}$-vector space. Instead of looking at the quite unmanageable terms $[\zeta_\lambda, U(\zeta_\lambda)]$ and $[\zeta_\lambda, V(\zeta_\lambda)]$ we rather consider the right hand side terms of the modified Lax equations (5.2). Since f from (5.8) and $I(a)$ satisfy the assumptions of Theorem 2.7 (ii) we know

$$T_{\zeta_\lambda}(I(a)) = \ker df(\zeta_\lambda),$$

so we are comfortably allowed to focus on the derivative's kernel (which is explicitly calculable) rather than on the abstract tangent space. As in the proof of Proposition 6.2 we regard the complexification f_c from (6.5) instead of f. As seen in Lemma 5.2, the right hand side terms $[\zeta_\lambda, U(\zeta_\lambda)]$ and $[\zeta_\lambda, V(\zeta_\lambda)]$ correspond to

$$\partial_x := \begin{pmatrix} \gamma^2 + \beta\gamma - \bar{\beta}\gamma^{-1} - \gamma^{-2} \\ \gamma^2 + \bar{\beta}\gamma - \beta\gamma^{-1} - \gamma^{-2} \\ -\alpha\beta + \bar{\alpha}\beta - 2\alpha\gamma + 2\bar{\alpha}\gamma^{-1} \\ -\bar{\alpha}\bar{\beta} + \alpha\bar{\beta} - 2\bar{\alpha}\gamma + 2\alpha\gamma^{-1} \\ -\alpha\gamma - \bar{\alpha}\gamma \end{pmatrix} \qquad \partial_y := i \begin{pmatrix} \gamma^{-2} + \beta\gamma - \bar{\beta}\gamma^{-1} - \gamma^2 \\ -\gamma^{-2} - \bar{\beta}\gamma + \beta\gamma^{-1} + \gamma^2 \\ -\alpha\beta - \bar{\alpha}\beta + 2\alpha\gamma + 2\bar{\alpha}\gamma^{-1} \\ \bar{\alpha}\bar{\beta} + \alpha\bar{\beta} - 2\bar{\alpha}\gamma - 2\alpha\gamma^{-1} \\ \bar{\alpha}\gamma - \alpha\gamma \end{pmatrix}$$

and the proof reduces to demonstrating the following two steps:

(a) ∂_x and $\partial_y \in \ker df_c((\alpha, \bar{\alpha}, \beta, \bar{\beta}, \gamma)^t)$ and

(b) ∂_x and ∂_y are linearly independent.

The derivative $df_c((\alpha, \bar{\alpha}, \beta, \bar{\beta}, \gamma)^t)$ is given by the Jacobian matrix (6.6) and we obtain

$$\begin{aligned} J_{f_c}\partial_x &= \begin{pmatrix} 0 & -2\bar{\alpha} & -\gamma^{-1} & -\gamma & (\beta\gamma^{-2} - \bar{\beta}) \\ -2\alpha & 0 & -\gamma & -\gamma^{-1} & (\bar{\beta}\gamma^{-2} - \beta) \\ 2\bar{\alpha} & 2\alpha & \bar{\beta} & \beta & 2(\gamma - \gamma^{-3}) \end{pmatrix} \begin{pmatrix} \gamma^2 + \beta\gamma - \bar{\beta}\gamma^{-1} - \gamma^{-2} \\ \gamma^2 + \bar{\beta}\gamma - \beta\gamma^{-1} - \gamma^{-2} \\ -\alpha\beta + \bar{\alpha}\beta - 2\alpha\gamma + 2\bar{\alpha}\gamma^{-1} \\ -\bar{\alpha}\bar{\beta} + \alpha\bar{\beta} - 2\bar{\alpha}\gamma + 2\alpha\gamma^{-1} \\ -\alpha\gamma - \bar{\alpha}\gamma \end{pmatrix} \\ &= \left(\begin{matrix} -2\bar{\alpha}\gamma^2 - 2\bar{\alpha}\bar{\beta}\gamma + 2\bar{\alpha}\beta\gamma^{-1} + 2\bar{\alpha}\gamma^{-2} + \alpha\beta\gamma^{-1} - \bar{\alpha}\beta\gamma^{-1} + 2\alpha - 2\bar{\alpha}\gamma^{-2} + \\ -2\alpha\gamma^2 - 2\alpha\beta\gamma + 2\alpha\bar{\beta}\gamma^{-1} + 2\alpha\gamma^{-2} + \alpha\beta\gamma - \bar{\alpha}\beta\gamma + 2\alpha\gamma^2 - 2\bar{\alpha} + \\ +2\bar{\alpha}\gamma^2 + 2\bar{\alpha}\beta\gamma - 2\bar{\alpha}\bar{\beta}\gamma^{-1} - 2\bar{\alpha}\gamma^{-2} + 2\alpha\gamma^2 + 2\alpha\bar{\beta}\gamma - 2\alpha\beta\gamma^{-1} - 2\alpha\gamma^{-2} - \alpha\beta\bar{\beta} + \bar{\alpha}\beta\bar{\beta} - \end{matrix} \right. \\ &\qquad \left. \begin{matrix} +\bar{\alpha}\bar{\beta}\gamma - \alpha\bar{\beta}\gamma + 2\bar{\alpha}\gamma^2 - 2\alpha - \alpha\beta\gamma^{-1} - \bar{\alpha}\beta\gamma^{-1} + \alpha\bar{\beta}\gamma + \bar{\alpha}\bar{\beta}\gamma \\ +\bar{\alpha}\bar{\beta}\gamma^{-1} - \alpha\bar{\beta}\gamma^{-1} + 2\bar{\alpha} - 2\alpha\gamma^{-2} - \alpha\bar{\beta}\gamma^{-1} - \bar{\alpha}\bar{\beta}\gamma^{-1} + \alpha\beta\gamma + \bar{\alpha}\beta\gamma \\ -2\alpha\bar{\beta}\gamma + 2\bar{\alpha}\bar{\beta}\gamma^{-1} - \bar{\alpha}\beta\bar{\beta} + \alpha\beta\bar{\beta} - 2\bar{\alpha}\beta\gamma + 2\alpha\beta\gamma^{-1} - 2\alpha\gamma^2 - 2\bar{\alpha}\gamma^2 + 2\alpha\gamma^{-2} + 2\bar{\alpha}\gamma^{-2} \end{matrix} \right) \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

and

$$J_{f_c}\partial_y = \begin{pmatrix} 0 & -2\bar{\alpha} & -\gamma^{-1} & -\gamma & (\beta\gamma^{-2}-\bar{\beta}) \\ -2\alpha & 0 & -\gamma & -\gamma^{-1} & (\bar{\beta}\gamma^{-2}-\beta) \\ 2\bar{\alpha} & 2\alpha & \bar{\beta} & \beta & 2(\gamma-\gamma^{-3}) \end{pmatrix} i \begin{pmatrix} \gamma^{-2}+\beta\gamma-\bar{\beta}\gamma^{-1}-\gamma^2 \\ -\gamma^{-2}-\bar{\beta}\gamma+\beta\gamma^{-1}+\gamma^2 \\ -\alpha\beta-\bar{\alpha}\beta+2\alpha\gamma+2\bar{\alpha}\gamma^{-1} \\ \bar{\alpha}\bar{\beta}+\alpha\bar{\beta}-2\bar{\alpha}\gamma-2\alpha\gamma^{-1} \\ \bar{\alpha}\gamma-\alpha\gamma \end{pmatrix}$$

$$= i \left(\begin{array}{l} 2\bar{\alpha}\gamma^{-2}+2\bar{\alpha}\bar{\beta}\gamma-2\bar{\alpha}\beta\gamma^{-1}-2\bar{\alpha}\gamma^2+\alpha\beta\gamma^{-1}+\bar{\alpha}\beta\gamma^{-1}-2\alpha-2\bar{\alpha}\gamma^{-2}- \\ -2\alpha\gamma^{-2}-2\alpha\beta\gamma+2\alpha\bar{\beta}\gamma^{-1}+2\alpha\gamma^2+\alpha\beta\gamma+\bar{\alpha}\beta\gamma-2\alpha\gamma^2-2\bar{\alpha}- \\ 2\bar{\alpha}\gamma^{-2}+2\bar{\alpha}\beta\gamma-2\bar{\alpha}\bar{\beta}\gamma^{-1}-2\bar{\alpha}\gamma^2-2\alpha\gamma^{-2}-2\alpha\bar{\beta}\gamma+2\alpha\beta\gamma^{-1}+2\alpha\gamma^2-\alpha\beta\bar{\beta}-\bar{\alpha}\beta\bar{\beta}+ \end{array} \right.$$

$$\left. \begin{array}{l} -\bar{\alpha}\bar{\beta}\gamma-\alpha\bar{\beta}\gamma+2\bar{\alpha}\gamma^2+2\alpha+\bar{\alpha}\beta\gamma^{-1}-\alpha\beta\gamma^{-1}-\bar{\alpha}\bar{\beta}\gamma+\alpha\bar{\beta}\gamma \\ -\bar{\alpha}\bar{\beta}\gamma^{-1}-\alpha\bar{\beta}\gamma^{-1}+2\bar{\alpha}+2\alpha\gamma^{-2}+\bar{\alpha}\bar{\beta}\gamma^{-1}-\alpha\bar{\beta}\gamma^{-1}-\bar{\alpha}\beta\gamma+\alpha\beta\gamma \\ +2\alpha\bar{\beta}\gamma+2\bar{\alpha}\bar{\beta}\gamma^{-1}+\bar{\alpha}\beta\bar{\beta}+\alpha\beta\bar{\beta}-2\bar{\alpha}\beta\gamma-2\alpha\beta\gamma^{-1}+2\bar{\alpha}\gamma^2-2\alpha\gamma^2-2\bar{\alpha}\gamma^{-2}+2\alpha\gamma^{-2} \end{array} \right)$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, part (a) is proven. For (b), we take the notation (6.4) from Proposition 6.2 and distinguish between the same two cases again. Thus, let first $\alpha \neq 0$ and consider $v, w \in \mathbb{C}$ such that

$$v\partial_x + w\partial_y = 0. \tag{6.8}$$

We need to demonstrate that $v = w = 0$. To do so, we explicitly focus on the system of five equations (6.8). The equation of the last component looks like

$$\begin{aligned} & -\alpha\gamma v - \bar{\alpha}\gamma v + i\bar{\alpha}\gamma w - i\alpha\gamma w \;=\; 0 \\ \Leftrightarrow\; & -\alpha(v+iw) - \bar{\alpha}(v-iw) \;=\; 0 \\ \Leftrightarrow\; & v - iw = \;-\alpha\bar{\alpha}^{-1}(v+iw). \end{aligned} \tag{6.9}$$

Analogously, the third equation from (6.8) provides (inserting (6.9) in the second implication)

$$\begin{aligned} & -\alpha\beta v + \bar{\alpha}\beta v - 2\alpha\gamma v + 2\bar{\alpha}\gamma^{-1}v - i\alpha\beta w - i\bar{\alpha}\beta w + 2i\alpha\gamma w + 2i\bar{\alpha}\gamma^{-1}w \;=\; 0 \\ \Rightarrow\; & (-\alpha\beta + 2\bar{\alpha}\gamma^{-1})(v+iw) + (\bar{\alpha}\beta - 2\alpha\gamma)(v-iw) \;=\; 0 \\ \Rightarrow\; & -2\bar{\alpha}\left(-\gamma^{-1} + \beta(\alpha\bar{\alpha}^{-1}) - \gamma(\alpha\bar{\alpha}^{-1})^2\right)(v+iw) \;=\; 0 \\ \Rightarrow\; & (v+iw)\, B(\alpha\bar{\alpha}^{-1}) = 0 \end{aligned} \tag{6.10}$$

and the fourth

$$\begin{aligned} & -\bar{\alpha}\bar{\beta}v + \alpha\bar{\beta}v - 2\bar{\alpha}\gamma v + 2\alpha\gamma^{-1}v + i\bar{\alpha}\bar{\beta}w + i\alpha\bar{\beta}w - 2i\bar{\alpha}\gamma w - 2i\alpha\gamma^{-1}w \;=\; 0 \\ \Rightarrow\; & (\alpha\bar{\beta} - 2\bar{\alpha}\gamma)(v+iw) + (2\alpha\gamma^{-1} - \bar{\alpha}\bar{\beta})(v-iw) \;=\; 0 \\ \Rightarrow\; & -2\bar{\alpha}\left(\gamma - \bar{\beta}(\alpha\bar{\alpha}^{-1}) + \gamma^{-1}(\alpha\bar{\alpha}^{-1})^2\right)(v+iw) \;=\; 0 \\ \Rightarrow\; & (v+iw)\, C(\alpha\bar{\alpha}^{-1}) = 0 \end{aligned} \tag{6.11}$$

with B, C from (6.4). If $B(\alpha\bar{\alpha}^{-1}) = C(\alpha\bar{\alpha}^{-1}) = 0$ then ζ_λ has a root in $\alpha\bar{\alpha}^{-1}$ which contradicts the assumption $a \in \mathcal{M}^1_2$. Therefore, at least one of them must be nonzero (by reality condition we even receive that both must be nonzero), so

$$v + iw = 0$$

and by (6.9) also

$$v - iw = 0.$$

In total, we obtain

$$v = \frac{1}{2}((v + iw) + (v - iw)) = 0$$

and

$$w = \frac{v + iw}{i} = 0.$$

When $\alpha = 0$, the last three entries of ∂_x and ∂_y equal zero and we can conduct the same approach as in the last part of the proof of Proposition 6.2: Consider the 2×2 matrix that consists of the remaining nonzero parts of ∂_x and ∂_y and calculate its determinant. If it happens to be nonzero, the vectors ∂_x and ∂_y are linearly independent.

$$\begin{aligned}
&\det\begin{pmatrix} \gamma^2 + \beta\gamma - \bar{\beta}\gamma^{-1} - \gamma^{-2} & i(\gamma^{-2} + \beta\gamma - \bar{\beta}\gamma^{-1} - \gamma^2) \\ \gamma^2 + \bar{\beta}\gamma - \beta\gamma^{-1} - \gamma^{-2} & i(-\gamma^{-2} - \bar{\beta}\gamma + \beta\gamma^{-1} + \gamma^2) \end{pmatrix} \\
&= i\Big((\gamma^2 + \beta\gamma - \bar{\beta}\gamma^{-1} - \gamma^{-2})(-\gamma^{-2} - \bar{\beta}\gamma + \beta\gamma^{-1} + \gamma^2) \\
&\quad -(\gamma^{-2} + \beta\gamma - \bar{\beta}\gamma^{-1} - \gamma^2)(\gamma^2 + \bar{\beta}\gamma - \beta\gamma^{-1} - \gamma^{-2})\Big) \\
&= i\Big(-1 - \bar{\beta}\gamma^3 + \beta\gamma + \gamma^4 - \beta\gamma^{-1} - \beta\bar{\beta}\gamma^2 + \beta^2 + \beta\gamma^3 + \bar{\beta}\gamma^{-3} + \bar{\beta}^2 - \beta\bar{\beta}\gamma^{-2} - \bar{\beta}\gamma \\
&\quad + \gamma^{-4} + \bar{\beta}\gamma^{-1} - \beta\gamma^{-3} - 1 - 1 - \bar{\beta}\gamma^{-1} + \beta\gamma^{-3} + \gamma^{-4} - \beta\gamma^3 - \beta\bar{\beta}\gamma^2 + \beta^2 + \beta\gamma^{-1} \\
&\quad + \bar{\beta}\gamma + \bar{\beta}^2 - \beta\bar{\beta}\gamma^{-2} - \bar{\beta}\gamma^{-3} + \gamma^4 + \bar{\beta}\gamma^3 - \beta\gamma - 1\Big) \\
&= 2i\Big(\beta^2 + \bar{\beta}^2 - \beta\bar{\beta}(\gamma^2 + \gamma^{-2}) + \gamma^4 + \gamma^{-4} - 2\Big) \\
&= 2i\,\mathrm{res}(B, C).
\end{aligned}$$

The last equality is due to our results in (6.7). The remaining argumentation goes strictly in line with Proposition 6.2: Since $a \in M^1_2$ by assumption, we obtain that ζ_λ has no roots, which is equivalent to $\mathrm{res}(B, C) \neq 0$ since $\alpha = 0$. Consequently, the determinant is nonzero and the proof is finished. **q.e.d.**

Definition 6.4.
Given a topological space V and two arbitrary points $x, y \in V$ a *path-component* of V is an equivalence class of V under the equivalence relation which makes x equivalent to y if there is a path from x to y. Any path-component is contained in a connected component.

Remark 6.5.
Obviously, a connected component which is path connected is a path-component. This criterion will be used in the following

Lemma 6.6.
Let $a \in \mathcal{M}_2^1$ and $\zeta_\lambda \in I(a)$. Then the orbit of the global flow ϕ from Corollary 5.10

$$A_{\zeta_\lambda} := \{\phi(x,y)(\zeta_\lambda) \mid (x,y) \in \mathbb{R}^2\}$$

is a path-component in the isospectral set $I(a)$.

Proof.
Since $\mathbb{R}^2$ is path-connected and

$$\phi_{\zeta_\lambda} : (x,y) \mapsto \phi(x,y)(\zeta_\lambda)$$

is a continuous function, the sets A_{ζ_λ} are path-connected. Thus, it remains to show that these sets are connected components. By using the same argument we obtain connectedness of the considered set, so the question now is whether the maximality condition is satisfied, i.e. whether there is no connected set containing A_{ζ_λ}. To do so, we demonstrate the fact that A_{ζ_λ} is open and closed with respect to the subspace topology of $I(a)$.
If openness is proven, we directly obtain closedness: Take any $\tilde{\zeta}_\lambda$ from the complement $I(a) \setminus A_{\zeta_\lambda}$. Then the respective orbit satisfies

$$A_{\tilde{\zeta}_\lambda} \cap A_{\zeta_\lambda} = \emptyset$$

and remains in the complement $I(a) \setminus A_{\zeta_\lambda}$ because if there were $x_1, x_2, y_1, y_2 \in \mathbb{R}$ such that

$$\phi(x_1, y_1)(\tilde{\zeta}_\lambda) = \phi(x_2, y_2)(\zeta_\lambda)$$

then

$$\begin{aligned} \tilde{\zeta}_\lambda &= \phi(-x_1, -y_1)(\phi(x_1, y_1)(\tilde{\zeta}_\lambda)) \\ &= \phi(-x_1, -y_1)(\phi(x_2, y_2)(\zeta_\lambda)) \\ &= \phi(x_2 - x_1, y_2 - y_1)(\zeta_\lambda) \end{aligned}$$

which implies $\tilde{\zeta}_\lambda \in A_{\zeta_\lambda}$ and contradicts the assumption. Thus, the complement can be written as

$$I(a) \setminus A_{\zeta_\lambda} = \bigcup_{\tilde{\zeta}_\lambda \in I(a) \setminus A_{\zeta_\lambda}} A_{\tilde{\zeta}_\lambda}$$

and since the sets $A_{\tilde{\zeta}_\lambda}$ are open in $I(a)$ by assumption, the entire complement is open in $I(a)$ as an infinite union of open sets. In particular, A_{ζ_λ} is closed in $I(a)$.
To complete the proof, we need to demonstrate openness of the sets A_{ζ_λ} with respect to the subspace topology of $I(a)$. We look at the restriction of the function f from (5.8) on $U := f^{-1}(\mathcal{M}_2^1)$:

$$\tilde{f} := f\big|_U : U \subset \mathbb{C} \times \mathbb{C} \times \mathbb{R}^+ \to \mathbb{C} \times \mathbb{R}^+, \quad \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \mapsto \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

Take an arbitrary $\tilde{\zeta}_\lambda \in A_{\zeta_\lambda}$ with $\phi_{\zeta_\lambda}(\tilde{x}, \tilde{y}) = \tilde{\zeta}_\lambda$ and $\det \tilde{\zeta}_\lambda = \lambda a(\lambda)$. We need to demonstrate that there exists a neighbourhood of $\tilde{\zeta}_\lambda$ in A_{ζ_λ} which is open in $I(a)$. Without loss of generality, we can assume $\tilde{\zeta}_\lambda = \zeta_\lambda$ and $\tilde{x} = \tilde{y} = 0$ since $A_{\tilde{\zeta}_\lambda} = A_{\zeta_\lambda}$ (or use an appropriate coordinate transfer). We define the space

$$X_{\zeta_\lambda} := \ker d\tilde{f}(\zeta_\lambda)$$

and as stated in the proof of Lemma 6.3 we know that

$$\dim(X_{\zeta_\lambda}) = \dim(\ker df(\zeta_\lambda)) = \dim(T_{\zeta_\lambda}(I(a))) = 2.$$

By Basis Extension Theorem we obtain an orthogonal complement Y_{ζ_λ} with $\dim(Y_{\zeta_\lambda}) = 3$ and $X_{\zeta_\lambda} \cap Y_{\zeta_\lambda} = \emptyset$ such that

$$X_{\zeta_\lambda} \oplus Y_{\zeta_\lambda} \cong \mathbb{R}^5.$$

Thus, we can consider U as a subset of $X_{\zeta_\lambda} \oplus Y_{\zeta_\lambda}$ and we denote the $X_{\zeta_\lambda}, Y_{\zeta_\lambda}$ components of ζ_λ by $\zeta_\lambda = (\zeta_X, \zeta_Y)$. As shown in the proof of Proposition 6.2, the determinant polynomial a is a regular value, so $\zeta_\lambda \in I(a)$ is a regular point, which means that, by construction of X_{ζ_λ} and Y_{ζ_λ}, the partial derivative

$$d\tilde{f}_Y(\zeta_\lambda) : Y_{\zeta_\lambda} \to \mathbb{C} \times \mathbb{R} \in \mathcal{L}(Y_{\zeta_\lambda}, \mathbb{R}^3)$$

is invertible and we can apply the Implicit Function Theorem A.2 in ζ_λ. Hence, there are open sets $U' \subset X_{\zeta_\lambda}$ containing ζ_X and $U'' \subset Y_{\zeta_\lambda}$ comprising ζ_Y as well as a continuously differentiable function

$$g : U' \to U''$$

such that

$$\zeta'_\lambda = (\zeta'_X, \zeta'_Y) \in (U' \times U'') \cap I(a) \Leftrightarrow \zeta'_X \in U' \text{ and } \zeta'_Y = g(\zeta'_X).$$

Consider now the projected flow with respect to the X_{ζ_λ}-coordinate

$$\phi^X_{\zeta_\lambda} : \mathbb{R}^2 \to X_{\zeta_\lambda},\ (x,y) \mapsto \left(\phi_{\zeta_\lambda}(x,y)\right)_X$$

and examine the conditions of the Inverse Function Theorem A.1 in the point $(0,0)$: $\phi^X_{\zeta_\lambda} = \pi_X \circ \phi_{\zeta_\lambda}$ where π_X is the projection with respect to the X_{ζ_λ}-coordinate. The global flow ϕ_{ζ_λ} is partially differentiable by assumption and the partial derivatives are continuous. Thus, the flow is continuously differentiable as well as the projection (as a linear map between finite dimensional vector spaces) and consequently, the entire function $\phi^X_{\zeta_\lambda}$ is continuously differentiable. Furthermore, by Lemma 6.3 we know that the right hand side terms of the Lax equations form a basis of $T_{\zeta_\lambda}(I(a)) = \ker d\tilde{f}(\zeta_\lambda) = X_{\zeta_\lambda}$ and hence, the derivative

$$d\phi_{\zeta_\lambda}(0,0) : \mathbb{R}^2 \to X_{\zeta_\lambda},\ (x,y) \mapsto x\,[\zeta_\lambda, U(\zeta_\lambda)] + y\,[\zeta_\lambda, V(\zeta_\lambda)]$$

is an isomorphism. In addition, the projection π_X is a linear map, so $d\pi_X(\zeta_\lambda) = \pi_X$, and acts on X_{ζ_λ} as the identity. Putting all together yields

$$d\phi^X_{\zeta_\lambda}(0,0) = d\pi_X(\zeta_\lambda) \circ d\phi_{\zeta_\lambda}(0,0) = \pi_X \circ \underbrace{d\phi_{\zeta_\lambda}(0,0)}_{\in X_{\zeta_\lambda}} = d\phi_{\zeta_\lambda}(0,0).$$

Since the right hand side is an isomorphism, so is the left hand side and all assumptions of the Inverse Function Theorem are satisfied. We obtain an open neigbourhood $U_0 \subset \mathbb{R}^2$ from $(0,0)$ such that

$$O := \phi^X_{\zeta_\lambda}(U_0) \subset X_{\zeta_\lambda}$$

contains ζ_λ and the restriction

$$\phi^X_{\zeta_\lambda}\big|_{U_0} : U_0 \to O$$

is a diffeomorphism. By making U_0 sufficiently small we can assume

$$O \subset U'.$$

Then the set

$$\Omega := O_0 \times U''$$

is open and consequently,

$$\{\phi_{\zeta_\lambda}(x,y) \mid (x,y) \in U_0\} = \Omega \cap I(a) \tag{6.12}$$

is open with respect to the subspace topology of $I(a)$ and a neigbourhood of ζ_λ that lies in A_{ζ_λ}. **q.e.d.**

Remark 6.7.
At first glance, one could think that the usage of the Implicit Function Theorem in the last proof was redundant and that it is possible to start directly with Inverse Function Theorem. Unfortunately, this approach would be wrong, the proof needs to be conducted in its full length. The last equality (6.12) is only guaranteed by the Implicit Function Theorem, because in general, just "$\subset$" holds true. Given any $(x,y) \in U_0$ there exists one unique $\phi_{\zeta_\lambda}(x,y) = \tilde{\zeta}_\lambda = (\tilde{\zeta}_X, \tilde{\zeta}_Y)$ on the left hand side, whereas on the right hand side there might be another element $\zeta'_\lambda = (\tilde{\zeta}_X, \zeta'_Y)$ besides $\tilde{\zeta}_\lambda$ with $\tilde{\zeta}_Y \neq \zeta'_Y$. The Implicit Function Theorem ensures that any element on the right hand side is uniquely defined by its X_{ζ_λ}-coordinate, so it is impossible that both $\tilde{\zeta}_\lambda$ and ζ'_λ are contained at the same time.

Theorem 6.8.
Let $a \in \mathcal{M}_2^1$ with roots $(\lambda_1, \bar{\lambda}_1^{-1}, \lambda_2, \bar{\lambda}_2^{-1})$ pairwise distinct. Then the flows ϕ act transitively on $I(a)$.

Proof.
Due to Lemma 6.6, the isospectral set $I(a)$ consists of possibly several path-components of the form

$$A_{\zeta_\lambda} = \{\tilde{\zeta}_\lambda(x,y) = \phi(x,y)(\zeta_\lambda) \mid (x,y) \in \mathbb{R}^2\}$$

with an appropriate $\zeta_\lambda \in I(a)$. We need to demonstrate that, indeed, the entire isospectral set is described by exactly one of these A_{ζ_λ}. The proof of Lemma 6.6 has shown that A_{ζ_λ} is open and closed in $I(a)$. By Proposition 5.9, the isospectral set $I(a)$ is compact and therefore, A_{ζ_λ} itself is compact as a closed subset of a compact set. Consequently, the restriction of the corresponding flows $\alpha(\phi(x,y)), \beta(\phi(x,y)), \gamma(\phi(x,y))$ on A_{ζ_λ} have compact images thanks to the continuity of the mappings. We focus on

$$\gamma\big|_{A_{\zeta_\lambda}} : A_{\zeta_\lambda} \to \mathbb{R}^+,\ \phi(x,y)(\zeta_\lambda) \mapsto \gamma(\phi(x,y)(\zeta_\lambda))$$

with compact image

$$\operatorname{im}(\gamma\big|_{A_{\zeta_\lambda}}) \subset \mathbb{R}^+.$$

Compact subsets of $\mathbb{R}$ possess a maximum value, so there is at least one $\zeta^*_\lambda \in A_{\zeta_\lambda}$ such that

$$\gamma(\zeta^*_\lambda) \geq \gamma(\zeta'_\lambda) \quad \text{for all } \zeta'_\lambda \in A_{\zeta_\lambda}.$$

So far, we have demonstrated that every path-component of $I(a)$ has a local maximum of the continuous function

$$\gamma\big|_{I(a)} : \zeta_\lambda \to \gamma(\zeta_\lambda).$$

Hence, we are interested in finding local maxima of this map. The structure for the rest of the proof is the following. We will see that there exists exactly one local maximum (which then must be ζ^*_λ). Consequently, any two path-components of $I(a)$ have at least ζ^*_λ as a common element and therefore, must be identical and we have proven the assertion. To show this, we first need to consider the critical points of γ. Notice that

$$\zeta_\lambda = \begin{pmatrix} 0 & B(\lambda) \\ \lambda C(\lambda) & 0 \end{pmatrix} \quad \Leftrightarrow \quad \frac{\partial \gamma}{\partial x} = \frac{\partial \gamma}{\partial y} = 0. \tag{6.13}$$

This is a direct consequence of the respective modified Lax equations (5.2)

$$\frac{\partial \gamma}{\partial x} = -\gamma(\alpha + \bar{\alpha}) \qquad \frac{\partial \gamma}{\partial y} = -i\gamma(\alpha - \bar{\alpha})$$

and the form of the diagonal elements

$$A(\lambda) = \alpha\lambda - \bar{\alpha}\lambda^2 = \lambda(\alpha - \bar{\alpha}\lambda)$$

because

$$A(\lambda) = 0 \text{ for all } \lambda \in \mathbb{C} \quad \Leftrightarrow \quad \alpha = 0 \quad \Leftrightarrow \quad \frac{\partial \gamma}{\partial x} = \frac{\partial \gamma}{\partial y} = 0.$$

The last equivalence holds since the condition $\alpha + \bar{\alpha} = 0$ implies $\alpha \in i\mathbb{R}$ and $\alpha - \bar{\alpha} = 0$ means $\alpha \in \mathbb{R}$ such that in total $\alpha = 0$. This relationship along with the reality condition (4.2) restricts the set of local maxima candidates on four different elements of $I(a)$: Either B has both roots inside the unit circle or both outside or one inside and one outside (Figure 3). In

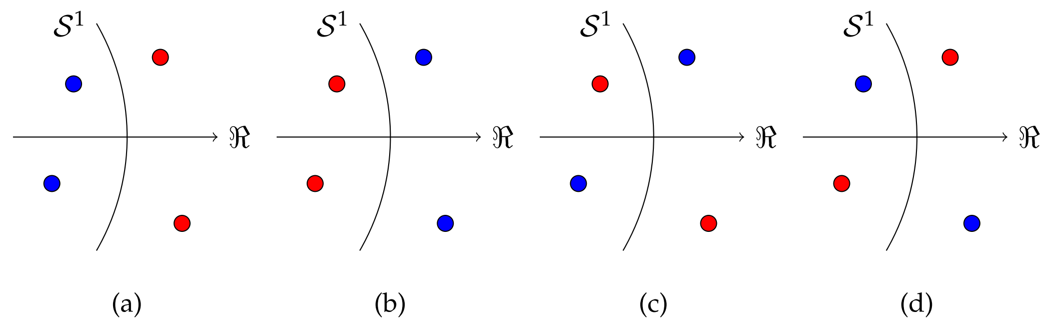


Figure 3: Possible off-diagonal elements in $I(a)$ given the roots of the determinant polynomial a. Blue ones belong to the upper right hand side entry $B(\lambda)$, red ones to the lower left hand side entry $C(\lambda)$.

the last case, there are only two scenarios imaginable, since due to the reality condition both roots must be part of different root groups $(\lambda_i, \bar{\lambda}_i^{-1})$, $i = 1, 2$.
Since $\gamma(\zeta^*_\lambda) > 0$ for any $\zeta'_\lambda \in \mathcal{P}_2$ and the logarithm function is strictly monotonous we will

consider $\log\gamma$ than γ for reasons of simplicity in the upcoming computations. We calculate the Hessian matrix $\mathcal{H}(\log\gamma)$ using the formulas of the modified Lax equations (5.2):

$$\begin{aligned}\frac{\partial}{\partial x}\log\gamma &= -\alpha-\bar{\alpha}\\ \frac{\partial^2}{\partial x^2}\log\gamma &= -(-2\gamma^{-2}+(\beta+\bar{\beta})\gamma-(\beta+\bar{\beta})\gamma^{-1}+2\gamma^2)\\ &= (\gamma^{-1}-\gamma)\left(2(\gamma+\gamma^{-1})+(\beta+\bar{\beta})\right).\end{aligned}$$

Analogously, the derivatives with respect to the y-component yield

$$\begin{aligned}\frac{\partial}{\partial y}\log\gamma &= i(\bar{\alpha}-\alpha)\\ \frac{\partial^2}{\partial y^2}\log\gamma &= i\left(-i(\gamma^{-2}+\bar{\beta}\gamma-\beta\gamma^{-1}-\gamma^2)-i(\gamma^{-2}+\beta\gamma-\bar{\beta}\gamma^{-1}-\gamma^2)\right)\\ &= 2\gamma^{-2}+(\beta+\bar{\beta})\gamma-(\beta+\bar{\beta})\gamma^{-1}-2\gamma^2\\ &= (\gamma^{-1}-\gamma)\left(2(\gamma+\gamma^{-1})-(\beta+\bar{\beta})\right).\end{aligned}$$

The mixed partial derivatives must be identical by Lemma 5.5, so we obtain

$$\begin{aligned}\frac{\partial^2}{\partial x\partial y}\log\gamma &= i(-\gamma^{-2}+\bar{\beta}\gamma-\beta\gamma^{-1}+\gamma^2+\gamma^{-2}-\beta\gamma+\bar{\beta}\gamma^{-1}-\gamma^2)\\ &= i(\bar{\beta}-\beta)(\gamma+\gamma^{-1})\\ \frac{\partial^2}{\partial y\partial x}\log\gamma &= i(\bar{\beta}-\beta)(\gamma+\gamma^{-1}).\end{aligned}$$

In order to figure out which elements of the four possibilites above define local maxima, we need to know the definite quadratic form of the Hessian $\mathcal{H}(\log\gamma)$. Due to its favourable symmetric, two-by-two form we know that both eigenvalues are real numbers. We will determine their sign by examining the determinant and the trace:

$$\begin{aligned}\det\left(\mathcal{H}(\log\gamma)\right) &= \frac{\partial}{\partial x}\log\gamma\frac{\partial}{\partial y}\log\gamma-(\frac{\partial^2}{\partial x\partial y}\log\gamma)^2\\ &= (\gamma-\gamma^{-1})^2\left(4(\gamma+\gamma^{-1})^2-(\beta+\bar{\beta})^2\right)+(\bar{\beta}-\beta)^2(\gamma+\gamma^{-1})^2\\ &= 4(\gamma^2-\gamma^{-2})^2+4\bar{\beta}^2+4\beta^2-4\beta\bar{\beta}(\gamma+\gamma^{-2}).\end{aligned}\tag{6.14}$$

In the last equation we used that

$$(\gamma-\gamma^{-1})^2(\gamma+\gamma^{-1})^2=(\gamma^2-\gamma^{-2})^2$$

and

$$\begin{aligned}(\gamma+\gamma^{-1})^2(\bar{\beta}-\beta)^2 &= (\gamma^2+2+\gamma^{-2})(\bar{\beta}^2-2\bar{\beta}\beta+\beta^2)\\ &= \bar{\beta}^2\gamma^2-2\bar{\beta}\beta\gamma^2+\beta^2\gamma^2+2\bar{\beta}^2-4\bar{\beta}\beta+2\beta^2+\bar{\beta}^2\gamma^{-2}-2\bar{\beta}\beta\gamma^{-2}+\beta^2\gamma^{-2}\\ (\gamma-\gamma^{-1})^2(\beta+\bar{\beta})^2 &= (\gamma^2-2+\gamma^{-2})(\beta^2+2\bar{\beta}\beta+\bar{\beta}^2)\\ &= \beta^2\gamma^2+2\bar{\beta}\beta\gamma^2+\bar{\beta}^2\gamma^2-2\beta^2-4\bar{\beta}\beta-2\bar{\beta}^2+\beta^2\gamma^{-2}+2\bar{\beta}\beta\gamma^{-2}+\bar{\beta}^2\gamma^{-2}\end{aligned}$$

which yields

$$(\gamma+\gamma^{-1})^2(\bar{\beta}-\beta)^2-(\gamma-\gamma^{-1})^2(\beta+\bar{\beta})^2=4\bar{\beta}^2+4\beta^2-4\beta\bar{\beta}(\gamma^2+\gamma^{-2}).$$

The trace of the Hessian looks like

$$\begin{aligned}\operatorname{tr}\left(\mathcal{H}(\log\gamma)\right)&=\frac{\partial^2}{\partial x^2}\log\gamma+\frac{\partial^2}{\partial y^2}\log\gamma\\&=4(\gamma^{-2}-\gamma^2).\end{aligned}\tag{6.15}$$

Both terms, the determinant and the trace, are quite unmanageable and besides, they do not provide any information about the location of the roots of $B(\lambda)$. Therefore, we need to substitute β and γ by expressions that only depend on the roots λ_1, λ_2. Since the product of the roots equals one, we immediately obtain

$$\lambda_1\lambda_2 \in \mathbb{R}\setminus\{0\}.$$

Thus, when writing λ_1 and λ_2 in polar coordinates, they have the following form

$$\lambda_1 = r_1 e^{i\varphi} \qquad \lambda_2 = r_2 e^{-i\varphi}$$

where $r_1, r_2 \in \mathbb{R}$, $\varphi \in [0, 2\pi]$. We compare both representations of the off-diagonal entry $B(\lambda)$

$$\begin{aligned}B(\lambda)&=-\gamma\lambda^2+\beta\lambda-\gamma^{-1}\\B(\lambda)&=-\gamma(\lambda-\lambda_1)(\lambda-\lambda_2)\\&=-\gamma\lambda^2+\gamma(\lambda_1+\lambda_2)\lambda-\gamma\lambda_1\lambda_2\end{aligned}$$

and obtain

$$\gamma\lambda_1\lambda_2=-\gamma^{-1} \quad\Leftrightarrow\quad \gamma=\frac{1}{\sqrt{\lambda_1\lambda_2}}=\frac{1}{\sqrt{r_1r_2}}$$

as well as

$$\beta=\gamma(\lambda_1+\lambda_2)=\frac{\lambda_1+\lambda_2}{\sqrt{r_1r_2}}=\sqrt{\frac{r_1}{r_2}}e^{i\varphi}+\sqrt{\frac{r_2}{r_1}}e^{-i\varphi}.$$

Furthermore, we can compute some of the expressions which appear in the determinant term:

$$\begin{aligned}\beta^2&=\frac{r_1}{r_2}e^{i2\varphi}+2+\frac{r_2}{r_1}e^{-i2\varphi}\\\beta\bar{\beta}&=\left(\sqrt{\frac{r_1}{r_2}}e^{i\varphi}+\sqrt{\frac{r_2}{r_1}}e^{-i\varphi}\right)\left(\sqrt{\frac{r_1}{r_2}}e^{-i\varphi}+\sqrt{\frac{r_2}{r_1}}e^{i\varphi}\right)\\&=\frac{r_1}{r_2}+2\cos(2\varphi)+\frac{r_2}{r_1}.\end{aligned}$$

Now we insert our findings in the determinant (6.14) and the trace (6.15)

$$\begin{aligned}
\operatorname{tr}\left(\mathcal{H}(\log\gamma)\right) &= 4\left(r_1r_2 - \frac{1}{r_1r_2}\right)\\
\det\left(\mathcal{H}(\log\gamma)\right) &= 4\left[\left(\frac{1}{r_1r_2} - r_1r_2\right)^2 + 4 + 2\left(\frac{r_1}{r_2} + \frac{r_2}{r_1}\right)\cos(2\varphi) - \right.\\
&\qquad \left. - \left(\frac{1}{r_1r_2} + r_1r_2\right)\left(\frac{r_1}{r_2} + \frac{r_2}{r_1} + 2\cos(2\varphi)\right)\right]\\
&= 4\left[\left(\frac{1}{r_1r_2} + r_1r_2\right)^2 + 2\left(\frac{r_1}{r_2} + \frac{r_2}{r_1} - \frac{1}{r_1r_2} - r_1r_2\right)\cos(2\varphi) - \right.\\
&\qquad \left. - \left(\frac{1}{r_1r_2} + r_1r_2\right)\left(\frac{r_1}{r_2} + \frac{r_2}{r_1} +\right)\right]\\
&= 4\left[\left(\frac{1}{r_1r_2} + r_1r_2\right)\left(-\frac{r_1}{r_2} - \frac{r_2}{r_1} + \frac{1}{r_1r_2} + r_1r_2\right)\right.\\
&\qquad \left. +2\left(\frac{r_1}{r_2} + \frac{r_2}{r_1} - \frac{1}{r_1r_2} - r_1r_2\right)\cos(2\varphi)\right]\\
&= 4\underbrace{\left(\frac{1}{r_1r_2} + r_1r_2 - \frac{r_1}{r_2} - \frac{r_2}{r_1}\right)}_{M}\underbrace{\left(\frac{1}{r_1r_2} + r_1r_2 - 2\cos(2\varphi)\right)}_{N}.
\end{aligned}$$

The determinant's sign is defined by the components M and N. Since the function

$$g : \mathbb{R}^+ \to \mathbb{R}^+,\ x \mapsto x + \frac{1}{x}$$

satisfies

$$g(x) \geq 2 \ \text{ for all } x \in \mathbb{R}^+$$

and additionally,

$$2\cos(2\varphi) \leq 2 \ \text{ for all } \varphi \in [0, 2\pi]$$

holds, we immediately receive

$$N = g(r_1r_2) - 2\cos(2\varphi) \geq 0$$

and the determinant's sign only depends on M. The function g strictly decreases on $(0, 1]$, increases on $[1, \infty)$ and satisfies

$$g(x) = g(x^{-1}) \quad \text{ for all } x \in \mathbb{R}^+.$$

Now we analyze which values of M are assumed in each of the four cases from Figure 3.

Cases 1 and 2:
If the roots of $B(\lambda)$ are both outside or both inside the unit circle, then $r_1, r_2 > 1$ or $r_1, r_2 < 1$ and consequently,

$$r_1r_2, r_1^2, r_2^2 > 1 \quad \text{ or } \quad \frac{1}{r_1r_2}, \frac{1}{r_1^2}, \frac{1}{r_2^2} > 1$$

holds. Therefore,

$$r_1 r_2 > \frac{r_1 r_2}{r_2^2} = \frac{r_1}{r_2} \qquad \text{and} \qquad r_1 r_2 > \frac{r_1 r_2}{r_1^2} = \frac{r_2}{r_1}$$

or

$$\frac{1}{r_1 r_2} > \frac{\frac{1}{r_1 r_2}}{\frac{1}{r_1^2}} = \frac{r_1}{r_2} \qquad \text{and} \qquad \frac{1}{r_1 r_2} > \frac{\frac{1}{r_1 r_2}}{\frac{1}{r_2^2}} = \frac{r_2}{r_1}.$$

Strict monotonicity of g yields

$$M = g(r_1 r_2) - g\left(\frac{r_1}{r_2}\right) > 0$$

or

$$M = g\left(\frac{1}{r_1 r_2}\right) - g\left(\frac{r_1}{r_2}\right) > 0.$$

This means that the determinant of the Hessian is positive in both cases which implies that both eigenvalues of the Hessian have the same sign.

Cases 3 and 4:
If the roots of $B(\lambda)$ are outside and inside the unit circle, then $r_1 > 1 > r_2$ or $r_2 > 1 > r_1$ and consequently,

$$\frac{r_1}{r_2} > 1,\ r_2^2 < 1,\ \frac{1}{r_1^2} < 1 \quad \text{or} \quad \frac{r_2}{r_1} > 1,\ r_1^2 < 1,\ \frac{1}{r_2^2} < 1$$

holds. Therefore,

$$\frac{r_1}{r_2} > \frac{r_1}{r_2} r_2^2 = r_1 r_2 \qquad \text{and} \qquad \frac{r_1}{r_2} > \frac{r_1}{r_2}\frac{1}{r_1^2} = \frac{1}{r_1 r_2}$$

or

$$\frac{r_2}{r_1} > \frac{r_2}{r_1} r_1^2 = r_1 r_2 \qquad \text{and} \qquad \frac{r_2}{r_1} > \frac{r_2}{r_1}\frac{1}{r_2^2} = \frac{1}{r_1 r_2}.$$

Strict monotonicity of g yields

$$M = g(r_1 r_2) - g\left(\frac{r_1}{r_2}\right) < 0$$

or

$$M = g\left(r_1 r_2\right) - g\left(\frac{r_2}{r_1}\right) < 0.$$

This means that the determinant of the Hessian is negative in cases 3 and 4 which implies that both eigenvalues have different signs and the Hessian is indefinite. It follows that these two cases describe saddle points and can be excluded as extremum candidates.

Consequently, we focus on cases 1 and 2 and take the trace of the Hessian into consideration

$$\operatorname{tr}\left(\mathcal{H}(\log \gamma)\right) = 4\tilde{g}(r_1 r_2)$$

with
$$\tilde{g} : \mathbb{R}^+ \to \mathbb{R},\, x \mapsto x - \frac{1}{x}.$$
This function $\tilde{g}$ is strictly increasing with
$$\lim_{x\to-\infty} \tilde{g}(x) = -\infty \qquad \text{and} \qquad \lim_{x\to\infty} \tilde{g}(x) = \infty$$
and root in $x = 1$. Therefore, the trace is negative when $B(\lambda)$ has both roots inside the unit circle. This implies that both eigenvalues are negative, so the Hessian is negative-definite and this element maximizes γ. Analogously, the other element in which $B(\lambda)$ has roots outside the unit circle minimizes γ. Since there are no other extremum-candidates left, we have demonstrated that there is exactly one maximizing element ζ_λ^* (shown in the first picture of Figure 3) and with the above argumentation, we have proven the assertion. **q.e.d.**

Remark 6.9.
We can find the sinh-Gordon equation in the trace expression (6.15) by substituting
$$u := \ln(\gamma).$$
This yields
$$\Delta u + 8\sinh(2u) = 0$$
After a suitable change of coordinates (as conducted in the end of Remark 5.7) we obtain the sinh-Gordon equation in its well-known form
$$\Delta u + \sinh(2u) = 0.$$

The last theorem is quite spectacular and entails the question whether it is possible to obtain a similar result when $a \notin \mathcal{M}_2^1$. We will see that the answer is yes, although the rationale of this assertion is not obvious at all at first glance. If a has at least one double root, we cannot apply the Implicit Function Theorem and the entire argumentation of the last pages breaks down. Consequently, we need a new or modified approach and we distinguish between the only remaining scenarios.

a) a has exactly one double root on $\mathcal{S}^1$.

b) a has two double roots on $\mathcal{S}^1$ (of course, the case that a has a quadruple root is included).

c) a has two different double roots which are swapped by the mapping $\lambda \mapsto \bar{\lambda}^{-1}$.

6.2 Case a): The polynomial $a(\lambda)$ has exactly one double root on $\mathcal{S}^1$

In this case, the determinant polynomial is of shape
$$a(\lambda) = (\lambda - \lambda_1)^2(\lambda - \lambda_2)(\lambda - \bar{\lambda}_2^{-1})$$
with $\lambda_1 \in \mathcal{S}^1$. By Theorem 4.5 we immediately obtain that any $\zeta_\lambda \in I(a)$ has the root λ_1. This suggests to divide ζ_λ by λ_1 with the hope of bringing ζ_λ in a form which resembles the standard case.
$$\tilde{\zeta}_\lambda = \frac{1}{\lambda - \lambda_1}\zeta_\lambda \tag{6.16}$$

and

$$\det\left(\tilde{\xi}_\lambda\right) = \lambda \frac{1}{(\lambda-\lambda_1)^2} a(\lambda) = \lambda \tilde{a}(\lambda). \tag{6.17}$$

Unfortunately, $\tilde{a}(\lambda)$ does in general not own any symmetry-property comparable to the ones of Theorem 4.3. However, notice that $\tilde{\xi}_\lambda$ is a matrix polynomial of degree two as well as the respective determinant polynomial $\tilde{a}(\lambda)$ is of degree two and has two different roots. This motivates us to analyze Polynomial Killing Fields of spectral genus $g = 1$, although yet it remains unclear how a relationship to $\tilde{\xi}_\lambda$ could be established.

Definition 6.10 (Potentials)**.**
The set of *potentials* is the following set of quadratic polynomials with matrix-valued coefficients:

$$\mathcal{P}_1 := \left\{ \xi_\lambda = \begin{pmatrix} 0 & \hat{\beta}^{-1} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} i\hat{\alpha} & \hat{\beta} \\ -\hat{\beta} & -i\hat{\alpha} \end{pmatrix} \lambda + \begin{pmatrix} 0 & 0 \\ -\hat{\beta}^{-1} & 0 \end{pmatrix} \lambda^2 \middle| \hat{\alpha} \in \mathbb{R}, \hat{\beta} \in \mathbb{R}^- \right\}.$$

Every $\xi_\lambda \in \mathcal{P}_1$ can be compactly written as

$$\xi_\lambda = \begin{pmatrix} i\hat{\alpha}\lambda & \hat{\beta}^{-1} + \hat{\beta}\lambda \\ -\hat{\beta}\lambda - \hat{\beta}^{-1}\lambda^2 & -i\hat{\alpha}\lambda \end{pmatrix}$$

and satisfies the *reality condition*

$$\lambda^2 \overline{\xi}^t_{\frac{1}{\bar{\lambda}}} = \lambda^2 \begin{pmatrix} -i\hat{\alpha}\lambda^{-1} & -\hat{\beta}\lambda^{-1} - \hat{\beta}^{-1}\lambda^{-2} \\ \hat{\beta}^{-1} + \hat{\beta}\lambda^{-1} & i\hat{\alpha}\lambda^{-1} \end{pmatrix} = -\xi_\lambda.$$

The determinant equals

$$\begin{aligned} \det\left(\xi_\lambda\right) &= \hat{\alpha}^2\lambda^2 + (\hat{\beta}^{-1} + \hat{\beta}\lambda)(\hat{\beta}\lambda + \hat{\beta}^{-1}\lambda^2) \\ &= \hat{\alpha}^2\lambda^2 + \lambda + (\hat{\beta}^2 + \hat{\beta}^{-2})\lambda^2 + \lambda^3 \\ &= \lambda\left(\lambda^2 + (\hat{\alpha}^2 + \hat{\beta}^2 + \hat{\beta}^{-2})\lambda + 1\right) \\ &= \lambda a_1(\lambda) \end{aligned}$$

where

$$a_1(\lambda) = \lambda^2 + (\hat{\alpha}^2 + \hat{\beta}^2 + \hat{\beta}^{-2})\lambda + 1 \tag{6.18}$$

and we state

Lemma 6.11.
The following sets are the same:

$$\begin{aligned} \mathcal{M}_1 :&= \{a_1 \in \mathbb{C}^2[\lambda] \mid \lambda a_1(\lambda) = \det(\xi_\lambda) \textit{ for a } \xi_\lambda \in \mathcal{P}_1\} \\ &= \{a_1 \in \mathbb{C}^2[\lambda] \mid a(0) = 1,\ \lambda^2\overline{a(\bar{\lambda}^{-1})} = a(\lambda),\ \lambda^{-1}a(\lambda) \geq 0 \textit{ for } \lambda \in \mathcal{S}^1\}. \end{aligned}$$

Proof.
This proof can be conducted in complete analogy to Theorem 4.3 and will just be sketched. For the forward implication, we use the reality condition of ξ_λ as it was done before and

notice the fact that $\lambda^{-1}\xi_\lambda$ is traceless and skew-Hermitian for $\lambda \in \mathcal{S}^1$. For the converse direction, we choose $\alpha = 0$ and by the first and second property of a, the determinant polynomial must be of the form

$$a(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_1^{-1})$$

with $\lambda_1 \in \mathbb{R}$. We denote by B and λC the off-diagonal entries of our future ξ_λ, choose

$$B(\lambda) = \hat{\beta}(\lambda - \lambda_1)$$
$$C(\lambda) = -\hat{\beta}^{-1}(\lambda - \lambda_1^{-1})$$

and check by comparison of the coefficients that

$$\hat{\beta} = \frac{1}{i\sqrt{\lambda_1}}$$

must hold. To prove $\hat{\beta} \in \mathbb{R}^-$ it suffices to demonstrate $\lambda_1 < 0$. Since

$$\lambda^{-1}a(\lambda) = \lambda - (\lambda_1 + \lambda_1^{-1}) + \lambda^{-1} \geq 0 \text{ for all } \lambda \in \mathcal{S}^1,$$

selecting $\lambda = i \in \mathcal{S}^1$ yields

$$\lambda_1 + \lambda_1^{-1} \leq 0$$

which directly implies $\lambda_1 < 0$ because λ_1 is a real, nonzero number. **q.e.d.**

Lemma 6.12.
Let $\xi_\lambda \in \mathcal{P}_1$ and $\det(\xi_\lambda) = \lambda a_1(\lambda)$ with $a_1 \in \mathcal{M}_1$.
If $\tilde{\lambda} \in \mathbb{C}$ is a root of ξ_λ then $\tilde{\lambda}$ is a double root of the determinant polynomial a.
Conversely, if $\tilde{\lambda} \in \mathcal{S}^1$ is a root of $a(\lambda)$, then $\tilde{\lambda}$ is a root of ξ_λ.

Proof.
This proof is completely analogous to the one of Theorem 4.5 with $\lambda^{-1}\xi_\lambda$ being traceless and skew-Hermitian for $\lambda \in \mathcal{S}^1$. **q.e.d.**

As before, we define

$$\mathcal{M}_1^0 := \{a \in \mathcal{M}_1 \mid \lambda^{-1}a(\lambda) > 0 \text{ for } \lambda \in \mathcal{S}^1\}$$
$$\mathcal{M}_1^1 := \{a \in \mathcal{M}_1 \mid a \text{ has two distinct roots}\}$$

with

$$\mathcal{M}_1^1 \subset \mathcal{M}_1^0 \subset \mathcal{M}_1.$$

In analogy to (5.8) we define the function

$$f_1 : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+, \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} \mapsto \hat{\alpha}^2 + \hat{\beta}^2 + \hat{\beta}^{-2} \tag{6.19}$$

and notice the pleasant fact

$$\begin{aligned} \nabla f \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = \begin{pmatrix} 2\hat{\alpha} \\ 2\hat{\beta} - 2\hat{\beta}^{-3} \end{pmatrix} = 0 &\Leftrightarrow \hat{\alpha} = 0 \text{ and } \hat{\beta}^4 = 1 \\ &\Leftrightarrow \hat{\alpha} = 0 \text{ and } \hat{\beta}^{-2} = \beta^2 \\ &\Leftrightarrow \hat{\alpha} = 0 \text{ and there is } \tilde{\lambda} \in \mathbb{C} \text{ such that} \\ &\quad\ \tilde{\lambda}\hat{\beta} + \hat{\beta}^{-1} = 0 \text{ and } \tilde{\lambda}\hat{\beta}^{-1} + \hat{\beta} = 0 \\ &\Leftrightarrow \xi_{\tilde{\lambda}} = 0. \end{aligned}$$

This relationship tremendously facilitates the proof of the upcoming

Theorem 6.13.
Let $a \in \mathcal{M}_1^1$. Then the isospectral set $I(a)$ is a compact, one-dimensional submanifold of $\mathbb{R} \times \mathbb{R}^-$.

Proof.
As in Proposition 6.2 we want to prove that $a \in \mathcal{M}_1^1$ is a regular value. Due to Lemma 6.12 any $\xi_\lambda \in I(a)$ has no roots at all and consequently, the above relation yields

$$df_1(\xi_\lambda) \neq 0 \ \text{ for all } \xi_\lambda \in I(a).$$

Therefore, the linear mapping $df_1(\xi_\lambda)$ is of full rank one and by Corollary (2.5), $I(a)$ is a submanifold of dimension $2-1=1$. Compactness can be proven analogously to the way it was done in Proposition 5.9. **q.e.d.**

Definition 6.14.
Polynomial Killing fields are maps $\xi_\lambda : \mathbb{R}^2 \to \mathcal{P}_1$, $(\hat{x},\hat{y}) \mapsto \xi_\lambda(\hat{x},\hat{y})$ which solve the *Lax equations*

$$\frac{\partial \xi_\lambda}{\partial \hat{x}} = [\xi_\lambda, U_1(\xi_\lambda)] \qquad \frac{\partial \xi_\lambda}{\partial \hat{y}} = [\xi_\lambda, V_1(\xi_\lambda)] \tag{6.20}$$

with $\xi_\lambda(0) = \xi_\lambda \in \mathcal{P}_1$ and

$$\begin{aligned} U_1(\xi_\lambda) &:= \begin{pmatrix} 0 & \hat{\beta}^{-1} \\ 0 & 0 \end{pmatrix} \lambda^{-1} + \begin{pmatrix} i\hat{\alpha} & \hat{\beta} \\ -\hat{\beta} & -i\hat{\alpha} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\hat{\beta}^{-1} & 0 \end{pmatrix} \lambda = \lambda^{-1}\xi_\lambda \\ &= \begin{pmatrix} i\hat{\alpha} & \hat{\beta}^{-1}\lambda^{-1} + \hat{\beta} \\ -\hat{\beta} - \hat{\beta}^{-1}\lambda & -i\hat{\alpha} \end{pmatrix} \\ V_1(\xi_\lambda) &:= \begin{pmatrix} 0 & \hat{\beta}^{-1} \\ 0 & 0 \end{pmatrix} i\lambda^{-1} + \begin{pmatrix} 0 & -\hat{\beta} \\ -\hat{\beta} & 0 \end{pmatrix} i + \begin{pmatrix} 0 & 0 \\ \hat{\beta}^{-1} & 0 \end{pmatrix} i\lambda \\ &= i \begin{pmatrix} 0 & \hat{\beta}^{-1}\lambda^{-1} - \hat{\beta} \\ -\hat{\beta} + \hat{\beta}^{-1}\lambda & 0 \end{pmatrix}. \end{aligned}$$

As before, we are interested in finding a more explicit representation of the Lax equations:

$$\begin{aligned} [\xi_\lambda, U_1(\xi_\lambda)] &= \xi_\lambda U_1(\xi_\lambda) - U_1(\xi_\lambda)\xi_\lambda \\ &= \lambda^{-1}\xi_\lambda\xi_\lambda - \lambda^{-1}\xi_\lambda\xi_\lambda \\ &= 0. \end{aligned}$$

Furthermore,

$$\begin{aligned} \xi_\lambda V_1(\xi_\lambda) &= \begin{pmatrix} i\hat{\alpha}\lambda & \hat{\beta}^{-1} + \hat{\beta}\lambda \\ -\hat{\beta}\lambda - \hat{\beta}^{-1}\lambda^2 & -i\hat{\alpha}\lambda \end{pmatrix} i \begin{pmatrix} 0 & \hat{\beta}^{-1}\lambda^{-1} - \hat{\beta} \\ -\hat{\beta} + \hat{\beta}^{-1}\lambda & 0 \end{pmatrix} \\ &= i \begin{pmatrix} (-1 + \hat{\beta}^{-2}\lambda - \hat{\beta}^2\lambda + \lambda^2) & (i\hat{\alpha}\hat{\beta}^{-1} - i\hat{\alpha}\hat{\beta}\lambda) \\ (i\hat{\alpha}\hat{\beta}\lambda - i\hat{\alpha}\hat{\beta}^{-1}\lambda^2) & (-1 + \hat{\beta}^2\lambda - \hat{\beta}^{-2}\lambda + \lambda^2) \end{pmatrix} \\ V_1(\xi_\lambda)\xi_\lambda &= \begin{pmatrix} 0 & \hat{\beta}^{-1}\lambda^{-1} - \hat{\beta} \\ -\hat{\beta} + \hat{\beta}^{-1}\lambda & 0 \end{pmatrix} i \begin{pmatrix} i\hat{\alpha}\lambda & \hat{\beta}^{-1} + \hat{\beta}\lambda \\ -\hat{\beta}\lambda - \hat{\beta}^{-1}\lambda^2 & -i\hat{\alpha}\lambda \end{pmatrix} \\ &= i \begin{pmatrix} (-1 - \hat{\beta}^{-2}\lambda + \hat{\beta}^2\lambda + \lambda^2) & (-i\hat{\alpha}\hat{\beta}^{-1} + i\hat{\alpha}\hat{\beta}\lambda) \\ (-i\hat{\alpha}\hat{\beta}\lambda + i\hat{\alpha}\hat{\beta}^{-1}\lambda^2) & (-1 - \hat{\beta}^2\lambda + \hat{\beta}^{-2}\lambda + \lambda^2) \end{pmatrix} \end{aligned}$$

holds and consequently, we obtain

$$\begin{aligned}
[\xi_\lambda, V_1(\xi_\lambda)] &= \xi_\lambda V_1(\xi_\lambda) - V_1(\xi_\lambda)\xi_\lambda \\
&= i\begin{pmatrix} (2\hat{\beta}^{-2}\lambda - 2\hat{\beta}^2\lambda) & (2i\hat{\alpha}\hat{\beta}^{-1} - 2i\hat{\alpha}\hat{\beta}\lambda) \\ (2i\hat{\alpha}\hat{\beta}\lambda - 2i\hat{\alpha}\hat{\beta}^{-1}\lambda^2) & (2\hat{\beta}^2\lambda - 2\hat{\beta}^{-2}\lambda) \end{pmatrix} \\
&= \begin{pmatrix} 0 & -2\hat{\alpha}\hat{\beta}^{-1} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2i(\hat{\beta}^{-2} - \hat{\beta}^2) & 2\hat{\alpha}\hat{\beta} \\ -2\hat{\alpha}\hat{\beta} & -2i(\hat{\beta}^{-2} - \hat{\beta}^2) \end{pmatrix}\lambda + \begin{pmatrix} 0 & 0 \\ -2\hat{\alpha}\hat{\beta}^{-1} & 0 \end{pmatrix}\lambda^2,
\end{aligned}$$

which directly proves

Lemma 6.15.
Let ξ_λ be a Polynomial Killing field. Then the entries $\hat{\alpha} : \mathbb{R}^2 \to \mathbb{R}$ and $\hat{\beta} : \mathbb{R}^2 \to \mathbb{R}^-$ satisfy the modified Lax equations:

$$\frac{\partial \hat{\alpha}}{\partial \hat{x}} = 0 \qquad \frac{\partial \hat{\alpha}}{\partial \hat{y}} = 2(\hat{\beta}^{-2} - \hat{\beta}^2)$$

$$\frac{\partial \hat{\beta}}{\partial \hat{x}} = 0 \qquad \frac{\partial \hat{\beta}}{\partial \hat{y}} = 2\hat{\alpha}\hat{\beta}.$$

Hence, we obtain a one-dimensional local flow $\phi(\hat{y})(\xi_\lambda)$.

Lemma 6.16.
The determinant polynomial $a(\lambda)$ from (6.18) is an integral of motion with respect to the Lax equations.

Proof.
Consider the coefficient $z = \hat{\alpha}^2 + \hat{\beta}^2 + \hat{\beta}^{-2}$ and calculate the derivative using the modified Lax equations

$$\frac{\partial z}{\partial \hat{y}} = 2\hat{\alpha}\frac{\partial \hat{\alpha}}{\partial \hat{y}} + 2\hat{\beta}\frac{\partial \hat{\beta}}{\partial \hat{y}} - 2\hat{\beta}^{-3}\frac{\partial \hat{\beta}}{\partial \hat{y}} = 4\hat{\alpha}\hat{\beta}^{-2} - 4\hat{\alpha}\hat{\beta}^2 + 4\hat{\alpha}\hat{\beta}^2 - 4\hat{\alpha}\hat{\beta}^{-2} = 0.$$

q.e.d.

If we combine Theorem 6.13 and Lemma 6.16 we directly obtain that the local flow $\phi(\hat{y})$ is global by means of Theorem 2.10.

Lemma 6.17.
Let $a_1 \in \mathcal{M}_1^1$, $I(a_1)$ its isospectral set and fix a tuple $(\hat{\alpha}, \hat{\beta}) \in \mathbb{R} \times \mathbb{R}^-$ such that the corresponding ξ_λ satisfies $\xi_\lambda \in I(a_1)$. Then the nonzero right hand side of the Lax equations

$$[\xi_\lambda, V_1(\xi_\lambda)]$$

forms a basis of the tangent space of $I(a_1)$ in ξ_λ.

Proof.
With Theorem 2.7 (i) we immediately obtain that $T_{\xi_\lambda}(I(a_1))$ is a one-dimensional $\mathbb{R}$-vector space because $I(a_1)$ is a one-dimensional submanifold. Since f_1 from (6.19) and $I(a_1)$ satisfy

the assumptions of Theorem 2.7 (ii) we look at the kernel of df_1 rather than at the tangent space itself and consider the right hand side terms of the modified Lax equations

$$\partial_y := \begin{pmatrix} 2\hat{\beta}^{-2} - 2\hat{\beta}^2 \\ 2\hat{\alpha}\hat{\beta} \end{pmatrix}.$$

It remains to prove that $\partial_y \in \ker df_1((\hat{\alpha}, \hat{\beta})^t)$. Simple computation yields

$$df_1((\hat{\alpha}, \hat{\beta})^t)\partial_y = \begin{pmatrix} 2\hat{\alpha} \\ 2\hat{\beta} - 2\hat{\beta}^{-3} \end{pmatrix}^t \begin{pmatrix} 2\hat{\beta}^{-2} - 2\hat{\beta}^2 \\ 2\hat{\alpha}\hat{\beta} \end{pmatrix} = 4\hat{\alpha}\hat{\beta}^{-2} - 4\hat{\alpha}\hat{\beta}^2 + 4\hat{\alpha}\hat{\beta}^2 - 4\hat{\alpha}\hat{\beta}^{-2} = 0.$$

q.e.d.

In analogy to Lemma 6.6 we state

Lemma 6.18.
Let $a \in \mathcal{M}_1^1$ and $\xi_\lambda \in I(a)$. Then the orbit of the global flow ϕ

$$A_{\xi_\lambda} := \{\phi(\hat{y})(\xi_\lambda) \,|\, \hat{y} \in \mathbb{R}\}$$

is a path-component in the isospectral set $I(a)$.

Proof.
The arguments seen in Lemma 6.6 hold true in this scenario as well. Again, the main work is composed of proving openness of the set A_{ξ_λ} with respect to the subspace topology of $I(a)$. We consider the restriction of function f_1 from (6.19) on $U := f_1^{-1}(\mathcal{M}_1^1)$:

$$\tilde{f}_1 := f_1\big|_U : U \subset \mathbb{R} \times \mathbb{R}^- \to \mathbb{R}^+, \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} \mapsto \hat{\alpha}^2 + \hat{\beta}^2 + \hat{\beta}^{-2}.$$

Then we define vector spaces $X_{\xi_\lambda} := \ker d\tilde{f}_1(\xi_\lambda)$ and Y_{ξ_λ} via the Basis Extension Theorem such that

$$X_{\xi_\lambda} \oplus Y_{\xi_\lambda} \cong \mathbb{R}^2.$$

Subsequently, we can apply the Implicit Function Theorem and the Inverse Function Theorem to deduce the assertion. For details check Lemma 6.6. **q.e.d.**

Finally, we state the main result for spectral genus $g = 1$:

Theorem 6.19.
Let $a \in \mathcal{M}_1^1$ with distinct roots λ_1, $\lambda_1^{-1} \in \mathbb{R}$. Then the flow ϕ acts transitively on $I(a)$.

Proof.
Since things are less complex under these circumstances, we will present a new proof instead of adapting the one from Theorem 6.8. First, we will show that $I(a)$ is path-connected. Then the assertion directly follows with Lemma 6.18. Again, we denote by

$$z = \hat{\alpha}^2 + \hat{\beta}^2 + \hat{\beta}^{-2}$$

the λ-coefficient of the determinant polynomial $a_1(\lambda)$ from (6.18). Notice that $z \geq 2$ since $\hat\beta^2 + \hat\beta^{-2} \geq 2$ and $\hat\alpha^2 \geq 0$. The condition that $\hat\alpha \in \mathbb{R}$ can be transformed into a requirement for $\hat\beta$ via

$$\begin{aligned}\hat\alpha = \pm\sqrt{z - \hat\beta^2 - \hat\beta^{-2}} \in \mathbb{R} &\Leftrightarrow z - \hat\beta^2 - \hat\beta^{-2} \geq 0 \\ &\Leftrightarrow \hat\beta^4 - z\hat\beta^2 + 1 \leq 0 \\ &\Leftrightarrow (\hat\beta^2 - \frac{1}{2}z)^2 - \frac{1}{4}z^2 + 1 \leq 0 \\ &\Leftrightarrow |\hat\beta^2 - \frac{1}{2}z| \leq \sqrt{\frac{1}{4}z^2 - 1}\end{aligned}$$

where the last equivalence is valid due to the fact that $z \geq 2$. The last expression is equivalent to

$$\hat\beta^2 \in \left[\frac{z}{2} - \sqrt{\frac{1}{4}z^2 - 1},\ \frac{z}{2} + \sqrt{\frac{1}{4}z^2 - 1}\right].$$

In order to have $\hat\beta \in \mathbb{R}^-$ we infer

$$\hat\beta \in P_z := [v, w]$$

with constants $v = v(z), w = w(z) \in \mathbb{R}$, $v \leq w$. Thus, given any $\hat\beta \in P_z$, the above equivalence yields two possible $\hat\alpha$:

$$\begin{aligned}\hat\alpha_1(\hat\beta) &= \sqrt{z - \hat\beta^2 - \hat\beta^{-2}} \\ \hat\alpha_2(\hat\beta) &= -\sqrt{z - \hat\beta^2 - \hat\beta^{-2}}.\end{aligned}$$

Since both mappings $\hat\alpha_1, \hat\alpha_2 : P_z \to \mathbb{R}$ are continuous in $\hat\beta$ and as the graph of a continuous mapping is path-connected, we have shown that the isospectral set consists of at most two path-components. If we now choose $\hat\beta \in \partial P_z$ we obtain $\hat\alpha_1(\hat\beta) = \hat\alpha_2(\hat\beta)$ by construction, so both graphs have a common element. Therefore, the isospectral set is path-connected.

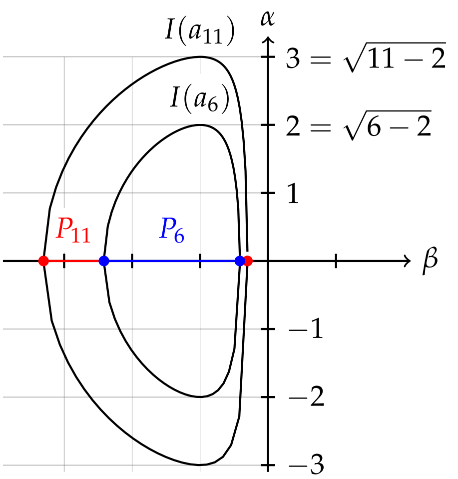


Figure 4: Level sets for $z = 6, 11$

q.e.d.

Remark 6.20.
In this proof we have explicitly calculated the function g whose existence is abstractly stated in the Implicit Function Theorem.

Now we return to our initial problem. The interesting question is whether $\tilde{\zeta}_\lambda$ from (6.16) can be associated - possibly after a certain transformation - with an element of $\mathcal{P}_1$. Moreover, if the answer is yes, is this transformation stable on Polynomial Killing fields? With other words, does a transformed Polynomial Killing field of spectral genus $g = 2$ solve the (modified) Lax equations (6.20) of spectral genus $g = 1$ and vice versa? For the first question, we want to construct a continuous bijection of the form

$$\hat{\zeta}_\lambda := u \begin{pmatrix} w & 0 \\ 0 & w^{-1} \end{pmatrix} \tilde{\zeta}_{v\lambda} \begin{pmatrix} w^{-1} & 0 \\ 0 & w \end{pmatrix} \tag{6.21}$$

with suitable $u, v, w \in \mathcal{S}^1$ such that $\hat{\zeta}_\lambda \in \mathcal{P}_1$. First, assume $\alpha \neq 0$. Then the root satisfies

$$\lambda_1 = \alpha\bar{\alpha}^{-1} \in \mathcal{S}^1$$

and we denote the entries of ζ_λ in multiplicative form by comparing the coefficients with (4.1) and using the relatity condition (4.2):

$$\begin{aligned} C(\lambda) &= \gamma^{-1}\left(\lambda - \alpha\bar{\alpha}^{-1}\right)\left(\lambda - \gamma^2\bar{\alpha}\alpha^{-1}\right) \\ B(\lambda) &= -\lambda^2\overline{C(\bar{\lambda}^{-1})} = -\lambda^2\gamma^{-1}\left(\lambda^{-1} - \gamma^2\alpha\bar{\alpha}^{-1}\right)\left(\lambda^{-1} - \bar{\alpha}\alpha^{-1}\right) \\ &= -\gamma^{-1}\left(1 - \gamma^2\alpha\bar{\alpha}^{-1}\lambda\right)\left(1 - \bar{\alpha}\alpha^{-1}\lambda\right) \\ &= -\gamma^{-1}\gamma^2\alpha\bar{\alpha}^{-1}\bar{\alpha}\alpha^{-1}\left(\gamma^{-2}\alpha^{-1}\bar{\alpha} - \lambda\right)\left(\bar{\alpha}^{-1}\alpha - \lambda\right) \\ &= -\gamma\left(\lambda - \gamma^{-2}\bar{\alpha}\alpha^{-1}\right)\left(\lambda - \alpha\bar{\alpha}^{-1}\right). \end{aligned}$$

By construction, both entries coincide with the structure (4.1). Thereafter we can explicitly compute

$$\tilde{\zeta}_\lambda = \left(\lambda - \alpha\bar{\alpha}^{-1}\right)^{-1}\zeta_\lambda = \begin{pmatrix} \lambda\bar{\alpha} & -\gamma\left(\lambda - \gamma^{-2}\bar{\alpha}\alpha^{-1}\right) \\ \lambda\gamma^{-1}\left(\lambda - \gamma^2\bar{\alpha}\alpha^{-1}\right) & -\lambda\bar{\alpha} \end{pmatrix}$$

and specify (6.21)

$$\begin{aligned} \hat{\zeta}_\lambda &= u\begin{pmatrix} v\bar{\alpha}\lambda & -w^2\gamma\left(v\lambda - \gamma^{-2}\bar{\alpha}\alpha^{-1}\right) \\ w^{-2}v\gamma^{-1}\lambda\left(v\lambda - \gamma^2\bar{\alpha}\alpha^{-1}\right) & -v\bar{\alpha}\lambda \end{pmatrix} \\ &= \begin{pmatrix} 0 & uw^2\gamma^{-1}\bar{\alpha}\alpha^{-1} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} uv\bar{\alpha} & -uvw^2\gamma \\ -uvw^{-2}\gamma\bar{\alpha}\alpha^{-1} & -uv\bar{\alpha} \end{pmatrix}\lambda + \begin{pmatrix} 0 & 0 \\ uv^2w^{-2}\gamma^{-1} & 0 \end{pmatrix}\lambda^2. \end{aligned}$$

In order to obtain $\hat{\zeta}_\lambda \in \mathcal{P}_1$, the following conditions must hold:

(i) $uw^2\gamma^{-1}\bar{\alpha}\alpha^{-1} = -uv^2w^{-2}\gamma^{-1} \quad \Leftrightarrow \quad w^4\bar{\alpha}\alpha^{-1} = -v^2$

(ii) $-u^{-1}v^{-1}w^{-2}\gamma^{-1} = uw^2\gamma^{-1}\bar{\alpha}\alpha^{-1} \quad \Leftrightarrow \quad u^2vw^4\bar{\alpha}\alpha^{-1} = -1$

(iii) $-uvw^2\gamma = uvw^{-2}\gamma\bar{\alpha}\alpha^{-1} \quad \Leftrightarrow \quad w^4\alpha\bar{\alpha}^{-1} = -1$

(iv) $uv\bar{\alpha} \in i\mathbb{R}$

(v) $-uvw^2\gamma \in \mathbb{R}^-$.

The values

$$u = i\left(\bar{\alpha}\alpha^{-1}\right)^{-\frac{3}{2}} = i\left(\bar{\lambda}_1\right)^{-\frac{3}{2}} \in \mathcal{S}^1 \tag{6.22}$$

$$v = -\bar{\alpha}\alpha^{-1} = -\bar{\lambda}_1 \in \mathcal{S}^1 \tag{6.23}$$

$$w = \left(-\bar{\alpha}\alpha^{-1}\right)^{\frac{1}{4}} = \left(-\bar{\lambda}_1\right)^{\frac{1}{4}} \in \mathcal{S}^1 \tag{6.24}$$

are deduced by (i)-(iii). Furthermore, they satisfy equations (iv) and (v)

$$uv\bar{\alpha} = -i\left(\bar{\alpha}\alpha^{-1}\right)^{-\frac{1}{2}}\bar{\alpha} = -i|\alpha| \in i\mathbb{R}$$

$$-uvw^2\gamma = i\left(\bar{\alpha}\alpha^{-1}\right)^{-\frac{1}{2}}\left(-\bar{\alpha}\alpha^{-1}\right)^{\frac{1}{2}}\gamma = -\gamma \in \mathbb{R}^-.$$

Hence, we conclude

$$\hat{\zeta}_\lambda = \begin{pmatrix} 0 & -\gamma^{-1} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -i|\alpha| & -\gamma \\ \gamma & i|\alpha| \end{pmatrix}\lambda + \begin{pmatrix} 0 & 0 \\ \gamma^{-1} & 0 \end{pmatrix}\lambda^2 \in \mathcal{P}_1.$$

When $\alpha = 0$ we conduct the same computations with an abstract $\lambda_1 \in \mathcal{S}^1$ and find the same $u, v, w \in \mathcal{S}^1$ as well as the same $\hat{\zeta}_\lambda$ (but off-diagonal since $\alpha = 0$).

The structure of the bijection (6.21) makes clear that $\hat{\zeta}_\lambda$ has a root if and only if $\tilde{\zeta}_\lambda$ has a root. By assumption and construction, $\tilde{\zeta}_\lambda$ has no root and therefore, $\hat{\zeta}_\lambda$ has no root either.

Now we want to check whether a Polynomial Killing field of spectral genus $g = 2$ is mapped on a Polynomial Killing field of spectral genus $g = 1$ under the transformation (6.21). For reasons of simplicity, we show the reverse assertion, thus, we start from a Polynomial Killing field $\xi_\lambda(\hat{x}, \hat{y})$ of spectral genus $g = 1$ which has no roots and consider the inverse transformation

$$\begin{aligned}
\hat{\xi}_\lambda &= u^{-1}(\lambda - \lambda_1)\begin{pmatrix} w^{-1} & 0 \\ 0 & w \end{pmatrix}\xi_{v^{-1}\lambda}\begin{pmatrix} w & 0 \\ 0 & w^{-1} \end{pmatrix} \qquad (6.25)\\
&= \left(-i\lambda_1^{-\frac{3}{2}}\lambda + i\lambda_1^{-\frac{1}{2}}\right)\begin{pmatrix} -i\hat{\alpha}\lambda_1\lambda & -i\lambda_1^{\frac{1}{2}}\hat{\beta}^{-1} + i\lambda_1^{\frac{3}{2}}\hat{\beta}\lambda \\ i\lambda_1^{\frac{1}{2}}\hat{\beta}\lambda - i\lambda_1^{\frac{3}{2}}\hat{\beta}^{-1}\lambda^2 & i\hat{\alpha}\lambda_1\lambda \end{pmatrix}\\
&= \begin{pmatrix} -\hat{\alpha}\lambda_1^{-\frac{1}{2}}\lambda^2 + \hat{\alpha}\lambda_1^{\frac{1}{2}}\lambda & \hat{\beta}\lambda^2 - (\lambda_1\hat{\beta} + \lambda_1^{-1}\hat{\beta}^{-1})\lambda + \hat{\beta}^{-1} \\ -\hat{\beta}^{-1}\lambda^3 + (\lambda_1^{-1}\hat{\beta} + \lambda_1\hat{\beta}^{-1})\lambda^2 - \hat{\beta}\lambda & \hat{\alpha}\lambda_1^{-\frac{1}{2}}\lambda^2 - \hat{\alpha}\lambda_1^{\frac{1}{2}}\lambda \end{pmatrix}\\
&= \begin{pmatrix} 0 & \hat{\beta}^{-1} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \hat{\alpha}\sqrt{\lambda_1} & -\hat{\beta}\lambda_1 - \hat{\beta}^{-1}\bar{\lambda}_1 \\ -\hat{\beta} & -\hat{\alpha}\sqrt{\lambda_1} \end{pmatrix}\lambda + \begin{pmatrix} -\hat{\alpha}\sqrt{\bar{\lambda}_1} & \hat{\beta} \\ \hat{\beta}\bar{\lambda}_1 + \hat{\beta}^{-1}\lambda_1 & \hat{\alpha}\sqrt{\bar{\lambda}_1} \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & 0 \\ -\hat{\beta}^{-1} & 0 \end{pmatrix}\lambda^3
\end{aligned}$$

We receive an embedding Z from the two-dimensional set $\mathcal{P}_1$ to the three-dimensional subset $Z(\mathcal{P}_1) \subset \mathcal{P}_2$ which is parametrised by $(\hat{\alpha}, \hat{\beta}, \lambda_1)$. From the form of $\hat{\xi}_\lambda$ we can see Z as a mapping

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \\ \lambda_1 \end{pmatrix} \mapsto \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \hat{\alpha}\sqrt{\lambda_1} \\ -\hat{\beta}\lambda_1 - \hat{\beta}^{-1}\bar{\lambda}_1 \\ -\hat{\beta} \end{pmatrix}.$$

We need to investigate the modified Lax equations from Lemma 5.2 of the transformed $\hat{\xi}_\lambda$. Unfortunately, the Lax equations of spectral genus $g = 1$ refer to coordinates $(\hat{x}, \hat{y})$ whereas the equations of spectral genus $g = 2$ consider coordinates (x, y). These coordinates are generally not the same. Thus, converting the Lax equations of both spectral genus into each other requires a change of coordinates. We will see that this change occurs with a rotation.
In order to make a transformed Polynomial Killing field from initial spectral genus $g = 1$ comparable with one of spectral genus $g = 2$, we translate the modified Lax equations from Lemma 5.2, which are equations in α, β, γ with coordinates (x, y), into equations in $\hat{\alpha}, \hat{\beta}, \lambda_1$ with coordinates (x, y) using the embedding Z.
The first transformation is fairly simple

$$\frac{\partial\hat{\beta}}{\partial x} = -\frac{\partial\gamma}{\partial x} = \alpha\gamma + \bar{\alpha}\gamma = -\hat{\alpha}\hat{\beta}(\sqrt{\lambda_1} + \sqrt{\bar{\lambda}_1}).$$

For the other one, we consider

$$\begin{aligned}\frac{\partial\beta}{\partial x} &= -\alpha\beta + \bar{\alpha}\beta - 2\alpha\gamma + 2\bar{\alpha}\gamma^{-1} \\ &= -\hat{\alpha}\sqrt{\lambda_1}(-\hat{\beta}\lambda_1 - \hat{\beta}^{-1}\bar{\lambda}_1) + \hat{\alpha}\sqrt{\bar{\lambda}_1}(-\hat{\beta}\lambda_1 - \hat{\beta}^{-1}\bar{\lambda}_1) + 2\hat{\alpha}\sqrt{\lambda_1}\hat{\beta} - 2\hat{\alpha}\sqrt{\bar{\lambda}_1}\hat{\beta}^{-1} \\ &= \left(\sqrt{\lambda_1} + \sqrt{\lambda_1^3}\right)\hat{\alpha}\hat{\beta} - \left(\sqrt{\bar{\lambda}_1} + \sqrt{\bar{\lambda}_1^3}\right)\hat{\alpha}\hat{\beta}^{-1}\end{aligned}$$

as well as

$$\begin{aligned}\frac{\partial\beta}{\partial x} &= -\frac{\partial\lambda_1}{\partial x}\hat{\beta} - \frac{\partial\hat{\beta}}{\partial x}\lambda_1 - \frac{\partial\bar{\lambda}_1}{\partial x}\hat{\beta}^{-1} + \frac{\partial\hat{\beta}}{\partial x}\bar{\lambda}_1\hat{\beta}^{-2} \\ &= -\frac{\partial\lambda_1}{\partial x}\hat{\beta} - \frac{\partial\bar{\lambda}_1}{\partial x}\hat{\beta}^{-1} + \left(\sqrt{\lambda_1} + \sqrt{\lambda_1^3}\right)\hat{\alpha}\hat{\beta} - \left(\sqrt{\bar{\lambda}_1} + \sqrt{\bar{\lambda}_1^3}\right)\hat{\alpha}\hat{\beta}^{-1}.\end{aligned}$$

Combining both results yields

$$\frac{\partial\lambda_1}{\partial x}\hat{\beta} + \frac{\partial\bar{\lambda}_1}{\partial x}\hat{\beta}^{-1} = 0$$

which implies (provided $\hat{\beta} \neq -1$)

$$\frac{\partial\lambda_1}{\partial x} = \frac{\partial\bar{\lambda}_1}{\partial x} = 0.$$

The case that $\hat{\beta} = -1$ will be discussed after this analysis. Finally, we have

$$\begin{aligned}\frac{\partial\alpha}{\partial x} &= \gamma^2 + \beta\gamma - \bar{\beta}\gamma^{-1} - \gamma^{-2} \\ &= \hat{\beta}^2 - (-\hat{\beta}\lambda_1 - \hat{\beta}^{-1}\bar{\lambda}_1)\hat{\beta} + (-\hat{\beta}\bar{\lambda}_1 - \hat{\beta}^{-1}\lambda_1)\hat{\beta}^{-1} - \bar{\beta}^{-2} \\ &= \lambda_1\left(\hat{\beta}^2 - \hat{\beta}^{-2}\right) + \left(\hat{\beta}^2 - \hat{\beta}^{-2}\right)\end{aligned}$$

as well as

$$\frac{\partial \alpha}{\partial x} = \frac{\hat{\alpha}}{2\sqrt{\lambda_1}} \frac{\partial \lambda_1}{\partial x} + \sqrt{\lambda_1} \frac{\partial \hat{\alpha}}{\partial x} = \sqrt{\lambda_1} \frac{\partial \hat{\alpha}}{\partial x}$$

and consequently,

$$\frac{\partial \hat{\alpha}}{\partial x} = \left(\hat{\beta}^2 - \hat{\beta}^{-2}\right) \left(\sqrt{\lambda_1} + \sqrt{\bar{\lambda}_1}\right)$$

holds. It remains to analyze the case in which $\hat{\beta} = -1$ in the calculation of $\frac{\partial \lambda_1}{\partial x}$. Here we cannot directly infer

$$\frac{\partial \lambda_1}{\partial x} = 0$$

so we stop examining this equation and focus on the derivative of α. With the same calculation, we obtain the equation

$$\lambda_1 \left(\hat{\beta}^2 - \hat{\beta}^{-2}\right) + \left(\hat{\beta}^2 - \hat{\beta}^{-2}\right) = \hat{\alpha} \frac{\partial \sqrt{\lambda_1}}{\partial x} + \sqrt{\lambda_1} \frac{\partial \hat{\alpha}}{\partial x}.$$

Multiplication of both sides with $\sqrt{\bar{\lambda}_1}$ yields

$$\sqrt{\lambda_1} \left(\hat{\beta}^2 - \hat{\beta}^{-2}\right) + \left(\hat{\beta}^2 - \hat{\beta}^{-2}\right) \sqrt{\bar{\lambda}_1} = \hat{\alpha} \sqrt{\bar{\lambda}_1} \frac{\partial \sqrt{\lambda_1}}{\partial x} + \frac{\partial \hat{\alpha}}{\partial x}.$$

The left hand side is real as it is the sum of a complex number and its complex conjugate and thus, the right hand side must be real as well. For the next finding, we denote $\sqrt{\lambda_1}$ in polar coordinates:

$$\begin{aligned} \sqrt{\lambda_1} = e^{i\varphi} \;\Rightarrow\; \frac{\partial \sqrt{\lambda_1}}{\partial x} &= \frac{\partial}{\partial x} e^{i\varphi} = i e^{i\phi} \left(\frac{\partial}{\partial x} \phi\right) \\ \Rightarrow \sqrt{\bar{\lambda}_1} \frac{\partial \sqrt{\lambda_1}}{\partial x} &= e^{-i\varphi} \left(\frac{\partial}{\partial x} e^{i\varphi}\right) = i \left(\frac{\partial}{\partial x} \phi\right) \in i\mathbb{R}. \end{aligned}$$

Therefore, the right hand side is real if and only if

$$\begin{aligned} & \hat{\alpha} = 0 \quad \text{or} \quad \frac{\partial \sqrt{\lambda_1}}{\partial x} = \frac{1}{2\sqrt{\lambda_1}} \frac{\partial \lambda_1}{\partial x} = 0 \\ \Leftrightarrow \quad & \hat{\alpha} = 0 \quad \text{or} \quad \frac{\partial \lambda_1}{\partial x} = 0. \end{aligned}$$

In the first case, $\hat{\alpha} = 0$ and $\hat{\beta} = -1$. But then, $\hat{\xi}_\lambda$ has a root in $\lambda = 1$ which is excluded by assumption. This is why we conclude with the same result as for $\hat{\beta} \neq -1$, namely

$$\frac{\partial \lambda_1}{\partial x} = 0.$$

Now we conduct the same computations in complete analogy with respect to the y-coordinate:

$$\frac{\partial \hat{\beta}}{\partial y} = -\frac{\partial \gamma}{\partial y} = -i \left(\bar{\alpha}\gamma - \alpha\gamma\right) = i\hat{\alpha}\hat{\beta} \left(\sqrt{\bar{\lambda}_1} - \sqrt{\lambda_1}\right).$$

Furthermore, we have the equations

$$\begin{aligned}\frac{\partial\beta}{\partial y} &= i\left(-\alpha\beta - \bar{\alpha}\beta + 2\alpha\gamma + 2\bar{\alpha}\gamma^{-1}\right)\\ &= i\left(\hat{\alpha}\sqrt{\lambda_1}(\hat{\beta}\lambda_1 + \hat{\beta}^{-1}\bar{\lambda}_1) + \hat{\alpha}\sqrt{\bar{\lambda}_1}(\hat{\beta}\lambda_1 + \hat{\beta}^{-1}\bar{\lambda}_1) - 2\hat{\alpha}\sqrt{\lambda_1}\hat{\beta} - 2\hat{\alpha}\sqrt{\bar{\lambda}_1}\hat{\beta}^{-1}\right)\\ &= \left(\sqrt{\lambda_1^3} - \sqrt{\lambda_1}\right) i\hat{\alpha}\hat{\beta} + \left(\sqrt{\bar{\lambda}_1^3} - \sqrt{\bar{\lambda}_1}\right) i\hat{\alpha}\hat{\beta}^{-1}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\beta}{\partial y} &= -\frac{\partial\lambda_1}{\partial y}\hat{\beta} - \frac{\partial\hat{\beta}}{\partial y}\lambda_1 - \frac{\partial\bar{\lambda}_1}{\partial y}\hat{\beta}^{-1} + \frac{\partial\hat{\beta}}{\partial y}\bar{\lambda}_1\hat{\beta}^{-2}\\ &= -\frac{\partial\lambda_1}{\partial y}\hat{\beta} - \frac{\partial\bar{\lambda}_1}{\partial y}\hat{\beta}^{-1} + \left(\sqrt{\lambda_1^3} - \sqrt{\lambda_1}\right) i\hat{\alpha}\hat{\beta} + \left(\sqrt{\bar{\lambda}_1^3} - \sqrt{\bar{\lambda}_1}\right) i\hat{\alpha}\hat{\beta}^{-1}\end{aligned}$$

which implies

$$\frac{\partial\lambda_1}{\partial y}\hat{\beta} + \frac{\partial\bar{\lambda}_1}{\partial y}\hat{\beta}^{-1} = 0.$$

So if $\hat{\beta} \neq -1$ holds, then

$$\frac{\partial\lambda_1}{\partial y} = \frac{\partial\bar{\lambda}_1}{\partial y} = 0$$

is valid. With an analogous argument as above, one can show that in case of $\hat{\beta} = -1$ we still infer the same result. The last equation yields

$$\begin{aligned}\frac{\partial\alpha}{\partial y} &= i\left(\gamma^{-2} + \beta\gamma - \bar{\beta}\gamma^{-1} - \gamma^2\right)\\ &= i\left(\hat{\beta}^{-2} + (\hat{\beta}\lambda_1 + \hat{\beta}^{-1}\bar{\lambda}_1)\hat{\beta} - (\hat{\beta}\bar{\lambda}_1 + \hat{\beta}^{-1}\lambda_1)\hat{\beta}^{-1} - \hat{\beta}^2\right)\\ &= i\lambda_1\left(\beta^2 - \beta^{-2}\right) - i\left(\beta^2 - \beta^{-2}\right)\end{aligned}$$

as well as

$$\frac{\partial\alpha}{\partial y} = \sqrt{\lambda_1}\frac{\partial\hat{\alpha}}{\partial y}$$

and therefore,

$$\frac{\partial\hat{\alpha}}{\partial y} = i\left(\sqrt{\lambda_1} - \sqrt{\bar{\lambda}_1}\right)\left(\beta^2 - \beta^{-2}\right).$$

All in all, we have proven the following

Lemma 6.21.
Let $a_1 \in \mathcal{M}_1^1$ and $\xi_\lambda(\hat{x}, \hat{y}) \in I(a_1)$ a Polynomial Killing field of spectral genus one. If the transformation $\hat{\xi}_\lambda$ from (6.25) is a Polynomial Killing field of spectral genus two, then it satisfies the following Lax equations (5.2)

$$\begin{aligned}\frac{\partial\hat{\alpha}}{\partial x} &= \left(\hat{\beta}^2 - \hat{\beta}^{-2}\right)\left(\sqrt{\lambda_1} + \sqrt{\bar{\lambda}_1}\right) & \frac{\partial\hat{\alpha}}{\partial y} &= i\left(\sqrt{\lambda_1} - \sqrt{\bar{\lambda}_1}\right)\left(\beta^2 - \beta^{-2}\right)\\ \frac{\partial\hat{\beta}}{\partial x} &= -\hat{\alpha}\hat{\beta}(\sqrt{\lambda_1} + \sqrt{\bar{\lambda}_1}) & \frac{\partial\hat{\beta}}{\partial y} &= i\hat{\alpha}\hat{\beta}\left(\sqrt{\bar{\lambda}_1} - \sqrt{\lambda_1}\right)\\ \frac{\partial\lambda_1}{\partial x} &= 0 & \frac{\partial\lambda_1}{\partial y} &= 0.\end{aligned}$$

Theorem 6.22.
Consider $a(\lambda) \in \mathcal{M}_2$ which has exactly one double root on $\mathcal{S}^1$. Then the isospectral set $I(a)$ is a one-dimensional compact subset of $\mathbb{C} \times \mathbb{C} \times \mathbb{R}^+$ and the flows ϕ act transitively on $I(a)$.

Proof.
It suffices to prove that the transformation (6.25) defines a one-to-one relationship between the flows ϕ and the flow $\hat{\phi}$ up to rotations in the coordinates. Then the claim directly follows with our spectral $g = 1$ analysis, in particular with Theorem 6.19. Due to our preconsiderations, it remains to demonstrate that the Lax equations from Lemma 6.21 with respect to the coordinates (x, y) are obtained by a rotation of the Lax equations from Lemma 6.15 with respect to the coordinates $(\hat{x}, \hat{y})$. This means, we show that there exists a $\varphi \in [0, 2\pi]$ such that

$$\begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix} \begin{pmatrix} \frac{\partial \hat{\alpha}}{\partial \hat{x}} \\ \frac{\partial \hat{\alpha}}{\partial \hat{y}} \end{pmatrix} = \begin{pmatrix} \frac{\partial \hat{\alpha}}{\partial x} \\ \frac{\partial \hat{\alpha}}{\partial y} \end{pmatrix} \tag{6.26}$$

and

$$\begin{pmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{pmatrix} \begin{pmatrix} \frac{\partial \hat{\beta}}{\partial \hat{x}} \\ \frac{\partial \hat{\beta}}{\partial \hat{y}} \end{pmatrix} = \begin{pmatrix} \frac{\partial \hat{\beta}}{\partial x} \\ \frac{\partial \hat{\beta}}{\partial y} \end{pmatrix}. \tag{6.27}$$

Remember from Lemma 6.15 that

$$\frac{\partial \hat{\alpha}}{\partial \hat{x}} = 0 \qquad \frac{\partial \hat{\alpha}}{\partial \hat{y}} = 2(\hat{\beta}^{-2} - \hat{\beta}^2)$$
$$\frac{\partial \hat{\beta}}{\partial \hat{x}} = 0 \qquad \frac{\partial \hat{\beta}}{\partial \hat{y}} = 2\hat{\alpha}\hat{\beta}.$$

From (6.27) we obtain the equations

$$\sin(\varphi) = -\frac{1}{2}\left(\sqrt{\lambda_1} + \sqrt{\bar{\lambda}_1}\right) = -\Re(\sqrt{\lambda_1})$$
$$\cos(\varphi) = -i\frac{1}{2}\left(\sqrt{\lambda_1} - \sqrt{\bar{\lambda}_1}\right) = \Im(\sqrt{\lambda_1}).$$

In order to understand, how φ is exactly described, we use the following formula

$$e^{i\varphi} = \cos(\varphi) + i\sin(\varphi) = \Im(\sqrt{\lambda_1}) - i\Re(\sqrt{\lambda_1}) = i\sqrt{\lambda_1}.$$

Therefore, φ is the angle of the complex number $i\sqrt{\lambda_1}$ and in particular, φ only depends on λ_1. We need to check whether φ satisfies the equations (6.26).

$$\sin(\varphi)\frac{\partial \hat{\alpha}}{\partial \hat{y}} = -\frac{1}{2}\left(\sqrt{\lambda_1} + \sqrt{\bar{\lambda}_1}\right) 2(\hat{\beta}^{-2} - \hat{\beta}^2) = \frac{\partial \hat{\alpha}}{\partial x}$$
$$\cos(\varphi)\frac{\partial \hat{\alpha}}{\partial \hat{y}} = -i\frac{1}{2}\left(\sqrt{\lambda_1} - \sqrt{\bar{\lambda}_1}\right) 2(\hat{\beta}^{-2} - \hat{\beta}^2) = \frac{\partial \hat{\alpha}}{\partial y}.$$

It results that the coordinates (x, y) are obtained by a rotation of the coordinates $(\hat{x}, \hat{y})$ about the angle φ and hence, the transformation (6.25) is stable on Polynomial Killing fields.

q.e.d.

6.3 Case b): The polynomial $a(\lambda)$ has two double roots on $\mathcal{S}^1$

Theorem 6.23.
Consider $a(\lambda) \in \mathcal{M}_2$ with two double roots $\lambda_1, \lambda_2 \in \mathcal{S}^1$ (possibly $\lambda_1 = \lambda_2$). Then the isospectral set $I(a)$ consists of one point.

Proof.
We use the same technique as in the previous subsection. The determinant polynomial $a(\lambda)$ must be of shape

$$a(\lambda) = (\lambda - \lambda_1)^2 (\lambda - \lambda_2)^2$$

and furthermore,

$$\begin{aligned} \lambda_1^2 \lambda_2^2 = 1 \quad &\Rightarrow \quad \lambda_2^2 = \bar{\lambda}_1^2 \\ &\Rightarrow \quad \lambda_2 = \pm\bar{\lambda}_1. \end{aligned}$$

Therefore, $\zeta_\lambda \in I(a)$ has two roots $\lambda_1, \pm\bar{\lambda}_1$ which implies that $\alpha = 0$ and ζ_λ is off-diagonal. In analogy to the previous discussions of the transformation (6.21) we know that the off-diagonal entries of ζ_λ must be

$$\begin{aligned} B(\lambda) &= -\gamma (\lambda - \lambda_1) \left(\lambda - \gamma^{-2}\bar{\lambda}_1\right) \\ C(\lambda) &= \gamma^{-1} (\lambda - \lambda_1) \left(\lambda - \gamma^{2}\bar{\lambda}_1\right). \end{aligned}$$

Consequently, in order to satisfy the above conditions,

$$\gamma = 1 \in \mathbb{R}^+ \quad \text{and} \quad \lambda_2 = \bar{\lambda}_1$$

must hold and β is uniquely given by

$$\beta = \lambda_1 + \bar{\lambda}_1 = 2\Re(\lambda_1).$$

Hence, $\zeta_\lambda \in I(a)$ is uniquely defined by

$$\zeta_\lambda = \left[\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \lambda\right] (\lambda - \lambda_1)(\lambda - \bar{\lambda}_1).$$

q.e.d.

6.4 Case c): The polynomial $a(\lambda)$ has two distinct double roots absent $\mathcal{S}^1$

Let $a(\lambda) \in \mathcal{M}_2$ with two distinct double roots $(\lambda_1, \bar{\lambda}_1^{-1})$ absent the unit circle. Then the isospectral set $I(a)$ falls into two disjunct subsets

$$I(a) = \underbrace{\{\zeta_\lambda \in I(a) \,|\, \zeta_{\lambda_1} \neq 0\}}_{K_a} \cup \underbrace{\{\zeta_\lambda \in I(a) \,|\, \zeta_{\lambda_1} = 0\}}_{L_a}.$$

Notice the fact that we can apply the same argumentation as in the proof of Lemma 6.1 and conclude that

$$\lambda_1 \in \mathbb{R} \setminus \{-1, 0, 1\}.$$

Thus, we can immediately indicate the determinant polynomial

$$a(\lambda) = (\lambda - \lambda_1)^2(\lambda - \lambda_1^{-1})^2.$$

Without loss of generality, we will assume λ_1 to be inside the unit circle for the rest of the chapter, so

$$\lambda_1 \in (-1,1) \setminus \{0\}$$

(otherwise switch notation).
First, we consider $\tilde{\zeta}_\lambda \in L_a$. By Theorem 4.5 and the reality condition (4.2), $\tilde{\zeta}_\lambda$ has single roots $\lambda_1, \lambda_1^{-1}$ and is hence - in analogy to case b) - uniquely defined as

$$\tilde{\zeta}_\lambda = \left[\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \lambda \right] (\lambda - \lambda_1)(\lambda - \lambda_1^{-1}). \tag{6.28}$$

In particular, L_a is a singleton. Now, we focus on K_a:

Lemma 6.24.
Let $a(\lambda) \in \mathcal{M}_2$ with two distinct double roots $(\lambda_1, \lambda_1^{-1})$, $\lambda_1 \in \mathbb{R} \setminus \{-1,0,1\}$. Then the set K_a is a two-dimensional submanifold of $\mathbb{C} \times \mathbb{C} \times \mathbb{R}^+$.

Proof.
All $\zeta_\lambda \in K_a$ have no roots and one can demonstrate in complete analogy to the proof of Proposition 6.2 that all elements of K_a are regular points. The only changing detail consists of the domain of the function f

$$U := \bigcup_a K_a = \bigcup_a (I(a) \setminus L_a).$$

This domain is an open set since the L_a is a singleton and the union is conducted over all a having the features assumed in the Lemma. The claim directly follows from Corollary 2.5. **q.e.d.**

Remark 6.25.
Notice that K_a is not a compact set due to the above structure of the compact isospectral set $I(a)$.

Theorem 6.26.
Let $a(\lambda) \in \mathcal{M}_2$ with two distinct double roots $(\lambda_1, \lambda_1^{-1})$, $\lambda_1 \in \mathbb{R} \setminus \{-1,0,1\}$. The flows $\phi(x,y)$ act transitively on both parts K_a, L_a of the isospectral set $I(a)$.

Proof.
We need to verify that ϕ remains in the part of the isospectral set it originated from:

$$\zeta_\lambda \in K_a \quad \Rightarrow \quad \phi(x,y)(\zeta_\lambda) \in K_a \text{ for all } (x,y) \in \mathbb{R}^2.$$

This statement holds true because the pointwise view on Lax equations (5.1) makes clear that whenever ζ_λ has a root in $\tilde{\lambda}$, the right hand side of the Lax equations equal zero in $\tilde{\lambda}$ and hence $\phi(x,y)(\zeta_\lambda)$ has a root $\tilde{\lambda}$ for all times. In particular, if $\zeta_\lambda \in K_a$ and $\phi(x,y)(\zeta_\lambda) \notin K_a$ we have $\phi(x,y)(\zeta_\lambda) \in L_a$ and so

$$\zeta_\lambda = \phi(-x,-y)\left(\phi(x,y)(\zeta_\lambda)\right) \in L_a$$

follows with the above argumentation. However, this contradicts the assumption. Consequently, the flow ϕ remains in the singleton L_a for all times if and only if its starting point is in L_a, so the flow ϕ acts transitively on L_a. For $\zeta_\lambda \in K_a$ we can conduct the same proofs as in Lemma 6.3 and Lemma 6.6 due to the favourable fact that ζ_λ is rootless in order to receive that the sets

$$M_\zeta := \{\phi(x,y)(\zeta_\lambda) \mid (x,y) \in \mathbb{R}^2\} \quad \text{with } \zeta_\lambda \in K_a$$

define path-components in K_a. By lack of compactness of K_a we consider the closure $\overline{M_\zeta}$ instead of M_ζ itself. Since $I(a)$ is compact, the closure $\overline{M_\zeta}$ either equals the set M_ζ itself or

$$\overline{M_\zeta} = M_\zeta \cup L_a.$$

The set $\overline{M_\zeta}$ is compact as a closed subset of the compact $I(a)$. The rest of the proof leans on the one from Theorem 6.8. The only deviation can be found in the number of possible critical points, as there are only three different off-diagonal elements in $I(a)$: The first with $B(\lambda)$ having a double root inside the unit circle, the second with $B(\lambda)$ having a double root outside the unit circle and the last with $B(\lambda)$ having mixed roots. The latter element corresponds to $\tilde{\zeta}_\lambda$ from (6.28), is located in L_a and is a saddle point. In particular, the arguments and computations of Theorem 6.8 show that there exists one unique maximizing element ζ_λ^* which is contained in every $\overline{M_\zeta}$ and does not equal $\tilde{\zeta}_\lambda$, so

$$\zeta_\lambda^* \in M_\zeta$$

and we conclude that K_a consists of exactly one path-component of the form M_ζ. **q.e.d.**

Remark 6.27.
This is the shortest proof we have developed using the results of the previous sections. However, there is another possibility to prove the above theorem with a very nice geometric argument. Starting from the point that the sets M_ζ form path-components of K_a, it remains to prove that K_a is path-connected in order to demonstrate the assertion. However, we prove the path-connectedness of $I(a)$ and gain path-connectedness of K_a from the result that K_a is a two-dimensional submanifold, as the subtraction of one element ($\tilde{\zeta}_\lambda \in L_a$) from $I(a)$ cannot destroy path-connectedness of the remaining two-dimensional set.
This will be done in two steps. First, one can show that given any element $\zeta_\lambda \in I(a)$ there is a continuous path in $I(a)$ connecting ζ_λ and an off-diagonal element by taking the convex combination of the diagonal entries - the continuity condition makes the choice of the off-diagonal entries unique. As mentioned above, there are only three types of off-diagonal elements in $I(a)$. Therefore, it remains to prove in the second step that these off-diagonal elements are path-connected. In fact, the elements with $B(\lambda)$ having a double root either inside or outside the unit circle are path-connected to the third type of mixed roots $\tilde{\zeta}_\lambda$ from L_a. This is the interesting part. There exists a diagonal entry $A(\lambda)'$ with resulting off-diagonal entries B', C' such that $B'C'$ has a double root on $\mathcal{S}^1$, a single root inside and a single root outside the unit circle. The resulting ζ_λ' is called *bifurcation point*. Given the off-diagonal ζ_λ with $B(\lambda)$ having both roots inside the unit circle, one can construct a continuous path in $I(a)$ to ζ_λ' as in the first step. At the bifurcation point, we switch the affiliation of the double root from $B'C'$ such that the root that once belonged to B_t now belongs to C_t and vice versa. Then we move the same root-path backwards (causing a new element-path!) and end up in $\tilde{\zeta}_\lambda \in L_a$ by construction (Figure 5). When ζ_λ is off-diagonal with $B(\lambda)$ having both roots outside the unit circle, $C(\lambda)$ has both inside and we conduct the same analogue argument.

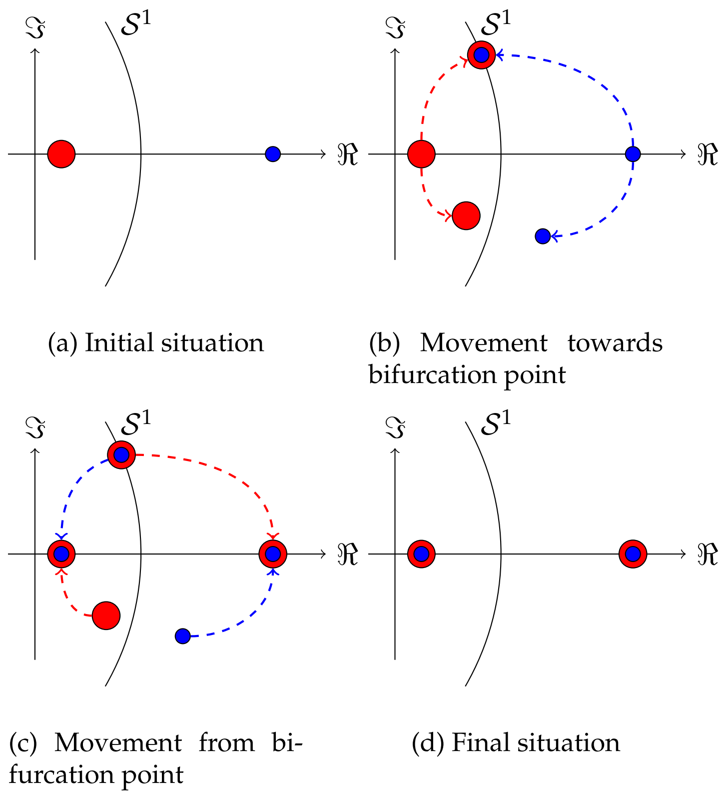


Figure 5: Path-connectedness of off-diagonal elements. Bullet points of the same color belong to the same off-diagonal entry.

We sum up this section by stating

Corollary 6.28.
Let $a \in \mathcal{M}_2$.

1. *If a has four pairwise distinct roots, the isospectral sets $I(a)$ are two-dimensional compact submanifolds of $\mathbb{C} \times \mathbb{C} \times \mathbb{R}^+$. The flows $\phi(x,y)(\zeta_\lambda)$ for given $\zeta_\lambda \in I(a)$ act transitively on the isospectral sets, i.e.*
$$I(a) = \{\phi(x,y)(\zeta_\lambda) \,|\, (x,y) \in \mathbb{R}^2\}.$$

2. *If a has one double root on $\mathcal{S}^1$ and two distinct single roots, the isospectral sets $I(a)$ are one-dimensional compact subsets of $\mathbb{C} \times \mathbb{C} \times \mathbb{R}^+$. The flows $\phi(x,y)(\zeta_\lambda)$ for given $\zeta_\lambda \in I(a)$ again act transitively on the isospectral sets.*

3. *If a has two double roots on $\mathcal{S}^1$, the isospectral set consists of one single element and the flows $\phi(x,y)(\zeta_\lambda)$ for given $\zeta_\lambda \in I(a)$ remain constant, i.e. they act transitively in a trivial way.*

4. *If a has two double roots $\lambda_1, \bar{\lambda}_1^{-1}$ absent $\mathcal{S}^1$, the isospectral set falls into two distinct subsets*

$$I(a) = \underbrace{\{\zeta_\lambda \in I(a) \,|\, \zeta_{\lambda_1} \neq 0\}}_{K_a} \cup \underbrace{\{\zeta_\lambda \in I(a) \,|\, \zeta_{\lambda_1} = \zeta_{\bar{\lambda}_1^{-1}} = 0\}}_{L_a}$$

where K_a is a two-dimensional submanifold of $\mathbb{C} \times \mathbb{C} \times \mathbb{R}^+$ and L_a is a singleton.
On both parts of the level set the flows $\phi(x, y)(\zeta_\lambda)$ act transitively for given $\zeta_\lambda \in I(a)$.

7 Lattices of Periods

In the last section, we have determined the analytic properties of the isospectral sets in dependence on the position of determinant polynomials' roots. In the classical case as well as in cases a) and b) (i.e. whenever a has no double roots absent the unit circle) the orbits

$$I(a) = \{\phi(x,y)(\zeta_\lambda) \mid (x,y) \in \mathbb{R}^2\}$$

are compact. We now focus on these determinant polynomials for the rest of this elaboration and define

$$\widetilde{\mathcal{M}}_2 := \{a \in \mathcal{M}_2 \mid a \text{ has no double root absent } \mathcal{S}^1\}.$$

First of all, we define for given $a \in \widetilde{\mathcal{M}}_2$ and initial value $\zeta_\lambda \in I(a)$ the set

$$\Gamma^a_\zeta := \{(x,y) \in \mathbb{R}^2 \mid \phi(x,y)(\zeta_\lambda) = \zeta_\lambda\}. \tag{7.1}$$

The upcoming lemma states its algebraic structure.

Lemma 7.1.
Let $a \in \widetilde{\mathcal{M}}_2$. The set $\Gamma^a_\zeta \subset \mathbb{R}^2$ from (7.1) is an additive, abelian subgroup of $\mathbb{R}^2$.
In particular, Γ^a_ζ is a normal subgroup and $\mathbb{R}^2/\Gamma^a_\zeta$ is well-defined.

Proof.
Given $(x_1,y_1),(x_2,y_2) \in \Gamma_\zeta$, the flow property (compare Remark 5.11) of ϕ yields

$$\phi(x_1+x_2,y_1+y_2)(\zeta_\lambda) = \phi(x_1,y_1)(\phi(x_2,y_2)(\zeta_\lambda)) = \phi(x_1,y_1)(\zeta_\lambda) = \zeta_\lambda$$

and therefore,

$$(x_1+x_2,y_1+y_2) \in \Gamma_\zeta.$$

Furthermore, when $(x,y) \in \Gamma^a_\zeta$, the additive inverse $(-x,-y)$ is also contained in Γ^a_ζ due to

$$\phi(-x,-y)(\zeta_\lambda) = \phi(-x,-y)(\phi(x,y)(\zeta_\lambda)) = \zeta_\lambda.$$

Commutativity is obviously inherited from $\mathbb{R}^2$. **q.e.d.**

Now we want to understand that for fixed $a \in \widetilde{\mathcal{M}}_2$ the subgroup Γ^a_ζ is independent of the choice of the initial value $\zeta_\lambda \in I(a)$ due to the flow property of ϕ (which is a consequence of the commutativity of the flows ϕ_E, ϕ_F as seen in Corollary 5.10) and hence, the index $'\zeta'$ can be omitted. To see this, let ζ_λ, $\tilde{\zeta}_\lambda \in I(a)$. We need to demonstrate

$$\Gamma^a_\zeta = \Gamma^a_{\tilde{\zeta}}.$$

We prove that any element of the left hand side is also contained in the right hand side (the reverse direction runs analogously). Thus, we consider $(x,y) \in \Gamma_\zeta$ and notice that by transitivity of ϕ, there is an element $(a,b) \in \mathbb{R}^2$ such that

$$\phi(a,b)(\tilde{\zeta}_\lambda) = \zeta_\lambda.$$

A quick computation yields

$$\begin{aligned}\tilde{\zeta}_\lambda &= \phi(-a,-b)(\zeta_\lambda)\\ &= \phi(-a,-b)(\phi(x,y)(\zeta_\lambda))\\ &= \phi(-a,-b)(\phi(x,y)(\phi(a,b)(\tilde{\zeta}_\lambda)))\\ &= \phi(-a,-b)(\phi(a,b)(\phi(x,y)(\tilde{\zeta}_\lambda)))\\ &= \phi(x,y)(\tilde{\zeta}_\lambda)\end{aligned}$$

which implies

$$(x,y) \in \Gamma^a_{\tilde{\zeta}}$$

and applying the above argumentation we can denote $\Gamma^a = \Gamma^a_\zeta$ for any $\zeta_\lambda \in I(a)$.

From now on we will focus on $a \in \mathcal{M}^1_2$, i.e. on determinant polynomials with four pairwise distinct roots. We will find out later that Γ^a defines a lattice under this assumption. However, we need to clarify the concept of discrete subgroups in order to come to this conclusion.

Definition 7.2.
A subgroup $\Gamma \subset \mathbb{R}^n$ is called *discrete* if there exists an open set $U \subset \mathbb{R}^n$ containing zero such that

$$U \cap \Gamma = \{0\}.$$

Lemma 7.3.
Let $a \in \mathcal{M}^1_2$. The factor group $\mathbb{R}^2/\Gamma^a$ is compact and the subgroup $\Gamma^a \subset \mathbb{R}^2$ from (7.1) is discrete.

Proof.
We consider the mapping

$$\begin{aligned} g_\zeta : \mathbb{R}^2/\Gamma^a &\to I(a),\\ [(x,y)] &\mapsto \phi(x,y)(\zeta_\lambda)\end{aligned}$$

where $[(x,y)] = (x,y) + \Gamma^a$ is the equivalence class of (x,y). This map is well-defined since all elements $(x+a, y+b) \in [(x,y)]$ with $(a,b) \in \Gamma^a$ satisfy

$$\phi(x+a,y+b)(\zeta_\lambda) = \phi(x,y)(\phi(a,b)(\zeta_\lambda)) = \phi(x,y)(\zeta_\lambda)$$

due to the flow property. We will first prove that g_ζ is a bijection. Due to transitivity of the flow ϕ we immediately obtain surjectivity. For injectivity, we consider two elements

$$[(x_1,y_1)]\,,\,[(x_2,y_2)] \in \mathbb{R}^2/\Gamma^a$$

with

$$\phi(x_1,y_1)(\zeta_\lambda) = \phi(x_2,y_2)(\zeta_\lambda)$$

and apply the flow property on the difference:

$$\phi(x_1-x_2,y_1-y_2)(\zeta_\lambda) = \phi(-x_2,-y_2)(\phi(x_1,y_1)(\zeta_\lambda)) = \zeta_\lambda.$$

Therefore, $(x_1-x_2, y_1-y_2) \in \Gamma^a$ and since $(x_1,y_1) = (x_2,y_2) + (x_1-x_2,y_1-y_2)$ we conclude

$$[(x_1,y_1)] = [(x_2,y_2)]\,.$$

In the next step, we will demonstrate with the help of the Inverse Function Theorem A.1 that g_ζ is not only a bijection but also a local diffeomorphism and therefore, a global diffeomorphism. In order to confirm the assertions, we are comfortably allowed to consider the mapping

$$\phi_\zeta : (x,y) \mapsto \phi(x,y)(\zeta_\lambda)$$

instead of g_ζ because any element of $\mathbb{R}^2/\Gamma^a$ is a subset of $\mathbb{R}^2$. As in the proof of Lemma 6.6, we directly obtain for any arbitrary $(x,y) \in \mathbb{R}^2$ with $\tilde{\zeta}_\lambda = \phi(x,y)(\zeta_\lambda)$ that

$$\begin{aligned} d\phi_\zeta(x,y) : \quad & \mathbb{R}^2 \to T_{\tilde{\zeta}_\lambda}(I(a)) \\ & (\tilde{x},\tilde{y}) \mapsto \tilde{x}[\tilde{\zeta}_\lambda, U(\tilde{\zeta}_\lambda)] + \tilde{y}[\tilde{\zeta}_\lambda, V(\tilde{\zeta}_\lambda)] \end{aligned}$$

is an isomorphism. Hence, there exists an open neighbourhood $U_0 \subset \mathbb{R}^2/\Gamma^a$ of $[(x,y)]$ such that $V_0 := g_\zeta(U_0) \subset I(a)$ is an open neighbourhood of $\tilde{\zeta}$ and the restriction $g_\zeta\big|_{U_0}$ is a diffeomorphism. Since differentiability is a local property, we herewith receive diffeomorphy of the entire mapping g_ζ. In particular, g_ζ owns a continuous inverse and consequently, compactness of $I(a)$ requires $\mathbb{R}^2/\Gamma^a$ to be compact as well. Lastly, the application of the Inverse Function Theorem A.1 on the specific element $(0,0) \in \mathbb{R}^2$ yields the first claim, because two diffeomorph open neighbourhoods $(0,0) \in U \subset \mathbb{R}^2$ and $\zeta_\lambda \in V = \phi_\zeta(U)$ exist. In particular, the presence of two distinct elements in U which are mapped to ζ_λ is impossible and thus,

$$U \cap \Gamma^a = \{(0,0)\}.$$

q.e.d.

The following lemma states an interesting characterization of discrete subgroups in $\mathbb{R}^n$.

Lemma 7.4 (Schmidt [8]).
The following statements are equivalent for a subgroup $\Gamma \neq \{0\}$ of $\mathbb{R}^n$:

i) Γ *is discrete.*

ii) All bounded subsets of $\mathbb{R}^n$ contain at most a finite number of elements in Γ.

iii) There exists an element in $\Gamma \setminus \{0\}$ of minimal length.

iv) The intersection of Γ and any linear subspace B of $\mathbb{R}^n$ is a discrete subgroup of B.

v) Given $\omega \in \Gamma \setminus \{0\}$ the set $\Gamma \cap \mathbb{R}\omega$ is a discrete subgroup of $\mathbb{R}\omega$.
If $\Gamma \not\subseteq \mathbb{R}\omega$, there exists an element in $\Gamma \setminus \mathbb{R}\omega$ with minimal distance to $\mathbb{R}\omega$.

vi) There is a finite number of linearly independent elements $\omega_1, \ldots, \omega_m \in \Gamma$ generating Γ.

Proof.
This proof originates Schmidt [8].
Let $\Gamma \neq \{0\}$ be a discrete subgroup of $\mathbb{R}^n$. Then there exists a number $\varepsilon > 0$ such that $B(0,\varepsilon) \cap \Gamma = \{0\}$. In particular, the difference of any two elements in Γ is either zero or has a length bigger than ε. This implies for arbitrary $x \in \mathbb{R}^n$ that the ball $B(x, \frac{\varepsilon}{2})$ contains at most one element of Γ. Consequently, if we consider any bounded subset $A \subset \mathbb{R}^n$, then the closure $\overline{A}$ is compact and owns a finite subcover of $\frac{\varepsilon}{2}$-balls. Hence, ii) follows from i).
Furthermore, let $a \in \Gamma \neq \{0\}$. Obviously, there exists a bounded subset containing both zero

and a. Due to ii) this subset possesses a finite number of elements in $\Gamma \setminus \{0\}$, one of which is of shortest length, and iii) is proven.
Conversely, if there exists an element in $\Gamma \setminus \{0\}$ of minimal length, there is an $\varepsilon > 0$ such that $B(0,\varepsilon) \cap \Gamma = \{0\}$ and i) follows from iii).
As the intersection of an open neighbourhood of zero in $\mathbb{R}^n$ and any linear subspace B yields an open neighbourhood of zero in B, iv) follows from i). In reverse, iv) implies i) since $\mathbb{R}^n$ is a linear subspace of $\mathbb{R}^n$.
Given i)-iv) we consider $\omega \in \Gamma \setminus \{0\}$. Clearly, $\mathbb{R}\omega$ is a linear subspace, so the first statement of v) follows from iv). Furthermore, if $\Gamma \not\subseteq \mathbb{R}\omega$, for every $x \in \mathbb{R}\omega$ there exists an integer $n \in \mathbb{Z}$ such that $x - n\omega \in B(0, \|\omega\|)$. Analogously, we can identify any element from $\Gamma \cap (\mathbb{R}^n \setminus \mathbb{R}\omega)$ whose distance to $\mathbb{R}\omega$ is smaller than $\delta > 0$ with an element from Γ in $B(0, \delta + \|\omega\|)$ having the same distance to $\mathbb{R}\omega$. Therefore, we immediately obtain with ii) (applied on the bounded set $B(0, \delta + \|\omega\|)$ with appropriate $\delta > 0$) the existence of an element in $\Gamma \setminus (\mathbb{R}\omega)$ having minimal distance to $\mathbb{R}\omega$. Conversely, if v) holds true and $\omega \in \Gamma \setminus \{0\}$ there exists an $\varepsilon > 0$ such that all elements from $(\Gamma \setminus \{0\}) \cap \mathbb{R}\omega$ and $\Gamma \setminus \mathbb{R}\omega$ have lengths bigger than ε. This implies i).
Under the assumptions i)-v), we proof vi) inductively. Let ω_1 be the shortest element of $\Gamma \setminus \{0\}$. Then $\Gamma \cap \mathbb{R}\omega_1 = \mathbb{Z}\omega_1$ holds because otherwise, there would exist a shorter element in $\mathbb{R}\omega_1$. We consider the orthogonal projection on the orthogonal complement of $\mathbb{R}\omega_1$

$$P_1 : \mathbb{R}^n \to \mathbb{R}^n,\ x \mapsto P_1(x) = x - \frac{\langle \omega_1, x\rangle}{\langle \omega_1, \omega_1\rangle}\omega_1.$$

The kernel of P_1 obviously equals $\mathbb{R}\omega_1$. Due to v), the image $P_1(\Gamma)$ is a discrete subgroup of $\mathbb{R}^n$ and $\Gamma \cong \mathbb{Z}\omega_1 \oplus P_1(\Gamma)$. We continue inductively and obtain linearly independent elements $\omega_1, \dots, \omega_m \in \Gamma$ which generate Γ. Conversely, if vi) holds true, there exist linearly independent mappings $l_1, \dots, l_m$ on $\mathbb{R}^n$ such that

$$\omega = l_1(\omega)\omega_1 + \cdots + l_m(\omega)\omega_m \quad \text{for all } \omega \in \Gamma.$$

Since the mappings $l_1, \dots, l_m$ are bounded on bounded sets, we obtain ii). **q.e.d.**

The last two lemmata are quite powerful because they precisely describe the subgroup Γ^a for $a \in \widetilde{\mathcal{M}}_2$. Due to Lemma 7.4, the only discrete subgroups of $\mathbb{R}^2$ (and therefore candidates for Γ^a) are the following ones:

i) $\Gamma^a = \{0\}$. In this case $\mathbb{R}^2/\Gamma^a \cong \mathbb{R}^2$, which contradicts the compactness condition of Lemma 7.3. Therefore, this scenario is impossible.

ii) $\Gamma^a = \omega\mathbb{Z}$, i.e. Γ^a is cyclic. Here, $\mathbb{R}^2/\Gamma^a \cong \mathbb{R} \times \mathcal{S}^1$ and the compactness condition is hurt again. This is also not feasible.

iii) $\Gamma^a = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$, where ω_1, ω_2 are linearly independent vectors of $\mathbb{R}^2$.
In this case,

$$\mathbb{R}^2/\Gamma^a \cong (\mathcal{S}^1)^2$$

is a compact torus.

Thus, we have proven that Γ^a is a so-called lattice.

Definition 7.5.
Given n linearly independent vectors $\omega_1, \dots, \omega_n \in \mathbb{R}^n$ the *(full-rank) lattice* generated by them is defined as the set

$$\left\{ \sum_{k=1}^{n} a_k \omega_k \mid a_k \in \mathbb{Z} \right\}.$$

Furthermore, the set $\{\sum_{k=1}^{n} r_k \omega_k \mid 0 \le r_k \le 1\}$ is called *fundamental region* of the lattice.

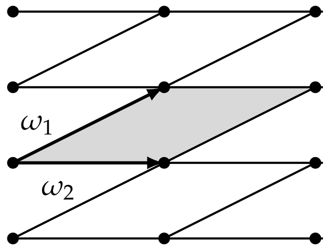


Figure 6: Example of a lattice in $\mathbb{R}^2$ with generators ω_1, ω_2 and shaded fundamental region.

To sum up, we have proven the following

Proposition 7.6.
Let $a \in \mathcal{M}_2^1$. Then the set

$$\Gamma^a = \{(x,y) \in \mathbb{R}^2 \mid \phi(x,y)(\zeta_\lambda) = \zeta_\lambda\}$$

does not depend on the choice of $\zeta_\lambda \in I(a)$ and defines a lattice in $\mathbb{R}^2$. In particular, there exist two linearly independent generators $\omega_1, \omega_2 \in \mathbb{R}^2$ such that

$$\Gamma = \omega_1 \mathbb{Z} + \omega_2 \mathbb{Z}.$$

7.1 Isomorphy of Lattices

In the last subsection we have seen that any $a \in \mathcal{M}_2^1$ induces a lattice

$$\Gamma^a = \omega_1^a \mathbb{Z} + \omega_2^a \mathbb{Z}$$

where ω_1^a, ω_2^a are complex, linearly independent generators. Of course, the choice of these generators is not uniquely defined, so there might exist several different pairs of complex numbers which all define the same lattice (Figure 7).
In order to make them unique up to sign, we consider for a given determinant polynomial $a \in \mathcal{M}_2^1$ a pair of generators (ω_1^a, ω_2^a) where $\omega_1^a \in \Gamma^a \setminus \{0\}$ has minimal length and $\omega_2^a \in \Gamma^a \setminus (\omega_1^a \mathbb{Z})$ is also of shortest length. Such a pair exists indeed according to Lemma 7.4. From now on, we will call this essential condition on the generators *minimality condition*.

Definition 7.7.
A *rotation-dilation* is a linear mapping $\mathbb{C} \to \mathbb{C}$ consisting of the composition of a rotation and the multiplication with a real number (dilation).
More precisely, a rotation-dilation maps $\omega \mapsto a\omega$ for a given $a \in \mathbb{C} \setminus \{0\}$.

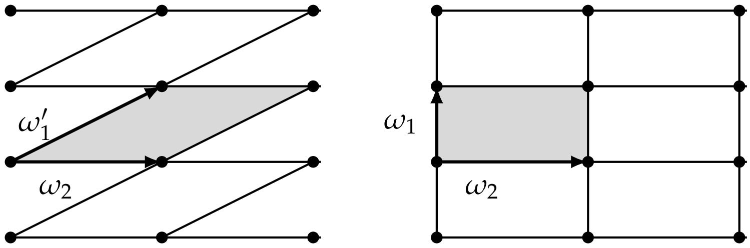


Figure 7: Example of two different pairs of generators (ω_1, ω_2), (ω_1', ω_2) producing the same lattice in $\mathbb{R}^2$. (ω_1, ω_2) satisfies minimality condition.

Definition 7.8.
Let $\Gamma, \Gamma' \subset \mathbb{R}^2$ be two lattices. We call them *isomorphic* if they originate from one another through a rotation-dilation.

By means of the rotation-dilation

$$\omega \mapsto \frac{1}{\omega_1^a}\omega$$

we can map the generator ω_1^a to $\tilde{\omega}_1^a = 1$ and ω_2^a to $\tilde{\omega}_2^a$. We can assume

$$\Im(\tilde{\omega}_2^a) > 0 \tag{7.2}$$

otherwise we consider $-\tilde{\omega}_2^a$ instead of $\tilde{\omega}_2^a$. This means, Γ^a is isomorphic to a lattice which is generated by $\tilde{\omega}_1^a = 1$ and some complex number in the upper half-plane. But we can even specify its location. As ω_2^a has at least the same length as ω_1^a by assumption,

$$\|\tilde{\omega}_2^a\| \geq \|\tilde{\omega}_1^a\| = 1 \tag{7.3}$$

so $\tilde{\omega}_2^a$ is outside the unit circle.
Moreover, if $|\Re(\tilde{\omega}_2^a)| > \frac{1}{2}$ held true we could add a suitable integer multiple of 1 to $\tilde{\omega}_2^a$ with the consequence that $|\Re(\tilde{\omega}_2^a)| \leq \frac{1}{2}$ and the imaginary part would remain unchanged. Hence, total length could be reduced which contradicts the minimality condition. Therefore,

$$|\Re(\tilde{\omega}_2^a)| \leq \frac{1}{2}. \tag{7.4}$$

Putting (7.2), (7.3) and (7.4) together yields the following

Proposition 7.9.
Let $a \in \mathcal{M}_2^1$ and Γ^a be the induced lattice with shortest generators (ω_1^a, ω_2^a) satisfying minimality condition. Then Γ^a is isomorphic to a lattice whose generators are $(1, \tau^a)$ with

$$\tau^a \in \mathcal{N}_1 := \left\{ \tau \in \mathbb{H}^+ \mid \|\tau\| \geq 1\,,\, |\Re(\tau)| \leq \frac{1}{2} \right\}.$$

Remark 7.10.
If two lattices are isomorphic they result from each other via a rotation-dilation as defined

above. But crucially, this does not necessarily imply that both pairs of generators are obtained by a rotation-dilation as well. This statement holds only true if we make the important restriction on pairs of shortest lengths (minimality condition). In general, two lattices are isomorphic if and only if their generators can be obtained by a *Moebius transformation* from each other (Freitag, Busam [2], Proposition V.7.4). Notice that rotation-dilations are contained in the set of Moebius transformations.
Therefore, one can also prove Proposition 7.9 considering an arbitrary pair of Γ^a generators (ω_1^a, ω_2^a) using the theory of Moebius transformations. This was proven, for instance, in Freitag, Busam [2], Propostion V.8.7.

Definition 7.11.
The set $\mathcal{N}_1$ is called *modular figure* and is sketched in Figure 8. It is a fundamental region for the so called modular group $SL(2,\mathbb{Z})$.

Remark 7.12.
The reference to the modular group $SL(2,\mathbb{Z})$ comes from the fact that any element from $SL(2,\mathbb{Z})$ is in one-to-one correspondence with a Moebius transformation which transfers generators of lattices into each other. For details, check Freitag, Busam [2], chapter V.7.

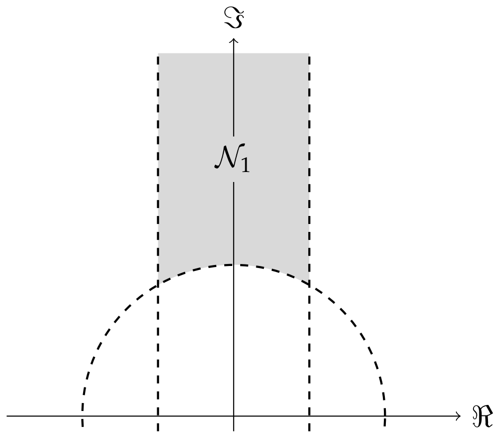


Figure 8: The set $\mathcal{N}_1$.

If we identify for appropriate $x, y \in \mathbb{R}$ the following elements of $\mathcal{N}_1$

$$\begin{aligned} -\frac{1}{2} + iy &\sim \frac{1}{2} + iy \\ -x + i\sqrt{1-x} &\sim x + i\sqrt{1-x} \end{aligned}$$

with each other, then the space $\mathcal{N}_1$ parametrizes the equivalent classes of isomorphic lattices. Hence, we can identify a determinant polynomial $a \in \mathcal{M}_2^1$ with an element from $\mathcal{N}_1$ due to Proposition 7.9. To be more precise, the following mapping is well defined:

$$\begin{aligned} g : \mathcal{M}_2^1 &\to \mathcal{N}_1 \\ a &\mapsto \tau^a. \end{aligned} \tag{7.5}$$

Notation 1.
For the rest of this elaboration we denote

$$\mathcal{M}_2^2 := \{a \in \mathcal{M}_2 \mid a \text{ has two double roots on } \mathcal{S}^1\}.$$

From now on, we discuss the following claims:

1. The mapping g is uniquely continuously extendable on $\mathcal{M}_2^2$ and on $\mathcal{M}_2 \setminus (\mathcal{M}_2^0 \cup \mathcal{M}_2^2)$ (the set of polynomials a having exactly one double root on $\mathcal{S}^1$).
2. The mapping g is surjective on this continuation.

We note that

$$\widetilde{\mathcal{M}}_2 = \mathcal{M}_2^1 \cup \mathcal{M}_2^2 \cup (\mathcal{M}_2 \setminus (\mathcal{M}_2^0 \cup \mathcal{M}_2^2)).$$

Before we get started, we need to familiarize ourselves with the concepts of frames and monodromies since they will build the fundament of the above mentioned continuation. The eigenvalues of the monodromies contain information about the periods which will enable us to explicitly calculate the generators (whose existence is so far just abstractly proven) on the continuation.

7.2 Frames and Monodromies

Throughout this subsection we assume $a \in \mathcal{M}_2^1$ and we have already seen that Γ^a from (7.1) defines a lattice in $\mathbb{R}^2$. The double periodicity of the flows ϕ is reflected in the coefficients $(\alpha, \beta, \gamma)(x, y)$ and therefore, the resulting $U(\zeta_\lambda), V(\zeta_\lambda)$ are doubly periodic as well:

$$U((x,y)+\omega_1) = U(x,y) = U((x,y)+\omega_2)$$
$$V((x,y)+\omega_1) = V(x,y) = V((x,y)+\omega_2).$$

We now focus on the following system of ordinary differential equations

$$\frac{\partial F_\lambda}{\partial x} = F_\lambda U \quad \frac{\partial F_\lambda}{\partial y} = F_\lambda V \quad F_\lambda(0,0) = \mathbb{1}. \tag{7.6}$$

The naturally arising question is whether there exists a kind of fundamental solution F_λ which is unique and solves both equations as well as the initial condition. The answer is yes as both equations satisfy the assumptions of Picard-Lindelöf Theorem 2.8 and the Maurer-Cartan equation (5.3) forms an integrability condition. More precisely, we solve

$$\frac{\partial F_\lambda(0,y)}{\partial y} = F_\lambda(0,y)V(0,y)$$

with initial condition

$$F_\lambda(0,0) = \mathbb{1}.$$

Then, for fixed y_0 we solve

$$\frac{\partial F_\lambda(x,y_0)}{\partial x} = F_\lambda(x,y_0)U(x,y_0)$$

with initial condition

$$F_\lambda(0,y_0).$$

Thus, we have defined a mapping $F_\lambda(x,y)$. Obviously,

$$\frac{\partial F_\lambda(x,y)}{\partial x} = F_\lambda(x,y)U(x,y)$$

holds true for all x, y. Due to the Maurer-Cartan equation (5.3) we have

$$\frac{\partial^2 F_\lambda(x,y)}{\partial xy} = \frac{\partial^2 F_\lambda(x,y)}{\partial yx}$$

which leads to

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\partial F_\lambda(x,y)}{\partial y} - F(x,y)V(x,y)\right) &= \frac{\partial^2 F_\lambda}{\partial y \partial x} - \frac{\partial F_\lambda}{\partial x}V - F_\lambda \frac{\partial V}{\partial x} \\
&= \frac{\partial}{\partial y}(F_\lambda U) - F_\lambda UV - F_\lambda \frac{\partial V}{\partial x} \\
&= \frac{\partial F_\lambda}{\partial y}U + F_\lambda\left(\frac{\partial}{\partial y}U - UV - \frac{\partial V}{\partial x}\right) \\
&= \left(\frac{\partial F_\lambda(x,y)}{\partial y} - F(x,y)V(x,y)\right)U(x,y).
\end{aligned}$$

Inserting the value $x = 0$ yields the initial condition

$$\frac{\partial F_\lambda(0,y)}{\partial y} - F(0,y)V(0,y) = F(0,y)V(0,y) - F(0,y)V(0,y) = 0.$$

By Picard-Lindelöf, this solution equals zero for all values x, y and we conclude

$$\frac{\partial F_\lambda(x,y)}{\partial y} = F(x,y)V(x,y).$$

Moreover, we define

$$M_\lambda^i := F(\omega_i),\ i = 1,2 \tag{7.7}$$

and consider

$$\tilde{F}_\lambda^i := (M_\lambda^i)^{-1}F_\lambda((x,y)+\omega_i),\ i = 1,2$$

which solves the following initial value problem:

$$\begin{aligned}
&\frac{\partial \tilde{F}_\lambda^i}{\partial x} = (M_\lambda^i)^{-1}F_\lambda((x,y)+\omega_i)U((x,y)+\omega_i) = \tilde{F}_\lambda^i U(x,y) \\
&\frac{\partial \tilde{F}_\lambda^i}{\partial y} = (M_\lambda^i)^{-1}F_\lambda((x,y)+\omega_i)V((x,y)+\omega_i) = \tilde{F}_\lambda^i V(x,y) \\
&\tilde{F}_\lambda^i(0,0) = \mathbb{1}.
\end{aligned}$$

By uniqueness of the fundamental solution F_λ, we immediately obtain $F_\lambda = \tilde{F}_\lambda^i$, $i = 1,2$ and conclude

$$F_\lambda((x,y)+\omega_i) = M_\lambda^i F_\lambda(x,y),\ i = 1,2. \tag{7.8}$$

Definition 7.13.
The fundamental solution F_λ from (7.6) is called *frame* and the matrices M_λ^1, M_λ^2 from (7.7) *monodromies*.

For reasons of simplicity, we will switch from real two-dimensional to complex notation and rather write $z = x + iy$ instead of (x, y) for the rest of this chapter.
First, we want to point out several pleasant features of these monodromies.

Lemma 7.14.
The monodromies M_λ^i, $i = 1, 2$ and the initial value of a Polynomial Killing field $\zeta_\lambda^0 := \zeta_\lambda(0,0)$ commute pairwise.

Proof.
Let us recall condition (7.8):

$$F_\lambda(z + \omega_1) = M_\lambda^1 F_\lambda(z) \quad \text{for all } z \in \mathbb{C}$$
$$F_\lambda(z + \omega_2) = M_\lambda^2 F_\lambda(z) \quad \text{for all } z \in \mathbb{C}.$$

If we insert $z = \omega_2$ in the first equation and $z = \omega_1$ in the second, we receive

$$M_\lambda^1 M_\lambda^2 = M_\lambda^1 F_\lambda(\omega_2) = F_\lambda(\omega_1 + \omega_2) = M_\lambda^2 F_\lambda(\omega_1) = M_\lambda^2 M_\lambda^1$$

which implies

$$[M_\lambda^1, M_\lambda^2] = 0.$$

For the commutators $[M_\lambda^i, \zeta_\lambda^0]$, $i = 1, 2$ we need to understand that

$$\zeta_\lambda(x, y) = F_\lambda^{-1} \zeta_\lambda^0 F_\lambda \tag{7.9}$$

holds true. Then a quick computation yields

$$\zeta_\lambda^0 = \zeta_\lambda(\omega_i) = F_\lambda^{-1}(\omega_i) \zeta_\lambda^0 F_\lambda(\omega_i) = (M_\lambda^i)^{-1} \zeta_\lambda^0 M_\lambda^i$$

which implies

$$[M_\lambda^i, \zeta_\lambda^0] = 0, \; i = 1, 2.$$

Thus, it remains to verify (7.9). We define

$$\tilde{\zeta}_\lambda := F_\lambda^{-1} \zeta_\lambda^0 F_\lambda.$$

From $F^{-1} F = \mathbb{1}$, one can quickly demonstrate that F^{-1} satisfies the following differential equations

$$\frac{\partial F^{-1}}{\partial x} = -U F^{-1} \qquad \frac{\partial F^{-1}}{\partial y} = -V F^{-1}. \tag{7.10}$$

Now we calculate the derivatives of $\tilde{\zeta}_\lambda$

$$\frac{\partial \tilde{\zeta}_\lambda}{\partial x} = \frac{\partial F^{-1}}{\partial x} \zeta_\lambda^0 F + F^{-1} \zeta_\lambda^0 \frac{\partial F}{\partial x} = \tilde{\zeta}_\lambda U - U \tilde{\zeta}_\lambda = [\tilde{\zeta}_\lambda, U]$$
$$\frac{\partial \tilde{\zeta}_\lambda}{\partial y} = [\tilde{\zeta}_\lambda, V].$$

Furthermore, $\tilde{\zeta}_\lambda(0,0) = \zeta_\lambda^0$ is valid. Due to uniqueness of the solutions of the Lax equations (according to Picard-Lindelöf) $\zeta_\lambda = \tilde{\zeta}_\lambda$ must be valid and (7.9) is justified. **q.e.d.**

Lemma 7.15.
The monodromies (7.7) satisfy $\det(M^i_\lambda) = 1$.

Proof.
Applying Theorem A.3 one can show that a fundamental solution of a differential equation $F'(t) = A(t)F(t)$, $F(0) = \mathbb{1}$ with continuous mapping $t \mapsto A(t)$ satisfies $\det(F(0)) = 1$ and

$$\begin{aligned} &\frac{d}{dt}\det(F(t)) = \operatorname{tr}(A(t))\det(F(t)) \\ \Rightarrow &\frac{d}{dt}\ln(\det(F(t))) = \operatorname{tr}(A(t)) \\ \Rightarrow &\ln(\det(F(t))) = \int_0^t \operatorname{tr}(A(s))ds \\ \Rightarrow &\det(F(t)) = \exp\left(\int_0^t \operatorname{tr}(A(s))ds\right). \end{aligned}$$

In our specific case, we have either $t = x$ and $A = U$ for given y or $t = y$ and $A = V$ for given x. Since U and V are both traceless, the fundamental solutions of each of the equations (7.6) have determinant one. Consequently, from the discussion after (7.6) referring to the construction of the frame F_λ, the assertion follows immediately. **q.e.d.**

Remark 7.16.
In accordance with the latter lemma, we will denote the eigenvalues of the matrices M^i_λ, $i = 1, 2$

$$\mu^i_\lambda, \ \frac{1}{\mu^i_\lambda}.$$

Furthermore, the eigenvalues of ζ_λ will be denoted $\nu_\lambda, -\nu_\lambda$ (recall the fact that $\operatorname{tr}(\zeta_\lambda) = 0$). We can put ν_λ in concrete terms:

$$\begin{aligned} 0 = \det(\zeta_\lambda - \nu_\lambda \mathbb{1}) &= \frac{1}{2}\left((\operatorname{tr}(\zeta_\lambda - \nu_\lambda\mathbb{1}))^2 - \operatorname{tr}((\zeta_\lambda - \nu_\lambda\mathbb{1})^2)\right) \\ &= \frac{1}{2}\left(4\nu^2 - (2\nu_\lambda^2 - 2\nu_\lambda \operatorname{tr}(\zeta_\lambda) + \operatorname{tr}(\zeta_\lambda^2))\right) \\ &= \nu_\lambda^2 - \frac{1}{2}\operatorname{tr}(\zeta_\lambda^2) \\ &= \nu_\lambda^2 - \det(\zeta_\lambda). \end{aligned}$$

Thus, we receive $\nu_\lambda^2 = \det(\zeta_\lambda) = \lambda a(\lambda)$, so

$$\nu_\lambda = \pm\sqrt{\lambda a(\lambda)}.$$

Lemma 7.17.
Let $a \in \mathcal{M}^1_2$ *and* $\tilde\lambda$ *an arbitrary root of* a. *Then the eigenvalues of the monodromies satisfy*

$$\mu^i_{\tilde\lambda} = \pm 1, \ \textit{for } i = 1, 2.$$

Proof.
Let $\zeta_\lambda \in I(a)$. Then $\zeta_{\tilde\lambda}$ has eigenvalues $\nu_{\tilde\lambda} = 0$ and the Jordan normal form is

$$J = Q^{-1}\zeta_{\tilde\lambda}Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(Theoretically, another candidate for J would be the matrix with zero entries, but then $\zeta_{\tilde{\lambda}}$ would equal zero, so $\tilde{\lambda}$ would be a root for ζ_λ and hence, a double root for a which is impossible by assumption.)
Due to Lemma 7.14

$$0 = [M^i_{\tilde{\lambda}}, \zeta_{\tilde{\lambda}}] = Q^{-1}[M^i_{\tilde{\lambda}}, \zeta_{\tilde{\lambda}}]Q = [Q^{-1}M^i_{\tilde{\lambda}}Q, J]$$

is valid and if we denote $Q^{-1}M^i_{\tilde{\lambda}}Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the above equation becomes

$$0 = \begin{pmatrix} -c & a-d \\ 0 & c \end{pmatrix}$$

which implies $c = 0$ and $a = d$. In particular, we have proven that there exist numbers α_i, β_i

$$Q^{-1}M^i_{\tilde{\lambda}}Q = \alpha_i\mathbb{1} + \beta_i J \quad \Leftrightarrow \quad M^i_{\tilde{\lambda}} = Q(\alpha_i\mathbb{1} + \beta_i J)Q^{-1} = \alpha_i\mathbb{1} + \beta_i\zeta_{\tilde{\lambda}}.$$

Lemma 7.15 yields

$$1 = \det(M^i_{\tilde{\lambda}}) = \det(Q(\alpha_i\mathbb{1} + \beta_i J)Q^{-1}) = \det(\alpha_i\mathbb{1} + \beta_i J) = \alpha_i^2$$

and thus, $\alpha_i = \pm 1$. Then we directly obtain with $\det(M^i_{\tilde{\lambda}}) = 1$ and $\operatorname{tr}(M^i_{\tilde{\lambda}}) = 2\alpha_i = \pm 2$ the assertion. **q.e.d.**

It will turn out that the figure $\ln(\mu^i_\lambda)$ is of particular interest for us. With its help we will find out that when a has multiple roots on the unit circle $\mathcal{S}^1$, we obtain a lattice structure on a certain subset. In addition, we can then define the following polynomials $b_i(\lambda)$:

$$\frac{\partial}{\partial\lambda}\ln(\mu^i_\lambda) =: \frac{b_i(\lambda)}{\lambda\nu_\lambda} \quad i = 1, 2. \tag{7.11}$$

At this point, the benefit of the polynomials b_i remains unclear and we cannot even see that b_i are indeed polynomials.

7.3 Example: The Vacuum Solution

As an example, we will explicitly compute the frame, monodromies, its eigenvalues and the polynomials b_i for the vacuum solution $u \equiv 0$. The symbol $\equiv$ indicates that the respective figure is constant in z (or, alternatively, in (x, y)). As in Remark 5.7 we define

$$u := \ln\gamma \equiv 0$$

and infer

$$\alpha = -\frac{\partial u}{\partial z} \equiv 0 \qquad \gamma = e^u \equiv 1.$$

This means, we are in the setting of case b), in which the determinant polynomial a has two double roots on $\mathcal{S}^1$ (the spectral genus $g = 0$ case). We notice that β is a real number (not necessarily zero) which also remains constant in (x, y) (or z). In particular, the resulting ζ_λ remains constant (as already shown in case b)). For calculating the frame F_λ, one could conduct the procedure which was explained in the discussion of (7.6). Unfortunately, this method turns out to have considerably complex terms and calculations, so we are interested in an alternative way exploiting the simple form of u. With the common formulas we can compute

$$U_\lambda \equiv \begin{pmatrix} 0 & -(1+\lambda^{-1}) \\ 1+\lambda & 0 \end{pmatrix} \qquad V_\lambda \equiv \begin{pmatrix} 0 & i(1-\lambda^{-1}) \\ i(1-\lambda) & 0 \end{pmatrix}.$$

Thus, the system of ordinary differential equations (7.6) is autonomous. Formula (5.5) from Remark 5.7 yields

$$[V_\lambda, U_\lambda] = i \begin{pmatrix} 2(e^{2u} - e^{-2u}) & 2e^{-u}u_{\bar{z}}\lambda^{-1} + 2e^{u}u_z \\ 2e^{u}u_{\bar{z}} + 2e^{-u}u_z\lambda & 2(e^{-2u} - e^{2u}) \end{pmatrix} = 0$$

and consequently, the matrices U_λ, V_λ can be diagonalized simultaneously. Having calculated the diagonalizing matrix P as well as the respective diagonal matrices we can quickly conclude what the frame F_λ looks like by taking advantage of the autonomy of the differential equations. Therefore, we first focus on the computation of P.

For the upcoming calculations, we substitute $k := \sqrt{\lambda}$.

Eigenvalues and eigenvectors of U_λ:
For the eigenvalues δ and $-\delta$ we consider the characteristic equation

$$\det(U_\lambda - \delta\mathbb{1}) = \delta^2 + \lambda + \lambda^{-1} + 2 = 0$$

and infer

$$\delta_{1/2} = \pm i\sqrt{(1+\lambda)(1+\lambda^{-1})} = \pm i\sqrt{\lambda^{-1}(1+\lambda)^2} = \pm i(k+k^{-1}).$$

The eigenspace of $\delta_1 := i(k+k^{-1})$ will be denoted E_1^U and results from

$$\begin{aligned}(U_\lambda - \delta\mathbb{1})v_1 &= \begin{pmatrix} -i(k+k^{-1}) & -(1+k^{-2}) \\ 1+k^2 & -i(k+k^{-1}) \end{pmatrix} v_1 = 0 \\ &\Rightarrow \begin{pmatrix} 1+k^2 & -i(k+k^{-1}) \\ 0 & 0 \end{pmatrix} v_1 = 0 \\ &\Rightarrow E_1^U = \left\langle \begin{pmatrix} i \\ k \end{pmatrix} \right\rangle.\end{aligned}$$

The eigenspace of $\delta_2 := -i(k+k^{-1})$ will be denoted E_2^U and follows from

$$\begin{aligned}&\begin{pmatrix} 1+k^2 & i(k+k^{-1}) \\ 0 & 0 \end{pmatrix} v_2 = 0 \\ &\Rightarrow E_2^U = \left\langle \begin{pmatrix} -i \\ k \end{pmatrix} \right\rangle.\end{aligned}$$

Eigenvalues and eigenvectors of V_λ:
For the eigenvalues ϵ and $-\epsilon$, we analogously infer

$$\epsilon_{1/2} = \pm i\sqrt{(1-\lambda)(1-\lambda^{-1})} = \pm(k^{-1} - k).$$

The eigenspace of $\epsilon_1 := k^{-1} - k$ will be denoted E_1^V and emerge from

$$(V_\lambda - \epsilon \mathbb{1})w_1 = \begin{pmatrix} -(k^{-1}-k) & i(1-k^{-2}) \\ i(1-k^2) & -(k^{-1}-k) \end{pmatrix} w_1 = 0$$
$$\Rightarrow \begin{pmatrix} i(1-k^2) & -(k^{-1}-k) \\ 0 & 0 \end{pmatrix} w_1 = 0$$
$$\Rightarrow E_1^V = \left\langle \begin{pmatrix} -i \\ k \end{pmatrix} \right\rangle .$$

The eigenspace of $\epsilon_2 := -(k^{-1} - k)$ will be denoted E_2^U and results from

$$\begin{pmatrix} i(1-k^2) & (k^{-1}-k) \\ 0 & 0 \end{pmatrix} w_2 = 0$$
$$\Rightarrow E_2^V = \left\langle \begin{pmatrix} i \\ k \end{pmatrix} \right\rangle .$$

All in all, we obtain

$$E_1^U = E_2^V \qquad E_2^U = E_1^V$$

and hence, the simultaneously diagonalizing matrix P is of shape

$$P := \begin{pmatrix} i & -i \\ k & k \end{pmatrix} \qquad P^{-1} = \frac{1}{2}\begin{pmatrix} -i & k^{-1} \\ i & k^{-1} \end{pmatrix} .$$

The emerging diagonal matrices are

$$\Delta^U := P^{-1}U_\lambda P = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix} = \begin{pmatrix} i(k+k^{-1}) & 0 \\ 0 & -i(k+k^{-1}) \end{pmatrix}$$
$$\Delta^V := P^{-1}V_\lambda P = \begin{pmatrix} \epsilon_2 & 0 \\ 0 & \epsilon_1 \end{pmatrix} = \begin{pmatrix} -(k^{-1}-k) & 0 \\ 0 & (k^{-1}-k) \end{pmatrix} .$$

Now we can directly specify the frame F_λ using theory of ordinary differential equations as the the system (7.6) is autonomous. The generated complicated matrix exponent will be tremendously simplified due to simultaneous diagonalization:

$$\begin{aligned} F(x,y) &= \exp(xU_\lambda + yV_\lambda)F(0,0) = P\exp(x\Delta^U + y\Delta^V)P^{-1} \\ &= P\begin{pmatrix} e^{ix(k+k^{-1})+y(k^{-1}-k)} & 0 \\ 0 & e^{-ix(k+k^{-1})-y(k^{-1}-k)} \end{pmatrix} P^{-1} \\ &= \begin{pmatrix} \cos(x(k+k^{-1}) - iy(k^{-1}-k)) & -k^{-1}\sin(x(k+k^{-1}) - iy(k^{-1}-k)) \\ k\sin(x(k+k^{-1}) - iy(k^{-1}-k)) & \cos(x(k+k^{-1}) - iy(k^{-1}-k)) \end{pmatrix} . \end{aligned}$$

Alternatively, in $z, \bar{z}$-coordinates, we have:

$$F(z) = P\begin{pmatrix} e^{ikz+ik^{-1}\bar{z}} & 0 \\ 0 & e^{-ikz-ik^{-1}\bar{z}} \end{pmatrix} P^{-1} \tag{7.12}$$
$$= \begin{pmatrix} \cos(kz+k^{-1}\bar{z}) & -k^{-1}\sin(kz+k^{-1}\bar{z}) \\ k\sin(kz+k^{-1}\bar{z}) & \cos(kz+k^{-1}\bar{z}) \end{pmatrix} . \tag{7.13}$$

Importantly, any $z \in \mathbb{C}$ defines a period due to $u \equiv 0$. Therefore, the expression of the frame (7.13) also defines the monodromies. Clearly, their eigenvalues can be found in the diagonal matrix of (7.12), which means

$$\mu_\lambda^z = e^{ikz+ik^{-1}\bar{z}} \quad \Rightarrow \quad \ln(\mu_\lambda^z) = ikz + ik^{-1}\bar{z}. \tag{7.14}$$

For the polynomials $b_z(\lambda)$, we need to specify the derivative of $\ln(\mu_\lambda^z)$ with respect to λ as well as the eigenvalues ν of ζ_λ. The derivative is a simple computation (with $k = \sqrt{\lambda}$)

$$\frac{\partial}{\partial\lambda}\ln(\mu_\lambda^z) = \frac{1}{2\sqrt{\lambda}}iz - \frac{1}{2\sqrt{\lambda^3}}\,i\bar{z}.$$

We have already proven in Remark 7.16 that $\nu = \sqrt{\lambda a(\lambda)}$ and by means of formula (4.3) we obtain (recalling that β is real)

$$\begin{aligned}
\nu_\lambda &= \sqrt{\lambda a(\lambda)} \\
&= \sqrt{\lambda}\sqrt{\lambda^4 - 2\beta\lambda^3 + (\beta^2+2)\lambda^2 - 2\beta\lambda + 1} \\
&= \sqrt{\lambda}\sqrt{(\lambda^2 - \beta\lambda + 1)^2} \\
&= \sqrt{\lambda}\,(\lambda^2 - \beta\lambda + 1).
\end{aligned}$$

Putting all together with formula (7.11) yields

$$\begin{aligned}
b_z(\lambda) &= \lambda\nu_\lambda \frac{\partial}{\partial\lambda}\ln(\mu_\lambda^z) \\
&= \lambda\sqrt{\lambda}\,(\lambda^2 - \beta\lambda + 1)\left(\frac{1}{2\sqrt{\lambda}}iz - \frac{1}{2\sqrt{\lambda^3}}\,i\bar{z}\right) \\
&= \frac{1}{2}i(\lambda^2 - \beta\lambda + 1)\,(z\lambda - \bar{z}) \\
&= \frac{\sqrt{a(\lambda)}}{2}i\,(z\lambda - \bar{z})\,.
\end{aligned}$$

Remark 7.18.
We see that $b_z(0) = -\frac{1}{2}i\bar{z}$, i.e. the evaluation of b at the point $\lambda = 0$ equals the complex conjugate of the period multiplied by some factor. This statment also holds true for b of other spectral genus and demonstrates the value of the polynomial b.
Furthermore, one can prove that b satisfies a reality condition. In the case of the vacuum solution, this can be verified quickly with the above explicit expression for b:

$$\begin{aligned}
\lambda^3\overline{b(\bar{\lambda}^{-1})} &= \lambda^3\left(-\frac{1}{2}i\sqrt{\overline{a(\bar{\lambda}^{-1})}}(\bar{z}\lambda^{-1} - z)\right) \\
&= \lambda^3\left(-\frac{1}{2}i\sqrt{\lambda^{-4}a(\lambda)}(\bar{z}\lambda^{-1} - z)\right) \\
&= b(\lambda).
\end{aligned}$$

Remark 7.19.
We deliberately avoided any terminology referring to lattices throughout the analysis of the

vacuum solution. The reason is that we have proven the lattice property of Γ^a only for $a \in \mathcal{M}_2^1$. In the situation of the vacuum solution, a has two double points on $\mathcal{S}^1$ and thus, we know from our findings in case b):

$$\Gamma^a = \{(x,y) \in \mathbb{R}^2 \,|\, \phi(x,y)(\zeta_\lambda) = \zeta_\lambda\} = \mathbb{R}^2.$$

This is clearly no lattice as any complex number $z \in \mathbb{C}$ defines a period. However, we will see that we can associate a lattice to determinant polynomials with multiple roots on $\mathcal{S}^1$, namely the limit of lattices belonging to polynomials of $\mathcal{M}_2^1$. This is highly non-trivial because it is not clear that the limit of lattices is well-defined and that it is indeed at lattice itself. This problem is investigated in the remaining chapter.

7.4 Limits of Lattices: The polynomial $a(\lambda)$ has two double roots on $\mathcal{S}^1$.

We return to our interesting mapping

$$g: \mathcal{M}_2^1 \to \mathcal{N}_1$$
$$a \mapsto \tau^a$$

from (7.5). As mentioned before, we have the following goals:

1. The mapping g is uniquely continuously extendable on $\mathcal{M}_2^2$ and on $\mathcal{M}_2 \setminus (\mathcal{M}_2^0 \cup \mathcal{M}_2^2)$ (the set of polynomials a having exactly one double root on $\mathcal{S}^1$).

2. The mapping g is surjective on this continuation.

The main work behind these numbers will consist of the association of an appropriate lattice to determinant polynomials having multiple roots on the unit sphere. We will analyze the cases $a \in \mathcal{M}_2^2$ and $\mathcal{M}_2 \setminus (\mathcal{M}_2^0 \cup \mathcal{M}_2^2)$ separately. For $a \in \mathcal{M}_2^2$ we have noticed in the analysis of the vacuum solution that the first point is highly non-trivial due to the apparent lack of a lattice structure ($\Gamma^a = \mathbb{R}^2 = \mathbb{C}$). We face the same problem when a has exactly one double root on $\mathcal{S}^1$. However, we will find out that the set of $\mathcal{M}_2^1$ period-limits forms a uniquely defined lattice and we will explicitly state the continuation of g in this situation. This is sufficient to prove 1. and 2. for $a \in \mathcal{M}_2^2$. For $\mathcal{M}_2 \setminus (\mathcal{M}_2^0 \cup \mathcal{M}_2^2)$ we will use common findings from elliptic theory and merely give a numerical intuition that 2. holds true.
Sticking to this plan, we first consider $a \in \mathcal{M}_2^2$. The idea is to approximate a by a sequence $(a_n)_{n\in\mathbb{N}} \in \mathcal{M}_2^1$ and transfer well-known properties of the sequence to its limit with the help of the quantity $\ln(\mu_\lambda^i)$. In this way, we will understand how the set of period limits explicitly looks like.

Lemma 7.20.
Let $a \in \mathcal{M}_2^2$ and $(a_n)_{n\in\mathbb{N}} \in \mathcal{M}_2^1$ be a sequence converging against a with bounded associated sequence of periods $(\omega_1^m, \omega_2^m)_{m\in\mathbb{N}}$. Then there exists a subsequence $(a_m)_{m\in\mathbb{N}} \in \mathcal{M}_2^1$ with the following properties:

i) The associated sequence of periods $(\omega_1^m, \omega_2^m)_{m\in\mathbb{N}}$ converges.

ii) The associated sequence of monodromies $(M_m^i)_{m\in\mathbb{N}}$ converges.

iii) The associated sequence of eigenvalues $(\mu_m^i)_{m\in\mathbb{N}}$ converges.

Proof.
For reasons of clarity, we will omit the index λ during this proof, although a great part of the mentioned quantities do still depend on λ.
Due to boundedness of the associated sequence of periods $(\omega_1^m, \omega_2^m)_{m\in\mathbb{N}}$, there exists a subsequence indexed by $k \in \mathbb{N}$ such that $(\omega_1^k, \omega_2^k)_{k\in\mathbb{N}}$ converges. The limit is denoted by (ω_1, ω_2). Now we choose for every a_k an arbitrary, but fixed initial value $\zeta_0^k \in I(a_k)$ and consider the sequence $(\zeta_0^k)_{k\in\mathbb{N}}$. Since convergence implies boundedness, the sequence $(a_k)_{k\in\mathbb{N}}$ must be bounded, in particular the polynomial-coefficients must be bounded and therefore - in analogy to the compactness proof of Proposition 5.9 - the sequence $(\alpha_k, \beta_k, \gamma_k)_{k\in\mathbb{N}}$ must be bounded. Hence, there exists a convergent subsequence indexed by $l \in \mathbb{N}$. To sum up, so far we have found a subsequence $(a_l)_{l\in\mathbb{N}} \in \mathcal{M}_2^1$ such that the associated sequence $(\omega_1^l, \omega_2^l)_{l\in\mathbb{N}}$ converges against (ω_1, ω_2) and $(\zeta_0^l)_{l\in\mathbb{N}}$ converges against $\zeta_0 \in I(a)$.
In the next step, we focus on the sequence $(\zeta^l)_{l\in\mathbb{N}}$ of Polynomial Killing fields that results from the sequence of initial values $(\zeta_0^l)_{l\in\mathbb{N}}$. In order to prove its convergence, we check the conditions of Arzela-Ascoli Theorem A.4:

1. By assumption, the sequence of periods is bounded, so there exists a real number $R > 0$ such that
$$\|\omega_i^l\| \leq R \text{ for all } l \in \mathbb{N},\ i = 1, 2.$$
We shift the center of the fundamental region's parallelogram to zero and determine its maximal expansion:
$$\frac{\|\omega_1 + \omega_2\|}{2} \leq \frac{\|\omega_1\|}{2} + \frac{\omega_2\|}{2} \leq R$$
$$\frac{\|\omega_1 - \omega_2\|}{2} \leq \frac{\|\omega_1\|}{2} + \frac{\omega_2\|}{2} \leq R.$$
Therefore, we can regard $(\zeta^l)_{l\in\mathbb{N}}$ as a sequence in $C(K, \mathbb{R}^2)$ where $K = B(0, R)$, because all fundamental regions of Γ^{a_l} are contained in K.

2. Let $(x, y) \in K$ be fixed. In complete analogy to the above argumentation, the convergence of $(a_l)_{l\in\mathbb{N}} \in \mathcal{M}_2^1$ implies boundedness of the polynomial coefficients independent of the choice of sequence and hence, uniformly boundedness of the sequence $(\alpha_l(x, y), \beta_l(x, y), \gamma_l(x, y))_{l\in\mathbb{N}}$ with upper bound independent of l or (x, y). In particular, the sequence $(\zeta^l(x, y))_{l\in\mathbb{N}}$ is bounded.

3. For equicontinuity, we apply the same argumentation as in 2. and infer that the right hand side of the Lax equations (5.1) is uniformly bounded, i.e. the partial derivatives of ζ^l, $l \in \mathbb{N}$ are uniformly bounded. With help of the Boundedness Theorem one can prove that any upper bound of the derivatives is a Lipschitz constant on the convex set $B(0, K)$ and furthermore, the Lipschitz constant is common (due to uniformly boundedness). Consequently, the sequence $(\zeta^l)_{l\in\mathbb{N}}$ is equicontinuous in all points $(x, y) \in K$.

Arzela-Ascoli provides a subsequence $(a_m)_{m\in\mathbb{N}} \in \mathcal{M}_2^1$ indexed by $m \in \mathbb{N}$ such that all previously associated sequences converge as well as the sequence of Polynomial Killing fields $(\zeta^m)_{m\in\mathbb{N}}$. Its limit will be denoted by $\zeta(x, y)$ and solves the Lax equations due to the Variation of Parameters A.5 applied in x- and y-direction. This implies the convergence of the respective entries
$$(\alpha_m(x, y), \beta_m(x, y), \gamma_m(x, y)) \to (\alpha(x, y), \beta(x, y), \gamma(x, y)) \text{ for } m \to \infty$$

and these, in turn, entail for the right hand side matrices of the Lax equations (5.1)

$$U^m \to U \quad \text{and} \quad V^m \to V \text{ for } m \to \infty.$$

Thanks to the powerful Variation of Parameters Theorem A.5, the solutions of the equations

$$\frac{\partial F^m}{\partial x} = F^m U \quad \frac{\partial F^m}{\partial y} = F^m V \quad F^m(0,0) = \mathbb{1} \tag{7.15}$$

depend continuously on the right hand sides on compact sets. By regarding $K = B(0, R)$ as in 1. of Arzela-Ascoli, we infer for the frames

$$F^m(x,y) \to F(x,y) \text{ for } m \to \infty.$$

This implies convergence of the monodromies

$$\lim_{m\to\infty} M^i_m = \lim_{m\to\infty} F^m(w^m_i) = F(w_i) = M^i \text{ for } i = 1, 2.$$

The second equality might be a bit confusing at first glance as we consider two sequences at once, but it holds true due to the following observation: Let a_n be an arbitrary sequence in $\mathbb{R}^d$ and b_n another one which converges against $b \in \mathbb{R}^d$. Then they have the same limit b if and only if the sequence $b_n - a_n$ converges against zero since we can denote $a_n = b_n + (a_n - b_n)$. In our case, we have $a_n = F^m(w^m_i)$ and $b_n = F^m(w_i)$ with limit $b = F(\omega_i)$ so we need to verify that $F^m(w^m_i) - F^m(w_i)$ converges against zero. This holds true as $(F^m)_{m\in\mathbb{N}}$ is equicontinuous (by completely analogous argumentation as conducted in the third part of Arzela-Ascoli Theorem application above). Finally, we have

$$\mu^i_m \to \mu^i \text{ for } m \to \infty.$$

q.e.d.

The previous lemma is quite powerful. Given $a \in \mathcal{M}^2_2$ with roots λ_0, $\bar{\lambda}_0 \in \mathcal{S}^1$, there can always be found a sequence $(a_n)_{n\in\mathbb{N}} \in \mathcal{M}^1_2$ converging against a (Figure 9).

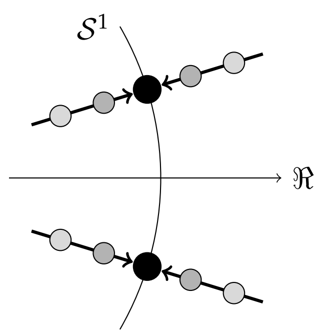


Figure 9: $\mathcal{M}^1_2$-sequence (gray) converging against a given element of $\mathcal{M}^2_2$ (black).

In accordance with Lemma 7.20, there is a subsequence such that the periods $(\omega^m_1, \omega^m_2)_{m\in\mathbb{N}}$ converge with limit (ω_1, ω_2) and the associated sequence of (λ-dependent) monodromy eigenvalues $(\mu_m^i)_{m\in\mathbb{N}}$ converge against μ_λ^i. From the latter property and Lemma 7.17 we can infer after inserting for λ the respective roots

$$\mu^i_{\lambda_0} = \pm 1,\ i = 1,2$$
$$\mu^i_{\bar\lambda_0} = \pm 1,\ i = 1,2$$

which implies

$$\ln(\mu^i_{\lambda_0}) \in i\pi\mathbb{Z},\ i = 1,2.$$
$$\ln(\mu^i_{\bar\lambda_0}) \in i\pi\mathbb{Z},\ i = 1,2.$$

At this point the quantity $\ln(\mu^i_{\lambda_0})$ displays its importance. Due to our computations in (7.14), we precisely know what it looks like and conclude with $k = \sqrt{\lambda_0}$

$$ik\omega_j + ik^{-1}\bar\omega_j \in i\pi\mathbb{Z}$$
$$ik^{-1}\omega_j + ik\bar\omega_j \in i\pi\mathbb{Z},\ j = 1,2.$$

The limits of periods satisfy two simple conditions defining a lattice and both conditions are generated by the figure $\ln(\mu^i_{\lambda_0})$. More precisely, even though there is no lattice structure given when $a \in \mathcal{M}_2^2$, we have proven - under the condition of a_n inducing a bounded sequence of periods - that the set of limits of periods is contained in the following lattice

$$\tilde\Gamma^a := \{\omega \in \mathbb{C} \,|\, k\omega + k^{-1}\bar\omega \in \pi\mathbb{Z},\ k^{-1}\omega + k\bar\omega \in \pi\mathbb{Z}\} \subset \Gamma^a = \mathbb{C}$$

with $k = \sqrt{\lambda_0}$ and λ_0 being a double root of a. It remains to demonstrate that the set of limits of periods is not only contained in but really equals $\tilde\Gamma^a$. Moreover, the boundedness hypothesis needs to be eliminated. However, we will refer to Hauswirth et al. [7] at this point so that we can see this as a

Fact 1.
For given $a \in \mathcal{M}_2$ we define $k := \sqrt{\lambda}$ and focus on a sliced set G consisting of the unit ball $\{k \in \mathbb{C} \,|\, |k| < 1\}$ with extracted line segments connecting k^* and $(\bar k^*)^{-1}$ with $a((k^*)^2) = 0$.

i) $v_\lambda^2 = \lambda a(\lambda)$ defines a holomorphic function $k \mapsto \nu_k$ on G which is unique up to sign.

ii) The expression $\frac{b(\lambda)}{\lambda\nu_\lambda}$ can be integrated as a quotient of holomorphic functions. We define

$$h(k) := \int \frac{b(k^2)}{k^2\nu_k} 2k dk = 2\int \frac{b(k^2)}{k\nu_k} dk$$

on G and the condition $h(-k) = -h(k)$ specifies the integration constant.

iii) ν_k and k as well as the polynomial $b(k^2)$ are continuously extendable on the roots of a.

iv) $h(k)$ is continuously extendable on the roots of a.

A complex number w is a period if and only if $h(k^*) \in i\pi\mathbb{Z}$ for all k^* satisfying $a((k^*)^2) = 0$ (Hauswirth et al. [7], Corollary 5.9, Definition 5.10).

Remark 7.21.
In our case, we have

$$h_j(k) = 2\int \frac{b(k^2)}{k\nu_k}dk = \int i(\omega_j - k^{-2}\bar{\omega}_j)dk = ik\omega_j + ik^{-1}\bar{\omega}_j + const. \quad j = 1,2.$$

The condition $h_j(-k) = -h(k)$ implies $const. = 0$. Furthermore, the polynomial a has two double roots λ_0 and λ_0^{-1} on $\mathcal{S}^1$ and therefore, we define $k_1^* := \sqrt{\lambda_0}$ and $k_2^* := \frac{1}{k_1^*}$.Hence, the requirement $h(k^*) \in i\pi\mathbb{Z}$ leads to (abbreviating $k = k_1^*$)

$$\begin{aligned} &ik\omega_j + ik^{-1}\bar{\omega}_j \in i\pi\mathbb{Z} \\ &ik^{-1}\omega_j + ik\bar{\omega}_j \in i\pi\mathbb{Z},\ j = 1,2. \end{aligned}$$

as in $\tilde{\Gamma}^a$.

Remark 7.22.
The function $\ln(\mu_\lambda)$ can be considered as the extension of the function h.

Before we start to focus on the mapping g again, we compute the generators (ω_1, ω_2) of $\tilde{\Gamma}^a$. We need to solve the following system of equations:

$$\begin{aligned} k\omega_1 + k^{-1}\bar{\omega}_1 &= \pi \\ k^{-1}\omega_1 + k\bar{\omega}_1 &= 0 \\ k\omega_2 + k^{-1}\bar{\omega}_2 &= 0 \\ k^{-1}\omega_2 + k\bar{\omega}_2 &= \pi. \end{aligned}$$

The solutions are

$$\begin{aligned} \omega_1 &= \frac{\pi}{k - k^{-3}} \\ \omega_2 &= \frac{\pi}{k^3 - k^{-1}} = k^{-2}\omega_1 = \lambda_0^{-1}\omega_1. \end{aligned} \tag{7.16}$$

Obviously, these computations are only valid for $\lambda_0 \notin \{-1, 1\}$, i.e. when a has no quadruple roots but two distinct double roots on $\mathcal{S}^1$. Now we can define the mapping g from (7.5) on $\mathcal{M}_2^2$ by taking $\tilde{\Gamma}^a$ rather than Γ^a into consideration and proceed as usual: We take two generators of $\tilde{\Gamma}^a$ satisfying minimality condition, transfer the shortest to one via rotation-dilation and define τ^a as the remaining generator in $\mathcal{N}_1$.
We will now examine how τ^a concretely looks like. If the generators ω_1, ω_2 from (7.16) satisfy minimality condition, i.e. if they are of shortest length, we transfer any of them to one via rotation-dilation (since both ω_1 and ω_2 have the same length), for example ω_2. In this case we obtain

$$\tau^a = \frac{\omega_1}{\omega_2} = \lambda_0 \in \mathcal{S}^1.$$

However, this τ^a does only result when the generators ω_1, ω_2 from (7.16) satisfy the minimality condition. We will now transform this condition into a restriction on λ_0 by having a closer look at Figure 7. Minimality condition is satisfied if and only if

$$\begin{aligned}
& \|\omega_2\| \overset{!}{\leq} \|\omega_2 - \omega_1\| = \|(1-\lambda_0)\omega_2\| = \|1-\lambda_0\|\|\omega_2\| \\
\Leftrightarrow \quad & 1 \overset{!}{\leq} (\Re(1-\lambda_0))^2 + (\Im(1-\lambda_0))^2 \\
\Leftrightarrow \quad & 1 \overset{!}{\leq} (1-\Re(\lambda_0))^2 + (\Im(\lambda_0))^2 \\
\Leftrightarrow \quad & 1 \overset{!}{\leq} 1 - 2\Re(\lambda_0) + \Re(\lambda_0)^2 + (\Im(\lambda_0))^2 \\
\Leftrightarrow \quad & 1 \overset{!}{\leq} 2 - 2\Re(\lambda_0) \\
\Leftrightarrow \quad & \Re(\lambda_0) \overset{!}{\leq} \frac{1}{2}
\end{aligned}$$

and analogously,

$$\begin{aligned}
& \|\omega_2\| \overset{!}{\leq} \|\omega_2 + \omega_1\| = \|(1+\lambda_0)\omega_2\| = \|1+\lambda_0\|\|\omega_2\| \\
\Leftrightarrow \quad & -\frac{1}{2} \overset{!}{\leq} \Re(\lambda_0).
\end{aligned}$$

Hence, ω_1 and ω_2 satisfy minimality condition if and only if $|\Re(\lambda_0)| \leq \frac{1}{2}$ and in this case, we obtain

$$\tau^a = \lambda_0 \in \mathcal{S}^1.$$

When $|\Re(\lambda_0)| > \frac{1}{2}$ either $\omega_2 - \omega_1$ or $\omega_1 + \omega_2$ is a shorter generator and needs to be transferred to 1 via rotation-dilation.
If $\omega_2 - \omega_1$ is the shortest generator, we consider the generators $(\omega_1, \omega_2 - \omega_1)$ (of course, one could also regard $(\omega_2, \omega_2 - \omega_1)$) and execute the respective rotation-dilation carrying $\omega_2 - \omega_1$ to 1. Then, we obtain

$$\tau^a = \frac{\omega_1}{\omega_2 - \omega_1} = \frac{\frac{\omega_1}{\omega_2}}{1 - \frac{\omega_1}{\omega_2}} = \frac{\lambda_0}{1-\lambda_0}.$$

In order to substantiate the geometrical location of τ^a, we substitute $\frac{\omega_1}{\omega_2} = \lambda_0 = e^{i\varphi}$ and compute

$$\begin{aligned}
\tau^a &= \frac{e^{i\varphi}}{1-e^{i\varphi}} = \frac{e^{i\varphi}(1-e^{-i\varphi})}{(1-e^{i\varphi})(1-e^{-i\varphi})} \\
&= \frac{e^{i\varphi} - 1}{2 - e^{i\varphi} - e^{-i\varphi}} \\
&= \frac{\cos(\varphi) + i\sin(\varphi) - 1}{2(1-\cos(\varphi))} \\
&= -\frac{1}{2} + i\,\frac{\sin(\varphi)}{2(1-\cos(\varphi))}.
\end{aligned}$$

This means that τ^a has real part of $-\frac{1}{2}$ and the absolute value of the imaginary part explodes when φ approaches 0 or 2π (Figure 10 (a)). With other words,

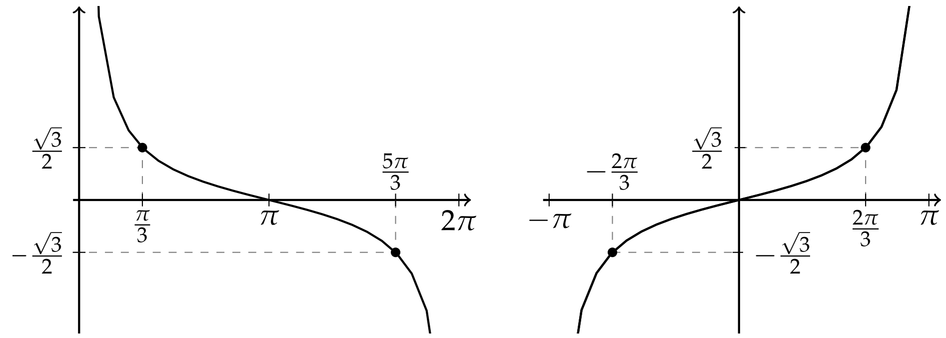
$$|\Im(\tau^a)| \to \infty \text{ when } \Re(\lambda_0) \uparrow 1.$$

If $\omega_1 + \omega_2$ is the shortest generator, we demonstrate analogously

$$\tau^a = \frac{\omega_1}{\omega_1 + \omega_2} = \frac{\frac{\omega_1}{\omega_2}}{\frac{\omega_1}{\omega_2} + 1} = \frac{\lambda_0}{1 + \lambda_0} = \frac{1}{2} + i\,\frac{\sin(\varphi)}{2(1 + \cos(\varphi))}$$

and conclude that τ^a has real part $\frac{1}{2}$ in this case as well as exploding absolute imaginary part whenever φ approaches π or $-\pi$ (Figure 10 (b)) or alternatively,

$$|\Im(\tau^a)| \to \infty \text{ when } \Re(\lambda_0) \downarrow -1.$$



(a) The function $\frac{\sin(\varphi)}{2(1-\cos(\varphi))}$. (b) The function $\frac{\sin(\varphi)}{2(1+\cos(\varphi))}$.

Figure 10: Plot of the imaginary values of τ^a in dependence of φ.

Finally, we will make some symmetry observations. We are allowed to restrict

$$\lambda_0 \in \mathcal{S}^1 \cap \mathbb{H}^+$$

instead of $\lambda_0 \in \mathbb{S}^1$ with $\mathbb{H}^+ := \{\omega \in \mathbb{C} \,|\, \Im(\omega) > 0\}$ because if we consider $\lambda_1 \in \mathcal{S}^1 \cap \mathbb{H}^-$ with corresponding τ_1 and $\lambda_2 := \bar{\lambda}_1 \in \mathcal{S}^1 \cap \mathbb{H}^+$ with τ_2 then one can verify with the above formulas for τ^a that

$$\tau_1 = \bar{\tau}_2$$

holds. In particular, the mapping $\lambda_0 \mapsto \tau^a$ is invariant under complex conjugation (which implies axis-symmetry to the real line) and the restriction is justified. Furthermore, the case of $a \in \mathcal{M}_2^2$ owning quadruple roots is excluded since $\pm 1 \notin \mathbb{H}^+$.

Remark 7.23.
Additionally, we could even restrict

$$\lambda_0 \in \mathcal{S}^1 \cap \mathbb{Q}^1$$

where $\mathbb{Q}^1 := \{\omega \in \mathbb{C} \,|\, \Re(\omega) \geq 0, \Im(\omega) > 0\}$ is the first quadrant of the complex plane because if we consider $\lambda_1 \in \mathcal{S}^1 \cap \mathbb{Q}^3$ from the third quadrant with corresponding τ_1 and $\lambda_2 := -\lambda_1 \in \mathcal{S}^1 \cap \mathbb{Q}^1$ with τ_2 then one can verify with the above formulas for τ^a that

$$\tau_1 = -\tau_2$$

holds. In particular, the mapping $\lambda_0 \mapsto \tau^a$ is invariant under sign changes (which implies point-symmetry to zero) and the restriction is justified.

To sum up, we have explicitly calculated the continuous extension of $\mathcal{M}_2^1$-lattices to polynomials having two double roots on $\mathcal{S}^1$:

Theorem 7.24.
Let $a \in \mathcal{M}_2^2$ with double roots $\lambda_0, \bar{\lambda}_0 \in \mathcal{S}^1$ and $\lambda_0 \in \mathcal{S}^1 \cap \mathbb{H}^+$. For the associated lattice of period limits

$$\tilde{\Gamma}^a := \{\omega \in \mathbb{C} \mid k\omega + k^{-1}\bar{\omega} \in \pi\mathbb{Z},\, k^{-1}\omega + k\bar{\omega} \in \pi\mathbb{Z}\}$$

with $k = \sqrt{\lambda_0}$ at least one of the generator pairs (ω_1, ω_2), $(\omega_1, \omega_2 - \omega_1)$ and $(\omega_1, \omega_1 + \omega_2)$ satisfies minimality condition whereby

$$\omega_1 = \frac{\pi}{k - k^{-3}} \qquad \omega_2 = \frac{\pi}{k^3 - k^{-1}} = k^{-2}\omega_1 = \lambda_0^{-1}\omega_1.$$

Furthermore, the mapping g from (7.5) can be extended on $\mathcal{M}_2^2$:

$$g: \mathcal{M}_2^2 \to \partial\mathcal{N}_1$$
$$a \mapsto \tau^a$$

where

$$\tau^a = \begin{cases} \lambda_0 & \text{for } |\Re(\lambda_0)| \leq \frac{1}{2} \\ \frac{\lambda_0}{1-\lambda_0} & \text{for } \Re(\lambda_0) > \frac{1}{2} \\ \frac{\lambda_0}{1+\lambda_0} & \text{for } \Re(\lambda_0) < -\frac{1}{2} \end{cases}$$

as shown in Figure 11.

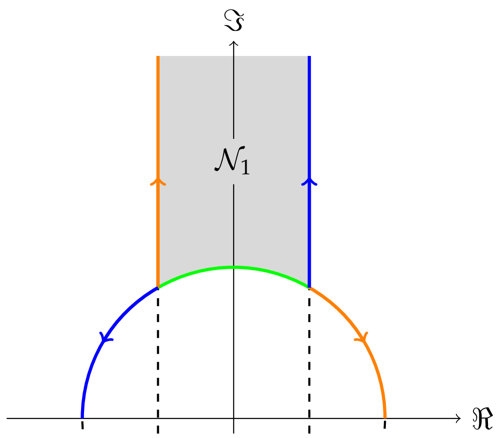


Figure 11: Roots $\lambda_0 \in \mathcal{S}^1 \cap \mathbb{H}^+$ and resulting τ^a are sketched in the same color.

Remark 7.25.
Due to our preconsiderations, the continuation on $\mathcal{M}_2^2$ of the mapping g from Theorem 7.24 is obviously surjective on $\partial\mathcal{N}_1$.

Remark 7.26.
Heuristically spoken, the case in which a has a quadruple root corresponds to the limit of the above mapping g where the absolute imaginary part equals infinity.

7.5 Limits of Lattices: The polynomial $a(\lambda)$ has one double root on $\mathcal{S}^1$.

We remember from Figure 1 that a is under the assumptions of this subsection uniquely defined by a radius $r > 0$ and an angle φ:

$$a(\lambda) = (\lambda - re^{i\varphi})(\lambda - r^{-1}e^{i\varphi})(\lambda - e^{-i\varphi})^2.$$

Similar to the previous subsection, we are interested in explicitly calculating the generators of the continuously extended lattice in this $\mathcal{M}_2 \setminus (\mathcal{M}_2^0 \cup \mathcal{M}_2^2)$ case. To do so, we need to figure out how $\ln(\mu_\lambda^i)$ looks like. It is well known, that $\ln(\mu_\lambda^1)$ can be expressed by a simple function. Therefore, we make the following ansatz:

$$\ln(\mu_\lambda^1) = \frac{\alpha i(1 - e^{-i\varphi})\nu_\lambda}{\lambda(\lambda - e^{-i\varphi})}, \ \alpha \in \mathbb{R}$$

where

$$\nu_\lambda^2 = \lambda a(\lambda).$$

Obviously, we have by construction

$$\ln(\mu^1_{re^{i\varphi}}) = \ln(\mu^1_{r^{-1}e^{i\varphi}}) = 0$$

as ν equals zero for these λ values. Before we calculate the period ω_1 by making use of the polynomial b we specify the parameter α, which is (up to sign) uniquely defined by r and φ. The condition

$$\ln(\mu^1_{e^{-i\varphi}}) = i\pi$$

from Fact 1 boils down to

$$\begin{aligned}
\pi^2 &= \alpha^2 e^{i\varphi}(1 - e^{-i\varphi})^2(e^{-i\varphi} - re^{i\varphi})(e^{-i\varphi} - r^{-1}e^{i\varphi}) \\
&= \alpha^2 e^{i\varphi}(1 - 2e^{-i\varphi} + e^{-2i\varphi})(e^{-2i\varphi} - r - r^{-1} + e^{2i\varphi}) \\
&= \alpha^2 (e^{i\varphi} - 2 + e^{-i\varphi})(e^{-2i\varphi} - r - r^{-1} + e^{2i\varphi}) \\
&= \alpha^2 \underbrace{(2\cos(\varphi) - 2)}_{\leq 0} \underbrace{(2\cos(2\varphi) - (r + r^{-1}))}_{\leq 0}
\end{aligned}$$

and therefore,

$$\alpha = \pm \frac{\pi}{\sqrt{(2\cos(\varphi) - 2)(2\cos(2\varphi) - (r + r^{-1}))}} \in \mathbb{R}. \tag{7.17}$$

In order to compute the polynomial $b(\lambda)$, we need to derive $\ln(\mu_\lambda^1)$ with respect to λ:

$$\begin{aligned}
\frac{\partial}{\partial\lambda}\ln(\mu_\lambda^1) &= \frac{\alpha i(1 - e^{-i\varphi})(a + \lambda\frac{\partial a}{\partial\lambda})}{2\nu_\lambda\lambda(\lambda - e^{-i\varphi})} - \frac{\alpha i(1 - e^{-i\varphi})\nu_\lambda(\lambda - e^{-i\varphi} + \lambda)}{\lambda^2(\lambda - e^{-i\varphi})^2} \\
&= \frac{\alpha i(1 - e^{-i\varphi})}{2\nu_\lambda\lambda}\left(\frac{a + \lambda\frac{\partial a}{\partial\lambda}}{\lambda - e^{-i\varphi}} - \frac{2a(2\lambda - e^{-i\varphi})}{(\lambda - e^{-i\varphi})^2}\right).
\end{aligned}$$

The derivative of a with respect to λ equals

$$\frac{\partial a}{\partial\lambda} = (\lambda - r^{-1}e^{i\varphi})(\lambda - e^{-i\varphi})^2 + (\lambda - re^{i\varphi})(\lambda - e^{-i\varphi})^2 + 2(\lambda - re^{i\varphi})(\lambda - r^{-1}e^{i\varphi})(\lambda - e^{-i\varphi})$$

and hence,

$$\begin{aligned}\frac{a+\lambda\frac{\partial a}{\partial\lambda}}{\lambda-e^{-i\varphi}} &= (\lambda-re^{i\varphi})(\lambda-r^{-1}e^{i\varphi})(\lambda-e^{-i\varphi})+\lambda(\lambda-r^{-1}e^{i\varphi})(\lambda-e^{-i\varphi})\\ &\quad+\lambda(\lambda-re^{i\varphi})(\lambda-e^{-i\varphi})+2\lambda(\lambda-re^{i\varphi})(\lambda-r^{-1}e^{i\varphi})\\ &= 5\lambda^3-(3e^{-i\varphi}+4(r^{-1}+r)e^{i\varphi})\lambda^2+(2(r^{-1}+r)+3e^{2i\varphi})\lambda-e^{i\varphi}\end{aligned}$$

as well as

$$\begin{aligned}\frac{2a(2\lambda-e^{-i\varphi})}{(\lambda-e^{-i\varphi})^2} &= 2(2\lambda-e^{-i\varphi})(\lambda-re^{i\varphi})(\lambda-r^{-1}e^{i\varphi})\\ &= 4\lambda^3-(4(r^{-1}+r)e^{i\varphi}+2e^{-i\varphi})\lambda^2+(4e^{2i\varphi}+2(r^{-1}+r))-2e^{i\varphi}.\end{aligned}$$

Thus, we conclude

$$\frac{\partial}{\partial\lambda}\ln(\mu^1_\lambda)=\frac{\alpha i(1-e^{-i\varphi})}{2\nu_\lambda\lambda}(\lambda^3-e^{-i\varphi}\lambda^2-e^{2i\varphi}\lambda+e^{i\varphi})$$

which implies

$$\begin{aligned}b_1(\lambda) &= \lambda\nu_\lambda\frac{\partial}{\partial\lambda}\ln(\mu^1_\lambda)\\ &= -\frac{i}{2}\alpha(e^{-i\varphi}-1)(\lambda^3-e^{-i\varphi}\lambda^2-e^{2i\varphi}\lambda+e^{i\varphi}).\end{aligned}$$

Remark 7.27.
Again, the polynomial b_1 satisfies the reality condition

$$\lambda^3\overline{b_1(\bar\lambda^{-1})}=\lambda^3\frac{i}{2}\alpha(e^{i\varphi}-1)(\lambda^{-3}-e^{i\varphi}\lambda^{-2}-e^{-2i\varphi}\lambda^{-1}+e^{-i\varphi})=b_1(\lambda).$$

Finally, as we have seen in Remark 7.18, the period ω_1 can be obtained from b_1 via

$$b_1(0)=-\frac{i}{2}\bar\omega_1.$$

This means

$$\omega_1=\alpha(1-e^{-i\varphi})=\alpha e^{-i\frac{\varphi}{2}}(e^{i\frac{\varphi}{2}}-e^{-i\frac{\varphi}{2}})=2i\alpha e^{-i\frac{\varphi}{2}}\sin(\frac{\varphi}{2}) \tag{7.18}$$

with α from (7.17).

For the second period ω_2, the approach is more difficult. It uses theory of elliptic functions, computations become more complex as well as less explicit. The upcoming concepts originate from Kilian et al. [12] (p.7ff.). As stated in this paper, we can make the following ansatz

$$\ln(\mu^2_\lambda)=s_1\ln(\mu^1_\lambda)+s_2 f(\lambda)$$

where s_1, s_2 are appropriate coefficients and f is a particular elliptic function which has been investigated in Kilian et al. [12]. The coefficients s_1, s_2 can be computed immediately in accordance with the conditions from Fact 1

$$i\pi\overset{!}{=}\ln(\mu^2_{re^{i\varphi}})=\ln(\mu^2_{r^{-1}e^{i\varphi}})=s_2$$

as

$$f(r^{-1}e^{i\varphi}) = f(re^{i\varphi}) = 1$$

and hence,

$$0 \stackrel{!}{=} \ln(\mu_{e^{-i\varphi}}) = s_1 i\pi + i\pi f(e^{-i\varphi}) \quad \Rightarrow s_1 = -f(e^{-i\varphi}).$$

Putting all together yields

$$\ln(\mu_\lambda^2) = -f(e^{-i\varphi})\ln(\mu_\lambda^1) + i\pi f(\lambda). \tag{7.19}$$

In order to calculate the polynomial $b_2(\lambda)$, we consider the derivative of f with respect to $\hat{\lambda}$ as stated in Kilian et al. [12], p.9:

$$\frac{\partial f}{\partial \hat{\lambda}} = \frac{2E' - rK'(\hat{\lambda} + \hat{\lambda}^{-1})}{4\pi \hat{\nu} i \hat{\lambda}} \tag{7.20}$$

The functions E, K are complete elliptic integrals

$$K(r) := \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}} \quad E(r) := \int_0^1 \sqrt{\frac{1-r^2x^2}{1-x^2}}\,dx$$

with derivatives

$$K' = K'(r) = K(\sqrt{1-r^2}) \quad E' = E'(r) = E(\sqrt{1-r^2})$$

for $0 \le r \le 1$. Unfortunately, the coordinates $\hat{\lambda}$ and λ do not coincide and thus, neither do $\hat{\nu}$ and ν_λ. From the mentioned paper it becomes clear that

$$\lambda = e^{i\varphi}\hat{\lambda} \quad \Leftrightarrow \quad \hat{\lambda} = e^{-i\varphi}\lambda. \tag{7.21}$$

Furthermore, the relationship (Kilian et al. [12], p.7)

$$\begin{aligned}
\hat{\nu}^2 &= \frac{1}{4}(\hat{\lambda} - r)(\hat{\lambda}^{-1} - r) \\
&= -\frac{r}{4}\hat{\lambda}^{-1}(\hat{\lambda} - r)(\hat{\lambda} - r^{-1}) \\
&= -\frac{r}{4}e^{-i\varphi}\lambda^{-1}(\lambda - re^{i\varphi})(\lambda - r^{-1}e^{i\varphi})
\end{aligned}$$

yields

$$\begin{aligned}
\nu_\lambda^2 &= \lambda(\lambda - re^{i\varphi})(\lambda - r^{-1}e^{i\varphi})(\lambda - e^{-i\varphi})^2 \\
&= -\frac{4}{r}e^{i\varphi}\lambda^2(\lambda - e^{-i\varphi})^2\hat{\nu}^2
\end{aligned}$$

and hence,

$$\nu_\lambda = \frac{2i}{\sqrt{r}}e^{i\frac{\varphi}{2}}\lambda(\lambda - e^{-i\varphi})\hat{\nu} \quad \Leftrightarrow \quad \hat{\nu} = -\frac{i\sqrt{r}}{2}e^{-i\frac{\varphi}{2}}\lambda^{-1}(\lambda - e^{-i\varphi})^{-1}\nu_\lambda. \tag{7.22}$$

Finally, with

$$\frac{\partial f}{\partial \lambda} = \frac{\partial f}{\partial \hat{\lambda}} \frac{\partial \hat{\lambda}}{\partial \lambda} = \frac{\partial f}{\partial \hat{\lambda}} e^{-i\varphi}$$

we obtain from (7.20) with inserted transformations (7.21) and (7.22)

$$\frac{\partial f}{\partial \lambda} = e^{-i\varphi} \frac{2E' - rK'(e^{-i\varphi}\lambda + e^{i\varphi}\lambda^{-1})}{2\pi\sqrt{r}e^{-i\frac{3}{2}\varphi}(\lambda - e^{-i\varphi})^{-1}\nu_\lambda}.$$

Now we can compute the polynomial $b_2(\lambda)$ with (7.19)

$$\begin{aligned} b_2(\lambda) &= \lambda\nu_\lambda \frac{\partial}{\partial \lambda} \ln(\mu_\lambda^2) \\ &= -f(e^{-i\varphi})b_1(\lambda) + i\pi\lambda\nu_\lambda e^{-i\varphi} \frac{2E' - rK'(e^{-i\varphi}\lambda + e^{i\varphi}\lambda^{-1})}{2\pi\sqrt{r}e^{-i\frac{3}{2}\varphi}(\lambda - e^{-i\varphi})^{-1}\nu_\lambda} \\ &= -f(e^{-i\varphi})b_1(\lambda) + \frac{i}{2}e^{-i\varphi}\left(\frac{2}{\sqrt{r}}E'e^{i\frac{3}{2}\varphi}(\lambda - e^{-i\varphi})\lambda - \sqrt{r}K'e^{i\frac{3}{2}\varphi}(\lambda - e^{-i\varphi})(e^{-i\varphi}\lambda + e^{i\varphi}\lambda^{-1})\lambda\right) \\ &= -f(e^{-i\varphi})b_1(\lambda) + \frac{i}{2}e^{-i\frac{\varphi}{2}}\left(\frac{2}{\sqrt{r}}E'(e^{i\varphi}\lambda - 1)\lambda - \sqrt{r}K'(\lambda^3 - e^{-i\varphi}\lambda^2 + e^{2i\varphi}\lambda - e^{i\varphi}\right). \end{aligned}$$

Remark 7.28.
One can quickly check that $b_2(\lambda)$ satisfies the reality condition

$$\lambda^3\,\overline{b_2(\bar{\lambda}^{-1})} = b_2(\lambda)$$

if and only if

$$f(e^{-i\varphi}) = \overline{f(e^{-i\varphi})}$$

i.e. $f(e^{-i\varphi})$ is a real number (which is true due to the analysis of Kilian et al.[12]).

Due to

$$-\frac{i}{2}\bar{\omega}_2 = b_2(0) = -f(e^{-i\varphi})b_1(0) - \frac{i}{2}(-\sqrt{r}K'e^{i\frac{\varphi}{2}})$$

we conclude

$$\omega_2 = -f(e^{-i\varphi})\omega_1 - \sqrt{r}K'e^{-i\frac{\varphi}{2}}. \tag{7.23}$$

The periods ω_1, ω_2 from (7.18) and (7.23) can be transferred continuously into the ones from (7.16) for $r \to 1$. Therefore, we are allowed to denote τ^a as the following quotient:

$$\begin{aligned} \tau^a = \frac{\omega_2}{\omega_1} &= -f(e^{-i\varphi}) - \sqrt{r}K'e^{-i\frac{\varphi}{2}}\omega_1^{-1} \\ &= -f(e^{-i\varphi}) + i\,\frac{\sqrt{r}K'}{2\alpha\sin(\frac{\varphi}{2})}. \end{aligned}$$

In particular, we obtain $\Re(\tau^a) = -f(e^{-i\varphi})$ as well as $\Im(\tau^a) = \frac{\sqrt{r}K'}{2\alpha\sin(\frac{\varphi}{2})}$.

We need to specify the elliptic function f on the unit sphere and thus, the real part, in order to investigate the behaviour of τ^a. As it turns out to be very complicated to express this function in terms of (r, λ) coordinates, we want to figure out which new coordinates might be suitable and transform the problem into these new coordinates. We start again with the common formula from Kilian et al. [12], p.7:

$$\begin{aligned}\hat{v}^2 &= \frac{1}{4}(\hat{\lambda} - r)(\hat{\lambda}^{-1} - r) \\ &= -r\hat{\lambda}^{-1}(\hat{\lambda} - r)(\hat{\lambda} - r^{-1})\end{aligned}$$

The set of all $(\hat{\lambda}, \hat{v})$ satisfying this equation defines an elliptic curve $\mathbb{C}/\Gamma$. We are interested in constituting this relationship in the following form

$$(\mathcal{P}')^2 = 4\mathcal{P}^3 - g_2\mathcal{P} - g_3$$

where $g_2 = g_2(r)$ and $g_3 = g_3(r)$ do only depend on r. Hence, we compute

$$\begin{aligned}\left(i4\sqrt{r^{-1}}\hat{v}\hat{\lambda}\right)^2 &= 4\hat{\lambda}(\hat{\lambda} - r)(\hat{\lambda} - r^{-1}) \\ &= 4(\hat{\lambda}^3 - (r + r^{-1})\hat{\lambda}^2 + \hat{\lambda}) \\ &= 4\left(\hat{\lambda} - \frac{1}{3}(r + r^{-1})\right)^3 - \left(\frac{4}{3}(r + r^{-1})^2 - 4\right)\left(\hat{\lambda} - \frac{1}{3}(r + r^{-1})\right) \\ &\quad - \left(\frac{8}{27}(r + r^{-1})^3 - \frac{4}{3}(r + r^{-1})\right).\end{aligned}$$

Therefore, we can state

$$\begin{aligned}g_2(r) &:= \frac{4}{3}(r + r^{-1})^2 - 4 \\ g_3(r) &:= \frac{8}{27}(r + r^{-1})^3 - \frac{4}{3}(r + r^{-1})\end{aligned}$$

as well as

$$\mathcal{P} = \hat{\lambda} - \frac{1}{3}(r + r^{-1}) \tag{7.24}$$

$$\mathcal{P}' = i4\sqrt{r^{-1}}\hat{v}\hat{\lambda}. \tag{7.25}$$

From theory of elliptic functions (Freitag et al.[2]) we know that there exists a biholomorphic mapping

$$\mathbb{C}^\times/\Gamma \to \mathbb{C},\ z \mapsto (\mathcal{P}, \mathcal{P}').$$

It will turn out to be beneficial to calculate with the coordinates (r, z) rather than (r, λ). Thus, we now want to transfer the old coordinates to the new ones before calculating f.
We consider the double root $\lambda_0 = e^{-i\varphi}$ of the determinant polynomial a. We have already seen in (7.21) that

$$\hat{\lambda}_0 = e^{-i\varphi}e^{-i\varphi} = e^{-2i\varphi}$$

and with (7.24) we infer

$$m(r, z) := \mathcal{P}(z) + \frac{1}{3}(r + r^{-1}) = e^{-2i\varphi}.$$

This gives us a relationship between r, z and φ. We now want to express the imaginary part of τ^a in terms of (r, z). Using the addition theorem $1 - \cos(\varphi) = 2\sin^2(\frac{\varphi}{2})$ we can simplify

$$\begin{aligned}2\alpha \sin\left(\frac{\varphi}{2}\right) &= \frac{2\pi \sin(\frac{\varphi}{2})}{\sqrt{(2\cos(\varphi) - 2)(2\cos(2\varphi) - (r + r^{-1}))}} \\ &= \frac{2\pi \sin(\frac{\varphi}{2})}{\sqrt{-4\sin^2(\frac{\varphi}{2})(2\cos(2\varphi) - (r + r^{-1}))}} \\ &= \frac{\pi}{\sqrt{((r + r^{-1}) - 2\cos(2\varphi))}}\end{aligned}$$

which leads to

$$\begin{aligned}\Im(\tau) &= \frac{1}{\pi}\sqrt{r}K'\sqrt{((r + r^{-1}) - 2\cos(2\varphi))} \\ &= \frac{1}{\pi}\sqrt{r}K'\sqrt{((r + r^{-1}) - e^{-2i\varphi} - e^{2i\varphi})} \\ &= \frac{1}{\pi}\sqrt{r}K'\sqrt{((r + r^{-1}) - m(r, z) - \frac{1}{m(r, z)}}.\end{aligned}$$

For the real part, we now focus on the elliptic function f. The Weierstrass P-function $\mathcal{P}$ is doubly periodic with half-periods $\omega, \omega' \in \mathbb{C}$. We define

$$\eta := \zeta(\omega) \quad \eta' := \zeta(\omega')$$

where ζ is the Weierstrass' Zeta function. Due to elliptic theory, f must be of shape

$$f(r, z) = \alpha(\zeta(z) + \zeta(z + \omega')) + \beta z + \gamma$$

for appropriate $\alpha, \beta, \gamma \in \mathbb{C}$ as well as satisfy the following conditions:

1. $f(r, -z) = -f(r, z)$ for all $z \in \mathbb{C}$
2. $f(r, z + 2\omega) - f(r, z) = 2$ for all $z \in \mathbb{C}$
3. $f(r, z + 2\omega') - f(r, z) = 0$ for all $z \in \mathbb{C}$.

These properties make the parameters α, β, γ unique. To see this, we make use of the following

Fact 2.
The Weierstrass' Zeta function has the following characteristics:

- *It is an odd function of its argument, i.e.* $\zeta(-z) = -\zeta(z)$.
- $\zeta(z + 2\omega) = \zeta(z) + 2\eta$ *as well as* $\zeta(z + 2\omega') = \zeta(z) + 2\eta'$.
- *It satisfies Legendre's relation:* $\eta\omega' - \eta'\omega = \frac{1}{2}\pi i$.

For details, check Bateman [13], p. 329.

Hence, condition 1 yields

$$\begin{aligned}0 &= f(r,-z) + f(r,z)\\ &= \alpha(\zeta(-z+\omega') + \zeta(z+\omega')) + 2\gamma\\ &= \alpha(\zeta(-z-\omega'+2\omega') + \zeta(z+\omega')) + 2\gamma\\ &= \alpha(-\zeta(z+\omega') + 2\eta' + \zeta(z+\omega')) + 2\gamma\\ &= 2\alpha\eta' + 2\gamma\end{aligned}$$

which implies

$$\gamma = -\alpha\eta'.$$

Furthermore, condition 2 transforms into

$$\begin{aligned}2 &= f(r,z+2\omega) - f(r,z)\\ &= \alpha(\zeta(z+2\omega) + \zeta(z+2\omega+\omega') - \zeta(z) - \zeta(z+\omega')) + 2\beta\omega\\ &= \alpha(\zeta(z) + 2\eta + \zeta(z+\omega') + 2\eta - \zeta(z) - \zeta(z+\omega')) + 2\beta\omega\\ &= 4\alpha\eta + 2\beta\omega\end{aligned}$$

and analogously, condition 3 yields

$$\begin{aligned}0 &= f(r,z+2\omega') - f(r,z)\\ &= 4\alpha\eta' + 2\beta\omega'.\end{aligned}$$

Thus, we need to solve the following system of equations

$$\begin{aligned}\gamma &= -\alpha\eta'\\ 1 &= 2\alpha\eta + \beta\omega\\ 0 &= 2\alpha\eta' + \beta\omega'.\end{aligned}$$

The solution is

$$\alpha = \frac{\omega'}{i\pi} \quad \beta = -\frac{2\eta'}{i\pi} \quad \gamma = -\frac{\omega'}{i\pi}\eta'$$

because the third equation enforces $\beta = -2\frac{\alpha\eta'}{\omega'}$ so the second turns into

$$1 = 2\alpha\eta - 2\frac{\alpha\eta'}{\omega'}\omega = 2\frac{\alpha}{\omega'}(\eta\omega' - \eta'\omega) = \frac{i\pi\alpha}{\omega'}$$

due to Legendre's relation. Putting all together yields

$$f(r,z) = \frac{\omega'}{i\pi}(\zeta(z) + \zeta(z+\omega') - \eta') - \frac{2\eta'}{i\pi}z.$$

Finally, we have seen that we are only interested in evaluating the function f at the unit sphere, i.e in the quantity $f(e^{-i\varphi})$, with respect to (r,λ) coordinates. This condition needs to be transfomed into (r,z) coordinates as well. The idea is to regard the unit sphere as a fixed point set of a particular antiholomorphic involution in (λ,ν) and then express the involution as well as the corresponding fixed point set in terms of z.

Definition 7.29.
Given a domain A, an *involution* is a function $j : A \to A$ which is its own inverse: $j \circ j = \text{id}$.

From elliptic theory we know $\Gamma = \mathbb{Z}2\omega + \mathbb{Z}2\omega'$ where $\omega \in \mathbb{R}$ and $\omega' \in i\mathbb{R}$ are half-periods. We transfer involutions on the spectral genus $g = 1$ curve (in (λ, ν) coordinates) to the torus (with z coordinates):

$$\nu^2 = \lambda(\lambda - r)(\lambda - r^{-1}) = \lambda a(\lambda) \quad \leftrightarrow \quad \mathbb{C}/\Gamma.$$

We know $z = 0$ corresponds to $(\lambda, \nu) = (\infty, \infty)$.

- First, we consider the antiholomorphic involution

  $$j : (\lambda, \nu) \mapsto (\bar{\lambda}, \bar{\nu}) \quad \leftrightarrow \quad j : z \mapsto \bar{z}.$$

  We investigate its fixed points both in the coordinates (λ, ν) and z. On the curve the fixed points are characterized by the conditions $\lambda, \nu \in \mathbb{R}$ which are equivalent to

  $$\lambda \in [0, r^{-1}] \cup [r, \infty].$$

  Thus, the fixed point set corresponds to the lines

  $$I := [0, r^{-1}] \qquad II := [r, \infty].$$

  On the torus, the fixed points satisfy

  $$z \overset{\text{mod } \Gamma}{=} \bar{z} \quad \Leftrightarrow \quad z - \bar{z} \overset{\text{mod } \Gamma}{=} 0 \quad \Leftrightarrow \quad z \in \mathbb{R} \cup (\omega' + \mathbb{R})$$

  because $\omega \in \mathbb{R}$ and $\omega' \in i\mathbb{R}$. Due to the fact that $z = 0$ corresponds to (∞, ∞), the roots of the curve satisfy

  $$(r, 0), (\infty, \infty) \in I \qquad (0, 0), (r^{-1}, 0) \in II.$$

- Now we consider the holomorphic involution

  $$k : (\lambda, \nu) \mapsto (\lambda^{-1}, \lambda^{-2}\nu) \quad \leftrightarrow \quad k : z \mapsto z + \omega'$$

  which transforms the following roots into each other

  $$(r, 0) \leftrightarrow (r^{-1}, 0)$$
  $$(0, 0) \leftrightarrow (\infty, \infty).$$

  Hence, the respective lines I and II are transformed into each other.

- Finally, the composition of j, k yields the following antiholomorphic involution

  $$l : (\lambda, \nu) \mapsto (\bar{\lambda}^{-1}, \bar{\lambda}^{-2}\bar{\nu}) \quad \leftrightarrow \quad l : z \mapsto \bar{z} + \omega'.$$

  Its fixed points on the curve are characterized by the conditions

  $$\lambda = \bar{\lambda}^{-1} \Leftrightarrow \lambda \in \mathcal{S}^1$$

as well as

$$\begin{aligned} &\bar\lambda^2\nu = \bar\nu \\ \Leftrightarrow\ &\bar\lambda^2\nu^2 = |\nu|^2 \\ \Leftrightarrow\ &\lambda^{-2}\nu^2 = \lambda^{-2}\bar\lambda^{-2}|\nu|^2 = |\nu|^2. \end{aligned}$$

This equation holds true for all $\lambda \in \mathcal{S}^1$ due to the following argumentation: As $a(\lambda) = (\lambda - r)(\lambda - r^{-1})$ is an element of $\mathcal{M}_1$ we have $\lambda^{-2}\nu^2 = \lambda^{-1}a(\lambda) \geq 0$ for $\lambda \in \mathcal{S}^1$ and hence, the left and the right hand sides are nonnegative real numbers of the same length. In particular, they must be identical, because

$$0 \leq \left|\lambda^{-2}\nu^2 - |\nu|^2\right| \leq \left||\lambda^{-2}\nu^2| - |\nu|^2\right| = 0.$$

Therefore, the set of fixed points equals the unit sphere in λ. A fixed point in z coordinates satisfies

$$\begin{aligned} z &\overset{\text{mod } \Gamma}{=} \bar z + \omega' \\ \Rightarrow\ z - \bar z &\overset{\text{mod } \Gamma}{=} \omega' \\ \Rightarrow\ i\Im(z) &\overset{\text{mod } \Gamma}{=} \frac{\omega'}{2} \text{ or } \frac{3\omega'}{2} \end{aligned}$$

and hence, the fixed point set of the torus equals $III \cup IV$ where

$$III := (\mathbb{R} + \frac{\omega'}{2}) \qquad IV := (\mathbb{R} + \frac{3\omega'}{2}).$$

Due to this analysis of involutions we have figured out to which set in z coordinates the unit sphere in λ coordinates corresponds:

$$\mathcal{S}^1 \text{ in } \lambda \quad \leftrightarrow \quad (\mathbb{R} + \frac{\omega'}{2}) \cup (\mathbb{R} + \frac{3\omega'}{2}) \text{ in } z.$$

Figure 12 summarizes the above findings.

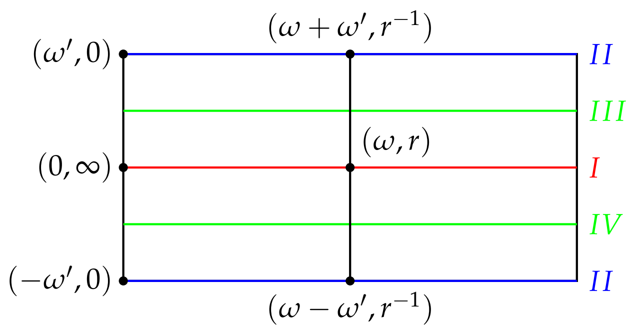


Figure 12: Fixed point sets I, II, III, IV on the torus $\mathbb{C}/\Gamma$ of the involutions j and l in z coordinates with corresponding λ values, denoted (z, λ).

All in all, we obtain for $z \in \mathbb{C}$ with $\Im(z) \in \{\frac{\omega'}{2}, \frac{3\omega'}{2}\}$

$$\Re(\tau^a) = -f(r,z) = -\frac{\omega'}{i\pi}(\zeta(z) + \zeta(z+\omega') - \eta') + \frac{2\eta'}{i\pi}z$$
$$\Im(\tau^a) = \frac{1}{\pi}\sqrt{r}K'\sqrt{((r+r^{-1}) - m(r,z) - \frac{1}{m(r,z)}}$$

where $m(r,z) = \mathcal{P}(z) + \frac{1}{3}(r+r^{-1}) = e^{-2i\varphi}$. We conduct a parametric plot of τ^a using Mathematica in Figure 13. Its algorithm can be found in section B.

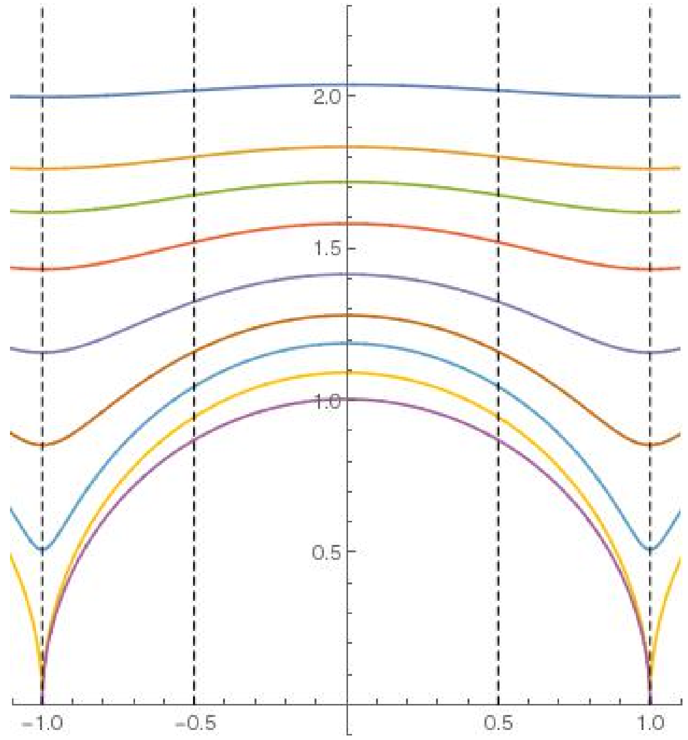


Figure 13: Parametric plots of τ for increasing $r > 1$.

Figure 13 suggests that τ^a is continuously moving upwards in $\mathcal{N}_1$ with increasing r and the described arc will be successively compressed until it resembles a horizontal line. Hereby, we should not forget that τ^a is the quotient of the two shortest generators and the φ-thresholds (or z-thresholds) at which τ^a turns into a vertical line (probably in dependence on r) are missed to be figured out. Finally, we conclude that g maps $\mathcal{M}_2 \setminus (\mathcal{M}_2^0 \cup \mathcal{M}_2^2)$ surjectively on $\mathcal{N}_1$.

8 Conclusion

In this section we will outline the most important findings of this work and conclude by suggesting possible future research.

We have seen that every continuously differentiable vector field defines a local flow. The commutator of two vector fields is a vector field itself and disappears if and only if the corresponding local flows commute.

All potentials $\zeta_\lambda \in \mathcal{P}_2$ satisfy the reality condition $\lambda^3 \overline{\zeta}^t_{1/\bar{\lambda}} = -\zeta_\lambda$ which is bequested on the determinant polynomial $a(\lambda) = \lambda^4 \overline{a(\bar{\lambda}^{-1})}$. Furthermore, every root of ζ_λ is a double root of $a(\lambda)$ and every root of $a(\lambda)$ from the unit sphere must be double and is also a root of ζ_λ.

Polynomial Killing fields are maps $\zeta_\lambda : \mathbb{R}^2 \to \mathcal{P}_2$ which solve the Lax equations

$$\frac{\partial \zeta_\lambda}{\partial x} = [\zeta_\lambda, U(\zeta_\lambda)] \qquad \frac{\partial \zeta_\lambda}{\partial y} = [\zeta_\lambda, V(\zeta_\lambda)].$$

These equations can be transformed into a system of ordinary differential equations in dependence on the matrix entries of the potentials. Crucially, the determinant polynomial forms an integral of motion with respect to the resulting flows. This implies the globality of these flows as their orbits are contained in compact isospectral sets.

If the determinant polynomial a has four pairwise distinct roots, the isospectral sets are two-dimensional compact submanifolds and the flows act transitively on them. If a has at least one double point on $\mathcal{S}^1$ we can divide a by this root and reduce the situation to the spectral genus $g = 1$ or $g = 0$ case. Hence, the flows act transitively on the isospectral sets as well. When a has two double roots absent the unit sphere, the isospectral set falls into a singleton and a non-compact set and on both parts the flows act transitively.

If a has four pairwise distinct roots, it induces a lattice

$$\Gamma^a = \{(x, y) \in \mathbb{R}^2 \mid \phi(x, y)(\zeta_\lambda) = \zeta_\lambda\}$$

which is isomorphic to a lattice with generators $(1, \tau^a)$ where τ^a is contained in the fundamental region $\mathcal{N}_1$ of the $SL(2, \mathbb{Z})$ group. The mapping g which transfers such an a to the generator τ^a can be continuously extended on those a having multiple roots on the unit sphere. In this context, we have shown that the limit of $\mathcal{M}^1_2-$ lattices is well-defined and is a lattice itself.

When a has two double roots on $\mathcal{S}^1$, g maps surjectively on $\partial\mathcal{N}_1$. Furthermore, we have seen a numerical motivation that the mapping is probably surjective on $\mathcal{N}_1$ when a has one double root on $\mathcal{S}^1$. This has to be verified analytically. To do so, the minimality condition needs to be transformed into conditions in the new coordinate z.

Having done this, one could probably obtain surjectivity of the original mapping g on $\mathcal{M}^1_2$ (without boundary). As it is already surjective on the boundary, an application of the Whitham deformation may lead to this conclusion.

A Basic Theorems

Theorem A.1 (Inverse Function Theorem).
Let V, W finite-dimensional, normed $\mathbb{K}$-vector spaces, $U \subset V$ an open subset and $f : U \to W$ a continuously differentiable mapping. If the differential $df(x_0) \in \mathcal{L}(V, W)$ in $x_0 \in U$ is invertible, then there is an open subset $U' \subset U$ containing x_0 such that $O := f(U') \subset W$ contains $f(x_0)$ and the restriction

$$f\big|_{U'} : U' \to O$$

is a diffeomorphism.

Proof.
The inverse function is constructed using Banach Fixed-Point Theorem and the Boundedness Theorem. Details can be studied in Königsberger [1]. **q.e.d.**

Theorem A.2 (Implicit Function Theorem).
Let V, W, Z be finite-dimensional, normed $\mathbb{K}$-vector spaces with $\dim V = \dim W$. Furthermore, let $U \subset Z \times V$ be an open subset containing an element (a_0, b_0) and $f : U \to W$ a continuously differentiable function with $f(a_0, b_0) = c_0 \in W$. When the linear mapping $\frac{\partial f}{\partial b}(a_0, b_0) \in \mathcal{L}(V, W)$ is invertible, then there exist open sets $U' \subset Z$ containing a_0, $U'' \subset V$ containing b_0 as well as a continuously differentiable mapping

$$g : U' \to U''$$

such that

$$f(a, g(a)) = c_0 \text{ for all } a \in U'.$$

With other words, given $c_0 \in W$, all solutions $(a, b) \in U' \times U''$ of the equation $f(a, b) = c_0$ are contained in the graph of the function g:

$$f(a, b) = c_0,\ (a, b) \in U' \times U'' \Leftrightarrow b = g(a),\ a \in U'.$$

Proof.
Consider the mapping

$$F : U \to Z \times W,\ (a, b) \mapsto (a, f(a, b))$$

and apply the Inverse Function Theorem A.1. For details, check Königsberger [1]. **q.e.d.**

Theorem A.3 (Trace and Determinant of Fundamental Solutions).
Let $A : I \to \mathbb{K}^{n\times n}$ be a continuous mapping from the open interval I to $\mathbb{K}$-valued $n \times n$ matrices. Then the fundamental solution

$$F : I \to \mathbb{K}^{n\times n} \text{ with } \frac{dF}{dt} = A(t)F(t) \text{ and } F(t_0) = \mathbb{1}$$

also satisfies

$$\frac{d}{dt}\det(F(t)) = \operatorname{tr}(A(t))\det(F(t)) \text{ with } \det(F(t_0)) = 1.$$

In particular, $\det(F(t))$ has no roots on I and $F(t)$ is invertible for all $t \in I$.

Proof.
Statement and proof of this theorem originate from Schmidt [10]. The determinant det : $\mathbb{K}^{n\times n} \to \mathbb{K}$ is a polynomial in the respective matrix' entries and therefore analytic. One calculates the derivative of a matrix B in direction of the matrix AB and uses the chain rule for F. For details, check Schmidt [10], p.23. **q.e.d.**

Theorem A.4 (Arzela-Ascoli)**.**
Let K be a compact metric space and V be a finite-dimensional Banach space. A sequence $(f_n)_{n\in\mathbb{N}}$ in $C(K,V)$ owns a convergent subsequence if

i) for any $x \in K$ the sequence $(f_n(x))_{n\in\mathbb{N}}$ is bounded and

ii) for any $x \in K$ the sequence $(f_n)_{n\in\mathbb{N}}$ is equicontinuous, i.e. for any $x \in K$ and any $\varepsilon > 0$ there exists an $\delta > 0$ such that

$$x' \in B(x,\delta) \subset K \quad \Rightarrow \quad f_n(x') \in B(f_n(x),\varepsilon) \subset V \text{ for all } n \in \mathbb{N}.$$

Proof.
Statement and proof of this theorem originate from Schmidt [10]. First, one proves that the sequence $(f_n)_{n\in\mathbb{N}}$ is even equicontinuous on K. Afterwards, a subsequence $(g_n)_{n\in\mathbb{N}}$ is constructed inductively which turns out to be Cauchy and hence, convergent. For details, check Schmidt [10], p.23. **q.e.d.**

Theorem A.5 (Variation of Parameters)**.**
Let $[\alpha,\beta] \subset \mathbb{R}$ be a compact interval and V a Banach space. Then the mapping

$$C([\alpha,\beta],\mathcal{L}(V)) \times C([\alpha,\beta],V) \times [\alpha,\beta] \times V \to C([\alpha,\beta],V)$$
$$(A,b,t_0,u_0) \mapsto u$$

with u being the unique solution of the initial value problem

$$u'(t) = A(t)u(t) + b(t) \text{ with } u(t_0) = u_0$$

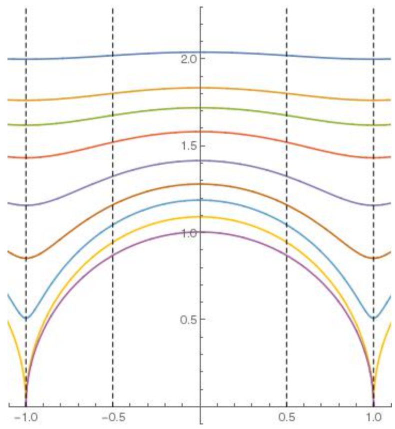
is continuous. The restriction of this mapping on fixed t_0 depends analytically from A, b and u_0. For each $(A,b,t_0) \in C([\alpha,\beta],\mathcal{L}(V)) \times C([\alpha,\beta],V) \times [\alpha,\beta]$ the respective restriction of the mapping is an affine isomorphism from $u_0 \in V$ to the set of solutions of the differential equation $u'(t) = A(t)u(t) + b(t)$.

Proof.
Statement and proof of this theorem originate from Schmidt [10]. Instead of considering the abstract solution u one denotes its integral form and modifies the above mapping respectively. This mapping turns out to be Lipschitz for fixed equation parameters which is the basis for all claims in the statement. For details, check Schmidt [10], p.16. **q.e.d.**

B Mathematica Algorithm of Figure 13

```
g2[r_]:=4/3*(r+1/r)^2-4

g3[r_]:=1/3*g2[r](r+1/r)-4/27*(r+1/r)^3

period[r_]:=WeierstrassHalfPeriods[{g2[r],g3[r]}].{1,0}

periodp[r_]:=WeierstrassHalfPeriods[{g2[r],g3[r]}].{0,1}

eta[r_]:=WeierstrassZeta[period[r],{g2[r],g3[r]}]

etap[r_]:=WeierstrassZeta[periodp[r],{g2[r],g3[r]}]

beta[r_]:=-2*eta[r]/(Pi*I)

betap[r_]:=-2*etap[r]/(Pi*I)

alpha[r_]:=period[r]/(Pi*I)

alphap[r_]:=periodp[r]/(Pi*I)

f[r_,z_]:=
alpha[r]*
(WeierstrassZeta[z,{g2[r],g3[r]}]+
WeierstrassZeta[z+period[r],{g2[r],g3[r]}]-eta[r])+beta[r]*z

fp[r_,z_]:=
alphap[r]*
(WeierstrassZeta[z,{g2[r],g3[r]}]+
WeierstrassZeta[z+periodp[r],{g2[r],g3[r]}]-etap[r])+betap[r]*z

Realteil[r_,z_]:=-f[r,z]

Realteilp[r_,z_]:=-fp[r,z]

Realteiltau[r_,z_]:=Re[Realteil[r,1/2*period[r]+z]]

Realteiltaup[r_,z_]:=Re[Realteilp[r,z+1/2*periodp[r]]]
```

```
m[r_, z_]:=WeierstrassP[z, {g2[r], g3[r]}] + 1/3 * (r + 1/r)

Kp[r_]:=EllipticK[Sqrt[1 − r^2]]

Imaginarteil[r_, z_]:=(1/Pi) * Kp[r] * Sqrt[r] * Sqrt[r + 1/r − m[r, z] − 1/(m[r, z])]

Imaginarteiltau[r_, z_]:=Re[Imaginarteil[r, 1/2 * period[r] + z]]

Imaginarteiltaup[r_, z_]:=Re[Imaginarteil[r, z + 1/2 * periodp[r]]]

Plot[Realteiltau[z]

ParametricPlot[{{Realteiltaup[0.01, z], Imaginarteiltaup[0.01, z]},
{Realteiltaup[0.02, z], Imaginarteiltaup[0.02, z]},
{Realteiltaup[0.03, z], Imaginarteiltaup[0.03, z]},
{Realteiltaup[0.05, z], Imaginarteiltaup[0.05, z]},
{Realteiltaup[0.1, z], Imaginarteiltaup[0.1, z]},
{Realteiltaup[0.2, z], Imaginarteiltaup[0.2, z]},
{Realteiltaup[0.4, z], Imaginarteiltaup[0.4, z]},
{Realteiltau[0.9, z], Imaginarteiltau[0.9, z]},
{Realteiltau[0.9999, z], Imaginarteiltau[0.9999, z]}}, {z, −4, 4},
PlotRange → {{−1.1, 1.1}, {−.1, 2.3}},
Epilog → {Dashed, Line[{{0.5, 0}, {0.5, 2.3}}], Line[{{−0.5, 0}, {−0.5, 2.3}}],
Line[{{1, 0}, {1, 2.3}}], Line[{{−1, 0}, {−1, 2.3}}]}]
```



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I, the undersigned, hereby declare that all material presented in this paper is my own work or fully or specifically acknowledged whereever adapted from other sources.

I declare that all statements and information contained herein are true, correct and accurate to the best of my knowledge and belief.

______________________	July 24th, 2015	Mannheim
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