UNIVERSITÄT MANNHEIM

# Diploma Thesis

# Constant mean curvature tori in the 3-hyperbolic space

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# 1 Introduction

One of the problems of differential geometry is to determine the structure and classification of constant mean curvature (CMC) surfaces in three dimensional manifolds. Common models used for those manifolds are the Euclidian Space $\mathbb{R}^3$, the 3-sphere $S^3$ and the 3-hyperbolic space $\mathbb{H}^3$.

It is well known that some nonlinear differential equations, which have been studied extensively during the last twenty five years by means of the inverse scattering method, were first obtained in the framework of surface theory. One of the most famous equations of this kind is the sinh-Gordon equation

$$u_{z\bar{z}} + sinh(u) = 0$$

, as it coincides with the Gauss-Codazzi system for surfaces of constant positive Gaussian curvature in the classic 3-spaces. The integrability of these equations, i.e. the zero curvature representations with spectral parameter, made it possible to develop the theory of exact solutions for the sinh-Gordon equation, which was then successfully applied to surface theory.

The solution of nonlinear equations as the sinh-Gordon equation is connected to Riemann surface theory via spectral theory of operators used in the inverse scattering method. The link between the theory of spectral theory of operators, the solution of the sinh-Gordon equation and Riemann surface theory is made through to the zero curvature condition. By this connection, the solution of the sinh-Gordon equations obtains the role as potential of the operators.

The operators define a special eigenvalue problem, which can be interpreted as hyperelliptic curve, the so-called spectral curve. This makes it possible to solve the problem with Riemann surface theory and the allied theory of theta-functions. This leads to the Baker-Akhiezer functions, functions with special properties on Riemann surfaces uniquely existing on them. The crucial point is that those Baker-Akhiezer functions also solve the special eigenvalue problem and are therefore the solution to the main problem. The Baker-Akhiezer functions can be constructed from theta-functions and one retrieves an explicit formula for the solution of the sinh-Gordon equation.

One is then in the situation to retrieve explicit formulas for the frame and immersion of a surface, (as this was for example conducted in [6]).Once the frame for a CMC surface is identified there exists the Sym-Bobenko formula for $\mathbb{R}^3, \mathbb{H}^3, S^3$ which makes it possible to retrieve a formula for the immersion without integration.

The latter is one of two main parts of the thesis. Publications studying this field are highly elaborated and difficult to understand even for a graduate student. The source of the theory of integrable systems had its development in Russia and a lot of publications being more specific and explaining are therefore not in English. Although there exist quite a few publications in English, they are mostly very brief, requiring the knowledge of many mathematics areas and not carrying out calculations. Consequently, the thesis tries to overcome this lack in literature, choosing a more intuitive approach and providind the reader with most of the calculations in detail.

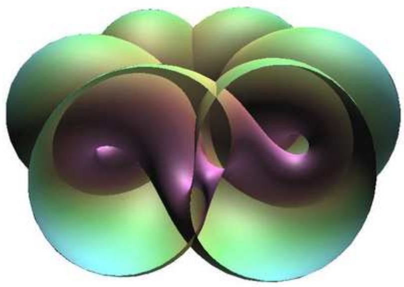

Figure 1.1: Cutaway view of the twisty torus

In this thesis we are especially interested in CMC tori in 3-spaces, which leads to the introduction of the concept of monodromy, emerging from the periodicty of the frame.

Studying CMC tori means, that we are interested in double periodic solutions of the sinh-Gordon equation. The one to one correspondence between the solution of the sinh-Gordon equation and hyperelliptic curves, i.e. vica versa one can obtain solutions of the sinh-Gordon equation from hyperelliptics curves as well es the sinh-Gordon equations defines one, offers a naturally way to study CMC tori via spectral curves. Equivalently one has to impose a supplementary condition on the monodromy of the immersion to retrieve immersions of CMC tori which leads back to the Sym Bobenko formula. The link to the spectral curve makes it possible to study, which data of the spectral curve need to be specified additionally to result in CMC tori. The second main purpose of this thesis is to describe by which data the spectral curve of a CMC torus is described in the 3-hyperbolic space. Deformations on the spectral curve can be described by a system of ordinary differential equations after having introduced a deformation parameter. In this way, a one-parameter family of spectral curves is obtained for every given spectral curve. Determining the spectral data of CMC tori in $\mathbb{H}^3$ as well as identifying the ordinary differentials equations describing the corresponding deformation has as far as we know nobody done before.

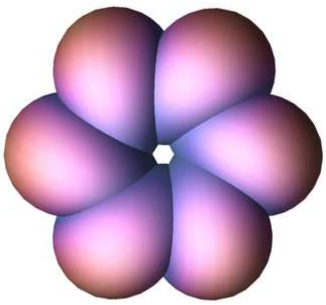

Figure 1.2: The twisty torus in $\mathbb{R}^3$

The thesis is structured as follows. In Chapter 2 we concentrate on classical surface theory. We review the basic concepts such as the Gauss map, the fundamental forms, and the shape operator. We introduce different definitions of curvature and define what we understand by constant mean curvature. We are then ready to answer the question of when an immersion is uniquely defined. We show that an immersion exists up to rigid motion uniquely if some functions satisfy a pair of equations, the Gausss-Codazzi equations. The immersion can be described by a so-called Lax pair, which satisfies the zero curvature condition. We start by constructing the Lax pair in terms of $3\times3$-matrices and rework them in $2 \times 2$-matrices for the Euclidean and the 3-hyperbolic space. We show that, regardless of which of both spaces that the integrability condition reduces to the sinh-Gordon equation.

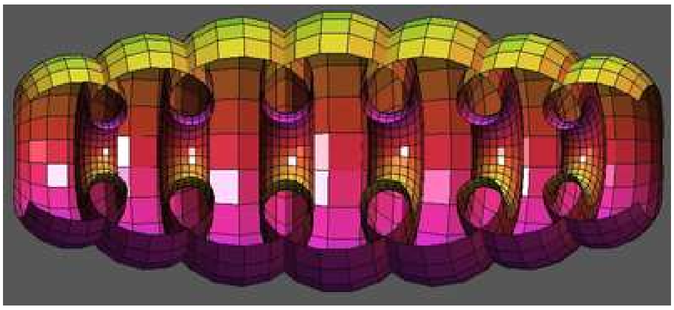

Figure 1.3: Cylinder in $\mathbb{H}^3$

In the third Chapter we give a brief introduction to Riemann surface theory as it is needed in Chapter 4 to derive an explicit solution of the sinh-Gordon equation and in Chapter 5 to study the spectral curve and its deformations. Because Riemann surface theory is only used as tool here, the overview given will be brief, concentrating on the necessary. As we mostly need theory about hyperelliptic curves, we will treat this case explictly in this chapter in preparation for the following chapters. In the second section of this chapter we introduce theta-functions which we will use in the next chapter.

In Chapter 4 we study solutions of nonlinear differential equations like the sinh-Gordon equation. In a first step we connect the sinh-Gordon equation to the spectral theory of operators. The connection between the operators and the spectral curve is then established via the eigenvalue problem of those operators and properties they fulfill. We reduce the problem of solving the sinh-Gordon equation to finding a special function on the spectral curve, the Baker-Akhiezer function. We are then in the situation to derive an explicit formula for the solution of the sinh-Gordon equation in terms of theta-functions introduced in the chapter before.

In Chapter 5 we turn our attention to the specific case of CMC tori, i.e. we impose a special closing condition on our immersion, which is equivalent to requiring the sinh-Gordon solution to be double periodic. We will introduce the concept of monodromy and

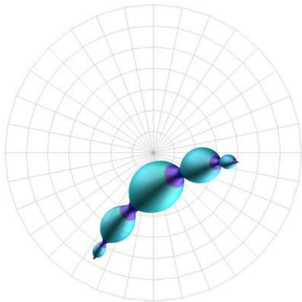

Figure 1.4: Delauny surface in $\mathbb{H}^3$

choose a slightly different, but equivalent, representation of the spectral curve. We study the closing condition of the immersion via the monodromy and the spectral curve and are then in the situation to manifest which conditions must be imposed on the spectral curve to be a CMC torus in the 3-hyperbolic space, the so called spectral data. This representation then allows us to study deformations and represent them by a system of ordinary differential equations.

Chapter 6 gives a summary about the different chapter of this thesis as well as the results. We highlight the main concepts used explaining our motivation for the chosen proceeding. Last but not least we provide some possible interesting directions on further research.

# 2 Classical surface theory

In this chapter we give an overview over classical surface theory. We begin with a brief introduction to the topic and revise concepts such as the Gauss map, the first and second fundamental forms, as well as the Weingarten Map, and the Gauss and mean curvature. In a next step we identify, which functions uniquely determine an immersion and derive the equations which have to be satisfied by those functions. We then derive explicitly those terms for the surfaces in $\mathbb{R}^3$ and $\mathbb{H}^3$. The goal of this chapter is to show that describing surfaces, or to be precise family of immersions, in $\mathbb{R}^3$ and $\mathbb{H}^3$ is reduced to the famous sinh-Gordon equation, which is well known in the theory of integrable systems. In this chapter we follow closely the works of [53], [54], [55] and [25], but try to carry out most of the calculations in detail.

## 2.1 Surface theory

In this section we recall basic concepts of elementary differential geometry. We derive the concepts in general with the purpose of using them for the Euclidean space $\mathbb{R}^3$ as well as the 3-hyperbolic space $\mathbb{H}^3$. We therefore assume that $M$ lies in a 4-dimensional Riemann or Lorentzian manifold with metric $\langle \cdot, \cdot \rangle_M$.

**Definition 2.1.** (Immersion) We consider $U \subset \mathbb{R}^2$, a non-empty set with the following properties: For each point $p \in U$ there exists an open neighborhood $U \subset M$ around $p$ and an open neighborhood $V$ of $p$ in $M$ and a differentiable map $f : U \longrightarrow M$ such that the following hold:

i. $f : U \longrightarrow V$ is a homeomorphism,

ii. $f(U) = V$,

iii. $(df)_p : U \longrightarrow M$ is injective for all $p \in U$.

Then $f$ is an immersion.

The last requirement is equivalent to the linear codependency of the vectors

$$f_x(p) = \frac{\partial f}{\partial x}(p) \quad f_y(p) = \frac{\partial f}{\partial y}(p).$$

**Definition 2.2.** (Tangent plane) A vector $v \in \mathbb{R}^3$ is tangent to the surface $M$ at $p \in M$ if there is a curve $\alpha : (-\varepsilon, \varepsilon) \longrightarrow M$ with $\alpha(0) = p$ and $\alpha'(0) = v$. The set of all tangent vectors to $M$ at a point $p$ will be denoted with $T_pM$ and is called the tangent plane of $M$ at the point $p$.

For a parameterization $f : \mathbb{R}^2 \supset U \longrightarrow M$ with $p \in f(U) \subset M$, we have $T_pM = (df)_{f^{-1}(p)}(\mathbb{R}^2)$. Consequently, one sees that $T_pM$ is a linear plane of $\mathbb{R}^3$ which is the tangent plane of $M$ at $p$. Furthermore, if the parameterization $f$ of $M$ covers $p$ and $q = f^{-1}(p)$, then $(df)_q(\mathbb{R}^2)$ does not depend on $f$ and $(f_x(q), f_y(q))$ is a basis of $T_pM$.

After having introduced the tangent plane, we can give a definition of how an immersion can be described:

**Definition 2.3.** (Immersion) Let $f : U \subset \mathbb{R}^2 \longrightarrow M$ be a smooth mapping, with differential $df_p : T_pU \longrightarrow T_{f(p)}M$. If the differential $df_p$ is injective for all $p \in U$ then $f$ is an immersion.

**Definition 2.4.** Let $f : U \longrightarrow M$ be an immersion. If $f$ is also a homeomorphism from $U$ onto $f(U)$ then $f$ is an embedding.

**Remark 2.5.** From now on we presume that $f$ is an immersion if not otherwise stated.

**Remark 2.6.** We can write $f$ in terms of its components

$$f(x, y) = (a(x, y), b(x, y), c(x, y))$$

and say that $f$ is differentiable if $a, b$, and $c$ are differentiable. The map $f$ will be called parameterization of $M$ and its variables $x, y$ local coordinates of $M$.

**Definition 2.7.** (First fundamental form) Let $f : U \longrightarrow M$ be an immersion and $p \in U$. The first fundamental form is the metric induced by an immersion

$$\langle v, w \rangle_M = \langle df(v), df(w) \rangle_M \tag{2.1}$$

for any two vectors $v, w \in T_pU$ with $\langle \cdot, \cdot \rangle_M$ the metric of $M$.

This means that the metric is pulled back by $df$.

**Remark 2.8.** For the sake of notional brevity we will omit the subscript $M$ at the metric since it will be clear which metric is meant. In the next section it will be standard Euclidean metric as from the next remark, and the metric of the 3-hyperbolic space will be introduced in the corresponding section.

**Remark 2.9.** The standard metric $\langle \cdot, \cdot \rangle$ of $\mathbb{R}^3$ is defined as

$$\langle v, w \rangle_p = \sum_{i,j=1,\ldots,3} v_i w_j g_{ij}$$

with $g_{ij} = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle$.

Since $(x, y)$ is a standard coordinate system on $\mathbb{R}^2$ and $f$ is an immersion, a basis for $T_pU$ can always be chosen as

$$f_x = \frac{\partial f}{\partial x}|_p, f_y = \frac{\partial f}{\partial y}|_p.$$

The first fundamental form with respect to the standard coordinate system is then represented by

$$g = \begin{pmatrix} E & F \\ F & G \end{pmatrix},$$

where

$$\begin{aligned} E = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial x} \rangle &= \langle f_x, f_x \rangle, \\ F &= \langle f_x, f_y \rangle, \\ G &= \langle f_y, f_y \rangle, \end{aligned}$$

or short

$$g_{ij} = \langle f_i, f_j \rangle.$$

**Remark 2.10.** $f$ is an immersion if and only if the determinant of the first fundamental form is positive.

**Remark 2.11.** In the next section it will be explained that in our case one can always choose coordinates $(x, y)$ on $U$ such that the metric becomes "conformal" which implies $F = 0$ and $E = G = 4e^{2u}$.

After having defined what to understand as a surface, we are now able to study the geometry of a surface which will lead to the concepts of curvature: Intuitively, this is connected to the question of how the tangent plane varies from a point to another on the surface. The Gaussian curvature is an intrinsic value as it is only determined by the first fundamental form $g$. One of the main focuses will be on the mean curvature. This is in comparison to the Gaussian curvature an extrinsic value and depends on the way the surface is immersed in the corresponding space. For the purpose of studying the curvature we introduce the following:

**Definition 2.12.** (Normal field) If $N(p) \perp T_pM,\ \forall p \in U$ we call $N$ normal field of $M$. If in addition $|N, (p)| = 1\ \forall p \in U$ we call it unit normal field $N$.

**Remark 2.13.** Let $f : U \longrightarrow M$ be a parameterization of the surface. At each $p \in U$ the vectors $f_x(p),\ f_y(p)$ form a basis of the tangent plane $T_{f(p)}M$. The unit normal vector $N$ in $M$ is therefore given by

$$N_f(p) = \frac{f_x(p) \times f_y(p)}{|f_x(p) \times f_y(p)|}.$$

**Remark 2.14.** The map $N : M \longrightarrow S^2$ with $|N| = 1$ maps to the unit sphere. Since $|N(p)|^2 = 1,\ \forall p \in U$ we have $N(M) \subset S^2$ so that each unit normal field $N$ on $M$ can be thought of as a map from $M$ to the sphere $S^2$. The map is called Gauss map. One obtains an explicit description of $N$ from various descriptions of $M$. $N$, or to be more specific $dN$, will play a central role for studying the shape of a surface.

**Definition 2.15.** (Frame) We define for the surface $f : U \longrightarrow \mathbb{R}^3$ $(f_x, f_y, N)$ as frame.

Using the unit normal vector $N$ we can now define the second fundamental form:

**Definition 2.16.** (Second fundamental form) The bilinear form

$$II_p : T_pM \times T_pM \longrightarrow \mathbb{R} \quad p \in U$$

defined by

$$II_p(v,w) = -\langle (dN)_p(v), w\rangle = \langle N(v), dw\rangle \quad v,w \in T_pM \tag{2.2}$$

is the second fundamental form.

With $f : U \longrightarrow M$ an immersion and $(x,y)$ a standard coordinate system on $\mathbb{R}^2$ the components of $II_p$ with respect to the standard coordinate system are given by

$$b = \begin{pmatrix} l & m \\ m & n \end{pmatrix}, \tag{2.3}$$

where

$$\begin{aligned} l = \langle -\frac{\partial N}{\partial x}, \frac{\partial f}{\partial x}\rangle &= \langle -N_x, f_x\rangle = \langle N, f_{xx}\rangle, \\ m &= \langle -N_x, f_y\rangle = \langle N, f_{xy}\rangle, \\ n &= \langle -N_y, f_y\rangle = \langle N, f_{yy}\rangle, \end{aligned}$$

or in short

$$l_{ij} = \langle -N_i, f_j\rangle = \langle N, f_{ij}\rangle. \tag{2.4}$$

**Definition 2.17.** (Weingarten map) The differential of the Gauss map at each point of the surface is an endomorphism of the tangent plane at this point. The map

$$-dN : T_pM \longrightarrow T_pM$$

is called Weingarten map (or shape operator).

We are now able to examine the connection between the first and second fundamental form and the Weingarten map.

**Theorem 2.18.** *The matrix of $-dN : T_pM \longrightarrow T_pM$ is represented by*

$$(g_{ij})^{-1}(l_{ij}) = \frac{1}{EG - F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} l & m \\ m & n \end{pmatrix}. \tag{2.5}$$

*Proof.* We can use the following: Let $(v_1, \dots, v_n)$ be basis for a vector space $V$ with an inner product $\langle\cdot,\cdot\rangle$. If $A$ is the matrix of the linear transformation $T : V \longrightarrow V$ with respect to the basis

$$B = \langle Tv_i, v_j \rangle \quad C = \langle v_i, v_j \rangle,$$

then we have

$$A = (BC^{-1})^t.$$

Setting $A$ the matrix of $-dN$ with respect to $f_x$ and $f_y$ and $B = (g_{ij})$ as well as $C = (l_{ij})$ we get

$$A = (BC^{-1})^t = C^{-1}B$$

since $B$ and $C$ are symmetric. □

We will now define the different types of curvature and establish their connection to the fundamental forms:

**Definition 2.19.** (Principal, mean and Gauss curvature) Let $\kappa_1(p), \kappa_2(p)$ be the principal curvature of a surface i.e. the eigenvalues of the Weingarten map.

We then set

$$K = \kappa_1 \kappa_2 \tag{2.6}$$

the Gauss curvature, and

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) \tag{2.7}$$

the mean curvature.

**Definition 2.20.** If $H = \textit{constant}$, the surface is called constant mean curvature surface, and if $H = 0$, $f$ is the immersion of a minimal surface.

So far all necessary tools have been derived and one can ask to what extent $(g_{ij}), (l_{ij})$ determine $f$ up to a proper Euclidean motion. In other words, we investigate which quantities uniquely determine $f$. In particular we will be interested in constant mean curvature surfaces. Eventually, we see that an immersion is uniquely determined by $u$, the Hopf differential $Q$ and the constant curvature $H$. We introduce a well known formula, the Sym-Bobenko formula, which enables one to retrieve an exact formula for the immersion having identified the frame. In a next step, we then extend one immersion to a whole family of immersions by introducing a spectral parameter. We will do this for $\mathbb{R}^3$ and $\mathbb{H}^3$ in the following two sections. There are some similarities between $\mathbb{R}^3$ and $\mathbb{H}^3$ in the way of proceeding, but differences for the Sym-Bobenko formula for the immersion, which we will see later is crucial.

## 2.2 Surfaces in $\mathbb{R}^3$

We start by computing the fundamental forms for the Euclidean space and then get to the fundamental theorem of surface theory, which tells us when an immersion exists. The immersion exists if $u, Q, H$ satisfy a pair of equations, the Gauss-Codazzi equations. Furthermore the theorem then tells us that, if $f$ exists, it is uniquely determined up to rigid motions. As a goal of this section, we will describe CMC surfaces in terms of linear first order partial differential equations, the so-called Lax pair. This Lax pair defines the frame of the surface. We will first formulate the Lax pair in terms of $3 \times 3$-matrices, and then rework the frame into $2 \times 2$ matrices and introduce the Sym-Bobenko formula with which one can retrieve an exact formula from the frame for an immersion without the need of integration.

### 2.2.1 Conformal parameterization

First we choose suitable coordinates. Like an immersion can be described in different parameterizations, one can also define domains in different local coordinates. Put differently, one can choose a more favorable representation (of many possible) for the surface which we will call coordinate system. All coordinate systems describe the same geometry, but some are better adapted to the geometry then others, which will lead us to conformal metrics as mentioned in the section before.

**Definition 2.21.** (Conformal metric) A metric $g$ is conformal if for an immersion

$$f : U \longrightarrow \mathbb{R}^3,$$

the vectors $f_x$ and $f_y$ are orthogonal and of equal positive length in $\mathbb{R}^3$ at every point $f(p)$.

If the dimension of a differentiable manifold is 2, as it is in our case, we have some special properties. This is a consequence of the coordinate charts here being maps from $\mathbb{R}^2$, and the fact that $\mathbb{R}^2$ can be thought of as the complex plane $\mathbb{C} \simeq \mathbb{R}^2$. We thus identify $\mathbb{R}^2$ with $\mathbb{C}$ by the following isomorphism:

$$\mathbb{R} \longrightarrow \mathbb{C}$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto x + iy$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is the matrix of the $\mathbb{R}$-linear map.

Setting

$$Az = lz \quad l = \alpha + i\beta$$

yields

$$A(x,y) = (\alpha x - \beta y, \beta x + \alpha y), \quad (z = (x,y)).$$

With respect to the canonical basis $1(= (1,0))$ and $i(= (0,1))$ the corresponding matrix has the following representation

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \quad (\alpha, \beta \in \mathbb{R}).$$

Setting

$$z = x + iy \qquad \overline{z} = x - iy$$

complexifies the coordinates of the surface.

This leads to the theory of Riemann surfaces, which contains the following important result. It ensures that, in our case, it will always be possible to choose a conformal environment.

**Theorem 2.22.** *Let $U$ be a 2-dimensional manifold with a family of coordinate charts that determines a differentiable structure as above and a positive metric $g$. Then there* *exists another family of coordinate charts which generates a complex structure, and with respect to which the metric $g$ is conformal.*

We are now going to study the implications of such a conformal environment.

Let $M$ be a smooth surface in $\mathbb{R}^3$:

$$M = f(U) \quad with \quad f : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3.$$

This makes $M$ a differentiable manifold with charts. In every point $p \in \mathbb{R}^2$ we have parameters $(x, y)$ such that $T_pM = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$. Theorem 2.22 tells us, that we can choose coordinates such that $g$ is conformal.

If $f$ is a conformal parameterization we have $E = G$ and $F = 0$.

We set

$$\langle f_x, f_x \rangle = 4e^{2u}$$

and thus have

$$\langle f_y, f_y \rangle = \langle f_x, f_x \rangle = 4e^{2u}, \quad \langle f_x, f_y \rangle = 0.$$

With $z = x+iy$ and $\overline{z} = x-iy$, a calculation shows that, in terms of complex coordinates, this results in:

$$\begin{aligned}
\langle f_x, f_x \rangle &= \langle f_z + f_{\overline{z}}, f_z + f_{\overline{z}} \rangle \\
&= \langle f_z, f_z \rangle + 2\langle f_z, f_{\overline{z}} \rangle + \langle f_{\overline{z}}, f_{\overline{z}} \rangle = 4e^{2u} \\
\langle f_y, f_y \rangle &= \langle i(f_z - f_{\overline{z}}), i(f_z - f_{\overline{z}}) \rangle \\
&= -(\langle f_z, f_z \rangle - 2\langle f_z, f_{\overline{z}} \rangle + \langle f_{\overline{z}}, f_{\overline{z}} \rangle) = 4e^{2u}.
\end{aligned}$$

Summation of 2.8 and 2.8 leads to

$$\langle f_z, f_{\overline{z}} \rangle = 2e^{2u}.$$

We also have

$$\langle f_x, f_y\rangle = \langle f_z + f_{\overline{z}}, i(f_z - f_{\overline{z}})\rangle = i\langle f_z, f_z\rangle - i\langle f_{\overline{z}}, f_{\overline{z}}\rangle = 0.$$

This leads us to

$$i(\langle f_z, f_z\rangle - \langle f_{\overline{z}}, f_{\overline{z}}\rangle) = 0 = \langle f_z, f_z\rangle + \langle f_{\overline{z}}, f_{\overline{z}}\rangle$$

and we get

$$\langle f_{\overline{z}}, f_{\overline{z}}\rangle = \langle f_z, f_z\rangle = 0.$$

### 2.2.2 Fundamental forms in $\mathbb{R}^3$

We are now ready to calculate the defined quantities of 2.1: $f_z, f_{\overline{z}}$ and the unit normal $N$ define a frame and we have

$$\langle f_z, N\rangle = \langle f_{\overline{z}}, N\rangle = 0 \quad \langle N, N\rangle = 1 \tag{2.8}$$

as $f_z, f_{\overline{z}} \perp N$, and the length of $N$ is equal to one.

Hence, the first fundamental form is given by

$$I = 4e^{2u}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

To get the second fundamental form one has to compute

$$l = \langle N, f_{xx}\rangle, m = \langle N, f_{xy}\rangle \quad and \quad n = \langle N, f_{yy}\rangle$$

as

$$II = \begin{pmatrix} l & m \\ m & n \end{pmatrix}.$$

To investigate $\langle f_{xx}, N\rangle, \langle f_{yy}, N\rangle$ and $\langle f_{xy}, N\rangle$ first compute

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} &= \frac{\partial^2}{\partial z^2} + 2\frac{\partial}{\partial z\partial \overline{z}} + \frac{\partial^2}{\partial \overline{z}^2}\\
\frac{\partial^2}{\partial y^2} &= -\frac{\partial^2}{\partial z^2} + 2\frac{\partial}{\partial z\partial \overline{z}} - \frac{\partial^2}{\partial \overline{z}^2}\\
\frac{\partial^2}{\partial x\partial y} &= i(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \overline{z}^2}).
\end{aligned}$$

We choose to set $\langle f_{zz}, N\rangle = Q$ and $\langle f_{z\overline{z}}, N\rangle = 2He^{2u}$

$$\begin{aligned}
\langle f_{xx}, N\rangle &= \langle f_{zz}, N\rangle + 2\langle f_{z\overline{z},N}\rangle + \langle f_{\overline{z}\overline{z},N}\rangle = Q + 4He^{2u} + \overline{Q}\\
\langle f_{yy}, N\rangle &= -\langle f_{zz}, N\rangle + 2\langle f_{z\overline{z},N}\rangle - \langle f_{\overline{z}\overline{z},N}\rangle = -Q + 4He^{2u} - \overline{Q}\\
\langle f_{xy}, N\rangle &= i(\langle f_{zz}, N\rangle - \langle f_{\overline{z}\overline{z},N}\rangle) = i(Q - \overline{Q}).
\end{aligned}$$

The second fundamental form then is of the form

$$II = \begin{pmatrix} Q + 4He^{2u} + \overline{Q} & i(Q-\overline{Q}) \\ i(Q-\overline{Q}) & -Q + 4He^{2u} - \overline{Q} \end{pmatrix}.$$

In Section 2.1 we defined the Gauss and mean curvature which we will now determine for this specific case. Let $\kappa_i (i = 1, 2)$ be the eigenvalues of the Weingarten map and $I$ and $II$ first and second fundamental forms.

We see that the mean curvature $H$ is well defined

$$\begin{aligned}
H = \frac{1}{2}(\kappa_1 + \kappa_2) &= \frac{1}{2}tr(I^{-1}II) = \frac{1}{8e^{2u}} tr \begin{pmatrix} Q + He^2u\overline{Q} & i(Q-\overline{Q}) \\ i(Q-\overline{Q}) & -Q + He^2u - \overline{Q} \end{pmatrix}\\
&= \frac{1}{8e^{2u}} 8He^2u = H.
\end{aligned}$$

The Gauss curvature reads

$$K = \kappa_1\kappa_2 = det(I^{-1}II) = H^2 - \frac{1}{4}Q\overline{Q}e^{-4u}.$$

.

### 2.2.3 Lax pair in terms of $3 \times 3$-matrices

We now formulate the first important result, identifying the Lax pair and the frame for an immersion in terms of $3 \times 3$-matrices:

**Proposition 2.23.** $f_z, f_{\overline{z}}$ and $N$ satisfy the following equations:

$$F_z = UF, \quad F_{\overline{z}} = VF, \quad with \ \ F = (f_z, f_{\overline{z}}, N)^T \tag{2.9}$$

$$U = \begin{pmatrix} 2u_z & 0 & Q \\ 0 & 0 & 2He2^u \\ -H & -\frac{1}{2}e^{-2u}Q & 0 \end{pmatrix} \quad V = \begin{pmatrix} 0 & 0 & 2He^{2u} \\ 0 & 2u_{\overline{z}} & \overline{Q} \\ -\frac{1}{2}e^{-2u}\overline{Q} & -H & 0 \end{pmatrix}. \tag{2.10}$$

$U$ and $V$ are called the Lax pair.

*Proof.* If $(e_1, e_2, e_3)$ is an orthonormal Basis of $\mathbb{R}^3$ then the following holds:

$$\frac{\partial f}{\partial x} = \langle\frac{\partial f}{\partial x}, e_1\rangle e_1 + \langle\frac{\partial f}{\partial x}, e_2\rangle e_2 + \langle\frac{\partial f}{\partial x}, e_3\rangle e_3.$$

Taking into consideration that $f_x, f_{\overline{y}}$ and $N$ is already an orthonormal basis, we obtain the five equations:

$$\begin{aligned}
f_{zz} &= \langle f_{zz}, N\rangle N + \langle f_{zz}, f_{\overline{z}}\rangle\frac{f_z}{2e^{2u}} + \langle f_{zz}, f_z\rangle\frac{f_{\overline{z}}}{2e^{2u}} \\
f_{\overline{z}\overline{z}} &= \langle f_{\overline{z}\overline{z}}, N\rangle N + \langle f_{\overline{z}\overline{z}}, f_{\overline{z}}\rangle\frac{f_z}{2e^{2u}} + \langle f_{\overline{z}\overline{z}}, f_z\rangle\frac{f_{\overline{z}}}{2e^{2u}} \\
f_{z\overline{z}} &= \langle f_{z\overline{z}}, N\rangle N + \langle f_{z\overline{z}}, f_z\rangle\frac{f_z}{2e^{2u}} + \langle f_{z\overline{z}}, f_{\overline{z}}\rangle\frac{f_{\overline{z}}}{2e^{2u}} \\
N_z &= \langle N_z, N\rangle N + \langle N_z, f_{\overline{z}}\rangle\frac{f_z}{2e^{2u}} + \langle N_z, f_z\rangle\frac{f_{\overline{z}}}{2e^{2u}} \\
N_{\overline{z}} &= \langle N_{\overline{z}}, N\rangle N + \langle N_{\overline{z}}, f_{\overline{z}}\rangle\frac{f_z}{2e^{2u}} + \langle N_{\overline{z}}, f_z\rangle\frac{f_{\overline{z}}}{2e^{2u}}.
\end{aligned}$$

We have $\langle f_{zz}, N\rangle = Q$.

With $\langle f_z, f_z\rangle = 0$ we get

$$\begin{aligned}
\partial_z\langle f_z, f_z\rangle &= 2\langle f_{zz}, f_z\rangle = 0\\
\langle f_z, f_{\overline{z}}\rangle &= 2e^{2u}\\
\partial_z\langle f_z, f_{\overline{z}}\rangle &= \langle f_{zz}, f_{\overline{z}}\rangle + \langle f_z, f_{\overline{zz}}\rangle = 4u_z e^{2u}.
\end{aligned}$$

In sum we get

$$f_{zz} = QN + 0 + 4u_z e^{2u}\frac{f_z}{2e^{2u}} = QN + 2u_z f_z.$$

In the same way we can compute the other equations. Using

$$\begin{aligned}
\langle N_z, N\rangle &= \langle N_{\overline{z}}, N\rangle = 0\\
\langle N_z, f_z\rangle &= -\langle f_{zz}, N\rangle\\
\langle N_{\overline{z}}, f_z\rangle &= -\langle f_{\overline{z}z}, N\rangle\\
\langle N_{\overline{z}}, f_{\overline{z}}\rangle &= -\langle f_{\overline{zz}}, N\rangle
\end{aligned}$$

yields to the so called Gauss-Weingarten equations

$$\begin{aligned}
f_{\overline{zz}} &= \overline{Q}N + 2u_{\overline{z}}f_{\overline{z}}\\
f_{zz} &= QN + 2u_z f_z\\
f_{z\overline{z}} &= f_{\overline{z}z} = 2He^{2u}N\\
N_z &= -\frac{1}{2}e^{-2u}Qf_{\overline{z}} - Hf_z\\
N_{\overline{z}} &= -\frac{1}{2}e^{-2u}\overline{Q}f_z - Hf_{\overline{z}}.
\end{aligned}$$

□

The Bonnet Theorem tells us that there exists an immersion $f : U \longrightarrow \mathbb{R}^3$ with a first fundamental form $I$ and second fundamental form $II$ (with respect to coordinates $x$ and $y$) if and only if $I$ and $II$ satisfy a pair of equations called the Gauss-Codazzi equations. Furthermore, the fundamental surface theorem tells us that when such an $f$ exists, it is uniquely determined by $I$ and $II$ up to rigid motions of $\mathbb{R}^3$.

The corresponding compatibility condition for the existence of a solution $f$, i.e. $f_{z\overline{z}} = f_{\overline{z}z}$, are the Gauss-Codazzi equations

$$\begin{aligned} 4u_{z\overline{z}} + 4H^2e^{2u} - Q\overline{Q}e^{-2u} &= 0 \\ Q_{\overline{z}} &= 2H_z e^{2u} \\ \overline{Q}_z &= 2H_{\overline{z}}e^{2u}. \end{aligned}$$

**Remark 2.24.** The Codazzi equation implies that $H$ is constant if and only if $Q$ is holomorphic, since $H$ is a real valued function.

The Gauss Codazzi equations are only written in terms of the functions $u$, $H$ and $Q$ and hence determine whether $f$ exists or not. We formulate this in a theorem.

**Theorem 2.25.** *(Bonnet Theorem) Given a metric $4e^{2u}dzd\overline{z}$, a quadratic differential $Qdz^2$ and a function $H$ satisfying the Gauss-Codazzi-equations, there exists an immersion*

$$f : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \tag{2.11}$$

*with the fundamental forms of $I$ and $II$.*

The next well known theorem establishes which conditions has to be fulfilled by the Lax pair $U$ and $V$ such that there exists a solution $F$ to the system $F_z = FU, F_{\overline{z}} = FV$.

**Theorem 2.26.** *For $U, V : \mathbb{C} \times \mathbb{C} \longrightarrow sl_n\mathbb{C}$ there exists a solution*

$$F = F(z, w) : \mathbb{C} \times \mathbb{C} \longrightarrow SL_n\mathbb{C}$$

*of the Lax pair*

$$F_z = FU, \quad F_{\overline{z}} = FV$$

*for any initial condition $F(0, 0) \in SL_n\mathbb{C}$ if and only if*

$$U_{\overline{z}} - Vz + [U, V] = 0 \quad with \quad [U, V] = UV - VU. \tag{2.12}$$

We can assume $H$ to be constant. The Gauss-Codazzi equations still hold if we replace

$Q$ with $\lambda^{-2}Q$ for any $\lambda \in S^1$. $\lambda$ is often referred to as spectral parameter. In this way we obtain a 1-parameter family of surfaces $f(z, \overline{z}, \lambda)$ all with CMC $H$, the same metric, and Hopf differential $\lambda^{-2}Q$. $f(z, \overline{z}, \lambda)$ is called the associated family.

### 2.2.4 Lax pair in terms of $2 \times 2$-matrices

We will now rework the $3 \times 3$ frame into a $2 \times 2$ frame. Define the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

with the $2 \times 2$ identity matrix $\mathbb{1}$. The Pauli matrices can then be used to form a basis

$$\{\mathbb{1}, -i\sigma_1, -i\sigma_2, -i\sigma_3\}.$$

Identify $\mathbb{R}^3$ with $su_2$, by identifying $x_1, x_2, x_3 \in \mathbb{R}^3$ with the matrix

$$-x_1\frac{i}{2}\sigma_1 + x_2\frac{i}{2}\sigma_2 + x_3\frac{i}{2}\sigma_3 = \frac{-i}{2}\begin{pmatrix} -x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_3 \end{pmatrix}.$$

Let $F = F(z, \overline{z}, \lambda) \in SU_2$ be the matrix that rotates $\frac{-i\sigma_1}{2}$, $\frac{-i\sigma_2}{2}$ and $\frac{-i\sigma_3}{2}$ to the $2 \times 2$-matrix forms of $e_1, e_2$ and $N$, respectively:

$$e_1 = F\frac{-i\sigma_1}{2}F^{-1}, \quad e_2 = F\frac{-i\sigma_2}{2}F^{-1} \quad N = F\frac{-i\sigma_3}{2}F^{-1}.$$

**Remark 2.27.** Those relations determine $F$ uniquely up to sign and $F^{-1}$ is a rotation of $\mathbb{R}^3$ as one can show that $F \in SU_2$.

Define

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} := F^{-1}F_z, \quad V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} := F^{-1}F_{\overline{z}}$$

and compute $U$ and $V$ in terms of $u, H$ and $Q$.

Making use of

$$e_1 = \frac{f_x}{|f_x|} = \frac{f_x}{2e^u} = F\frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} F^{-1},$$
$$e_2 = \frac{f_y}{|f_y|} = \frac{f_y}{2e^u} = F\frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} F^{-1}$$

we get

$$f_z = -ie^u F \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} F^{-1}, \quad f_{\overline{z}} = -ie^u F \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} F^{-1}.$$

We now recover the entries of the matrices $U$ and $V$. Differentiating $f_{\overline{z}}$ with respect to $z$ leads to

$$\begin{aligned} f_{\overline{z}z} &= u_z f_{\overline{z}} + (-ie^u)\left(F_z \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} F^{-1} + F \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} F_z^{-1}\right) \\ &= u_z f_{\overline{z}} + (-ie^u)\left(FU \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} F^{-1} + F \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} U^{-1}F^{-1}\right) \\ &= u_z f_{\overline{z}} + (-ie^u)\left(F \begin{pmatrix} -U_{21} & 2U_{11} \\ 0 & U_{21} \end{pmatrix} F^{-1}\right). \end{aligned}$$

We differentiate $f_z$ with respect to $\overline{z}$

$$\begin{aligned} f_{z\overline{z}} &= u_{\overline{z}} f_z + (-ie^u)\left(F_{\overline{z}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} F^{-1} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} F_{\overline{z}}^{-1}\right) \\ &= u_{\overline{z}} f_z + (-ie^u)\left(FV \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} F^{-1} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} V^{-1}F^{-1}\right) \\ &= u_{\overline{z}} f_z + (-ie^u)\left(F \begin{pmatrix} V_{12} & 0 \\ 2V_{22} & -V_{12} \end{pmatrix} F^{-1}\right). \end{aligned}$$

$f_{z\overline{z}} = f_{\overline{z}z}$ then implies

$$u_z f_{\overline{z}} + (-ie^u)\left(F \begin{pmatrix} -U_{21} & 2U_{11} \\ 0 & U_{21} \end{pmatrix} F^{-1}\right) = u_{\overline{z}} \quad f_z + (-ie^u)\left(F \begin{pmatrix} V_{12} & 0 \\ 2V_{22} & -V_{12} \end{pmatrix} F^{-1}\right)$$

$$\iff u_z f_{\overline{z}} - u_{\overline{z}} f_z = (-ie^u)\left(F\begin{pmatrix} V_{12} & 0 \\ 2V_{22} & -V_{12}\end{pmatrix}F^{-1} - F\begin{pmatrix} -U_{21} & 2U_{11} \\ 0 & U_{21}\end{pmatrix}F^{-1}\right)$$
$$\iff u_z f_{\overline{z}} - u_{\overline{z}} f_z = (-ie^u)\left(F\begin{pmatrix} V_{12}+U_{21} & -2U_{11} \\ 2V_{22} & -V_{12}-U_{21}\end{pmatrix}\right).$$

For $u_z f_{\overline{z}} - u_{\overline{z}} f_z$ we get

$$\begin{aligned} u_z f_{\overline{z}} - u_{\overline{z}} f_z &= (-ie^u)\left(F\begin{pmatrix} 0 & u_z \\ -u_{\overline{z}} & 0\end{pmatrix}\right) \\ &= (-ie^u)\left(F\begin{pmatrix} V_{12}+U_{21} & -2U_{11} \\ 2V_{22} & -V_{12}-U_{21}\end{pmatrix}\right). \end{aligned}$$

Hence

$$U_{11} = -\frac{1}{2}u_z \quad V_{22} = \frac{1}{2}u_{\overline{z}}, \quad U_{21} = -V_{12}.$$

To recover the next coefficients we compute $f_{zz}$

$$\begin{aligned} f_{zz} &= u_z f_z + (-ie^u)\left(F_z\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}F^{-1} + F\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}F_z^{-1}\right) \\ &= u_z f_z + (-ie^u)\left(FU\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}F^{-1} + F\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}U^{-1}F^{-1}\right) \\ &= u_z f_z + (-ie^u)\left(F\begin{pmatrix} U_{12} & 0 \\ 2U_{22} & -U_{12}\end{pmatrix}F^{-1}\right). \end{aligned}$$

We know that $f_{zz} = 2u_z f_z + \lambda^{-2}QN$ and with $N = F\frac{-i\sigma_3}{2}F^{-1}$ this yields

$$\begin{aligned} & u_z f_z + (-ie^u)\left(F\begin{pmatrix} U_{12} & 0 \\ 2U_{22} & -U_{12}\end{pmatrix}F^{-1}\right) = 2u_z f_z + \lambda^{-2}QN \\ \iff \quad & -u_z f_z + (-ie^u)\left(F\begin{pmatrix} U_{12} & 0 \\ 2U_{22} & -U_{12}\end{pmatrix}F^{-1}\right) = \lambda^{-2}QN \\ \iff \quad & \left(F\begin{pmatrix} U_{12} & 0 \\ 0 & -U_{12}\end{pmatrix}F^{-1}\right) = \frac{1}{2}\left(F\begin{pmatrix} \lambda^{-2}Qe^{-u} & 0 \\ 0 & -\lambda^{-2}Qe^{-u}\end{pmatrix}F^{-1}\right). \end{aligned}$$

Hence

$$U_{12} = \frac{1}{2}\lambda^{-2}Qe^{-u}.$$

In the same way we then get $V_{21} = -e^{-u}\lambda^2\overline{Q}$.

We know that

$$f_{\overline{z}z} \quad = \quad u_z f_{\overline{z}} + (-ie^u)\left(F\begin{pmatrix} -U_{21} & 2U_{11} \\ 0 & U_{21} \end{pmatrix} F^{-1}\right).$$

With $U_{11} = \frac{1}{2}u_z$ this reduces to

$$f_{\overline{z}z} \quad = \quad (-ie^u)\left(F\begin{pmatrix} -U_{21} & 0 \\ 0 & U_{21} \end{pmatrix} F^{-1}\right).$$

From $N = F\frac{-i\sigma_3}{2}F^{-1}$ and $f_{\overline{z}z} = 2He^{2u}N$ we obtain

$$2He^{2u}N = (-ie^u)\left(F\begin{pmatrix} -U_{21} & 0 \\ 0 & U_{21} \end{pmatrix} F^{-1}\right)$$

$$\iff \quad F\left(\begin{pmatrix} He^u & 0 \\ 0 & -He^u \end{pmatrix}\right) = \left(F\begin{pmatrix} -U_{21} & 0 \\ 0 & U_{21} \end{pmatrix} F^{-1}\right)$$

and therefore $U_{21} = -He^u$. Similarly one finds $V_{12} = He^u$. In sum we have recovered $U$ and $V$.

$$U = \frac{1}{2}\begin{pmatrix} -u_z & e^{-u}\lambda^{-2}Q \\ -2He^u & u_z \end{pmatrix} \quad V = \frac{1}{2}\begin{pmatrix} u_{\overline{z}} & 2He^u \\ -e^{-u}\lambda^2\overline{Q} & -u_{\overline{z}} \end{pmatrix}.$$

$U$ and $V$ are the Lax pair in terms of $2 \times 2$-matrices of the frame $F$. A computation of the zero curvature condition show that the corresponding Gauss-Codazzi equations reads

$$4u_{z\overline{z}} - e^{-u}Q\overline{Q} + 4H^2e^{2u}, \quad Q_{\overline{z}} = 2H_z e^{2u}.]$$

Given a frame $F$ we then know $f_z, f_{\overline{z}}$ and would expect to need to integrate in order to find $f$. The Sym-Bobenko formula

$$f(z,\overline{z},\lambda) = \frac{1}{2H}\left[F\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}F^{-1} - i\lambda(\partial_\lambda)F^{-1})\right] \tag{2.13}$$

provides a formula for the immersion $f$ in dependence of $F$.

**Remark 2.28.** This formula avoids integration, using the derivative of $F$ with respect to the spectral parameter $\lambda$. Another advantage is that the Sym-Bobenko formula gives a meaning for understanding special immersions and under which conditions they will be well defined. We come back to this in Chapter 5.

We summarize the latest results in a Lemma.

**Lemma 2.29.** The CMC $H$ surfaces $f(z,\overline{z},\lambda)$ with $H \neq 0$ as in the $3\times 3$ case and the surfaces derived from the Sym-Bobenko formula differ only by rigid motion of $\mathbb{R}^3$. Thus the Sym-Bobenko formula produces the associated family of any CMC H surface from a frame $F$ solving

$$F_z = FU \quad , \quad F_{\overline{z}} = FV \quad with$$
$$U = \frac{1}{2}\begin{pmatrix} -u_z & e^{-u}\lambda^{-2}Q \\ -2He^u & u_z\end{pmatrix} \quad , \quad V = \frac{1}{2}\begin{pmatrix} u_{\overline{z}} & 2He^u \\ -e^{-u}\lambda^2\overline{Q} & -u_{\overline{z}}\end{pmatrix}. \tag{2.14}$$

Conversely, for any $u$ and $Q$ satisfying the Gauss-Codazzi equations

$$4u_{z\overline{z}} - Q\overline{Q}e^{-2u} + 4H^2e^{2u} = 0, \quad Q_{\overline{z}} = 2H_z e^{2u}, \tag{2.15}$$

and any solution $F$ of $U$ and $V$ satisfying 2.14 such that $F \in SU_2$ for all $\lambda \in S^1$, $f$ defined by the Sym-Bobenko formula is a conformal CMC H immersion into $\mathbb{R}^3$ with metric $4e^{2u}(dx^2+dy^2)$ and Hopf differential $\lambda^{-2}Q$.

The proof of the lemma can be found in [25].

**Remark 2.30.** Another formula is given for example in [18] where one has to integrate in terms of Baker-Akhiezer functions (see Chapter 4 for more information on Baker-Akhiezer functions).

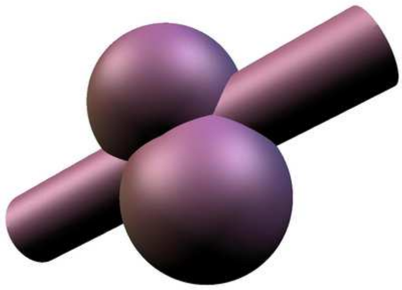

Figure 2.1: Two-Lobed Bubbleton with round cylinder ends in $\mathbb{R}^3$

## 2.3 The sinh-Gordon equation and CMC tori

We give a brief summary of the results obtained so far. Let $f : U \subset \mathbb{C}^2 \longrightarrow \mathbb{R}^3$ be an immersion. The surface $f(z, \overline{z})$ with $(z, \overline{z}) \in U$ is uniquely defined up to rigid motion by the first and second fundamental forms. Let $N(z, \overline{z})$ be the normal vector field at each point of the surface $f(z, \overline{z})$. Then the triple $(f_z, f_{\overline{z}}, N)$ defines a basis $T_pM$, where $M$ is the surface parameterized by $f(z, \overline{z})$ and $p$ a point in $M$. The motion of the basis on $M$ is characterized by the Gauss-Weingarten equations. The compatibility condition of these equations are the well known Gauss-Mainardi-Codazzi equations which are coupled to nonlinear differential equations for the coefficients of the first and second fundamental forms, respectively. As we will see in this section, these equations reduce to a single system of an integrable equation, the sinh-Gordon equation.

From now on we focus on tori with constant mean curvature (CMC) and set, without loss of generality, $H = \frac{1}{2}$. As $Q$ is of finiteness $Q = const.$. We fix a conformal coordinate with $Q = e^{i\varphi}$, where $\varphi = const. \in \mathbb{R}$.

A simple computation shows that the Gauss-Codazzi equations transform to the sinh-Gordon equation

$$2u_{z\overline{z}} + (e^{2u} - e^{-2u}) = 2u_{z\overline{z}} + sinh(2u) = 0. \tag{2.16}$$

We now consider a lattice $\Lambda$ in $\mathbb{C}$. It can be written as

$$\Lambda = \{a\omega_1 + b\omega_2 | a, b \in \mathbb{Z}, \quad \omega_1, \omega_2 \in \mathbb{C}\}.$$

We compute the conditions for deciding periodic points $\omega_1$ and $\omega_2$ for which the resulting finite type CMC surfaces $f$ will close to become a torus

$$f(z) = f(z + \omega_1) = f(z + \omega_2).$$

The immersion $f$ will then be well defined on the torus $\mathbb{C} \setminus \Lambda$ .

Note that this arguments holds for $\mathbb{R}^3$ as well as $\mathbb{H}^3$ and $S^3$. The difference between the three spaces is due to the closing condition which is derived from the different Sym-Bobenko formulas for the three spaces.

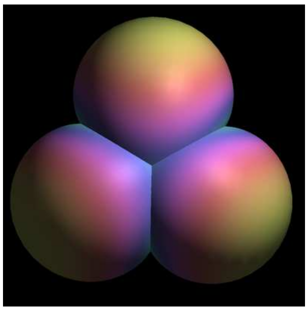

Figure 2.2: The Wente torus in $\mathbb{R}^3$

Regarding CMC tori we have the following result:

**Proposition 2.31.** CMC tori do not have umbilical points.

*Proof.* $Q(z)$ is an elliptic function without any singularities as $Q(z)(dz)^2$ is holomorphic. Hence $Q(z) = const.$ $Q(z)$ is not zero, otherwise it would follow from the Hopf theorem that the surface is a sphere. We therefore have $Q = const. \neq 0$ which proves the proposition. □

In the next section we will apply the concepts and formulas derived so far for the hyperbolic 3-space, not changing much compared to the $\mathbb{R}^3$ case. We proceed exactly the same way and will see that in $\mathbb{H}^3$, eventually the Gauss-Codazzi equations also transform to the sinh Gordon equation. The periodicity arguments for CMC toris in $\mathbb{H}^3$ are the same as in $\mathbb{R}^3$, but will lead to different closing conditions due to the different formula for the immersion in terms of the frame.

## 2.4 Surfaces in $\mathbb{H}^3$

### 2.4.1 The model for the hyperbolic 3-space

We begin this section by introducing the hyperbolic 3-space and then proceed in the same way as for $\mathbb{R}^3$.

The hyperbolic 3-space can be described by various models, each one with its own advantages. We will identify $\mathbb{H}^3$ with the Minkowski 4-space $\mathbb{R}^{3,1}$ and its Lorentzian metric

$$\{(x_0, x_1, x_2, x_3 \in \mathbb{R}^{3,1} | x_0^2 - \sum_{j=1}^{3} x_j^2 = 1, x_0 > 0\}. \tag{2.17}$$

We call this the Minkowski model for hyperbolic 3-space. Although the metric $g_{\mathbb{R}^{3,1}}$ is Lorentzian and therefore not positive definite, the restriction $g$ to the upper sheet is actually positive definite, so $\mathbb{H}^3$ is a Riemannian manifold.

The isometry group of $\mathbb{H}^3$ can be described using the special orthogonal group

$$O_+(3,1) = \{A = (a_{ij})_{i,j=1}^4 \in O(3,1) | a_{44} > 0\} \tag{2.18}$$

of $\mathbb{R}^{3,1}$.

The isometry of $\mathbb{R}^{3,1}$ for each $A \in O_+(3,1)$ is the map

$$\vec{x} \in \mathbb{R}^{3,1} \Longrightarrow (A\vec{x}^t)^t \in \mathbb{R}^{3,1},$$

which preserves the Minkowski model for $\mathbb{H}^3$.

The above definition for the Minkowski model for hyperbolic 3-space does not immediately imply that it has all required properties. The following lemma tells us that the Minkowski model for the 3-hyperbolic space is indeed the true 3-hyperbolic space.

**Lemma 2.32.** The Minkowski model $\mathbb{H}^3$ for the 3-hyperbolic space is a simple connected 3-dimensional Riemann manifold.

For the proof the reader may refer to [25].

The only difference int the computation of the values is that one uses the metric $g_{\mathbb{R}^{3,1}}$, which we will denote by $\{\cdot,\cdot\}$, instead of using the standard metric of $\mathbb{R}^3$.

As was stated before, it is no problem here to fix a conformal parameterization. Then

$$\{f,f\} = -1.$$

By definition of the tangent space $T\mathbb{H}^3$, by definition of the normal $N$ and the conformality of $f$ and because of $\{f,f\}=-1$ we have

$$\begin{aligned}
\{N,f\} &= \{N,f_z\} = \{N,f_{\overline{z}}\} \\
&= \{f_z,f_z\} = \{f_{\overline{z}},f_{\overline{z}}\} = \{f,f_z\} = \{f,f_{\overline{z}}\} = 0 \\
\{N,N\} &= 1 \\
\{f_z,f_{\overline{z}}\} &= 2e^{2u}.
\end{aligned}$$

We set

$$Q = \{f_{zz},N\}.$$

### 2.4.2 Lax pair in terms of $4\times 4$-matrices

As before we fix a frame with the orthonormal basis $f, f_x, f_y, N$.

**Proposition 2.33.** $f, f_z, f_{\overline{z}}$ and $N$ fulfill the equations.

$$F_z = FU \quad , \qquad F_{\overline{z}} = FV, \quad \textit{with} \;\; F = (f, f_z, f_{\overline{z}}, N)^T \tag{2.19}$$

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 2u_z & 0 & Q \\ 2e^{2u} & 0 & 0 & 2He^{2u} \\ 0 & -H & -\frac{1}{2}Qe^{-2u} & 0 \end{pmatrix} \tag{2.20}$$

$$V = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 2e^{2u} & 0 & 0 & 2He^{2u} \\ 0 & 2u_{\overline{z}} & 0 & \overline{Q} \\ 0 & -\frac{1}{2}\overline{Q}e^{-2u} & -H & 0 \end{pmatrix}. \tag{2.21}$$

*Proof.* We use the same argument as before:

$$\begin{aligned}
f_z &= \{f_z, f\}f + \{f_z, f_{\overline{z}}\}\frac{f_z}{2e^{2u}} + \{f_z, f_z\}\frac{f_{\overline{z}}}{2e^{2u}} + \{f_z, N\}N \\
f_{\overline{z}} &= \{f_{\overline{z}}, f\}f + \{f_{\overline{z}}, f_{\overline{z}}\}\frac{f_z}{2e^{2u}} + \{f_{\overline{z}}, f_z\}\frac{f_{\overline{z}}}{2e^{2u}} + \{f_{\overline{z}\overline{z}}, N\}N \\
f_{zz} &= \{f_{zz}, f\}f + \{f_{zz}, f_{\overline{z}}\}\frac{f_z}{2e^{2u}} + \{f_{zz}, f_z\}\frac{f_{\overline{z}}}{2e^{2u}} + \{f_{zz}, N\}N \\
f_{z\overline{z}} &= \{f_{z\overline{z}}, f\}f + \{f_{z\overline{z}}, f_{\overline{z}}\}\frac{f_z}{2e^{2u}} + \{f_{z\overline{z}}, f_z\}\frac{f_{\overline{z}}}{2e^{2u}} + \{f_{z\overline{z}}, N\}N \\
f_{\overline{z}\overline{z}} &= \{f_{\overline{z}\overline{z}}, f\}f + \{f_{\overline{z}\overline{z}}, f_{\overline{z}}\}\frac{f_z}{2e^{2u}} + \{f_{\overline{z}\overline{z}}, f_z\}\frac{f_{\overline{z}}}{2e^{2u}} + \{f_{\overline{z}\overline{z}}, N\}N \\
N_z &= \{N_z, f\}f + \{N_z, f_{\overline{z}}\}\frac{f_z}{2e^{2u}} + \{N_z, f_z\}\frac{f_{\overline{z}}}{2e^{2u}} + \{N_z, N\}N \\
N_{\overline{z}} &= \{N_{\overline{z}}, f\}f + \{N_{\overline{z}}, f_{\overline{z}}\}\frac{f_z}{2e^{2u}} + \{N_{\overline{z}}, f_z\}\frac{f_{\overline{z}}}{2e^{2u}} + \{N_{\overline{z}}, N\}N.
\end{aligned}$$

The following holds:

$$\begin{aligned}
\{f_z, f_z\} &= \{f_{\overline{z}}, f_{\overline{z}}\} = \{N, F\} = \{N, F_z\} = \{N, F_{\overline{z}}\} = 0 \\
\{f_z, f_{\overline{z}}\} &= 2e^{2u} \\
\{f_z, N\} &= \{f_{\overline{z}}, N\} = 0 \\
\{N, N\} &= 1 \\
\{f_{zz}, N\} &= Q \\
\{f_{z\overline{z}}, N\} &= 2He^{2u}.
\end{aligned}$$

Examining $f_{zz}$:

$$\begin{aligned}
\{f_{zz}, f\} &= 0 \quad as \quad \partial_z\{f, f_z\} = \{f, f_{zz}\} + \{f_z, f_z\} = 0 \\
\{f_{zz}, f_z\} &= 0 \quad as \quad \partial_z\{f_z, f_z\} = \{f_{zz}, f_z\} = 0 \\
\{f_{zz}, f_{\overline{z}}\} &= 4u_z e^{2u} \quad as \quad 4u_z e^{2u} = \partial_z\{f_z, f_{\overline{z}}\} = \{f_{zz}, f_{\overline{z}}\} + \{f_z, f_{z\overline{z}}\} \; and \; \{f_z, f_{z\overline{z}}\} = 0 \\
\{f_{zz}, N\} &= Q \\
\Rightarrow f_{zz} &= 2u_z f_z + QN.
\end{aligned}$$

For $f_{\overline{z}z}$ compute:

$$\begin{aligned}
\{f_{z\overline{z}}, f\} &= 2e^{2u} \quad as \quad 0 = \partial_{\overline{z}}\{f_z, f\} = \{f_{z\overline{z}}, f\} + \{f_z, f_{\overline{z}}\} = 2e^{2u} \\
\{f_{z\overline{z}}, f_z\} &= 0 \quad as \quad \partial_{\overline{z}}\{f_z, f_z\} = 2\{f_{z\overline{z}}, f_z\} = 0 \\
\{f_{z\overline{z}}, f_{\overline{z}}\} &= 0 \quad as \quad \partial_z\{f_{\overline{z}}, f_{\overline{z}}\} = s\{f_{\overline{z}z}, f_{\overline{z}}\} = 0 \\
\{f_{\overline{z}z}, N\} &= 2He^{2u} \\
\Rightarrow f_{\overline{z}z} &= 2e^{2u} f + 2He^{2u} N.
\end{aligned}$$

We now look at $f_{\overline{z}\overline{z}}$:

$$\begin{aligned}
\{f_{\overline{z}\overline{z}}, f\} &= 0 \quad as \quad \partial_{\overline{z}}\{f_{\overline{z}}, f\} = \{f_{\overline{z}\overline{z}}, f\} + \{f_{\overline{z}}, f_{\overline{z}}\} = 0 \\
\{f_{\overline{z}\overline{z}}, f_z\} &= 4u_{\overline{z}}e^{2u} \quad as \quad 4u_{\overline{z}}e^{2u} = \partial_{\overline{z}}\{f_{\overline{z}\overline{z}}, f_z\} + \{f_{\overline{z}}, f_{\overline{z}z}\} \; and \; \{f_{\overline{z}}, f_{\overline{z}z}\} = 0 \\
\{f_{\overline{z}\overline{z}}, f_{\overline{z}}\} &= 0 \quad as \quad \partial_{\overline{z}}\{f_{\overline{z}}, f_{\overline{z}}\} = \{f_{\overline{z}\overline{z}}, f_{\overline{z}}\} = 0 \\
\{f_{\overline{z}\overline{z}}, N\} &= \overline{Q} \\
\Rightarrow f_{\overline{z}\overline{z}} &= 2u_{\overline{z}}f_{\overline{z}} + \overline{Q}N.
\end{aligned}$$

For $N_z$ and $N_{\overline{z}}$ one has:

$$\begin{aligned}
\{N_z, f\} &= \{N_z, N\} = \{N_{\overline{z}}, f\} = \{N_{\overline{z}}, N\} = 0 \\
\{N_z f_z\} &= -\{N, f_{zz}\} = -Q \\
\{N_z, f_{\overline{z}}\} &= -2He^{2u} \\
\{N_{\overline{z}}, f_{\overline{z}}\} &= -\overline{Q} \\
\{N_{\overline{z}}, f_z\} &= -2He^{2u} \\
\Rightarrow N_z &= -\frac{1}{2}e^{-2u}Qf_{\overline{z}} - Hf_z \\
\Rightarrow N_{\overline{z}} &= -\frac{1}{2}e^{-2u}f_z - Hf_{\overline{z}}.
\end{aligned}$$

Summarizing the results yields:

$$\begin{aligned}
f_z &= f_z \\
f_{\overline{z}} &= f_{\overline{z}} \\
f_{zz} &= 2u_z f_z + QN \\
f_{\overline{z}z} &= 2e^{2u}f + 2He^{2u}N \\
N_z &= -\frac{1}{2}e^{-2u}Qf_z - Hf_{\overline{z}} \\
f_{z\overline{z}} &= 2e^{2u}f + 2He^{2u}N \\
f_{\overline{z}\overline{z}} &= 2u_{\overline{z}}f_{\overline{z}} + \overline{Q}N \\
N_{\overline{z}} &= -\frac{1}{2}e^{-2u}f_z - Hf_{\overline{z}}
\end{aligned}$$

which concludes the proof. □

The compatibility condition is again

$$U_{\overline{z}} - V_z + [U, V] = 0.$$

As in $\mathbb{R}^3$, $u, Q$ and $H$ determine uniquely if the immersion $f$ exists and the Gauss-Codazzi equations reads

$$u_{z\overline{z}} - e^{2u}(H^2 - 1) + \frac{1}{4}Q\overline{Q}e^{-2u} = 0, \quad Q_{\overline{z}} = 2H_z e^{2u}.$$

Computing the compatibility equation leads to the latter Gauss-Codazzi equations.

### 2.4.3 Lax pair in terms of $2 \times 2$-matrices

In the next step we rework in $\mathbb{H}^3$, $U$ and $V$ in $2\times2$ matrices with the help of the hermitian matrix model.

Recall the following definitions: The group $SL_2\mathbb{C}$ consists of all $2 \times 2$-matrices with complex entries and determinant 1, with matrix multiplication as the group operation. The vector space $sl_2\mathbb{C}$ consists of all $2 \times 2$ complex matrices with trace 0, with the vector space operations being matrix addition and scalar multiplication. $SL_2\mathbb{C}$ is a 6-dimensional Lie group and $sl_2\mathbb{C}$ is the associated 6-dimensional Lie algebra, and thus the tangent space of $SL_2\mathbb{C}$ at the identity matrix.

$SU_2$ is the subgroup of matrices $F \in SL_2\mathbb{C}$ such that $FF^*$ is the identity matrix, where $F^* = \overline{F}^t$. Equivalently,

$$F = \begin{pmatrix} p & -\overline{q} \\ q & \overline{p} \end{pmatrix}$$

for some $p, q \in \mathbb{C}$ with $\|p\|^2 + \|q\|^2 = 1$. $SU_2$ is a 3-dimensional Lie subgroup.

Finally, we define Hermitian symmetric matrices as matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} \\ \overline{a_{12}} & a_{22} \end{pmatrix}$$

where $a_{12} \in \mathbb{C}$ and $a_{11}, a_{22} \in \mathbb{R}$. Hermitian symmetric matrices with determinant 1 have the additional condition that $a_{11}a_{22} - a_{12}\overline{a_{12}} = 1$.

The Lorentz 4-space $\mathbb{R}^{3,1}$ can be mapped into the space of $2 \times 2$ Hermitian symmetric matrices by

$$\psi : \vec{x} = (x_0, x_1, x_2, x_3) \longrightarrow \psi(\vec{x}) = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix}.$$

For $x \in \mathbb{R}^3$ the metric in the Hermitian matrix form is given by $\{x, x\} = -det(\psi(x))$. $\psi$ maps the Minkowski model for $\mathbb{H}^3$ to the set of Hermitian symmetric matrices with determinant 1.

Any Hermitian symmetric matrix with determinant 1 can be written as the product $F\overline{F}^t$ for some $F \in SL_2\mathbb{C}$, and $F$ is determined uniquely up to right-multiplication by elements in $SU_2$:

$$\begin{aligned} F, \hat{F} \in SL_2\mathbb{C} \qquad &\Longrightarrow \qquad F\overline{F}^t = \hat{F}\overline{\hat{F}}^t \\ &\Longleftrightarrow \quad F = \hat{F}B, \;\; B \in SU_2. \end{aligned}$$

Therefore there Hermintian model $H$ is given by

$$\mathcal{H} = \{FF^* | F \in SL_2\mathbb{C}\}, F^* := \overline{F}^t$$

and $\mathcal{H}$ is given the metric such that $\psi$ is an isometry from the Minkowski model of $\mathbb{H}^3$ to $\mathcal{H}$.

There exists an $F \in SL_2\mathbb{C}$ such that

$$f = F\overline{F}^t, \quad e_1 := F\sigma_1\overline{F}^t, \quad e_2 := F\sigma_2\overline{F}^t, \quad N := F\sigma_3\overline{F}^t$$

and it follows

$$f_z = 2e^u F \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \overline{F}^t, \quad f_z = 2e^u F \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \overline{F}^t.$$

As before define

$$U := \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}, \quad V := \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}.$$

Start for instance by differentiating $f_z$ with respect to $\overline{z}$:

$$\begin{aligned} f_{z\overline{z}} &= u_{\overline{z}} f_z + 2e^u F_{\overline{z}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \overline{F}^t + 2e^u F \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \overline{F}_z^t \\ &= u_{\overline{z}} f_z + 2e^u F V \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \overline{F}^t + 2e^u F \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \overline{U}^t \overline{F}^t \\ &= u_{\overline{z}} f_z + 2e^u F \begin{pmatrix} V_{12} & 0 \\ V_{22} + \overline{U}_{11} & \overline{U}_{21} \end{pmatrix} \overline{F}^t. \end{aligned}$$

In the next step differentiate, $f_{\overline{z}}$ with respect to $z$:

$$\begin{aligned} f_{\overline{z}z} &= u_z f_{\overline{z}} + 2e^u F_z \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \overline{F}^t + 2e^u F \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \overline{F_{\overline{z}}}^t \\ &= u_z f_{\overline{z}} + 2e^u F U \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \overline{F}^t + 2e^u F \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \overline{V}^t \overline{F}^t \\ &= u_z f_{\overline{z}} + 2e^u F U \begin{pmatrix} \overline{V}_{21} & U_{11} + \overline{V}_{22} \\ 0 & U_{21} \end{pmatrix} \overline{F}^t. \end{aligned}$$

Since $f_{z\overline{z}} = f_{\overline{z}z}$ we get

$$\begin{aligned} & u_{\overline{z}} f_z - u_z f_{\overline{z}} = 2e^u F U \begin{pmatrix} \overline{V}_{21} & U_{11} + \overline{V}_{22} \\ 0 & U_{21} \end{pmatrix} \overline{F}^t - 2e^u F \begin{pmatrix} V_{12} & 0 \\ V_{22} + \overline{U}_{11} & \overline{U}_{21} \end{pmatrix} \overline{F}^t \\ \iff \quad & 2e^u F \begin{pmatrix} 0 & -u_z \\ u_{\overline{z}} & 0 \end{pmatrix} \overline{F}^t = F U \begin{pmatrix} \overline{V}_{21} - V_{12} & U_{11} + \overline{V}_{22} \\ -V_{22} - \overline{U}_{11} & U_{21} - \overline{U}_{21} \end{pmatrix} \overline{F}^t. \end{aligned}$$

This implies

$$\overline{V}_{21} = V_{12}, \quad \overline{U}_{21} = U_{21}, \quad U_{11} + \overline{V}_{22} = -u_z, \quad -V_{22} - \overline{U}_{11} = u_{\overline{z}}.$$

We know that $f_{\bar{z}z} = 2e^u f + 2He^{2u}N$ and with $f = F\overline{F}^t, N = F\sigma_3\overline{F}^t$ we get

$$\begin{aligned} f_{\bar{z}z} &= 2e^u f + 2He^{2u}N \\ &= 2e^{2u}F \begin{pmatrix} 1+H & 0 \\ 0 & 1-H \end{pmatrix} \overline{F}^t. \end{aligned}$$

We then get

$$\begin{aligned} & 2e^{2u}F \begin{pmatrix} 1+H & 0 \\ 0 & 1-H \end{pmatrix} \overline{F}^t = u_z f_{\bar{z}} + 2e^u FU \begin{pmatrix} \overline{V}_{21} & U_{11}+\overline{V}_{22} \\ 0 & U_{21} \end{pmatrix} \overline{F}^t \\ \iff \quad & F \begin{pmatrix} e^u(1+H) & -u_z \\ 0 & e^u(1-H) \end{pmatrix} \overline{F}^t = \begin{pmatrix} \overline{V}_{21} & U_{11}+\overline{V}_{22} \\ 0 & U_{21} \end{pmatrix} \overline{F}^t \end{aligned}$$

$$\begin{aligned} & V_{12} = (1+H)e^u, \quad U_{21} = (1-H)e^u \\ & U_{11} + \overline{V}_{22} = -u_z. \end{aligned}$$

Now differentiate $f_z$ with respect to $z$:

$$\begin{aligned} f_{zz} &= u_z f_z + 2e^u F_z \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \overline{F}^t + 2e^u F \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \overline{F_{\bar{z}}}^t \\ &= u_z f_z + 2e^u FU \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \overline{F}^t + 2e^u F \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \overline{V}^t \overline{F}^t \\ &= u_z f_z + 2e^u F \begin{pmatrix} U_{12} & 0 \\ U_{22}+\overline{V}_{11} & \overline{V}_{21} \end{pmatrix} \overline{F}^t. \end{aligned}$$

With $f_{zz} = u_z f_z + QN$ and $N = F\sigma_3\overline{F}^t$:

$$\begin{aligned} & u_z f_z + 2e^u F \begin{pmatrix} U_{12} & 0 \\ U_{22}+\overline{V}_{11} & \overline{V}_{21} \end{pmatrix} \overline{F}^t \\ = \quad & u_z f_z + F \begin{pmatrix} Q & 0 \\ 0 & -Q \end{pmatrix} \end{aligned}$$

we know that

$$U_{12} = \frac{1}{2}e^{-u}Q, \quad V_{21} = -\frac{1}{2}e^{-u}\overline{Q}.$$

$U$ and $V$ are trace free because $detF = 1$. It follows that $U_{11} = -U_{22}$ and $V_{11} = -V_{22}$ and we have recovered $U$ and $V$ to be

$$U = \frac{1}{2}\begin{pmatrix} -u_z & e^{-u}Q \\ 2(1-H)e^u & u_z \end{pmatrix} \quad V = \frac{1}{2}\begin{pmatrix} u_{\overline{z}} & 2(1+H)e^u \\ -e^{-u}\overline{Q} & -u_{\overline{z}} \end{pmatrix}.$$

We can again summarize the collected information.

**Theorem 2.34.** *Let $\sum$ be a simple connected domain in $\mathbb{C}$ with complex coordinate $z$. Choose $\psi \in \mathbb{R}$ and $q \in \mathbb{R} \setminus \{0\}$. Let $u$ and $Q$ solve*

$$4u_{z\overline{z}} + e^{2u} - Q\overline{Q}e^{-2u} = 0, \quad Q_{\overline{z}} = 0$$

*and let $F = F(z, \overline{z}, \lambda)$ be a solution of*

$$F_z = FU, \quad F_{\overline{z}} = FV \tag{2.22}$$

*with*

$$U = \frac{1}{2}\begin{pmatrix} u_z & -\lambda^{-1}e^u \\ \lambda^{-1}Qe^{-u} & -u_z \end{pmatrix} \quad V = \frac{1}{2}\begin{pmatrix} -u_{\overline{z}} & -\lambda\overline{Q}e^{-u} \\ \lambda e^u & -u_{\overline{z}} \end{pmatrix}. \tag{2.23}$$

*Suppose that $det(F) = 1 \forall \lambda, z$. Set $F_0 = F|_{\lambda = e^{q/2}e^{i\psi}}$. Then define*

$$f(z, \overline{z}) = \tilde{F}\overline{\tilde{F}^t}, \quad \tilde{F} := F_0\begin{pmatrix} e^{q/4} & 0 \\ 0 & e^{-q/4} \end{pmatrix}, \quad N = \tilde{F}\sigma_3\overline{\tilde{F}^t}. \tag{2.24}$$

*$f$ is a CMC $H = coth(-q)$ surface in $\mathbb{H}^3$ with normal $N$.*

The proof using the Sym-Bobenko formula and there is no need to integrate.

In Chapter 5 we work with a slightly other formula for the immersion, and therefore want to get rid of the term $\begin{pmatrix} e^{\frac{q}{2}} & 0 \\ 0 & e^{\frac{q}{2}} \end{pmatrix}$ and therefore have to transform the Lax pair $U$

and $V$. We make the following Ansatz with $c \in \mathbb{R}$

$$U = \frac{1}{2}\begin{pmatrix} -u_z & e^{-u}Q \\ \lambda^{-1}ce^u & u_z \end{pmatrix} \quad V = \frac{1}{2}\begin{pmatrix} u_{\overline{z}} & -\lambda ce^u \\ -e^{-u}\overline{Q} & -u_z \end{pmatrix}.$$

Next step is to recover the Gauss-Codazzi equation. We therefore calculate the zero curvature condition with the $U$ and $V$ from the Ansatz.

$$[U,V] = \frac{1}{4}\begin{pmatrix} -e^{-2u}\overline{Q}Q + c^2e^{2u} & -2u_{\overline{z}}e^uQ \\ -2u_ze^{-u}\overline{Q} & -c^2e^{2u} + e^{-u}\overline{Q}Q \end{pmatrix}.$$

With $U_{\overline{z}}$ and $V_z$, in sum we get

$$U_{\overline{z}} - V_z + [U,V] = u_{z\overline{z}} + \frac{1}{4}e^{-2u}Q\overline{Q} - \frac{1}{4}c^2e^{2u} = 0.$$

Comparing this to the original Gauss-Codazzi equation resulting from 2.4.3

$$u_{z\overline{z}} + \frac{1}{4}e^{-2u}Q\overline{Q} - (H^2-1)e^{2u} = 0$$

yields

$$\begin{aligned} c^2 &= 4(H^2-1) \\ \Leftrightarrow c &= 2\sqrt{H^2-1} \end{aligned}$$

Comparing the coefficients of

$$U = \frac{1}{2}\begin{pmatrix} -u_z & e^{-u}Q \\ \lambda^{-1}2\sqrt{H^2-1}e^u & u_z \end{pmatrix}$$

with

$$U = \frac{1}{2}\begin{pmatrix} -u_z & e^{-u}Q \\ 2(1-H)e^u & u_z \end{pmatrix}$$

we get

$$\lambda = \frac{2\sqrt{H^2-1}}{2(1-H)} = \sqrt{\frac{H+1}{H-1}}.$$

We then define

$$\hat{U} = \frac{1}{2}\begin{pmatrix}\sqrt{\lambda}^{-1} & 0\\ 0 & \sqrt{\lambda}\end{pmatrix}\begin{pmatrix}u_z & -\lambda^{-1}e^{u}\\ \lambda^{-1}Qe^{-u} & -u_z\end{pmatrix}\begin{pmatrix}\sqrt{\lambda} & 0\\ 0 & \sqrt{\lambda}^{-1}\end{pmatrix} \tag{2.25}$$

$$= \frac{1}{2}\begin{pmatrix}u_z & -\lambda^{-2}e^{u}\\ Qe^{-u} & -u_z\end{pmatrix} \tag{2.26}$$

$$\hat{V} = \frac{1}{2}\begin{pmatrix}\sqrt{\lambda}^{-1} & 0\\ 0 & \sqrt{\lambda}\end{pmatrix}\begin{pmatrix}-u_{\overline{z}} & -\lambda\overline{Q}e^{-u}\\ \lambda e^{u} & u_{\overline{z}}\end{pmatrix}\begin{pmatrix}\sqrt{\lambda} & 0\\ 0 & \sqrt{\lambda}^{-1}\end{pmatrix} \tag{2.27}$$

$$= \frac{1}{2}\begin{pmatrix}-u_{\overline{z}} & -\overline{Q}e^{-u}\\ \lambda^2 e^{u} & u_{\overline{z}}\end{pmatrix} \tag{2.28}$$

$$\hat{F} = \begin{pmatrix}\sqrt{\lambda}^{-1} & 0\\ 0 & \sqrt{\lambda}\end{pmatrix} F \begin{pmatrix}\sqrt{\lambda} & 0\\ 0 & \sqrt{\lambda}^{-1}\end{pmatrix}. \tag{2.29}$$

We calculate

$$f = \hat{F}\overline{\hat{F}}^{t} \tag{2.30}$$

$$= \hat{F}\begin{pmatrix}u_z & -\lambda^2 e^{u}\\ Qe^{-u} & -u_z\end{pmatrix}\begin{pmatrix}u_z & -\lambda^2 e^{u}\\ Qe^{-u} & -u_z\end{pmatrix}\overline{\hat{F}}^{t} \tag{2.31}$$

$$= F\begin{pmatrix}\sqrt{\lambda}\sqrt{\overline{\lambda}} & 0\\ 0 & \sqrt{\lambda}^{-1}\sqrt{\overline{\lambda}}^{-1}\end{pmatrix}\overline{\hat{F}}^{t}. \tag{2.32}$$

With

$$\lambda_0 = e^{\frac{q}{2}}e^{i\psi}$$

we see that

$$\sqrt{\lambda_0}\sqrt{\overline{\lambda_0}} = e^{\frac{q}{4}}.$$

This yields

$$f = F\begin{pmatrix}\sqrt{e^{\frac{q}{4}}} & 0\\ 0 & e^{\frac{q}{4}}\end{pmatrix}\overline{\hat{F}}^{t}.$$

With

$$F = \tilde{F} = F_0\begin{pmatrix}e^{\frac{q}{4}} & 0\\ 0 & e^{-\frac{q}{4}}\end{pmatrix}$$

we have

$$f = \tilde{F}\overline{\tilde{F}}^t.$$

### 2.4.4 CMC tori in $\mathbb{H}^3$

To close this chapter we show that as mentioned before we can reduce in $\mathbb{H}^3$ the Gauss-Codazzi equation as in $\mathbb{R}^3$ to the sinh-Gordon equation. Fixing the conformal coordinate $z$ by the condition $Q = 2\sqrt{H^2 - 1}e^{\varphi}$, and plugging it in the Gauss-Codazzi equations, we get

$$\begin{aligned} u_{z\bar{z}} + 2(H^2-1)e^u - \frac{1}{2}Q\overline{Q}e^{-u} & \\ &= u_{z\bar{z}} + 2(H^2-1)e^u - \frac{1}{2}4(H^2-1)e^{-u} \\ &= u_{z\bar{z}} + 2(H^2-1)sinh(u) \\ &= 0. \end{aligned}$$

The same arguments as in the as in the previous section apply to the 3-hyperbolic space: In order to obtain a CMC torus we will have to look for double-periodic solutions of the sinh-Gordon equation. We will be able to find analog conditions for the immersion to close with respect to the lattice due to the Sym-Bobenko formula for an immersion in the 3-hyperbolic space.

We do not need an explicit solution of the sinh-Gordon equation in order to study closing conditions leading to a CMC torus in $\mathbb{H}^3$ or other 3-spaces and deformations. Nevertheless, we will do it as there is a theory developed in the 70‘s in the course of nonlinear differential equations integrable in an algebra-geometric sense. We will therefore need some theory of Riemann surface, especially hyperelliptic ones. We will see that with theta-functions one will be able to get exact solutions in terms of Riemann theta-functions in connection with algebraic/spectral curves.

But not only for that purpose we need Riemann surface theory and, once again, hyperelliptic curves in particular. In Chapter 5 we will study the closing conditions and deformations and will come back again to Riemann surface theory. As we are only introducing basic Riemann theory, we will skip most of the proofs which are stated in standard books dealing with Riemann surface theory (see for example [19] or [23]).

# 3 Riemann surface theory and theta-functions

Before proceeding with the explanation of how to obtain solutions of the sinh-Gordon equation, we include for reasons of completeness a chapter about classical Riemann surface theory and theta-functions. For the reasons stated above we will only hthe theory in brief. For more extensive theory see for example [19], [23] and for algebraic curves in particular we recommend [45]. We will put an emphasis on hyperelliptic curves as preparation for the next Chapters.

## 3.1 Riemann surface theory

### 3.1.1 Basics

**Definition 3.1.** (Riemann surface) A Riemann surface $R$ is a connected one-dimensional analytic manifold, that is a two-real dimensional connected manifold with a complex structure on it.

**Remark 3.2.** The complex structure implies that for each point $p \in R$ there is a homeomorphism $\varphi : U \longrightarrow V$ of some neighborhood $p \in U$ onto an open subset $V \subset \mathbb{C}$ with holomorphic transition function $\varphi \circ \bar{\varphi}$. $\varphi$ will be referred to as local parameter.

A nontrivial example of a Riemann surface is given by an algebraic curve:

**Definition 3.3.** (Algebraic curve) An algebraic curve $C$ is a subset in $\mathbb{C}^2$

$$C = \{(\mu, \lambda) \in \mathbb{C}^2 | P(\mu, \lambda) = 0, P(\mu, \lambda) = \sum_{i=1}^{N} \sum_{j=1}^{M} p_{ij} \mu^i \lambda^j, p_{ij} \neq 0 \forall i, j\}. \tag{3.1}$$

**Definition 3.4.** (non-singular) A curve $C$ is called non-singular if

$$grad_{\mathbb{C}} P_{|P=0} = \left(\frac{\partial P}{\partial \mu}, \frac{\partial P}{\partial \lambda}\right)_{|P=0} \neq 0.$$

A complex-analytic structure on the algebraic curve is introduced as follows: The variable $\lambda$ is taken as a local parameter in the neighborhood of the points where $\frac{\partial P}{\partial \mu} \neq 0$ and the variable $\mu$ is a local parameter in the neighborhood of the points where $\frac{\partial P}{\partial \lambda} \neq 0$.

It turns out that all compact Riemann surfaces $\hat{C}$ can be described as compactifications of algebraic curves $C$. The mapping defines a holomorphic covering $\hat{C} \longrightarrow \bar{\mathbb{C}}$. If $N$ is the degree of the polynomial $P(\mu, \lambda)$ in $\mu$

$$P(\mu, \lambda) = \mu^N p_N(\lambda) + \mu^{N-1} p_{N-1}(\lambda) + ... + p_0(\lambda)$$

where all $p_i(\lambda)$ are polynomials, then $\lambda : \hat{C} \longrightarrow \bar{\mathbb{C}}$ is an $N$-sheeted covering. We call $\hat{C}$ the Riemann surface of the curve $C$.

In the case of the hyperelliptic curve

$$\mu^2 = \prod_{j=1}^{N} (\lambda - \lambda_j) \quad N \in \mathbb{N}, \lambda_j \in \mathbb{C}, \lambda_j \neq \lambda_k \quad j, k = 1, ..., N, \tag{3.2}$$

we assume them to be compactified by joining points at infinity. For a hyperellitptic curve there are thus two points at infinity if $N = 2g + 1$, and one such point when $N = 2g + 2$.

In the first case we have the points $\infty^+$ and $\infty^-$. These points are distinguished by the condition

$$P \equiv (\mu, \lambda) \to \infty^\pm \Longleftrightarrow \lambda \to \infty, \mu \sim \pm \lambda^g$$

with a local parameter in the neighborhood of both points given by the homeomorphism $(\lambda, \mu) \to \frac{1}{\lambda}$.

In the $N = 2g + 2$-case, the point $\infty$ is distinguished by the condition

$$P \equiv (\mu, \lambda) \to \infty \Longleftrightarrow \lambda \to \infty, \mu \sim \lambda^{\frac{g+1}{2}}$$

and the local parameter in its neighborhood is $\sqrt{\lambda}$.

**Remark 3.5.** The curve is non-singular if for all points $\lambda_j \neq \lambda_i$ holds.

The local parameterization in the neighborhoods of the points $(\mu_0, \lambda_0)$ with $\lambda_0 \neq \lambda_j \forall j$ is defined by the homeomorphism $(\mu, \lambda) \longrightarrow \lambda$ and in the neighborhood of each $(0, \lambda_j)$ by $(\mu, \lambda) \longrightarrow \sqrt{\lambda - \lambda_j}$.

A function which is defined on the hyperelliptic curve and is holomorphic in a neighborhood of $(\mu_0, \lambda_0)$ with $\lambda_0 = \lambda_j$, can be represented by a convergent Taylor series in integral powers of the variable $\sqrt{\lambda - \lambda_j}$.

**Theorem 3.6.** *Any compact Riemann surface is homeomorphic to a sphere with g handles.*

**Definition 3.7.** (Genus) $g \in \mathbb{N}$ is called genus of the Riemann surface.

**Remark 3.8.** For the hyperlliptic curves 4.10 with $N = 2g+1$ or $N = 2g+2$, the genus is equal to $g$.

**Definition 3.9.** (Holomorphic, meromorphic) A mapping

$$f : M \longrightarrow N$$

between Riemann surfaces $M.N$ is called holomorphic if for every local parameter $(U, z)$ on M and every local parameter $(V, w)$ on N with $U \cap f^{-1}(V) \neq 0$ the mapping

$$w \circ f \circ z^{-1} : z(U \cap f^{-1}(V))) \longrightarrow w(V)$$

is holomorphic.

A holomorphic mapping into $\bar{\mathbb{C}} = \mathbb{C} \cup \infty$ is called a meromorphic function.

**Remark 3.10.** Non-constant holomorphic mappings of Riemann surfaces are called holomorphic coverings. The general properties of holomorphic coverings imply that the meromorphic function $f$ takes every value $c \in \bar{\mathbb{C}}$ the same finite number of times (counting multiplicities).

**Lemma 3.11.** The following three assertions are equivalent:

i. The Riemann surface $R$ is given by 4.10.

ii. There is a meromorphic function on $R$ that defines a two-sheeted covering of $\bar{\mathbb{C}}$.

iii. There is a function on $R$ that has its unique singularity (a second-order pole) at some point $P_0$.

**Definition 3.12.** (Branch point) Let $f : M \longrightarrow N$ be a holomorphic covering between to Riemman surfaces. A point $P \in M$ is called branch point of $f$ if it has no neighborhood $P \in V$ such that $f|_V$ is injective. A covering without branch points is called unramified.

**Remark 3.13.** One can characterize the branch points of a holomorphic covering $f$ as the points with multiplicity $k > 1$. The number $b_f(P) = k - 1$ is called branch number of $f$ at $P \in M$.

There exists $m \in \mathbb{N}$ such that every $Q \in N$ is assumed by $f$ exactly $m$ times, counting multiplicities. That is

$$\sum_{P \in f^{-1}(Q)} (b_f(P) + 1) = m, \quad b_f = k - 1, \tag{3.3}$$

so $k$ is the multiplicity of the point. The number $m$ is called degree of $f$ and the covering $f$ is called $m$-sheeted covering.

**Proposition 3.14.** A non-constant meromorphic function on a compact Riemann surface assumes every one of its value in $\bar{\mathbb{C}}$ $m$ times where $m$ is the number of its poles (counting multiplicities).

We return to the case of a hyperelliptic curve

$$\mu^2 = \prod_{j=1}^{N}(\lambda - \lambda_j) \quad N \in \mathbb{N}, \lambda_j \in \mathbb{C}, \lambda_j \neq \lambda_k \quad j,k = 1, ..., N.$$

The compactifications $\hat{C}$ of the special case of a hyperelliptic curve $C$ is a two-sheeted covering of the extended complex plane $\lambda : \hat{C} \longrightarrow \bar{\mathbb{C}}$. The branch points of this covering are

$$\begin{aligned} (0, \lambda_i), i &= 1, ..., N \quad \text{and} \quad \infty \quad \text{for} \quad N = 2g+1 \\ (0, \lambda_i), i &= 1, ..., N \quad \text{for} \quad N = 2g+2 \end{aligned}$$

with branch numbers $b_\lambda = 1$ at these points.

**Remark 3.15.** The hyperelliptic curves obey a holomorphic involution

$$h : (\mu, \lambda) \longrightarrow (-\mu, \lambda),$$

which interchanges the sheets of the covering $\lambda : \hat{C} \longrightarrow \bar{\mathbb{C}}$. The branch points of the covering are fixed points of $h$. The factor $\hat{C}/h$ is therefore a Riemann sphere. The covering $\hat{C} \longrightarrow \hat{C}/h = \bar{\mathbb{C}}$ is ramified at the points $\lambda = \pm\lambda_n$.

**Definition 3.16.** (Canonical basis of cycles) A homology basis of cycles

$$a_1, b_1, ..., a_g, b_g$$

of a compact Riemann surface of genus $g$ with the following intersection numbers

$$a_i \circ b_j = \delta_{ij} \quad a_i \circ a_j = b_i \circ b_j = 0, \quad i, j = 1, ..., g$$

is called canonical basis of cycles.

**Remark 3.17.** The canonical intersection means that one can take different loops such that there is no intersection between the loops $a_i$ and $a_j$ or $b_j$ with $i \neq j$, while $a_i$ and $b_i$ intersect at only one point $p$. At $p$, the tangent vectors $a_i$‘ and $b_i$‘ form a positively oriented basis of the tangent space at $p$.

### 3.1.2 Abelian differentials

Our next review of classical Riemann surface theory is concerned with Abelian differentials. The main goal is to construct functions on compact Riemann surfaces with prescribed analytical properties (for example meromorphic functions with prescribed singularities). Therefore we introduce Abelain differentials which are easier to handle as those functions and are a basic concept to investigate and construct such functions.

**Definition 3.18.** (0-form, 1-form and 2-form) Let $(z, \overline{z})$ be a local coordinate on the compact Riemann surface. If there are functions

$$f(z, \overline{z}), p(z, \overline{z}), q(z, \overline{z}), s(z, \overline{z})$$

with

$$\begin{aligned} f &= f(z, \overline{z}) \\ \omega &= p(z, \overline{z})dz + q(z, \overline{z})d\overline{z} \\ S &= s(z, \overline{z})dz \wedge d\overline{z}, \end{aligned}$$

being invariant under coordinate changes, then the function $f$ is called a 0-form, $\omega$ a 1-form and $S$ a 2-form on the Riemann surface (where $dz \wedge d\overline{z}$ denotes the exterior product).

**Remark 3.19.** One can integrate as follows:

i. 0-forms over a finite set $\{P_\alpha\}_{\{\alpha\}}$ of points $P_\alpha \in R$:

$$\sum_\alpha f(p_\alpha).$$

2. 1-forms over paths $\gamma$:

$$\int_\gamma \omega.$$

3. 2-forms over finite unions of domains $D$:

$$\int_D S.$$

**Remark 3.20.** Due to the invariance under coordinate changes, the differential operator $d$ which transforms a $k$-form in a $(k+1)$-form is defined by

$$\begin{aligned} df &= f_z dz + f_{\overline{z}} d\overline{z} \\ d\omega &= (q_z - p_{\overline{z}}) dz \wedge d\overline{z} \\ dS &= 0. \end{aligned}$$

**Definition 3.21.** (Closed and exact differential) A differential $df$ is called exact. A differential $\omega$ with $d\omega = 0$ is called closed.

**Remark 3.22.** One can easily check that every exact form is closed.

**Definition 3.23.** (Period of a differential) Let $\gamma_1, ..., \gamma_n$ be a canonical basis of the Riemann surface. Periods of $\omega$ are defined by

$$\Lambda_i = \int_{\gamma_i} \omega.$$

**Remark 3.24.** For any closed curve $\gamma$, $\int_\gamma \omega = \sum n_i \Lambda_i$ holds.

**Definition 3.25.** (Abelian differential of the first kind) A differential $\omega$ on a Riemann surface is called Abelian differential of the first kind if in any local chart, it is represented as

$$\omega = pdz + qd\overline{z} = h(z)dz,$$

where $h(z)$ is holomorphic.

**Remark 3.26.** Abelian differentials of the first kind form a complex vector space $H(R, \mathbb{C})$. Its dimension is equal to the genus of the Riemann surface.

**Theorem 3.27.** *Let $R$ be a Riemann surface described by a hyperelliptic curve 4.10. The differentials*

$$\omega_j = \frac{\lambda^{j-1} d\lambda}{\mu}$$

*form a basis of holomorphic differentials.*

**Remark 3.28.** If all $a$-periods of the holomorphic differential $\omega$ are zero, then $\omega \equiv 0$. Then the matrix of $a$-periods

$$A_{ij} = \int_{a_i} \omega$$

of any basis $\omega_j, j = 1, ...g$ of $H(R, \mathbb{C})$ is invertible. We can therefore normalize the basis as follows: Let $a_j, b_j, j = 1, ..., g$ be a canonical basis of $H(R, \mathbb{C})$. Then

$$\int_{a_j} \omega_k = 2\pi i \delta_{jk}, \quad k = 1, ..., g.$$

**Definition 3.29.** (Abelian (meromorphic) differential , Abelian integral) A differential $\Omega$ is called Abelian or meromorphic differential if in any local chart it is of the form

$$\Omega = g(z)dz.$$

The integral

$$\int_{P_0}^{P} \Omega$$

is called Abelian integral.

**Definition 3.30.** (Abelian differential of the second kind) An Abelian differential with singularities is called Abelian differential of the second kind if the residues are equal to zero at all singular points.

**Remark 3.31.** Abelian differentials are usually divided into three kinds: holomorphic differentials (first kind), meromorphic differentials with residues equal to zero at all singular points (second kind), and meromorphic differentials of the general form (third kind).

**Theorem 3.32.** *Let $R$ be a Riemann surface of genus $g$. Then*

*i. The dimension of the space of differential holomorphic on $R$ is equal to $g$.*

*ii. For any finite set of points $P_j \in R$ there exists a holomorphic Abelian differential homomorphic on $R \setminus \{P_j\}$ with poles at the points $P_j$.*

**Remark 3.33.** Abelian differentials of the second kind have only one singularity of the form

$$\Omega_R^{(N)} = \left(\frac{1}{z^{N+1}} + O(1)\right) dz.$$

This implies for the corresponding Abelian integral

$$\int^P \Omega_R^{(N)} = -\frac{1}{Nz^N} + O(1) \quad P \longrightarrow R.$$

**Remark 3.34.** Abelian differentials of the second kind can be normalized as follows:

$$\int_{a_j} \Omega_R^{(N)} = 0.$$

They are called normalized Abelian differentials of the second kind.

Before we return to the case of a hyperelliptic curve 4.10, we describe the Abelian differentials of the second kind in more detail:

Let $R = (\mu_R, \lambda_R),\ Q = (\mu_Q, \lambda_Q)$ denote the coordinates of two points with $\lambda_R \neq \infty$, $\lambda_Q \neq \infty$. If $R$ is not a branch point to get a proper singularity, we multiply $\frac{d\lambda}{\mu}$ by $\frac{1}{\lambda-\lambda_R}^n$ and cancel the singularity at the point $\pi R = (-\mu_R, \lambda_R)$ by multiplication by a linear function $\mu$.

The differentials of the second kind

$$\Omega_R^{(N)} = \frac{\mu + \mu_R^{[N]}}{(\lambda - \lambda_R)} \frac{d\lambda}{2\mu} \quad \mu_R \neq 0$$

where

$$\mu_R^{[N]} = \mu + \frac{\partial \mu}{\partial \lambda} \mid_R (\lambda - \lambda_R) + ... + \frac{1}{N!} \frac{\partial^N \mu}{\partial^N \lambda^N} \mid_R (\lambda - \lambda_R)^N$$

have singularities of the form

$$((z^{-N-1}) + o(^{-N-1}))dz$$

with $z = \lambda - \lambda_R$.

If $R$ is a branch point $\mu_R = 0$, the singularity is the same. The form of the differentials is then

$$\begin{aligned}\Omega_R^{(N)} &= \frac{d\lambda}{2(\lambda-\lambda_R)^n\mu}\sqrt{\prod_{i=1,i\neq R}^{N}(\lambda_R-\lambda_i)} \quad \text{for} \quad N=2n-1\\ \Omega_R^{(N)} &= \frac{d\lambda}{2(\lambda-\lambda_R)^n} \quad \text{for} \quad N=2n-2.\end{aligned}$$

**Definition 3.35.** (Period matrix of a Riemann surface) Let $a_j, b_j, j=1,...,g$ be a canonical homology basis of the Riemann surface and $\omega_k, k=1,...,g$ the basis of $H^1(R,\mathbb{C})$. The matrix

$$B_{ij} = \int_{b_i}\omega_j$$

is called period matrix of the Riemann surface.

**Remark 3.36.** In the case of normalized Abelian differentials of the second kind the periods are equal to

$$\int_{b_j}\Omega_R^{(N)} = \frac{1}{N}\alpha_{N-1,j}.$$

**Definition 3.37.** (Jacobian variety) The complex torus

$$\begin{aligned}Jac(R) &= \mathbb{C}_g/\Lambda \quad with\\ \Lambda &= \{2\pi iN + BM, N, M\in\mathbb{R}^g\}\end{aligned}$$

is called Jacobian variety, where $\Lambda$ is the lattice generated by the periods of $R$.

**Definition 3.38.** (Abel map) Let $\omega = (\omega_1,...,\omega_g)$ be the canonical basis of holomorphic differentials and $P_0\in R$. The map

$$A: R \longrightarrow Jac(R), \tag{3.4}$$
$$A(P) = \int_{P_0}^{P}\omega \tag{3.5}$$

is called the Abel map.

**Proposition 3.39.** Given a compact Riemann surface with a canonical basis of cycles

$a_1, b_1, ..., a_g, b_g$, there exist unique $2g$ harmonic differentials $h_1, ..., h_{2g}$ with

$$\int_{a_j} h_i = \int_{b_j} h_{g+i} = \delta_{ij}, \quad \int_{a_{g+i}} h_{g+1} = \int_{b_j} h_i = 0, \quad i = 1, ..., g.$$

### 3.1.3 Divisor and Abels theorem

When analyzing functions and differentials on Riemann surfaces, one characterizes them in terms of their zeros and poles. It is convenient to consider formal sums of points on $R$ which leads to theory of divisors.

**Definition 3.40.** (Divisor and degree) The formal linear combination

$$D = \sum_{j=1}^{N} n_j P_j, \quad n_j \in \mathbb{R}, P_j \in R$$

is called divisor on the Riemann surface R.

The sum

$$deg(D) = \sum_{j=1}^{N} n_j$$

is called the degree of $D$.

**Definition 3.41.** (Divisor of a meromorphic function) Let $f$ be a meromorphic function on the Riemann surface $R$, $P_i, i = 1, ..., M$ its zeros with multiplicities $p_i$, and $Q_j, j = 1, .., N$ the poles with multiplicities $q_j$. The divisor of $f$ is defined as

$$D = (f) = p_1 P_1 + ... + p_M P_M - q_1 Q_1 - ... - q_M Q_M.$$

A divisor is called principle if there exists a function $f$ such that $(f) = D$.

**Remark 3.42.** If a divisor is principle, then the function $f$ has exactly the zeros and poles prescribed by the divisor. If $R$ is a compact Riemann surface then a divisor is principle if and only if $deg(D) = 0$.

The divisor of an Abelian differential is well defined.

**Definition 3.43.** (Divisor of an Abelian differential) The divisor of an Abelian differential $\Omega$ is

$$(\Omega) = \sum_{P \in R} N(P)P$$

where $N(P)$ is the order of the point $P$ of $\Omega$.

**Definition 3.44.** (Abel map for divisors) The Abel map for a divisor is defined naturally by

$$A(D) = \sum_{j=1}^{N} n_j \int_{P_0}^{P} \omega.$$

**Remark 3.45.** If the divisor is principal (i.e. $deg(D) = 0$), then $A(D)$ is independent of $P_0$

$$D = P_1 + ... + P_N - Q_1 - ... - Q_M$$
$$A(D) = \sum_{j=1}^{N} \int_{Q_i}^{P_i} \omega.$$

**Theorem 3.46.** *(Abel theorem) The divisor $D$ is principal if and only if*

1. $deg(D) = 0$
2. $A(D) \equiv 0.$

### 3.1.4 Riemann Roch theorem

We now investigate the problem of describing the vector space of meromorphic functions with prescribed poles. Consider the vector space

$$L(D) = \{f \ meromorphic \ function \ on \ R | (f) \geq D \ or \ f \equiv 0\}$$

with dimension

$$l(D) = dim L(D).$$

Similarly, we define the corresponding vector space of differentials

$$H(D) = \{\Omega \ Abelian \ differential \ on \ R | (\Omega \geq 0 \ or \ \Omega \equiv 0)\}$$

with dimension

$$h(D) = dimH(D).$$

**Theorem 3.47.** *(Riemann Roch) Let $R$ be a compact Riemann surface of genus $g$ and $D$ divisor on $R$. Then*

$$l(-D) = deg(D) - g + 1 + h(D) \tag{3.6}$$

**Remark 3.48.** The Riemann-Roch theorem gives exactly the number of linearly independent meromorphic functions with prescribed zeros and poles on a Riemann surface.

We will once more investigate in detail the case of a hyperelliptic Riemann surface and we will see that a hyperelliptic Riemann surface is the same as a hyperelliptic curve. We first define the hyperelliptic Riemann surface.

**Definition 3.49.** (Hyperellitptic Riemann surface) A compact Riemann surface $R$ of genus $g \geq 2$ is called hyperelliptic if there exists a positive divisor (i.e. $\sum_j n_j \geq 0$) on $R$ with $deg(D) = 2$.

**Remark 3.50.** In other words, $R$ is hyperellitptic if and only if there exists a non-constant meromorphic function $\Lambda$ on $R$ with precisely two poles counting multiplicities. If this is the case, $R$ defines a two-sheeted covering of the complex sphere

$$\Lambda : R \longrightarrow \bar{\mathbb{C}}.$$

One can prove that this covering is unique up to fractional linear transformations. All branch points have branch number 1 and they are all Weierstrass points.

**Proposition 3.51.** The definition of a hyperelliptic Riemann surface is equivalent to the definition of a hyperelliptic curve (4.10).

**Proposition 3.52.** Any Riemann surface of genus 2 is hyperelliptic.

## 3.2 Theta-functions

We now introduce theta-functions and some properties of them, as we will need them in the following chapter.

**Definition 3.53.** (Riemann matrix) A symmetric $g \times g$-matrix $B = B_{jk}$ with negative definite real part $Re(B) = (Re(B_{jk}))$ is called Riemann Matrix.

**Definition 3.54.** (Theta function) Let $B$ be a Riemann Matrix. The Riemann theta-function is then defined by its multi-dimensional Fourier series

$$\theta_{z,N} = \sum_{m \in Z^g} exp\{\frac{1}{2}\langle BN, N\rangle + \langle N, z\rangle\}$$

with $z = (z_1, ..., z_g) \in \mathbb{C}^g$ a complex vector, the diamond brackets denoting the Euclidean scalar product, and

$$\begin{aligned} \langle N, z\rangle &= \sum_{i=1}^{g} N_i z_i \\ \langle BN, N\rangle &= \sum_{j,i=1}^{g} B_{ij} N_i N_j, \end{aligned}$$

with $N = (N_1, ..., N_g)$ integer vectors.

**Remark 3.55.** The theta-function shifted by $2\pi i$ obeys the following transformation law

$$\theta(z + 2\phi i N) = \theta(z), .$$

We immediately get the following transformation law too:

$$\theta(z + 2\pi i N + BM) = exp\left(-\frac{1}{2}\langle BM, M\rangle - \langle M, z\rangle\right)\theta(z). \tag{3.7}$$

This follows from the simple calculation

$$\begin{aligned} &\theta(z + 2\pi i N + BM) \\ &\quad = \theta(z + BM) \\ &\quad = \sum_{m \in \mathbb{Z}^g} exp(\frac{1}{2}B\langle m + M, m + M\rangle \\ &\quad + \langle z, m + M\rangle - \langle m, M\rangle - \frac{1}{2}\langle BM, M\rangle) \\ &\quad = \sum_{m \in \mathbb{Z}^g} exp\left(-\frac{1}{2}\langle BM, M\rangle - \langle z, M\rangle\right)\theta(z). \end{aligned}$$

**Lemma 3.56.** The $\theta$-function is analytic.

*Proof.* The summation in 3.54 is taken over the lattice of integer vectors $N = (N_1, ..., N_g)$. The general term of this series depends only on the symmetric part of the matrix $B$. We can estimate

$$Re(\{BN, N\}) \leq -b\{N, N\}, b > 0,$$

where $-b$ is the largest eigenvalue of the matrix $ReB$. Hence the theta-function is absolutely convergent, uniformly on compact sets, and $\theta$ is analytic. □

We will now construct a function $F$ from the theta-function which will be useful later on. We can then make a statement about the roots of this constructed function. It is a Lemma which is widely used to solve the so called Jacobian inversion problem.

**Lemma 3.57.** Let $e = (e_1, ..., e_g) \in \mathbb{C}^g$ be a fixed arbitrary vector. $D = P_1 + ... + P_g$ a non-special divisor. Then the function

$$F(P) = \theta(A(P) - e)$$

is single-valued and analytic on the cut surface of $R$ and has exactly $g$ zeros at $P_1, ..., P_g$.

The proof of this theorem can be found for example in [45].

**Remark 3.58.** Under analytic continuation along $a$-cycles and $b$-cycles on the Riemann surface, the map $F(P)$ is transformed

$$\begin{aligned} M_{a_k}F(P) &= F(P) \\ M_{b_k}F(P) &= exp\left(-\frac{1}{2}B_{kk} - \int_{P_0}^{P} \omega_k + e_k\right) F(P), \end{aligned}$$

where $M$ denotes the monodromy.

**Remark 3.59.** One can show that either $F(P)$ vanishes identically on $R$ or it has exactly $g$ zeros (counting multiplicities).

We have so far introduced the basic theory we need to proceed. In the next chapter we will put this together with the theory of integrable systems and spectral theory of differential commuting operators in order to solve the sinh-Gordon equation. We have now all ingredients to construct the solution of the sinh-Gordon equation. We begin reviewing the theory of integrable systems by means of the inverse scattering method, a technique used to recover solutions of nonlinear equations by the spectral data of operators.

# 4 The sinh-Gordon equation

As was shown in the previous chapter, the solution of the integration of the Gauss-Weingarten equations and the compatibility condition, the zero curvature condition, is related to the sinh-Gordon equation. In generalizing the connection of nonlinear differential equations and a system of linear differential operators with their zero curvature condition, a powerful machinery involving many areas of mathematics began to emerge in the 70´s to solve such nonlinear differential equations.

There is only little literature giving simple access to this subject. Most published work are articles in scientific journals, short and difficult to understand, or even only available in Russian. What makes it difficult to get an access to this field is not only that it is connected with many particularly mathematical areas, but also that there is no really generalized method for approaching different nonlinear differential equations. One has thus to decide from case to case which approach to choose. That is why we began with an introduction to the basic concepts in the previous chapter, and is the reason why this chapter tries to get by with as less theory involved in integrable systems as possible and to derive a more intuitive approach. However, we will not be able to completely avoid some theory on spectral theory, differential commuting operators and integrable systems.

We omit an extended literature review, as there exists a large body of interdependent and dependent literature regarding the field of integrable systems, and it would be beyound the scope of this thesis to x-ray the different approaches from different areas of mathematics and physics engaged in this field. Note that most of the literature which was responsible for new approaches to nonlinear differential equations was published between 1970 and 1995. We will summerize the main developments and proceedings limiting our review to a selected number of key publications. The interested reader is referred to [17], [38], [36], [39], [14], [13],[6], [5], [2] and the most elaborate work [3]. For a brief and general overview with many examples we propose [16].

The solution of nonlinear equations, like the sinh-Gordon equation, is connected to Riemann surfaces (algebraic curves) via spectral theory of linear operators used in the inverse scattering method. This was for example done by Dubrovin Matveev Novikov in [17]. Krichever extended this approach without spectral theory [36], [37] and [41] for example. The connection between solutions of nonlinear equations of a certain type and Riemann surfaces is commonly formulated in an algebro-geometric language.

The next sections illustrate that we can link the solution $u$ of the sinh-Gordon equation to operators fulfilling the zero curvature condition. By this connection the solution $u$ will take the role as potential of the operator and will be connected to some eigenvalue problem of the operator.

Until the end of 1973 there were practically no examples where the spectrum and the eigenfunctions of such operators could be explicitly computed in terms of some special functions. There were also no effective methods for finding the coefficients of operators on the basis of spectral data. In 1974 the situation changed with the class of finite zone periodic and quasiperiodic potentials of the Schrödinger operator. It was then when different studies showed the complex connection between the spectral theory of operators with periodic coefficients, algebraic geometry, the theory of finite-dimensional completely integrable systems and the theory of nonlinear equations (see [38] for an extended review).

Almost all nonlinear equations integrable by the method of inverse scattering are associated with the spectral theory of matrix linear differential operators, which frequently are not even self-adjoint. The spectral properties of (non-self-adjoint) operators with periodic coefficients is connected to a Riemann surface: The corresponding Riemann surfaces of the spectrum of the associated matrix linear operators turn out to be plane, nonsingular real algebraic curves. An important improvement was the discovery of the possibility of applying the spectral theory of matrix operators to the problem of the classification of plane real curves.

The crucial role is played by the function which is the simulations solution of two linear operators and has on the Riemann surface essential singularities at prescribed points. The form of the singularities depends on the order of the operator. This function is then constructed by using theory of Abelian integrals. In our case, we will see that this function with prescribed properties turns out to be the Baker-Akhiezer function, which is then used to construct solutions of the sinh-Gordon equation. With this method, one obtains explicit formulas in terms of theta-functions for the sinh-Gordon equation.

In general, in the inverse scattering transformation there are four key steps in the solution method: First one sets up an appropriate linear scattering (eigenvalue) problem in the space variable where the solution of the nonlinear evolution equations plays the role of the potential. As a next step one chooses the time dependence of the eigenfunctions in such a way that the eigenvalues remain time invariant as the potential evolves according to the equation. The third step consists in solving the direct scattering problem at initial time and determining the time dependence of the scattering data. Then solve the inverse scattering problem at later times by knowing the discrete eigenvalues corresponding to the bound states and the time dependence of the other scattering data reconstruct the potential. In the final step, the solution can be written in terms of linear integral equations from which one can compute the solution to the evolution equation all the time.

As we have seen, studying CMC tori in $\mathbb{R}^3$ or $\mathbb{H}^3$ is related to double-periodic solutions of the sinh-Gordon equation. We will establish the connection between two commuting differential operators (our Lax pair) and the sinh-Gordon equation. It can be shown that, starting from this, we arrive at one stationary $t$-evolution and two corresponding stationary equations. Due to the special properties associated, a special eigenvalue problem which in turn defines the so called spectral curve, a Riemann surface, arises. We will then introduce the Baker-Akhiezer function and show that it exactly solves the special eigenvalue problem. The Baker-Akhiezer function is determined by a specific set of properties and one can show that under those it is uniquely determined on a Riemann surface. Baker-Akhiezer functions can be constructed in terms of theta-functions which we introduced earlier. With the help of the Baker-Akhiezer function we are then in the situation to solve the sinh-Gordon equation and get a formula in terms of the theta-function. Now one is able to retrieve exact formulas for the frame and the immersion. This was done in [6] and [5] for $\mathbb{R}^3$, $\mathbb{H}^3$, and $S^3$.

## 4.1 Lax pair and finite gap solutions

The sinh-Gordon equation has an appealing: In contrast to many other nonlinear differential equations, all real smooth-double periodic solutions are automatically finite gap here [49]. This is a consequence of the observation that all isospectral flows from the corresponding hierarchy are zero eigenfunctions of the linearized problem. Two linear operators are called isospectral if they have the same spectrum. Roughly speaking, they are supposed to have the same sets of eigenvalues when those are counted with multiplicity. The linearized system may have only finite-dimensional space of double periodic zero eigenfunctions. This means that the hierarchy contains only finitely many linearly independent flows at this point. It follows that the spectral curve has finite genus.

**Proposition 4.1.** Equation 2.16 is the compatibility condition for

$$U_{\overline{z}} - V_z + [U, V] = 0 \tag{4.1}$$

for a system of two linear differential equations

$$\Psi_z = U\Psi \quad , \qquad \Psi_{\overline{z}} = V\Psi, \tag{4.2}$$

$$\tag{4.3}$$

$$U = \frac{1}{2}\begin{pmatrix} -u_z & -i\nu \\ -i\nu & u_z \end{pmatrix} \quad , \qquad V = \frac{1}{2\nu i}\begin{pmatrix} 0 & e^{-u} \\ e^{u} & 0 \end{pmatrix} \tag{4.4}$$

with an auxiliary parameter $\nu$.

*Proof.* Computing $U_{\overline{z}} - V_z + [U, V] = 0$ shows that

$$\begin{aligned} U_{\overline{z}} - V_z + [U, V] & \\ &= \begin{pmatrix} -\frac{1}{2}u_{z\overline{z}} + \frac{1}{4}(e^{-u} + e^{u}) & 0 \\ 0 & \frac{1}{2} + \frac{1}{4}(e^{u} - e^{-u}) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

In order to satisfy the last equation, the following must hold:

$$\begin{aligned} & -\frac{1}{2}u_{z\bar{z}} + \frac{1}{4}(e^{-u} + e^{u}) = \frac{1}{2}u_{z\bar{z}} + \frac{1}{4}(e^{u} - e^{-u}) = 0 \\ \Longleftrightarrow \quad & u_{z\bar{z}} + sinh(u) = 0. \end{aligned}$$

□

**Remark 4.2.** $U$ and $V$ are the Lax pair. The modern developments of integrable systems rely on the notion of Lax pairs, see [42] for their origin. In general, there does not exist an algorithm how to construct them. However, Zakharov and Shabat provide a general method to construct the Lax pair depending on a spectral parameter and fulfilling a special form of the Lax equation (see for example [1]).

**Remark 4.3.** The pair $U, V$ satisfies the following reductions:

$$\begin{aligned} \Psi(\nu) \quad &\Rightarrow \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi(\nu) \\ \Psi(\nu) \quad &\Rightarrow \quad \begin{pmatrix} 0 & e^{-\frac{u}{2}} \\ e^{\frac{u}{2}} & 0 \end{pmatrix} \overline{\Psi(\overline{\nu^{-1}})}, \end{aligned}$$

which means that both sides are solutions of the system 4.2.

## 4.2 Higher commuting flows

Equation 2.16 is integrable and has infinite conservation laws which determine commuting "higher" flows. For an extended review on how to introduce higher flows see [43]. We only give a short outline closely following [5].

Let $z = z_1$, where we then consider an infinite series of new variables $z_2, ..., z_n, ...$.Then $\Psi$ satisfies an infinite series of differential equations

$$\Psi_{z_n} U_n \Psi.$$

$U_n$ is a matrix polynomial of degree $2n - 1$. The coefficients of $U_n$ are derivatives of $u_z, u_{zz}, ..., u_z^n$ of polynomial nature. The exact form of $U_n$ is not of interest here. The second reduction implies

$$\Psi_{z_n} = V_n \Psi.$$

$V_n$ is a matrix polynomial of degree $2n-1$. We use the normalization

$$V_n(\nu = \infty) = 0,$$

and putting $\nu = 0$ in

$$U_n(\nu) = R_{z_n} R^{-1} + R\overline{V_n(\overline{\nu^{-1}})} R^{-1}$$

with

$$R = \begin{pmatrix} & e^{-\frac{u}{2}} \\ e^{\frac{u}{2}} & 0 \end{pmatrix},$$

we obtain the higher sinh-Gordon equation

$$u_{z_n} = P_n(u_z, ..., u_z^n)$$

with $P_n$ coefficients in $U_n$.

If we now take real variables $x_n, y_n$ with $(z_n = x_n + iy_n)$, we obtain a nonsingular solution $u(x_1, y_1, ..., x_n, y_n)$ of all higher sinh-Gordon equations. The higher flows commute and therefore the set of solutions with respect to some flow $u_{t_i} = 0$, where $t_i$ denotes higher times, is invariant with respect to other flows including $x, y$ flows. Those solutions are called finite gap solutions.

**Theorem 4.4.** *All nonsingular double-periodic solutions of 2.16 are stationary solutions of a higher elliptic sinh-Gordon flow.*

*Proof.* Looking at the partial derivatives

$$v_i = \frac{\partial}{\partial t_i} u(t_1, ...),$$

we have

$$(\partial_z \partial_{\overline{z}} + coshu) v_i = 0.$$

The $v_i$ are linearly dependent as the set of eigenfunctions of

$$(\partial_z \partial_{\overline{z}} + coshu)v_i = 0$$

on the torus is discrete. Higher times t exists with respect to which $u(t_1, ...)$ is stationary $u_t = 0$ and hence $u$ is of finite gap type. □

## 4.3 The associated problem

We now present the simplest ideas of the method of finite-zone integration application to matrix operators of second order.

Let $u(z, \overline{z})$ be the finite gap solution of the equation 2.16 which is stationary with respect to higher time $t$ as sketched in the previous section. The $t$-evolution of the $\Psi$-function is determined by the polynomial $W(v)$ of degree $2N - 1$ in both $v$ and $1/v$,

$$\Psi_t = W\Psi, \tag{4.5}$$

with the corresponding stationary equations

$$-W_z + [U, W] = 0, \quad -W_{\overline{z}} + [V, W] = 0. \tag{4.6}$$

**Theorem 4.5.** *The corresponding stationary equations 4.6 are equivalent to finding a solution* $\Psi$ *of the special eigenvalue problem*

$$W\Psi = \mu\Psi, \quad \Psi_z = U\Psi, \quad \Psi_{\overline{z}} = V\Psi. \tag{4.7}$$

*Proof.* Let $W, U, V$ be a solution of 4.7. Furthermore,

$$\Psi \neq 0, \quad \mu_z = \mu_{\overline{z}} = 0$$

for a higher time $z, \overline{z}$. Since $W_z + [W, U] = 0$ is isospectral, the eigenvalues of $W$ are time independent.

Derive the derivate of $W\Psi = \mu\Psi$ with respect to $z, \overline{z}$:

$$\begin{aligned} W_z\Psi + W\Psi_z &= \mu_z\Psi + \mu\Psi_z, \\ W_{\overline{z}}\Psi + W\Psi_{\overline{z}} &= \mu_{\overline{z}}\Psi + \mu\Psi_{\overline{z}}. \end{aligned}$$

The following then holds

$$\begin{aligned} &\iff && W_z\Psi + WU\Psi = \mu_z\Psi + \mu U\Psi, \quad W_{\overline{z}}\Psi + WV\Psi = \mu_{\overline{z}}\Psi + \mu V\Psi \\ &\iff && W_z\Psi + WU\Psi = \mu_z\Psi + UW\Psi, \quad W_{\overline{z}}\Psi + WV\Psi = \mu_{\overline{z}}\Psi + VW\Psi \\ &\iff && W_z\Psi + WU\Psi - UW\Psi = \mu_z\Psi, \quad W_{\overline{z}}\Psi + WV\Psi - VW\Psi = \mu_{\overline{z}}\Psi \\ &\overset{\Psi\neq 0, \mu_z, \mu_{\overline{z}}=0}{\iff} && W_z + WU - UW = 0, \quad W_{\overline{z}} + WV - VW = 0 \\ &\iff && W_z + [W,U] = 0 \quad W_{\overline{z}} + [W,V] = 0 \\ &\iff && -W_z + [U,W] = 0 \quad -W_{\overline{z}} + [V,W] = 0. \end{aligned}$$

□

**Remark 4.6.** The common eigenfunctions $\Psi$ of $U$ and $V$ determine the 2-dimensional holomorphic bundle over a Riemann manifold $R$.

The goal of this method is to determine the analytical properties of the eigenvector $\Psi$ and see how much of $W$ can then be reconstructed from these information. For any two pair of commuting ordinary differential operators, there exists an algebraic relation which determines an algebraic curve, the so called spectral curve:

**Lemma 4.7.** The eigenvalue problem $W\psi = \mu\psi$ defines a hyperelliptic curve and therefore a Riemann surface $R$.

*Proof.* $U$ and $V$ as in 4.2 are traceless. Consequently, one can always fix $W$ to be traceless. The characteristic polynomial of $W$ is given by

$$det(W(v) - \mu\mathbb{1}) = 0).$$

For a traceless matrix we have

$$det(A+B) = det(A) + det(B).$$

We get

$$\begin{aligned} det(W(v) - \mu\mathbb{1}) &= det(W(v)) - det(\mu\mathbb{1}) \\ \Longleftrightarrow det(W(v)) &= \mu^2, \end{aligned}$$

which is called the spectral curve. □

The spectral curve is the characteristic equation for the eigenvalues of the Lax matrix. Since $W_z + [W, U] = 0$ is isospectral, the eigenvalues of $W$ are time independent and so is the spectral curve. Hence, at any point of the spectral curve there exists an eigenvector of $W$ with eigenvalue $\mu$. This defines an analytic line bundle on the curve with prescribed chern class. Chern classes are characteristic classes. They are topological invariants associated to vector bundles on a smooth manifold. A bundle is a topological construction which makes precise the idea of a family of vector spaces parameterized by another space.

The spectral curve is the characteristic equation for the eigenvalues of $W$. One can then reconstruct the eigenvalue by the properties of the spectral curve. In particular, the information is contained in the divisor of the spectral curve poles. The spectral curve $det(W(v)) = \mu^2$ possess because of the reduction

$$\sigma_3 W(-\nu)\sigma_3 = W(\nu)$$

(see [6]) the involution

$$\pi : (\nu, \mu) \longrightarrow (-\nu, \mu).$$

## 4.4 The Baker-Akhiezer function

We will now give a general definition of the Baker-Akhiezer function and see that it solves 4.7, i.e. we construct a matrix of the Baker-Akhiezer function. In a next step, we then state a special definition of the Baker-Akhiezer function, which solves the sinh-Gordon equation. It is the same as the general definition, except that we have its vector valued 2-point version, and that we have a special condition as we are in the situation of an hyperelliptic Riemann surface, i.g. we have some additional restrictions on the function being single-valued.

Baker-Akhiezer functions are special functions with essential singularities on Riemann surfaces. They provide a very natural parameterization of eigenvectors of the linear system.

**Definition 4.8.** (Baker-Akhiezer function) An $n$-point Baker-Akhiezer function $\psi$ on $R$ of genus $g$ corresponding to $Q$, to the local parameter $z = \frac{1}{k}$ at $Q$, to the polynomial $q(k)$ and divisor $D$, is a function $\psi(P)$ such that:

i. $\psi(P)$ is meromorphic on $R$ outside the points $P_\alpha$ (i.e. except $P = Q$) and has on $R \diagdown Q$ poles only at the points $P_1, ..., P_g$ of $D$ (this means $(\psi) + D \geq 0$).

ii. $\psi(P)e^{-q(k)}$ is analytic in a neighborhood $P = Q$.

Instead of ii. we can also say that $\psi(P)$ has at $P = Q$ an essential singularity of the form $\psi(P) \sim ce^{q(k)}$.

**Remark 4.9.** From pure algebro-geometric arguments it follows that there exists a unique function $\psi$ with the prescribed properties. We come back to this in the next section.

**Remark 4.10.** The Baker-Akhiezer function is determined by its algebraic properties with respect to the variable $Q$. It is defined up to multiplicative constant with the following set of parameters:

i. $R$ a compact Riemann surface of genus g.

ii. $P_\infty \in R$ a marked fixed point.

iii. A fixed choice of local parameters $p = k^{-1}$ near $P_\infty$ with $k \longrightarrow \infty$ as $P \longrightarrow P_\infty$.

iv. $Q(k)$ a polynomial.

v. $D = P_1 + ... + P_g$ a non-special divisor on $R$.

Through the equivalence of a set of solutions we have seen that the sinh-Gordon equation generates a spectral curve of the form

$$\mu^2 = \lambda^{-2n+1} \sum_{i=1}^{2g} (\lambda - \lambda_i), \quad g = 2n-1, \quad or \quad g = 2n-2$$

of genus $g$ with contour $L$ fixing the branch $\sqrt{\lambda}$ on $R \setminus L$.

We will now state the corresponding vector valued definition of the Baker-Akhiezer function:

**Definition 4.11.** Let $D = P_1, ..., P_g$ be a positive divisor of degree $g$ on $R$. The vector valued Baker Akhiezer function $\psi = (\psi_1, \psi_2)^T$ associated to the solution of the sinh-Gordon equation is analytic on $R$ and

i. $\psi$ is meromorphic on $R \setminus \{\lambda = 0, \infty\}$, its polar divisor is non-special of degree $g$ and does not depend on $z, \bar{z}$. Or, equivalently, $\psi$ has on $R \setminus \{\lambda = 0, \infty\}$ poles only at the points $P_1, ..., P_g$ of $D$.

ii. The functions $\psi$ and $\sqrt{\lambda}\psi_2$ are one-valued on $R$.

iii $\psi$ has essential singularities of the kind

$$\psi_{1,2} = (1 + O(1)) exp\left(-\frac{i}{2} z\sqrt{\lambda}\right), \quad \lambda \to \infty, \tag{4.8}$$

$$\psi_{1,2} = O(1) exp\left(-\frac{i}{2\sqrt{\lambda}}\bar{z}\right), \quad \lambda \to \infty. \tag{4.9}$$

The Baker-Akhiezer function is uniquely defined by its analytic properties and can be explicitly constructed by theta-functions and Abelian integrals which is done in the next section.

## 4.5 Explicit formula

We now turn our attention to the problem of a formula for the Baker-Akhiezer function. We begin with the introduction of the necessary tools to derive the formula and then claim that it fulfills the above definitions and therefore solves the problems associated with the sinh-Gordon equation. We will return to the functions and theorems of the previous chapter.

Now consider the Riemann Surface of the hyperelliptic curve, defined by

$$\mu^2 = \lambda^{-2n+1} \prod_{i=1}^{4n-2} (\lambda - \lambda_i) \quad ; \lambda = \mu^2. \tag{4.10}$$

**Remark 4.12.** $C \longrightarrow C/\pi, \pi : (\lambda, \mu) \longrightarrow (\lambda, -\mu)$ is a double cover. $C$ is a hyperelliptic curve of genus $2n-1$ where $\lambda = 0, \infty$ are branch points. The points $\infty^{\pm}$ denote the points of the surface with $\mu \rightarrow \pm\nu^{2n-1}, \nu \rightarrow \infty$ and $O^{\pm}$ the two points with $\nu = 0$. The involution interchanges them: $\infty^{+} \overset{\pi}{\longleftrightarrow} \infty^{-}, O^{+} \overset{\pi}{\longleftrightarrow} O^{-}$.

**Remark 4.13.** We assume $C$ not to be singular. One can show (see [5]) that the singular case does not lead to CMC tori.

Let be $C$ a hyperellitptic Riemann Surface of genus $g$ as in 4.10 with branch points $\lambda = 0, \infty$ and let $L$ be a contour defining a one-valued branch of the function $\sqrt{\lambda}$ on $C \setminus L$.

Set a canonical basis of cycles

$$a_n, b_n, n = 1, ..., g$$

chosen such that

$$L = a_1 + ... + a_g.$$

A contour $L$ fixes the sheet of the covering that contains $\infty$ and $O^{+}$ $C \setminus L$. On $C \setminus L$, there is singled out a one-valued branch of the function $\nu$, two valued on $C$, and a local parameter $\lambda$ at the points $\lambda = 0, \infty \in C$ are chosen so that $\sqrt{\lambda} = v, \lambda \rightarrow 0, \infty$.

The normalized holomorphic Abelian differentials define a period matrix

$$B_{mn} = \int_{\delta_m} du_n.$$

By normalized we mean as usual

$$\int_{a_m} du_n = 2\pi i \delta_{nm}.$$

Define the theta-function, as introduced earlier as

$$\theta(z) = \sum_{z \in \mathbb{Z}^g} exp\left(\frac{1}{2}\langle BM, M\rangle + \langle z, M\rangle\right), \quad z \in \mathbb{C}^g,$$

periodic with periods $i\pi\mathbb{Z}^g$.

We introduce differentials $\Omega_1, \Omega_2$ of the second kind, normalized by

$$\int_{a_n} d\Omega_i = 0, \quad i = 1, 2, \quad n = 1, ..., g,$$

with singularities of the following form:

$$d\Omega_1 \to d\sqrt{\lambda}, \lambda \to \infty, \tag{4.11}$$

$$d\Omega_2 \to -\frac{d\sqrt{\lambda}}{\lambda}, \lambda \to 0. \tag{4.12}$$

**Remark 4.14.** Note that any Abelian differential of the second kind (or third kind) with zero $a$-periods or with all purely imaginary cyclic periods, is uniquely defined by its principal parts at singular points.

Set their periods to

$$U_n = \int_{b_m} d\Omega_1, \quad V_n = \int_{b_n} d\Omega_2.$$

**Lemma 4.15.** The BA-function is given by the following formulas:

$$\psi_1 = \frac{\theta(u + \Omega)\theta(D)}{\theta(u + D)\theta(\Omega)} exp\{-\frac{i}{2}(\Omega_1 z + \Omega_2 \overline{z})\} \tag{4.13}$$

$$\psi_2 = \frac{\theta(u + \Omega + \Delta)\theta(D)}{\theta(u + D)\theta(\Omega + \Delta)} exp\{-\frac{i}{2}(\Omega_1 z + \Omega_2 \overline{z})\} \tag{4.14}$$

where

$$\begin{aligned}
\Omega &= -\frac{i}{2}(Uz + V\overline{z}) + D \\
U &= (U_1, ..., U_g), \\
V &= (V_1, ..., V_g) \\
\Delta &= \pi i(1, ..., 1) \\
\Omega_i &= \int_{\infty}^{P} d\Omega_i \\
u &= \int_{\infty}^{P} du, \quad P = (\lambda, \mu) \in X
\end{aligned}$$

is the Abel map and

$$D \in \mathbb{C}^g$$

is arbitrary such that $\theta(u + D)$ has $D$ as a null divisor.

*Proof.* We have to check that all the properties of 4.11 are fulfilled and that the defined function is unique:

i. $\psi_{1,2}$ is meromorphic on $R \setminus \lambda = 0, \infty$, which follows directly from the property that $\theta$ is an analytic function 3.56. From 3.57 it follows that $\theta(u + D)$ has zeros at $P_1, ..., P_g$. Obviously, these are the poles of $\psi_{1,2}$.

ii. This requirement is equivalent to the property that $\psi_1$ has no jump on $L$ and that $\psi_2$ changes sign when its arguments crosses sign. Put differently, $\psi_1$ has to be invariant when the point $P$ goes around an arbitrary cycle $\gamma$ and $\psi_2$ changes sign. We denote by $M_\gamma$ the monodromy operator that corresponds to the cycle $\gamma$ being traversed.

We first examine $\psi_1$. Changing path of integration we have

$$\begin{aligned}
-\frac{i}{2}(\Omega_1 z + \Omega_2 \overline{z}) &\rightarrow -\frac{i}{2}(\Omega_1 z + \Omega_2 \overline{z}) \left( \langle M, -\frac{i}{2} \left(Uz + V\overline{z}\right) \rangle \right) \\
u &\rightarrow u + 2\pi i N + BM
\end{aligned}$$

where $M = (m_1, ..., m_g)$ and $N = (n_1, ..., n_g)$.

We compute

$$
\begin{aligned}
&M_\gamma\left[\psi_1\right]\\
&= \frac{\theta(u+2\pi iN+BM+\Omega)\theta(D)}{\theta(u+2\pi iN+BM+D)\theta(\Omega)} exp\left(-\frac{i}{2}\left(\Omega_1 z+\Omega_2\overline{z}\right)\right) exp\left(\langle M,-\frac{i}{2}\left(Uz+V\overline{z}\right)\rangle\right)\\
&= \frac{exp\left(-\frac{1}{2}\langle B,M\rangle-\langle M,u+\Omega\rangle\right)}{exp\left(-\frac{1}{2}\langle B,M\rangle-\langle M,u+D\rangle\right)} exp\left(\langle M,-\frac{i}{2}(Uz+V\overline{z})\rangle\right)\psi_1\\
&= \frac{exp\left(-\langle M,u\rangle-\langle M,\Omega\rangle\right)}{exp\left(-\langle M,u\rangle-\langle M,D\rangle\right)} exp\left(\langle M,-\frac{i}{2}\left(Uz+V\overline{z}\right)\rangle\right)\psi_1\\
&= \frac{exp\left(-\langle M,\Omega\rangle\right)}{exp\left(-\langle M,D\rangle\right)} exp\left(\langle M,-\frac{i}{2}(Uz+V\overline{z})\rangle\right)\psi_1\\
&= \frac{exp\left(-\langle M,-\frac{i}{2}\left(Uz+V\overline{z}\right)+D\rangle\right)}{exp\left(-\langle M,D\rangle\right)} exp\left(\langle M,-\frac{i}{2}\left(Uz+V\overline{z}\right)\rangle\right)\psi_1\\
&= \psi_1.
\end{aligned}
$$

Regarding $\psi_2$, we have for changing the path of integration

$$
\begin{aligned}
-\frac{i}{2}(\Omega_1 z+\Omega_2\overline{z}) &\rightarrow -\frac{i}{2}(\Omega_1 z+\Omega_2\overline{z})\left(\langle M,-\frac{i}{2}(Uz+V\overline{z})\rangle\right)\\
u &\rightarrow u+2\pi iBM
\end{aligned}
$$

where $M=(1,0,...,0)$.

Calculate

$$
\begin{aligned}
&M_\gamma\left[\psi_2\right]\\
&= \frac{\theta(u+\Omega+\Delta+2\pi iBM)\theta(D)}{\theta(u+2\pi iBM+D)\theta(\Omega+\Delta)} exp\left(-\frac{i}{2}(\Omega_1 z+\Omega_2\overline{z})\left(\langle M,-\frac{i}{2}(Uz+V\overline{z})\rangle\right)\right)\\
&= \frac{exp\left(-\frac{1}{2}\langle 2\pi iB,M\rangle-\langle M,u+\Omega+\Delta\rangle\right)}{exp\left(-\frac{1}{2}\langle 2\pi iB,M\rangle-\langle M,u+D\rangle\right)} exp\left(\langle M,-\frac{i}{2}(Uz+V\overline{z})\rangle\right)\psi_2\\
&= \frac{exp\left(-\langle M,\Omega+\Delta\rangle\right)}{exp\left(-\langle M,D\rangle\right)} exp\left(\langle M,-\frac{i}{2}(Uz+V\overline{z})\rangle\right)\psi_2\\
&= exp\left(-\langle M,\Delta\rangle\right)\psi_2\\
&= -\psi_2.
\end{aligned}
$$

iii. This follows directly from the chosen normalization of $\psi_{1,2}$ and 4.11.

It remains to be shown that $\psi_{1,2}$ is unique: Suppose now there exists another Baker-Akhiezer function $\psi$. From the definition of the Baker-Akhiezer function it follows that the ratio $\psi_{1,2}/\psi$ is a meromorphic function on $C$, which is equal to one at the puncture and with the only possible poles at the zeros of the function $\psi$. According to 3.57, the zeros of $\psi$ are zeros of the function $\theta(u+\Omega)$. Hence, $\psi$ has $g$ zeros. The simplest form of the Riemann-Roch theorem implies that a function on a Riemann surface with at most $g$ poles at a generic set of points is a constant. Therefore, $\psi_{1,2} = \psi$ and the existence and uniqueness of the Baker-Akhiezer function is proved. □

**Proposition 4.16.** The solution of 2.16 is given by

$$u = 2ln\frac{\theta(\Omega)}{\Omega+\Delta}. \tag{4.15}$$

*Proof.* Let $\hat{C}$ be the Riemann surface which is a two-sheeted covering of $C$ 4.10. The hyperelliptic curve $C$ consisting of two sheets of $\hat{C}$ is defined as above. There exists an involution $\pi : \hat{C} \longrightarrow \hat{C}$, interchanging the sheets of the covering and hence chose $\Psi$ to be

$$\Psi = \begin{pmatrix} \psi_1 & \psi_1^* \\ \psi_2 & \psi_2^* \end{pmatrix}, \tag{4.16}$$

the solution of 4.2. We have

$$\begin{aligned} \Psi_{\overline{z}} &= V\Psi \\ \Leftrightarrow \begin{pmatrix} \psi_{1\overline{z}} & \psi_{1\overline{z}}^* \\ \psi_{2\overline{z}} & \psi_{2\overline{z}}^* \end{pmatrix} &= \frac{1}{2vi}\begin{pmatrix} 0 & e^{-u} \\ e^u & 0 \end{pmatrix}\begin{pmatrix} \psi_1 & \psi_1^* \\ \psi_2 & \psi_2^* \end{pmatrix} \\ &= \begin{pmatrix} e^{-u}\psi_2 & e^{-u}\psi_2^* \\ e^u\psi_1 & e^u\psi_1^* \end{pmatrix}. \end{aligned}$$

For instance, we can take

$$\begin{aligned} \psi_{1\overline{z}} &= e^u \psi_2 \\ \Longleftrightarrow e^u &= \frac{\psi_2}{\psi_{1\overline{z}}} \\ \Longleftrightarrow u &= ln(\frac{\psi_2}{\psi_{1\overline{z}}}). \end{aligned}$$

We now take into consideration that

$$\int_{\infty}^{0} du = \Delta, \quad \int_{\infty}^{0} d\Omega_i = 0$$

and hence compute

$$\begin{aligned} & \frac{\psi_2}{\psi_{1\overline{z}}} = \frac{\theta(\Omega)\theta(D)}{\theta(u+D)\theta(\Omega+\Delta)} \frac{\theta(u+D)\theta(\Omega)}{\theta(u+\Omega)+\theta(D)} = \frac{\theta^2(\Omega)}{\theta^2(u+\Omega)} \\ \Longrightarrow \quad & u = ln\left(\frac{\theta^2(\Omega)}{\theta^2(u+\Omega)}\right) \\ & = 2ln\left(\frac{\theta(\Omega)}{\theta(u+\Omega)}\right). \end{aligned}$$

□

We can now connect the Baker-Akhiezer function with the original problem:

**Lemma 4.17.** The vector valued Baker-Akhiezer function is a solution for 4.7.

*Proof.* For this proof it suffices to consider the matrix $\Psi(z, \overline{z}, \lambda)$ with the vectors $\psi(z, \overline{z}, \lambda), P_j = (\lambda, \mu_j)$ as columns. This matrix depends on the numbering of the columns (i.e. of the points $P_j$). The matrices

$$(\partial_z \Psi \Psi)^{-1}, \quad (\partial_{\overline{z}} \Psi \Psi)^{-1}, \quad \Psi \hat{\mu} \Psi^{-1}$$

are well defined, as they do not depend on the numbering. $\hat{\mu}$ is a diagonal matrix equal to $\hat{\mu}_{ij} = -i\delta_{ij}$. By virtue of the analytic properties of $\psi$, the matrices are rational functions of $\lambda$ and are denoted by $U, V, W$ respectively. □

**Remark 4.18.** All real-valued solutions of 2.16 are given by 4.14. For the proof see for example [5].

We summarize the results accomplished so far: We showed that the sinh-Gordon equation is equivalent to the solution of 4.6, differential equations of a pair of operators satisfying the zero-curvature condition. We approached the problem from two sides. On the one hand, we showed that 4.6 corresponds to an eigenvalue problem of the form 4.7, and defines a spectral curve. On the other hand, we defined the Baker-Akhiezer function, which is uniquely determined on an algebraic curve. The Baker-Akhiezer function also solves 4.7, which concludes that finding the Baker-Akhiezer function solves the sinh-Gordon equation. Therefore we have shown that solving problem 4.5 and determining the Baker-Akhiezer function is evidently the same.

We are not going to derive the detailed formula for the immersions in $\mathbb{R}^3, \mathbb{H}^3$, as this was done in [6] and [5] already.

We will now turn our attention to CMC tori in $\mathbb{H}^3$ and we study by which spectral data they are uniquely determined and how to obtain deformations. Very similar results for deformations of CMC tori in $\mathbb{R}^3, S^3$ were obtained in [31], [32] and [33], and we closely follow those works in the proceeding.

# 5 Deformations

Soon after the development of finite gap integration of nonlinear differential equations by Novikov [1974], Dubrovin et al. [1976], Its and Matveev [1975], Lax [1975], and McKean [1975], there were already new ideas linked to that new method: The spectral data of the Lax pair characterizing the associated Lax-type operators consist of a Riemann surface (algebraic curve or also called spectral curve), equipped with a selected set of points (divisor of points, infinities). In the finite gap case, this Riemann surface has finite genus and the number of selected points is also finite. As was shown before the aglebro-geometric approach in particular allows one to write down explicit solutions in terms of theta-functions. But the problem is not solved by simply retrieving exact formulas. It is often necessary to select geometrically relevant classes of solutions corresponding to the source problem: For instance solutions satisfying a certain reality condition, or regular solutions, or bounded solutions. To solve this problem one must deal with the following questions:

Problem 1: How to select solutions that are real? Problem 2: How to select real nonsingular solutions? Problem 3: How select periodic solutions with a given period (or also quasi-periodic solutions)?

In the previous chapters we derived the link between CMC tori and the sinh-Gordon equation. We observed that CMC tori are connected to a spectral curve via spectral theory of commuting differential operators. In this chapter we want to look more closely at the periodicity of solutions of the sinh-Gordon equation. We want to specify the data of the spectral curve which need to be fulfilled in order to abtain CMC tori in the 3-hyperbolic space. We will introduce the concept of monodromy and also use another description of the spectral curve (compared to the one of the chapter before). This new description will be equivalent to the spectral curve introduced in the chapter before.

The imposed periodicity conditions are a result of the so called closing conditions of the monodromy, or to be specific, the closing conditions derivable from the Sym Bobenko formula. In a first part we will introduce the monodromy and derive some useful properties.

The second part of the chapter will deal with the question of how to encode the spectral curve with additional functions to obtain a solution of the sinh-Gordon equation to a set of complex numbers. We will call this set the spectral data. This representation of the spectral curve will allow us to define a deformation on the spectral curve and represent this deformation by a system of ordinary differential equations. In the proceeding we closely follow [32], [33] and [31].

## 5.1 Spectral curve, monodromy and closing condition

### 5.1.1 The closing condition

Recall from the first chapter that for $F = F(z, \overline{z}, \lambda)$ being a solution of

$$F_z = FU, \quad F_{\overline{z}} = FV \tag{5.1}$$

with

$$U = \frac{1}{2}\begin{pmatrix} u_z & -\lambda^{-2}e^u \\ Qe^{-u} & -u_z \end{pmatrix} \quad V = \frac{1}{2}\begin{pmatrix} -u_{\overline{z}} & -\overline{Q}e^{-u} \\ \lambda^2 e^u & u_{\overline{z}} \end{pmatrix} \tag{5.2}$$

and

$$\begin{aligned} &det(F) = 1, \quad \forall \lambda, z \\ &\lambda_0 = e^{\frac{q}{2}} e^{i\psi} \\ &\lambda = \frac{\sqrt{1+H}}{\sqrt{H-1}} \end{aligned}$$

we have with

$$f = F\overline{F}^t$$

that $f$ is the immersion of a CMC surface (Note that for the sake of brevity we omitteed here the hat).

As the solution of the sinh-Gordon equation $u$ is double periodic, so are $U$ and $V$. We then define:

**Definition 5.1.** (Monodromy) Let $\tau_{i=1,2}$ be the two periods. We call $M_1, M_2$ for which

$$F(\tau_{i=1,2}+z, \overline{\tau}_{i=1,2}+\overline{z}, \lambda) = M_{i=1,2}(\lambda)F(z,\overline{z},\lambda)$$

the Monodromies of $F_\lambda$.

**Remark 5.2.** The monodromy $\mathbb{C}^* \longrightarrow SL_2(\mathbb{C}), \lambda \longrightarrow M_\lambda$ is a holomorphic map with essential singularities at $\lambda = 0, \infty$. By construction the monodromy takes values in $SU_2$ for $|\lambda| = 1$.

**Theorem 5.3.** *Let $F$ be a unitary frame as in 5.1 with monodromy $M_\tau(\lambda)$. Then the closing conditions is given by $M_\tau(\lambda_0) = M_\tau(\frac{1}{\bar{\lambda 0}}) = \pm\mathbb{1}$.*

*Proof.* The proof follows directly from the Sym-Bobenko formula: As $F \in SL_2(\mathbb{C})$ we have $\bar{F}_\lambda^t = F_{\frac{1}{\bar{\lambda}}}^{-1}$.

$$\begin{aligned} f(z,\overline{z},\lambda) &= F_{\lambda_0}\bar{F}_{\lambda_0}^T \\ \Longleftrightarrow f(z,\overline{z},\lambda) &= F_{\lambda_0}F_{\lambda_1}^{-1} \quad \lambda_1 = \bar{\lambda_0}^{-1}. \end{aligned}$$

As the frame $F$ is periodic so is $f$

$$\begin{aligned} \Longrightarrow f(z+\tau,\overline{z}+\overline{\tau},\lambda) &= M_\tau(\lambda_0)F_{\lambda_0}F_{\lambda_1}^{-1}M_\tau^{-1}(\lambda_1) \\ &= M_\tau(\lambda_0)f(z)M_\tau^{-1}(\lambda_1) \\ \Longrightarrow M_\tau(\lambda_0) &= M_\tau(\lambda_1) = \pm\mathbb{1}. \end{aligned}$$

□

The additional restriction on the monodromy thus ensures that the immersion closes to become a torus.

**Definition 5.4.** Let $\tau_{i=1,2}$ be the periods and $M_1(\lambda), M_2(\lambda)$ the monodromies of $F_\lambda$ with corresponding eigenvalues $\mu_1, \mu_2$. Then the spectral curve of the CMC torus is given by

$$\sum_f = \{(\lambda, \mu_1, \mu_2) : det(\mu_1 \mathbb{1} - M_1(\lambda)) = det(\mu_2 \mathbb{1} - M_2(\lambda)) = 0\}. \tag{5.3}$$

**Remark 5.5.** This description of CMC tori differs slightly from the one we used before. The description used before is related to polynomial Killing fields and is equivalent to the one used here, see for example [32].

### 5.1.2 Properties of the monodromy

In the propositions below we will derive some important properties of the monodromy. The argument is always the same: If $(Udz + Vd\overline{z})$ fulfills a certain property, then both $F$ and $M$ do as well as $dF = (Udz + Vd\overline{z})F$.

**Proposition 5.6.** The monodromy satisfies

$$M(\frac{1}{\overline{\lambda}}) = (\overline{M}^t(\lambda))^{-1}. \tag{5.4}$$

*Proof.* We have to show that with

$$U = \frac{1}{2}\begin{pmatrix} u_z & -\lambda^{-2}e^u \\ Qe^{-u} & -u_z \end{pmatrix}, \quad V = \frac{1}{2}\begin{pmatrix} -u_{\overline{z}} & -\overline{Q}e^{-u} \\ \lambda^2 e^u & u_{\overline{z}} \end{pmatrix},$$

$$(Udz + V\overline{z})(\frac{1}{\overline{\lambda}}) = -\overline{(Udz + Vd\overline{z})}^t$$

holds. We compute

$$\begin{aligned}
(Udz+Vd\overline{z})(\frac{1}{\overline{\lambda}}) &= \begin{pmatrix} u_z dz - u_{\overline{z}} d\overline{z} & -\overline{\lambda}^2 e^u dz - \overline{Q} e^{-u} d\overline{z} \\ Q e^{-u} dz + \overline{\lambda}^{-2} e^u d\overline{z} & -u_z dz + u_{\overline{z}} d\overline{z} \end{pmatrix} \\
\overline{(Udz+Vd\overline{z})}(\lambda) &= \begin{pmatrix} u_{\overline{z}} d\overline{z} - u_z dz & -\overline{\lambda}^2 e^u d\overline{z} - Q e^{-u} dz \\ \overline{Q} e^{-u} d\overline{z} \overline{\lambda}^{-2} e^u dz & -u_{\overline{z}} d\overline{z} + u_z dz \end{pmatrix} \\
-\overline{(Udz+Vd\overline{z})}^t(\lambda) &= \begin{pmatrix} -u_{\overline{z}} d\overline{z} + u_z dz & \overline{Q} e^{-u} d\overline{z} + \overline{\lambda}^2 e^u dz \\ -\overline{\lambda}^2 e^u d\overline{z} - Q e^{-u} dz & u_{\overline{z}} d\overline{z} - u_z dz \end{pmatrix}.
\end{aligned}$$

As $dF = (Udz + Vd\overline{z})F^{-1}$, we have

$$F(\frac{1}{\overline{\lambda}}) = (\overline{F}^t)^{-1}(\lambda).$$

The same then holds for the monodromy. □

**Remark 5.7.** Consequently, $M$ is of the form

$$M(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ -\overline{b(\frac{1}{\overline{\lambda}})} & \overline{a(\frac{1}{\overline{\lambda}})} \end{pmatrix}.$$

**Remark 5.8.** We set

$$M^*(\lambda) = \overline{(M(\frac{1}{\overline{\lambda}}))^{t^{-1}}}. \tag{5.5}$$

**Proposition 5.9.** Define

$$e = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SU(2).$$

Then

$$eM(\lambda)e^{-1} = M(\lambda)^{t^{-1}}$$

holds.

*Proof.* Using the proposition before we have to show that

$$e(Udz + Vd\overline{z})(\lambda)e^{-1} = \overline{(Udz + Vd\overline{z})}(\frac{1}{\overline{\lambda}})$$

holds:

$$
\begin{aligned}
e(Udz+Vd\bar{z})(\lambda)e^{-1} \quad &= \quad \begin{pmatrix} -u_z dz + u_{\bar{z}} d\bar{z} & \frac{1}{\lambda e^u} Q dz + \lambda e^u d\bar{z} \\ -\frac{e^u}{\lambda} dz - \frac{\lambda}{e^u} \overline{Q} d\bar{z} & u_z dz - u_{\bar{z}} d\bar{z} \end{pmatrix}^{-1} \\
\overline{(Udz+Vd\bar{z})(\frac{1}{\bar{\lambda}})} \quad &\overbrace{=}^{det(Udz+Vd\bar{z})=1} \quad \begin{pmatrix} u_{\bar{z}} d\bar{z} - u_z dz & -\lambda^2 e^u d\bar{z} - Q e^{-u} dz \\ \overline{Q} e^{-u} d\bar{z} + \lambda^{-2} e^u dz & -u_{\bar{z}} d\bar{z} + u_z dz \end{pmatrix}.
\end{aligned}
$$

□

**Proposition 5.10.** Set

$$
e = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SU(2).
$$

Then

$$
e\overline{M(\bar{\lambda}^{-1})}e^{-1} = M(\lambda)
$$

holds.

*Proof.* We have to show that

$$
e(overline(Udz+Vd\bar{z})(\frac{1}{\bar{\lambda}})e^{-1} = (Udz+Vd\bar{z})(\lambda)
$$

holds

$$
\begin{aligned}
\overline{(Udz+Vd\bar{z})(\frac{1}{\bar{\lambda}})} \quad &= \quad \begin{pmatrix} u_{\bar{z}} d\bar{z} - u_z dz & -\lambda^2 e^u d\bar{z} - Q e^{-u} dz \\ \overline{Q} e^{-u} d\bar{z} + \lambda^{-2} e^u dz & -u_{\bar{z}} d\bar{z} + u_z dz \end{pmatrix} \\
e\overline{(Udz+Vd\bar{z})(\frac{1}{\bar{\lambda}})}e^{-1} \quad &= \quad \begin{pmatrix} -u_{\bar{z}} d\bar{z} + u_z dz & -\overline{Q} e^{-u} d\bar{z} - \lambda^{-2} e^u dz \\ \lambda^2 e^u d\bar{z} + Q e^{-u} dz & u_{\bar{z}} d\bar{z} - u_z dz \end{pmatrix}.
\end{aligned}
$$

□

## 5.2 Conditions on the $\mathbb{H}^3$ spectral curve

In the previous section we carried out the concepts and tools we will need for this section. We will now study which spectral data determine a CMC torus in the hyperbolic 3-space. In a first part, we will prove some propositions for later use. Note that most of the proofs follow from the properties of the monodromy, which are in turn determined by the closing condition derived from the Sym Bobenko formula.

**Proposition 5.11.** Let $C$ be an hyperelliptic Riemann surface with branch points over $\lambda = 0(y^+)$ and $\lambda = \infty(y^-)$ and $M_{i=1,2}$, the monodromy as in 5.3. Then the surface $C$ has one holomorphic involution and two anti-holomorphic involutions:

$$\sigma : (\lambda, \mu_1, \mu_2) \longrightarrow (\lambda, \frac{1}{\mu_1}, \frac{1}{\mu_2})$$
$$\eta : (\lambda, \mu_1, \mu_2) \longrightarrow (\frac{1}{\overline{\lambda}}, \overline{\mu}_1, \overline{\mu}_2)$$
$$\rho : (\lambda, \_1, \mu_2 \longrightarrow (\frac{1}{\overline{\lambda}}, \frac{1}{\overline{\mu}_1}, \frac{1}{\overline{\mu}_2})$$

with $\rho = \eta \circ \sigma = \sigma \circ \eta$ such that $\eta$ has no fixpoints, and $\eta(y^+) = y^-$.

*Proof.* The existence of these three involutions follows immediately from 5.11, 5.12 and 5.13. Obviously $\rho = \eta \circ \sigma = \sigma \circ \eta$ holds as well $\eta(y^+) = y^-$.

To complete the proof we have to check that $\eta$ has no fixpoints:

Let $\nu$ be an eigenvector of $M$. If $\nu$ is an eigenvector of $M(\lambda)$ then $\overline{\nu}$ is an eigenvector of $\overline{M}(\frac{1}{\overline{\lambda}})$ because

$$\begin{aligned} det(\mu\mathbb{1} - M(\lambda)) &= det(\mu\mathbb{1} - e\overline{M}(\frac{1}{\overline{\lambda}})) \\ &= det(e(\mu\mathbb{1} - \overline{M}(\frac{1}{\overline{\lambda}}))e^{-1}) \\ &= det(\mu\mathbb{1} - \overline{M}(\frac{1}{\overline{\lambda}})) \\ &= 0. \end{aligned}$$

We further have
$$\overline{M}(\frac{1}{\overline{\lambda}})\overline{\nu} = \overline{M}(\frac{1}{\overline{\lambda}})\overline{\nu} = \overline{\overline{\mu}\nu} = \mu\overline{v} = \mathbb{1}.$$

With
$$\overline{M}(\frac{1}{\overline{\lambda}}) = eM(\lambda)e^{-1}$$

we get

$$
\begin{aligned}
\overline{M}(\frac{1}{\lambda})\overline{\nu} &= \mu\overline{\nu} \\
\Longleftrightarrow eM(\lambda)e^{-1}\overline{\nu} &= \mu\overline{\nu} \\
\Longleftrightarrow M(\lambda)e\overline{\nu} &= \mu e\overline{\nu}
\end{aligned}
$$

and see that $e\overline{\nu}$ is an eigenvector of $M_\lambda$. If $\eta$ had fix points, the eigenvectors of $M$ would not be linearly independent, so

$$
\begin{aligned}
e\overline{\nu} &= \gamma\nu \quad \hat{}|^2 \\
\Longrightarrow e^2\overline{\nu} &= \gamma e\nu = \gamma\overline{\gamma\nu} \\
\Longrightarrow \gamma\overline{\gamma} &= -1.
\end{aligned}
$$

Since this is a contradiction the two eigenvectors are linearly independent and $\eta$ has no fixpoints. □

**Proposition 5.12.** Let $C$ be an hyperelliptic Riemann surface with branch points over $\lambda = 0(y^+)$, and $\lambda = \infty(y^-)$ and $M_{i=1,2}$ the monodromy as in 5.3. Then there exist two non-zero holomorphic functions $\mu_1, \mu_2$ on $C \setminus \{y^+, y^-\}$ for $i = 1, 2$, with

$$\sigma * \mu_i = \mu_i^{-1}, \quad \eta * \bar{\mu}_i = \mu_i, \quad \rho * \bar{\mu}_i = \mu_i^{-1}.$$

*Proof.* With the help of 5.11, 5.12 and 5.13 we compute:

$$\begin{aligned}\overline{P(\frac{1}{\overline{\lambda}},\overline{\mu})} &= \overline{det(\mu\mathbb{1} - M(\frac{1}{\overline{\lambda}}))} = det(\mu\mathbb{1} - M^*(\lambda)) \\ &= det(\mu\mathbb{1} - M(\lambda)) = P(\mu,\lambda)\end{aligned}$$

$$\begin{aligned}P(\lambda,\frac{1}{\mu}) &= det(\frac{1}{\mu}\mathbb{1} - M(\lambda)) = det(\frac{1}{\mu}\mathbb{1} - M^{t^{-1}}(\lambda)) \\ &= det(\frac{1}{\mu}\mathbb{1} - M^t(\lambda)) = det(\mu^2\frac{1}{\mu}\mathbb{1} - M^t(\lambda)\mu^2) = P(\lambda,\mu) \\ \overline{P(\frac{1}{\overline{\lambda}},\frac{1}{\overline{\mu}})} &= det(\frac{1}{\mu}\mathbb{1} - \overline{M(\frac{1}{\overline{\lambda}})}) = det(\frac{1}{\mu}\mathbb{1} - M^*(\lambda)) \\ &= det(\frac{1}{\mu}\mathbb{1} - M(\lambda)) = P(\lambda,\mu).\end{aligned}$$

□

**Proposition 5.13.** Let $C$ be an hyperelliptic Riemann surface with branch points over $\lambda = 0(y^+)$, and $\lambda = \infty(y^-)$ and $M_{i=1,2}$ the monodromy as in 5.3. Then the forms $dln\mu_i$ are meromorphic differentials of the second kind with double poles at $y^{\pm}$. The singular parts of these two differentials are linearly independent at $y^+$ and $y^-$ .

*Proof.* From the theory of ordinary differential equations (the so called fundamental solution) we get that

$$F(z_0 + \tau_{i=1,2}, \lambda) = M_{\tau_{i=1,2}}(\lambda)F(z_0, \lambda).$$

Since we know from the previous chapter that the Baker-Akhiezer function is uniquely defined, we have

$$F(z_0 + \tau_{i=1,2}, \lambda) = \mu F(z_0, \lambda).$$

Replacing $F(z_0 + \tau_{i=1,2}, \lambda)$ by $M_{\tau_{i=1,2}}(\lambda)F(z_0, \lambda)$ we get

$$M_{\tau_{i=1,2}}(\lambda)F(z_0, \lambda) = \mu F(z_0, \lambda),$$

and observe that the Baker-Akhiezer function is an eigenfunction of $M$ with eigenvalue $\mu$.

We consider $\Omega$ to be defined as in 4.15. Since

$$\Omega(z+\tau_{i=1,2},\overline{z}+\overline{\tau}_{i=1,2}) = \Omega(z,\overline{z}) + 2\pi i N,$$

and the transformation law 3.7 of the $\theta$-function, we get

$$F(z_0+\tau_{i=1,2},\lambda) = exp\left(-\frac{i}{2}\left(\Omega_1\tau_{i=1,2}+\Omega_2\overline{\tau}_{i=1,2}\right)\right)F(z_0,\lambda).$$

Consequently, we obtain

$$\mu_1 = exp\left(-\frac{i}{2}\left(\Omega_1\tau_1+\Omega_2\overline{\tau}_1\right)\right)$$
$$\mu_2 = exp\left(-\frac{i}{2}\left(\Omega_1\tau_2+\Omega_2\overline{\tau}_2\right)\right)$$

or, equivalently,

$$ln\ \mu_1 = -\frac{i}{2}\left(\Omega_1\tau_1+\Omega_2\overline{\tau}_1\right)$$
$$ln\ \mu_2 = -\frac{i}{2}\left(\Omega_1\tau_2+\Omega_2\overline{\tau}_2\right).$$

It now follows from the discussion of the previous chapter that

$$dln\ \mu_i = -\frac{i}{2}\left(d\Omega_1\tau_i+d\Omega_2\overline{\tau}_i\right), \quad i=1,2$$

fulfills the claimed properties. □

**Theorem 5.14.** *Let $C$ be an hyperelliptic Riemann surface with branch points over $\lambda = 0(y^+)$ and $\lambda = \infty(y^-)$ and $M_{i=1,2}$ the monodromy as in 5.3. Let the conditions 5.11, 5.12 and 5.13 be valid. Additionally there are points with $y_1, \sigma(y_1) = y_2, y_3, \sigma(y_3) = y_4$ and $\rho(y_1) = y_4, \rho(y_2) = y_3$ such that $\mu = \pm 1$. Then $C$ is the spectral curve of an immersed torus in $\mathbb{H}^3$.*

*Proof.* From the previous chapter we know of the existence of a unique meromorphic function with poles at the branch points 0 and $\infty$. By construction, it is ensured that this function has the right behavior at the poles (see previous the chapter). From proposition 5.13 we have that $dln\ \mu_i = -\frac{i}{2}\left(d\Omega_1\tau_i+d\Omega_2\overline{\tau}_i\right),\ i=1,2$. Due to [19] one can show that an Abelian differential of the second kind with vanishing $a$-periods and exitsing principal parts at their singularities is uniquely determined.

Set

$$U_k = \int_{b_k} d\Omega_1, \quad V_k = \int_{b_k} d\Omega_2$$

as before. We then have

$$\int_{b_k} -\frac{i}{2}\left(d\Omega_1 \tau_i + d\Omega_2 \overline{\tau}_i\right)$$
$$= -\frac{i}{2}(\tau U_k + \overline{\tau} V_k) \in 2\pi i \mathbb{Z}^g.$$

Hence the solution is double periodic.

The additional restrictions on the points ensures that the closing condition holds. I.g. for $\lambda_0$ lying in the unit circle one may consider $y_1, y_2$ and for $\frac{1}{\lambda_0} = \lambda_1$ take the points $y_3, y_4$ where then $\rho$ interchanges $y_2, y_3$ and $y_1, y_4$. One can retrieve then the immersion of a CMC torus via the Sym-Bobenko formula. □

## 5.3 Spectral data describing CMC tori in $\mathbb{H}^3$

We shall now instigate the spectral curves of CMC tori in $\mathbb{H}^3$. We describe the spectral curves of the periodic finite type sinh-Gordon solution by a hyperelliptic curve $a$ and a meromorphic differential on this curve $b$. Not all polynomials correspond to spectral curves of periodic solutions of the sinh-Gordon equation. In the next section we will investigate which ones do. Instead of working with $\lambda$, we now introduce the transformed spectral parameter $\kappa = i\frac{1-\lambda}{1+\lambda}$ which will make our work easier. Let $\kappa = \pm i$ at $y^{\pm}$ with $\sigma * \kappa = \kappa, ta * \overline{\kappa} = \kappa, \rho * \overline{\kappa} = \kappa$. Define $a(\kappa)$ by

$$a(\kappa) = \sum_{i=1}^{g} (\kappa - \alpha_i)(\kappa - \overline{\alpha_i}), \tag{5.6}$$

with pairwise different branch points $a_i, i, ..., g \in \{\kappa \in \mathbb{C} : Im(\kappa) < 0\}$. This is equivalent to $a_i, i, ..., g \in \{\lambda \in \mathbb{C} : |\lambda|\langle 1\}$. We have $\eta * \overline{a} = a, \rho * \overline{a} = a$. The hyperellitptic curve is then described by

$$\nu^2 = (\kappa^2 + 1)a(\kappa). \tag{5.7}$$

Note that for $Im(\kappa) = 0$ $a(\kappa) > 0$ and $\eta * \overline{\nu} = -\nu, \rho * \overline{\nu}, \sigma * \nu = -\nu$. We now introduce two polynomials with real coefficients $b_i$,

$$b_i(\kappa) = \frac{1}{\pi i}\nu(\kappa^2 + 1)\frac{\partial}{\partial\kappa} ln\mu_i, i = 1, 2 \tag{5.8}$$

of degree $g + 1$. As $b_i$ clearly satisfy

$$\eta * \overline{b_i} = -b_i, \quad ord_{\kappa=\pm i} = \frac{g}{2}$$

we can write $d\, ln(\mu)_i$ as

$$d\, ln(\mu_i) = \pi i \frac{b_i(\kappa)}{\nu(\kappa^2 + 1)} d\kappa. \tag{5.9}$$

We still have a freedom in the choice. To overcome this we choose $\kappa_0 \in C_s, \kappa_1 = \overline{\kappa_0}$ and take the unique $\kappa$ corresponding to theorem 5.14. Then $y_1$ and $y_2 = \sigma(y_1)$ correspond to the two points over $\kappa = \kappa_0$ and $y_3$ and $y_4 = \sigma(y_3)$ to the two points over $\kappa = \kappa_1 = \overline{\kappa_0}$.

**Remark 5.15.** We choose $\kappa = \kappa_1 = \overline{\kappa_0}$ because we know that, for $\lambda_0$, we have $\lambda_1 = 1/\overline{\lambda_0}$. $\kappa = i\frac{1-\lambda}{1+\lambda}$ implies

$$\begin{aligned} \kappa_0 &= i\frac{1 - \lambda_0}{1 + \lambda_0} \\ \kappa_1 &= i\frac{1 - \frac{1}{\overline{\lambda_0}}}{1 + \frac{1}{\overline{\lambda_0}}} \\ &= i\frac{\frac{\overline{\lambda_0}-1}{\overline{\lambda_0}}}{\frac{\overline{\lambda_0}+1}{\overline{\lambda_0}}} \\ &= i\frac{\overline{\lambda_0} - 1}{\overline{\lambda_0} + 1} = \overline{\kappa_0}. \end{aligned}$$

Summarizing the above conclusions, we can exactly formulate which data is sufficient to describe a torus in $\mathbb{H}^3$.

**Theorem 5.16.** *By the choice of the parameter* $\kappa$*,*$a(\kappa)$ *and two polynomials* $b_i(\kappa), i = 1, 2$*, as well as one point* $\kappa_0$ *with* $\kappa_1 = \overline{\kappa_0}$*, the resulting spectral curve is the spectral curve of a torus in* $\mathbb{H}^3$*.*

*Proof.* Follows from the observations before. □

## 5.4 Deformation on spectral data

In this section, the set of data describing a CMC in tori in the 3-hyperbolic space is studied. We show that one is able to induce deformations on this spectral data and obtain a one-parameter family of spectral curves for every given spectral curve. The deformation can be represented by a system of ordinary linear differential equations which we retrieve based on the results of the last section. With other words, the deformation is defined by a system of ordinary differential equations on the spectral data. We follow closely the work of [32], which established period preserving deformations of the spectral curve.

Recall from the last section that

$$d\ ln(\mu_i) = \pi i \frac{b_i(\kappa)}{\nu(\kappa^2 + 1)} d\kappa.$$

For the purpose of parameterizing families of spectral data of periodic finite type solutions, we introduce the deformation parameter $t$. From now on we view all functions derived before as functions of $\kappa$ and $t$. From 5.14 we can conclude that $\partial_t ln \mu_i$ is meromorphic on the the corresponding family with possible poles only at the branch points of $C$. Or, equivalently at the zeros of $a$ and at $\kappa = \pm i$.

We make the Ansatz

$$\partial_t\ ln(\mu_i) = \pi i \frac{c_i(\kappa)}{\nu} \tag{5.10}$$

where $c_i$ are real polynomials of degree of at most $g + 1$.

Now define the differential $\omega$ as

$$\begin{aligned}\omega &= \partial_t\, ln(\mu_1) d\, ln(\mu_2) - \partial_t\, ln(\mu_2) d\, ln(\mu_1)\\ &= -\pi^2 \frac{c_1(\kappa)}{\nu}\frac{b_2(\kappa)}{\nu(\kappa^2+1)}d\kappa - \frac{c_2(\kappa)}{\nu}\frac{b_1(\kappa)}{\nu(\kappa^2+1)}d\kappa\\ &= -\pi^2 \frac{c_1(\kappa)b_2(\kappa) - c_2(\kappa)b_1(\kappa)}{\nu^2(\kappa^2+1)}d\kappa.\end{aligned}$$

**Remark 5.17.** For a similar, but more general, approach see [26].

$\omega$ is a meromorphic 1-form on $Y$ of the form

$$\omega \backsim \frac{(\kappa-\kappa_0)(\kappa+\overline{\kappa_0})}{(\kappa^2+1)^2}d\kappa,$$

as $\omega$ has roots at $\kappa = \kappa_0$ and $\kappa = -\overline{\kappa_0}$, and poles at $\kappa = \pm i$.

Pulling those results together we obtain

$$-\pi^2 \frac{c_1(\kappa)b_2(\kappa) - c_2(\kappa)b_1(\kappa)}{\nu^2(\kappa^2+1)} = \frac{(\kappa-\kappa_0)(\kappa+\overline{\kappa_0})}{(\kappa^2+1)^2}. \tag{5.11}$$

We now use the definition $\nu^2(\kappa^2+1)a(\kappa)$ and differentiate

$$\partial_t\, ln(\mu_i) = \pi i \frac{c_i(\kappa)}{\nu}$$

(5.10) with respect to $\kappa$ (denoting differentiation with respect to $\kappa$ by "′"):

$$\begin{aligned}\partial^2_{t\kappa} ln(\mu_i) &= \pi i \frac{c_i'(\kappa^2+1)a - 4\kappa a c_i - 2c_i(\kappa^2 a')}{\nu^3} &(5.12)\\ &= \pi i \frac{(\kappa^2+1)(ac_i' - 2a'c_i) - 4\kappa a c_i}{\nu^3}. &(5.13)\end{aligned}$$

We repeat this with

$$\partial_k\, ln(\mu_i) = \pi i \frac{b_i(\kappa)}{\nu(\kappa^2+1)}$$

(5.9), but differentiate with respect to $t$ (denoting this by "˙"):

$$\partial^2_{\kappa t}\, ln(\mu_i) = \pi i \frac{2a\dot{b}_i - \dot{a}b_i}{\nu^3}.$$

Since $\partial^2_{t\kappa} = \partial^2_{\kappa t}$ must hold to insure integrability, we have

$$2a(\kappa)\dot{b_i(\kappa)} - b_i(\kappa)\dot{a(\kappa)} = (\kappa^2+1)(a(\kappa)c_i'(\kappa) - 2a'(\kappa)c_i(\kappa)) - 4\kappa a(\kappa)c_i(\kappa). \tag{5.14}$$

From 5.11 we get

$$c_1(\kappa)b_2(\kappa) - c_1(\kappa)b_2(\kappa) = -\frac{1}{\pi^2}(\kappa-\kappa_0)(\kappa+\overline{\kappa_0})a(\kappa). \tag{5.15}$$

Finally we have to determine the requirements on $\dot{\kappa}(t)$, such that the closing conditions are preserved during deformation, i.e.

$$\partial_t(ln(\mu_i)(\kappa_j,t))|_{\kappa_j} = \partial_t\, ln(\mu_i) + \dot{\kappa_j}\partial_\kappa\, ln(\mu_i)|_{\kappa_j} = 0.$$

With

$$\partial_t\, ln(\mu_i) = \pi\frac{c_i(\kappa)}{\nu} \quad and \quad b_i(\kappa) = \frac{1}{\pi i}\nu(\kappa^2+1)\partial_\kappa\, ln(\mu_i)$$

we get

$$\dot{\kappa_j} = -\frac{\partial_t\, ln(\mu_i)}{\partial_\kappa\, ln(\mu_i)}|_{\kappa=\kappa_j} = -(\kappa_j^2+1)\frac{c_i(\kappa_j)}{b_i(\kappa_j)}.$$

In order for the last expression to be well defined, we make the following proposition:

**Proposition 5.18.** The ratio $\frac{c_i(\kappa)}{b_i(\kappa)}$ is well defined as $d\; ln(\mu_{i=1,2})$ have no common roots.

*Proof.* If $d\; ln(\mu_{i=1,2})$ has no common roots, neither do $b_{i=1,2}$. In this case $c_{i=1,2}$ are uniquely determined by (5.15), and $c_i$ admits a representation of the form

$$c_i(\kappa,t) = \sum_{j=0}^{g+1}\gamma_{i,j}\kappa^j.$$

Let $\alpha_j$ be a root of $a$, then 5.14 reads

$$b_i\dot{a} = (\kappa^2+1)2a'c$$

and, at this roots (5.15) becomes

$$c_1(\kappa)b_2(\kappa) - c_1(\kappa)b_2(\kappa) = 0.$$

As $b_{i=1,2}$ has no common roots, one ratio is well defined. Then, by (5.15), the other ratio also must exist, and both must coincide. □

We are now ready to formulate the last proposition based on the previous considerations, close with a short proof.

**Proposition 5.19.** Let $C$ be a spectral curve of a CMC torus in the 3-hyperbolic space. The above deformation (5.14) and (5.15) is well defined if the differentials $d\ ln(\mu_{i=1,2})$ do not have any common roots.

*Proof.* In order to prove this proposition, we have to show that, given

$$(a, b_{i=1,2}, \kappa_{j=0,1}),$$

the equations (5.14) and (5.15) uniquely determine

$$(\dot{a}, \dot{b}_{i=1,2}, c_{i=1,2}).$$

We have already shown that if the differentials

$$d\ ln(\mu_{i=1,2})$$

have no common roots, the ratio

$$\frac{c_i}{b_i}$$

exists. Therefore

$$b_i\dot{a} = (\kappa^2 + 1)2a'c_i$$

is also well defined and uniquely determines $\dot{a}$. Then $\dot{b}_{i=1,2}$ are also unambiguously defined. □

Hence we see that deformations are described by a system of ordinary differential equations.

# 6 Conclusions and outlook

The thesis first started with an introduction to differential geometry. We revisited commonly known concepts and tools as the fundamental forms, Weingarten map, and the different curvature concepts.

In particular we were interested in CMC surfaces and therefore set $H$ as constant. A surface is parameterized by an immersion and we saw that it is uniquely determined by $u$, $Q$ and $H$. The results were carried out for $\mathbb{R}^3$ as well as for $\mathbb{H}^3$. In both cases we saw that the corresponding compatibility equation leads to the Gauss-Codazzi equations and the immersion can be recovered by the Sym-Bobenko formula avoiding integration. The concept of moving frames was introduced and we calculated the Lax pair in terms of $3 \times 3$-matrices as well as $2 \times 2$-matrices for $\mathbb{R}^3$ and $\mathbb{H}^3$.

Although we reviewed commonly known results and concepts in Chapter 2, we tried to prove most the of claimed results and not shorten any calculations with the intention that a following thesis pursuing possible further research directions would not need a great amount of time to get familiar with the area.

CMC surfaces in $\mathbb{R}^3$ and $\mathbb{H}^3$ are linked to the sinh-Gordon equation, which motivated Chapter 4 were solutions of the sinh-Gordon were studied. Therefore Chapter 3 reviewing Riemann surface theory and introducing theta-functions was included first. In Chapter 3 we started with reviewing basic Riemann surface theory and then introduced Abelian differentials. In the next section we studied divisors and stated Abels theorem and the Riemann Roch theorem. Having the concept of divisors one can determine the dimension of meromorphic functions on a Riemann surface. We focused on regarding results on the special case of hyperelliptic curves to facilitate the comprehension for Chapters 4 and 5. Eventually, we presented the theta-function and reviewed all the important properties, especially the transformation laws that the theta-function obeys. We already introduced a special constructed function and stated an important result regarding the zeros of this functions, which we could then use in the next chapter.

With the necessary tools of Riemann surface theory and theta-functions, Chapter 4 investigated Baker-Akhiezer functions, spectral theory of operators, hyperelliptic curves and the associated problems. In a first step it was shown that arriving at the stationary equations of the commuting flows, one obtains a special eigenvalue problem and a hyperellitptic Riemann surface due to the characteristic polynom of the operator. In the section of higher commuting flows we closely followed [6] and included it for the sake of completeness. We reworked then the problem of finding the solution of the sinh-Gordon equation from another side. Namely, we introduced the Baker-Akhiezer function, a function with special properties, which uniquely exists on the hyperelliptic Riemann surface. Parallell, this Baker-Akhiezer function solves the special eigenvalue problem of the operator and is therefore the tool to solve the sinh-Gordon equation. The Baker-Akhiezer function is constructed by terms of theta-functions and we could prove that it has the required properties, introducing Abelian differentials. With the help of the Riemann-Roch theorem we could prove the uniquely existenceof the Baker-Akhiezer function. Using the transformation laws the theta-function obeys some of the properties of the Baker-Akhiezer function could be proven as well. The right behavior at the singularities could be reduced to the behavior of the Abelian differentials there. At the end we could deducted an exact formula solving the sinh-Gordon equation closely following [6].

As mentioned in the introduction, Chapter 4 is related to the field of integrable systems, a highly sophisticated area. A lot of literature availabe is either in Russian or very brief in terms of the lack of detailed calculations and explanations. This is why we had chosen the most intuitive approach, modifying the approach of [6], and carrying out mostly every prof as well as calculation. For a better understanding we gave a long verbal introduction, trying to explain without formulas the approach chosen in general solving nonlinear equations as well as our case of the sinh-Gordon equation.

The other main research direction of this thesis was given by the purpose of studying CMC tori, leading to double-periodic solutions of the sinh-Gordon equation. Immersions of CMC tori obey a special closing condition in order to close to become a torus. We therefore introduced the concept of monodromy, on which we imposed the closing condition in $\mathbb{H}^3$ in order to retrieve a CMC torus from the Sym-Bobenko formula. We then introduced a slightly different description of the spectral curve (compared to the previous chapter) and once more the research was reduced to the investigation of the spectral curve. We included a section proposing useful properties of the monodromy. Afterwards we turned our attention to the condition one has to impose on the spectral curve in order to study CMC tori in $\mathbb{H}^3$. We could narrow this task down by four propositions5.11-5.14. It is worth mentioning that the first three propositions, i.e. 5.11, 5.12 and 5.13 are the same for $\mathbb{R}^3$ and $S^3$. Only proposition 5.14 differs in the three cases $\mathbb{R}^3, S^3, \mathbb{H}^3$, which is due to the differing Sym-Bobenko formulas. After we identified the constraints to impose on the spectral curve we shifted our attention to the spectral data describing the spectral curve. Following the Ansatz of [32] we saw that the spectral curve is described, by a hyperelliptic curve, two meromorphic differentials and a spectral parameter.

Eventually, we were able to study deformations on the spectral data and showed how to get a one-parameter family of spectral curves for every spectral curve. This was realized by introducing a deformation parameter and a calculation then showed that deformations can be represented by a system of ordinary differential equations on the spectral data. From our knowledge this has not be done for $\mathbb{H}^3$ in this notation so far, a very similar proceeding can be found in [32] for $S^3$.

In a following work it would one could investigate a classification of CMC tori in $\mathbb{H}^3$ in the notation of this thesis as it was done for $S^3$ in [32]. One could expect similar results regarding onesighted embedded Alexandrov tori.

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# Declaration

I declare that this thesis is my own original work, conducted under the supervision of Prof. Dr. M. Schmidt written without help of a third party. It is submitted for the degree of Diplom-Mathematikerin at the Department of Mathemtices and Informatics at the University of Mannheim, Germany. No part of this research has been submitted in the past or is being submitted, for a degree or examination at any other university. It has been my intention throughout this work to give reference to stated results and credit to the work of others.

Mannheim, 22.09.2008