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Diplomarbeit

Simply periodic solutions of the Sinh-Gordon equation and numerical computations of the spectral curve

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Contents

1 Introduction

Surfaces of constant mean curvature (CMC-surfaces for short) may be thought as the mathematical description of soap films [3]. The study of CMC-surfaces into one of the three space-forms $\mathbb{R}^3$, $\mathbb{H}^3$ and $\mathbb{S}^3$ has been of big interest in the past decades and there have been mainly two approaches to classify and construct these surfaces in the three ambient spaces. Pinkall and Sterling use finite type solutions of the Sinh-Gordon equation to classify CMC-tori in $\mathbb{R}^3$ [9] [1]. Another approach was developed by F. Pedit, H. Wu and J. Dorfmeister and is named after them the DPW method [2] [3] [1]. It allows the costruction of CMC-immersions from a meromorphic and a holomorphiv function, for which reason it can be called a Weierstrass type representation of CMC-immersions [2].

In this work we will investigate isoperiodic deformations of the spectral data of a conformally embedded CMC surface into $\mathbb{S}^3$. In this setting the Sinh-Gordon equation arises naturally: Any solution of the Sinh-Gordon equation determines a conformal CMC-immersion into $\mathbb{S}^3$ up to isometry. The particular solution of the Sinh-Gordon equation arises as the (induced and conformal) metric of the immersion. In this work we will only consider simply periodic solutions of the Sinh-Gordon equation that are, in addition, of finite type.

Pinkall and Sterling [9] and independently Hitchin [**?**] proved that doubly periodic solutions of the Sinh-Gordon equation are of finite type. Thus all CMC tori are of finite type. By relaxing one period we define [7]

Definition 1.1. *The CMC cylinders with constant Hopf differential and whose metric is a periodic solution of finite type of the Sinh-Gordon equation will be called CMC cylinders of finite type. We call the corresponding solution of the Sinh-Gordon equation a simply periodic solution.*

Under cetrain assumptions one has a $1 : 1$ correspondence between a CMC-immersion (into $\mathbb{S}^3$), a solution of the Sinh-Gordon equation and the spectral data of a (simply) periodic solution of the Sinh-Gordon equation of finite type. The spectral data parametrize the space of all periodic solutions of finite type of the Sinh-Gordon equation. In this work we finally want to show how one could start with the spectral data of a most simple solution of the Sinh-Gordon equation, the vacuum solution, and deform these data continuously by preserving certain properties of solution. To say it more precisely: When deforming the spectral data we want to ensure that we do not leave the moduli space, which means that the deformed spectral data also correspond to a solution of the

Sinh-Gordon equation with certain properties.
In this diploma thesis we mainly follow [4] and [7]. We now give short descriptions of the contents of the chapters.
In chapter one we explain the relationship between the zero curvature condition, the Gauß-Codazzi equations and the Sinh-Gordon equation. The main goal of this chapter is to deduce a λ-dependent one form α_λ that also depends on u and whose integrability condition, the Maurer Cartan equation, is fullfilled if and only if u solves the Sinh-Gordon equation. The properties of this one form will be described in theorem 2.23. We also introduce the concept of (extended) moving frames corresponding to one forms that fullfill the Maurer-Cartan equation. They are, roughly spoken, the integral of such one-forms and will represent immersions into $\mathbb{S}^3 \subset \mathbb{R}^4$ with certain properties. In the context of the Maurer-Cartan equation as an integrability condition we will make use of the concepts of Lie groups and Lie algebras.
In chapter two we give attention to the periodicity condition of the solution u of the Sinh-Gordon equation. We introduce the Monodromy operator that encodes how the moving frame F_λ varies when traversing a period of u and derive certain conditions for the Monodromy that must hold. We then characterize finite type solutions of the Sinh-Gordon equation and asign the spectral curve to the monodromy. We characterize properties of the spectral curve that must hold if the monodromy belongs to a CMC cylinder of finite type.
In chapter three we define the spectral data of a CMC cylinder of finite type in $\mathbb{S}^3$ and we define isoperiodic deformations of these data. We introduce local coordinates to deform these data isoperiodically. And we derive ordinary differential equations that meet the conditions that were set up.
In chapter five we discuss how to deform the spectral data numerically. One example will be calculated explicitely with the software Wolfram Mathematica.

2 CMC-surfaces in $\mathbb{S}^3$

Our goal is to describe surfaces of constant mean curvature in $\mathbb{S}^3$ (or CMC surface for short). These are given by conformal immersions $f : \mathbb{R}^2 \cong \mathbb{C} \to \mathbb{S}^3 \subset \mathbb{R}^4$ with the additional feature to have constant mean curvature H.
We may think of $\mathbb{S}^3 \subset \mathbb{R}^4$ to be equipped with the standard Euclidean metric and therefore with the euclidean metric tensor g. Then the condition that the immersion $f : \mathbb{R}^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$ is conformal may be expressed as $f^*g = \rho g$ where ρ is some positive function defined on $\mathbb{R}^2 \cong \mathbb{C}$.
We are going to identify $\mathbb{S}^3$ with $SU(2)$ which is a Lie group so that we may describe the tangent space of $SU(2)$ by the Cartesian product $SU(2) \times su_2$, where the second component denotes the Lie algebra of $SU(2)$. In this setting the canonical one-form is the **Maurer-Cartan form** which is the linear map $\theta_a : T_aSU(2) \to su_2$ into the Lie algebra of $SU(2)$. We will pull back θ via the immersion f and obtain the **connection form** $f^*\theta =: \omega$ on $\mathbb{R}^2 \cong \mathbb{C}$.
The flatness of this connection ω is expressed by the **zero-curvature condition**. It corresponds to the vanishing of the **curvature form**

$$d\omega + \omega \wedge \omega.$$

It ensures the integrability of ω and is equivalent to the **Gauß-Codazzi equations**. By the main theorem of surface theory the equations of Gauß and Codazzi ensure the existence of an immersion f up to rigid motions. They read

$$4u_{z\bar{z}} - Q\bar{Q}e^{-2u} - 4(H^2 + \kappa)e^{2u} = 0 \qquad \kappa \in \{-1, 0, 1\}$$
$$Q_{\bar{z}} = e^{2u}H_z$$

κ describes the **sectional curvature**. The value of κ depends on the ambient space: If $\kappa = -1, 0, 1$ then f describes an immersion into $\mathbb{H}^3, \mathbb{R}^3, \mathbb{S}^3$ and the functions e^{2u}, Q, H denote the conformal factor, the Hopf differential und the mean curvature of the immersion f. In the case where H is constant the Codazzi equation gives $Q_{\bar{z}} = 0$, that is Q must be holomorphic.
One result we will achieve in this chapter is that the zero curvature condition for CMC-immersions into $\mathbb{S}^3$ equals the sinh-Gordon equation under certain conditions:

$$\frac{1}{2}\triangle u + \sinh(2u) = 0 \qquad (2.0.1)$$

with $u = u(x, y)$ being equivalent to

$$\partial\bar{\partial}2u + \sinh(2u) = 0 \tag{2.0.2}$$

with $u = u(z, \bar{z})$. This may be easily proved by using the definition of the Wirtinger operators

$$\partial := \partial_z := \frac{1}{2}(\partial_x - i\partial_y) \qquad \bar{\partial} := \partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y)$$

This is the reason why the sinh-Gordon equation appears quite naturally in this setting. One of the main goals of this chapter is to construct a λ-dependent family of matrix-valued one-forms, $\lambda \in \mathbb{C}^\times$, that satisfies the zero-curvature condition as well. We will denote the final one-form by α_λ. This will be the starting point for the next chapter.

2.1 Basics of Lie groups and Lie algebras

Definition 2.1 (Lie group). *A Lie group is a group G being also a smooth manifold in the way that it is compatible with the group-structure in the following sense: For $g, h \in G$ both*

1. *(left-) multiplication:* $L_g : G \times G \to G \; ; \; h \overset{L_g}{\mapsto} gh$
2. *inversion:* $(\cdot)^{-1} : G \to G \; ; \; g \overset{(\cdot)^{-1}}{\mapsto} g^{-1}$

are smooth operations on the manifold.

We now will identify $\mathbb{S}^3$ with $SU_2(\mathbb{C})$ via the isomorphism

$$\begin{aligned} \{ \quad x & \in \mathbb{R}^4 | \|x\| = 1\} \cong \{x \in \mathbb{C}^2 | \|x\| = 1\} \\ & \cong \quad SU_2 = \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \middle| \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\} \end{aligned}$$

From this isomorphism one can see that the group $SU(2)$ is compact and connected. With

$$U_2(\mathbb{C}) = U(2) = \{M \in Gl(2, \mathbb{C}) | M^* M = \mathbb{1}\} \text{ with } M^* := \bar{M}^T \tag{2.1.1}$$

$$Sl_2(\mathbb{C}) = Sl(2) = \{M \in Gl(2, \mathbb{C}) | \det(M) = 1\} \tag{2.1.2}$$

we may write $SU_2(\mathbb{C})$ as $SU(2) = U_2(\mathbb{C}) \cap Sl_2(\mathbb{C})$. To any Lie group there corresponds a Lie algebra and a 1-form called the Maurer-Cartan form. We state the correspondence between Lie groups and Lie algebras in the following lemma.

Lemma 2.2 (Lie group - Lie algebra relation)**.** *The tangent space at the identity of a Lie group has the structure of a Lie algebra, and this Lie algebra determines the local structure of the Lie group via the exponential map [11].*

Lemma 2.3 (The Lie algebra of $SU(2)$)**.** *The Lie algebra of $SU(2)$ consists of the traceless and anti-hermitian 2×2 matrices*

$$su_2 = \left\{ \begin{pmatrix} ix & -\bar{\beta} \\ \beta & -ix \end{pmatrix} \middle| x \in \mathbb{R}; \beta \in \mathbb{C} \right\} \tag{2.1.3}$$

It has the following basis, often denoted as ***Pauli matrices***

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.1.4}$$

Proof. We will calculate the Lie algebras of $U(2)$ and $Sl(2)$ explicitly.
For $U(2)$ we define the map $F : M^{2\times 2} \to M^{2\times 2}$ by $F(A) = \bar{A}^T \cdot A$. Then $U(2) = F^{-1}(\mathbb{1})$.
For any tangent element v of $SU(2)$ at the identity we may calculate

$$\begin{aligned} F^{'}(A)v &= \lim_{h\to 0} \frac{1}{h}(F(A+hv) - F(A)) = \lim_{h\to 0} \frac{1}{h}\left((\overline{(A+hv)}^T (A+hv)) - \bar{A}^T A) \right) \\ &= \lim_{h\to 0} \frac{1}{h}\left(h\overline{v}^T A + \overline{A}^T hv + h^2 \overline{v}^T v \right) = \overline{v}^T A + \overline{A}^T v \end{aligned}$$

and in the identity element $A = \mathbb{1}$ the equation reads $0 = \overline{v}^T + v$ which gives the condition for the Lie algebra of $U(2)$.
For $Sl(2)$ as in (2.1.2) we consider the map $F : M^{2\times 2} \to \mathbb{C}$ given by $F(M) = \det(M)$. Then $F^{-1}(1)$ is the Lie group $Sl(2)$. Now we make use of the fact that for any (real or complex) matrix A the following equation holds:

$$\det(e^A) = e^{\text{trace}(A)}$$

So in our case the following equation must hold:

$$1 = \det(e^A) = e^{\text{trace}(A)}$$

For any compact and connected Lie group we have a unique map which is one to one and that reads $e^g = G$. So the Lie algebra of $Sl(2)$ may be characterized by the traceless 2 by 2 matrices.
Putting things together we find that $su_2 = sl_2 \cap u_2$ are the anti-Hermitean and traceless 2 by 2 matrices. One verifies that (2.1.4) is a basis of $sl_2(\mathbb{C})$ by taking into account equation (2.1.3). Any element of (2.1.3) may be lineary combined by (2.1.4). ☐

Remark 2.4 (The Lie algebras sl_2 and u_2)**.** *The Lie algebras $sl_2(\mathbb{C})$ and $u_2(\mathbb{C})$ may be characterized by*

$$u_2(\mathbb{C}) = \{M \in \mathbb{C}^{2\times2} | -\overline{M}^T = M\} \tag{2.1.5}$$

and

$$sl_2(\mathbb{C}) = \{M \in \mathbb{C}^{2\times2} | \sum_1^2 M_{ii} = 0\} \tag{2.1.6}$$

Definition 2.5 (Killing form)**.** *The Killing form is an inner product on a finite dimensional Lie algebra g defined by*

$$B(X, Y) = trace(ad(X)ad(Y)) \tag{2.1.7}$$

Definition 2.6 (Killing form for su_2)**.** *The Killing form for su_2 is $\frac{1}{2}tr(X \cdot Y)$ for elements X, Y in su_2.*

Now we may identify the tangent space of $SU(2)$ with $SU(2) \times su_2$.

Lemma 2.7 (The tangent spaces of SU_2 with the Killing form and $\mathbb{S}^3 \subset \mathbb{R}^4$ are isometric)**.** *$TSU_2 = SU_2 \times su_2$ and $T\mathbb{S}^3 = \mathbb{S}^3 \times \mathbb{R}^3 \subset \mathbb{R}^4 \times \mathbb{R}^3$. The following diagram commutes:*

$$\begin{array}{ccc} \mathbb{S}^3 \times \mathbb{R}^3 & \xrightarrow{\cong} & SU(2) \times su_2 \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{S}^3 & \xrightarrow{\cong} & SU(2) \end{array}$$

We consider $\mathbb{R}^3$ to be equipped with the Euclidean standard metric. su_2 is 3-dimensional and is spanned by the basis of traceless matrices (2.1.4)*. If we identify e_i with σ_i the isometry results from a simple computation.*

2.2 Maurer-Cartan form and complexified tangent space

For any Lie group with associated Lie algebra we have the corresponding (left-) **Maurer-Cartan-form**

$$\theta : TSU_2 \cong SU_2 \times su_2 \longrightarrow su_2, \; v_g \mapsto dL_g^{-1}(v_g) \tag{2.2.1}$$

Here g denotes the element of the group and v_g is an element of the tangent space of $G = SU(2)$ at $g \in SU(2)$. So the Maurer-Cartan form maps elements of the tangent space at any element $g \in G$ to the tangent space of the identity element of G.
The Maurer-Cartan form satisfies the **Maurer-Cartan equation**

$$2d\theta + [\theta \wedge \theta] = 0 \tag{2.2.2}$$

with

$$[\alpha \wedge \beta](X,Y) := [\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)] = [\alpha(X), \beta(Y)] + [\beta(X), \alpha(Y)].$$

If N is a connected and simply connected manifold, then for any map $f : M \subset N \longrightarrow SU_2$ the pullback $\omega = f^*\theta,\ \omega \in \Omega^1(M, su_2)$ where M denotes an open subset of N also satisfies the Maurer-Cartan-equation (2.2.2). But also the converse statement holds:

Theorem 2.8. *Any $\omega \in \Omega^1(N, su_2)$ satisfying (2.2.2) integrates to a smooth map $f : N \longrightarrow SU_2$ with $\omega = f^*\theta$.*

So we may interpret the Maurer-Cartan equation (2.2.2) as an **integrability condition** as follows:
The system

$$df = f\omega\ ;\ f(0) = \mathbb{1} \tag{2.2.3}$$

has a unique solution $f : M \subset \mathbb{C} \longrightarrow SU(2)$. We will see in remark (2.22) that we may transform the Maurer-Cartan equation into a **Lax-pair equation**. In this form the integrability condition will read

$$f_{z\bar{z}} = f_{\bar{z}z}$$

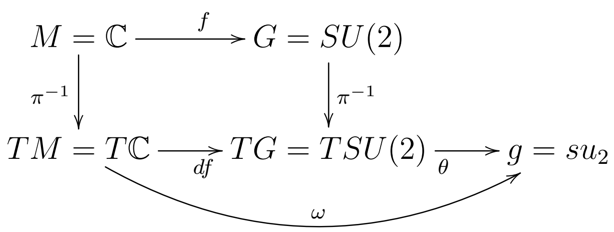
We will be interested in the case where $M = \mathbb{C} \simeq \mathbb{R}^2$.
If we work in complex coordinates $(z, \bar{z})$ where $M = \mathbb{C}$ it is useful to complexify the tangent space of $M, TM^{\mathbb{C}} = TM' \oplus TM''$ where TM' denotes the $(1,0)$ part and TM'' denotes the $(0,1)$ part of the complexified tangent space.
Writing the (complex total) differential as $d = \partial + \bar{\partial}$ we may dually decompose the space of one-forms

$$\Omega^1(M, g^{\mathbb{C}}) = \Omega^{'}(M, g^{\mathbb{C}}) \oplus \Omega^{''}(M, g^{\mathbb{C}}).$$

So in these coordinates the form ω splits into $(1,0)$ and $(0,1)$ parts, $\omega = \omega^{'} + \omega^{''}$.
The tangent space of $SU(2)$ is su_2 and $su_2^{\mathbb{C}} = sl_2(\mathbb{C})$. For the complexified tangent space of $SU(2)$ we fix the following basis of $sl_2(\mathbb{C})$:

$$\epsilon_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\ ,\ \epsilon_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\ ,\ \epsilon = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{2.2.4}$$

We now resume this section with the following diagram:



If the group G is Gl_n the Maurer-Cartan form at a point $v \in T_g(G)$ is $\omega(g,v) = L_{g^{-1}*}(g,v) = (e, g^{-1}v)$. So for the choice of any base point $(g,\cdot) \in T_g(G)$ the Maurer-Cartan form is trivial in the sense, that this base point is mapped to the identity element e.
Therefore the pulled back Maurer-Cartan form ω is a map from $\mathbb{C}$ (being the second component of $T\mathbb{C} = \mathbb{C} \times \mathbb{C}$) to su_2 and we may write ω in the form

$$\omega = f^{-1}df \tag{2.2.5}$$

Remark 2.9. *One may easily see that the form ω satisfies the Maurer-Cartan equation by differentiating*

$$0 = d(df) = df \wedge \omega + f d\omega = f(\omega \wedge \omega + d\omega)$$

Remark 2.10 (Outlook)**.** *In the previous situation we have $\omega = f^{-1}df$. If we split ω into its complex conjugated parts dz and $d\bar{z}$, we may write $\omega = \omega' dz + \omega'' d\bar{z}$. Then for ω' and ω'' the **Lax-equation** holds:*

$$\omega'_{\bar{z}} - \omega''_z - [\omega', \omega''] = 0$$

This equivalence will be proved in 2.21.

2.3 Some remarks on classical differential geometry in Euclidean coordinates

In this section we will consider immersions from $M \subset \mathbb{R}^2 \simeq \mathbb{C}$ to $SU(2) \cong \mathbb{S}^3 \subset \mathbb{R}^4$. We will denote the immersions by $f : M \longrightarrow SU(2)$ and the immersed surface we shall call Y.

Definition 2.11 (The first fundamental form and induced metric)**.** *Let f be an immersion from $M \subset \mathbb{R}^2 \longrightarrow SU(2) \simeq \mathbb{S}^3 \subset \mathbb{R}^4$and let $q = f(p)$ be a point of $Y = f(M)$, $p \in M$. By the **induced metric** we denote the 2×2 matrix*

$$g(p) = g_{ij}(p) = \begin{pmatrix} \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial x} \rangle_p & \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle_p \\ \langle \frac{\partial f}{\partial y}, \frac{\partial f}{\partial x} \rangle_p & \langle \frac{\partial f}{\partial y}, \frac{\partial f}{\partial y} \rangle_p \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

*It defines a **positive definite bilinear form on** T_pM. This metric is also called the induced metric since it is induced by the standard Euclidean metric on $\mathbb{R}^4$ via the differential df. Here we use the isomorphism explained in lemma 2.7. It is defined on the tangent space T_pM of $p \in M$. For any tangent vectors v, w at $p \in M$ the first fundamental form acts as*

$$\langle v, w \rangle_p = v^T df^T \cdot df w = \langle df_p(v), df_p(w) \rangle_{\langle \cdot,\cdot \rangle_{\mathbb{R}^4}}$$

*The quadratic differential form that is associated to the induced metric $g = g_{ij}$ may be written as $ds^2 = \sum_{i,j=1}^2 g_{ij}(p) dx^i dx^j$ and denotes the **first fundamental form**.*

We call an immersion conformal if its first fundamental form is **conformally equivalent** to the Euclidean standard metric, that is

$$\langle\ ;\ \rangle = e^{2u} \cdot \mathbb{1} = v^2 \mathbb{1}$$

where $e^{2u} = v^2$ is the **conformal factor**. So f is a conformal immersion if the entries of the matrix g are of the form

$$E = \langle f_x, f_x \rangle = G = \langle f_y, f_y \rangle = e^{2u}, F = 0\ ;\ \|f_x\| = \sqrt{\langle f_x, f_x \rangle} = \sqrt{e^{2u}} = e^u$$

and in this case we may write the metric as

$$g = Edx^2 + Gdy^2 = e^{2u}dx^2 + e^{2u}dy^2 \tag{2.3.1}$$

A conformal immersion may be called an orthogonal immersion because it ensures that the partial derivatives spanning the tangent space of $T_{f(p)}Y$ are orthogonal everywhere. If we use complex parameters conformality translates as follows:

Proposition 2.12 (complex conformal immersion)**.** *In complex parameters an immersion $f(z, \overline{z})$ is conformal if*

$$\langle\ ;\ \rangle_{\mathbb{C}} = e^{2u} \cdot \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = g_{ij}^{\mathbb{C}} \tag{2.3.2}$$

Using the notation of the first fundamental form the last statement may be achieved by calculating $dz \cdot d\bar{z} = (dx + i \cdot dy) \cdot (dx - i \cdot dy) = dx^2 + dy^2$. One has

$$\begin{pmatrix} dz & d\bar{z} \end{pmatrix} \cdot \begin{pmatrix} 0 & \frac{e^{2u}}{2} \\ \frac{e^{2u}}{2} & 0 \end{pmatrix} \cdot \begin{pmatrix} dz \\ d\bar{z} \end{pmatrix} = 2 \cdot (\frac{e^{2u}}{2}dx^2 + \frac{e^{2u}}{2}dy^2)$$

So both, the matrix and the conformal factor, change when one uses complex parameters.

Definition 2.13 (Gauß map and Weingarten operator)**.** *Suppose M is oriented. The unit normal vector field to $df_p(T_pM), p \in M$ in $\mathbb{S}^3 \subset \mathbb{R}^4$ defines a mapping (Gauß map) $\nu_p : M \to T_{f(p)}^{\perp}Y \subset \mathbb{R}^4$.*
If we consider the unit normal vector field ν of M as a vector valued function over M we can define the Weingarten map to be the differential

$$W_p := d\nu_p : T_pM \longrightarrow T_{f(p)}Y \subset \mathbb{R}^4. \tag{2.3.3}$$

The tangent space at $\nu(p)$ consists of all vectors perpendicular to $\nu(p)$, and so it can be identified with $T_{f(p)}Y$.

We are now in the situation to define the second fundamental form.

Definition 2.14 (The second fundamental form). *The second fundamental form is defined to be the symmetric bilinear form on T_pM for any two tangent vectors $v, w \in T_pM$. It is given by*

$$II_p(v,w) := -\langle W_p v, df_p w\rangle = -\langle df_p v, W_p w\rangle.$$

We again may assign a 2×2 matrix to the second fundamental form, usually denoted by b, and we may calculate its coefficients at any point $p \in M$ by means of

$$b_{ij} = -\langle \partial_i \nu, \partial_j f\rangle = \langle \nu, \partial_i \partial_j f\rangle = \begin{pmatrix} L & M \\ M & N \end{pmatrix}.$$

We derived the last equation by differentiating the identity $\langle \nu, \partial_i f\rangle = 0$ at any point $p \in M$. So the coefficients (considered as functions depending on $p \in M$) L, M and N are given by the inner producs $L = \langle -f_x, \nu_x\rangle, M = \langle -f_x, \nu_y\rangle$ and $N = \langle -f_y, \nu_y\rangle$.
In terms of L, M, N the second fundamental form is often written as

$$II = Ldx^2 + 2Mdxdy + Ndy^2.$$

From the symmetry of the operator W_p it follows, that W_p is diagonizable with real eigenvalues. The eigenvalues $k_i(p), i = 1, 2$ of the Weingarten map are called the **principal curvatures of f at the point** p.

Remark 2.15. *The Weingarten map may equally be defined as*

$$W_p = g_p^{-1} b_p.$$

Definition 2.16 (Gaussian curvature and mean curvature). *The determinant and the half-trace of the Weingarten map $g^{-1}b$ of an immersion f are defined to be the Gaussian curvature K and the mean curvature H of the immersion f.*
We call an immersed surface of ***constant mean curvature*** *(in short: CMC-surface) if the function H is constant.*

Having defined the first and second fundamental form of an immersion f we may state the

Theorem 2.17 (Fundamental theorem of surface theory [6]). *To any two immersions f_1 and f_2 that induce the same first and second fundamental form there exists an Euclidean motion $\alpha \in E(4), \alpha(x) = A(x) + b$ with $A \in O(4)$, such that $f_1 = \alpha \circ f_2$.*

Translated in complex coordinates $(x, y) \to (z, \bar{z}) = (x + iy, x - iy)$ we may write the second fundamental form as

$$b = Qdz^2 + \tilde{H}dzd\bar{z} + \bar{Q}d\bar{z}^2. \tag{2.3.4}$$

The symmetric 2-differential Qdz^2 is called the **Hopf differential** of the immersion f. In terms of the coefficients of the second fundamental form the functions $\tilde{H}$ and Q are given by

$$\tilde{H} = \frac{1}{2}(b_{11} + b_{22}) \ , \ Q = \frac{1}{4}(b_{11} - b_{22} - ib_{12} - ib_{21}) \ , \ \tilde{H} = e^{2u} \cdot H \tag{2.3.5}$$

The formula for the mean curvature H we derived by calculating the halftrace of the Weingarten map $g^{-1}b$, g and b both in complex coordinates, which gives

$$H = \text{trace}\left(\frac{g^{-1}b}{2}\right) = \frac{\text{trace}}{2}\left(\begin{pmatrix} 0 & \frac{2}{e^{2u}} \\ \frac{2}{e^{2u}} & 0 \end{pmatrix}\begin{pmatrix} Q & \frac{\tilde{H}}{2} \\ \frac{\tilde{H}}{2} & \bar{Q} \end{pmatrix}\right) = \frac{\tilde{H}}{e^{2u}}.$$

By now we know, that the three functions u, Q and H completely determine the first and second fundamental form in case of a conformal immersion f. And by the main theorem of surface theory we therefore know that these three functions determine an immersion f up to rigid motions.
The existence of such an immersion is equivalent to the vanishing of the curvature form $d\omega + \omega \wedge \omega = 0$ with $\omega = f^{-1}df$. In the following sections we will investigate the one-form ω and the integrability condition. We will argue how the conformal factor and the functions Q and H appear in this more abstract setting.

2.4 Conformality in terms of the form ω

Let now f be the (conformal) immersion (with constant mean curvature) we are looking for:

$$f : M \subset \mathbb{C} \cong \mathbb{R}^2 \longrightarrow SU_2 \cong \mathbb{S}^3 \subset \mathbb{R}^4.$$

We are now in the situation to translate the properties of our immersion $f : M \subset \mathbb{C} \longrightarrow SU(2)$ to the one-form ω. In particular we translate the properties of being conformal and to be of constant mean curvature to the 1-form $\omega = f^{-1}df$ which is the Maurer-Cartan form associated to f. In the next section we will see how the condition of being of constant mean curvature translates into the form ω.

Proposition 2.18. *f is conformal if and only if the $(1,0)$ part is isotropic, $\langle w', w' \rangle = 0$. Then the conformal factor is given by $v^2 = 2\langle \omega', \omega'' \rangle$.*

Proof. From equation (2.3.2) we can deduce the first part of the proposition. Since our immersion is complex-valued, equation (2.3.2) yields $\langle w', w' \rangle = 0$.
We have to take two times the conformal factor in complex coordinates to get the conformal factor in real coordinates as it was defined. This yields $v^2 = e^{2u} = 2\langle \omega', \omega'' \rangle$ □

2.5 The mean curvature of a conformal immersion

We may associate to any immersion $f : M \subset \mathbb{C} \to \mathbb{S}^3$ the mean curvature. In this section we are going to show that the mean curvature H for such an immersion is given by

$$2d * \omega = H[\omega \wedge \omega] \ , \ \omega = f^{-1}df \tag{2.5.1}$$

Here $*\omega$ is defined via the Hodge star operator, i.e.

$$*\omega = *(\omega' + \omega'') = -i\omega' + i\omega''.$$

If we consider the immersion $f : M \to \mathbb{S}^3 \subset \mathbb{R}^4$ we may associate to each element of $\mathbb{S}^3$ an **orthonormal frame** of $\mathbb{R}^4$ consisting of the four normalized vectors

$$SO(4) \ni \mathcal{F} = \left(f, \frac{f_x}{\|f_x\|}, \frac{f_y}{\|f_y\|}, N\right)$$

Here the vectors $f, \frac{f_x}{\|f_x\|}, \frac{f_y}{\|f_y\|}$ are orthonormal bedause of the conformality of f. So we may chose N to be the vector orthonormal to those.
We will denote by (e_1, e_2, e_3, e_4) the standard Euclidean ONB. The action of $\mathcal{F}$ on the standard basis is

$$\mathcal{F} \cdot e_1 = f \ ; \ \mathcal{F} \cdot e_2 = \frac{f_x}{\|f_x\|} \ ; \ \mathcal{F} \cdot e_3 = \frac{f_y}{\|f_y\|} \ ; \ \mathcal{F} \cdot e_4 = N$$

We may represent any point in $\mathbb{R}^4$ by a 2×2 matrix via the representation $\phi : \mathbb{R}^4 \to M^{2\times2}$ by the following map:

$$\phi(x_1, x_2, x_3, x_4) = \begin{pmatrix} x_1 + ix_4 & x_3 + ix_2 \\ -x_3 + ix_2 & x_1 - ix_4 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \tag{2.5.2}$$

If $\|\alpha\|^2 + \|\beta\|^2 = 1$, then the matrix represents a point in $\mathbb{S}^3 \cong SU(2)$.

Lemma 2.19 ($\mathbb{S}^3 \subset \mathbb{R}^4$ and $SU(2)$ are isometric)**.** *We show the stronger result that $\mathbb{R}^4$ is isometric to the matrix-representation of $\mathbb{R}^4$ via ϕ.*

Proof. Let X, Y be arbitrary elements of the form $\phi(x), \phi(y), x, y \in \mathbb{R}^4$. Then $\langle x, y\rangle_{\mathbb{R}^4} = \langle X, Y\rangle = \frac{1}{2} \cdot \mathrm{tr}(X\sigma_2 Y^T \sigma_2) = \frac{1}{2} \cdot \mathrm{tr}(X\bar{Y}^T)$.
Now $\langle x, y\rangle = \sum_{i=1}^4 x_i y_i$ and

$$\begin{aligned} \langle X, Y\rangle &= \frac{1}{2} \cdot \mathrm{tr}\left(\begin{pmatrix} x_1 + ix_4 & x_3 + ix_2 \\ -x_3 + ix_2 & x_1 - ix_4 \end{pmatrix}\begin{pmatrix} y_1 - iy_4 & -y_3 - iy_2 \\ y_3 - iy_2 & y_1 + iy_4 \end{pmatrix}\right) \\ &= \sum_{i=1}^4 x_i y_i = \langle x, y\rangle. \end{aligned}$$

□

We have seen that $\mathbb{S}^3 \subset \mathbb{R}^4$ and $SU(2)$ are isometric. There is also a commutative action diagram which translates the action of $\mathcal{F}$ on $\mathbb{R}^4$ into a group action of $SU(2) \times SU(2)$ on $\phi(\mathbb{R}^4)$. $SU(2) \times SU(2)$ is the double cover of the group $SO(4)$. That is, two pairs of matrices (F, G) in $SU(2) \times SU(2)$ represent the same group action of an element $\mathcal{F}$ in $SO(4)$. For $F, G \in SU(2)$ representing the action of an $\mathcal{F} \in SO(4)$ the pair $-F, -G$ represents the same action.
So the following diagram of group actions holds:

$$\begin{array}{ccc} SO(4) & \xrightarrow{\tilde{\phi}} & SU(2) \times SU(2) \\ \downarrow & & \downarrow \\ \mathbb{R}^4 & \xrightarrow{\phi} & \phi(\mathbb{R}^4) \end{array}$$

With the representation ϕ we have

$$\phi(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \phi(e_2) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{2.5.3}$$

$$\phi(e_3) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \phi(e_4) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{2.5.4}$$

In complex parameters we have $f_z = \frac{1}{2} \cdot (f_x - i \cdot f_y); f_{\bar{z}} = \frac{1}{2} \cdot (f_x + i \cdot f_y)$. We get the following result on how the group action of $\mathcal{F}$ on $\mathbb{S}^3 \subset \mathbb{R}^4$ translates into the group action of $SU(2) \times SU(2)$ on $SU(2)$:

$$\mathcal{F}(e_1) = \mathcal{F}_1 = f = F\phi(e_1)G^{-1} = F\mathbb{1}G^{-1} = FG^{-1}$$

$$\mathcal{F}(e_2) = \mathcal{F}_2 = \frac{f_x}{e^u} = F\phi(e_2)G^{-1} = F \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} G^{-1}$$

$$\mathcal{F}(e_3) = \mathcal{F}_3 = \frac{f_y}{e^u} = F\phi(e_3)G^{-1} = F \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} G^{-1}$$

$$\mathcal{F}(e_4) = \mathcal{F}_4 = F \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} G^{-1}$$

If we use complex coordinates we calculate

$$f_z = e^u \cdot \frac{1}{2} F(\phi(e_2) - i\phi(e_3))G^{-1} = ie^u F \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} G^{-1} = ie^u F \epsilon_+ G^{-1}$$

$$f_{\bar{z}} = e^u \cdot \frac{1}{2} F(\phi(e_2) + i\phi(e_3))G^{-1} = ie^u F \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} G^{-1} = ie^u F \epsilon_- G^{-1}$$

and for the differential of f we get

$$df = ie^u F(\epsilon_+ dz + \epsilon_- d\bar{z})G^{-1}$$

If we set $\alpha = F^{-1}dF$ and $\beta = G^{-1}dG$ we may write the form ω as $\omega = G(\alpha - \beta)G^{-1}$, since

$$f = FG^{-1} \Rightarrow df = (dF)G^{-1} - FG^{-1}(dG)G^{-1} = F\alpha G^{-1} - FG^{-1}G\beta G^{-1}$$
$$\omega = f^{-1}df = (FG^{-1})^{-1}F(\alpha - \beta)G^{-1} = G(\alpha - \beta)G^{-1}.$$

With H and of Q as in equation (2.3.5) a computation gives

$$\alpha = \begin{pmatrix} -\frac{1}{2}iv^{-1}(v_z dz - v_{\bar{z}} d\bar{z}) & iv^{-1}Q dz + \frac{1}{2}vi(H - i)d\bar{z} \\ \frac{1}{2}iv(H + i)dz - v^{-1}\bar{Q}d\bar{z} & \frac{1}{2}iv^{-1}(v_z dz - v_{\bar{z}} d\bar{z}) \end{pmatrix} \tag{2.5.5}$$

and

$$\beta = \begin{pmatrix} -\frac{1}{2}iv^{-1}(v_z dz - v_{\bar{z}} d\bar{z}) & iv^{-1}Q dz + \frac{1}{2}vi(H + i)d\bar{z} \\ \frac{1}{2}iv(H - i)dz - v^{-1}\bar{Q}d\bar{z} & \frac{1}{2}iv^{-1}(v_z dz - v_{\bar{z}} d\bar{z}) \end{pmatrix} \tag{2.5.6}$$

Now $\omega = G(\alpha - \beta)G^{-1}$ leads to $d * \omega = iv^2 HG\epsilon G^{-1}dz \wedge d\bar{z}$ and on the other hand one has $[\omega \wedge \omega] = 2iv^2 G\epsilon G^{-1}dz \wedge d\bar{z}$ which proves the formula [10].

2.6 The form α and the parameter λ

We assume the mean curvature H to be constant in the following, $H = H_0$. We may combine the Maurer-Cartan equation (2.2.2) and the formula for the mean curvature (2.5.1) to get

$$d\omega + H_0^{-1} d * \omega = 0 \Longleftrightarrow d\omega' + id\omega'' + H_0^{-1}d(-i\omega' + i\omega'') = 0 \tag{2.6.1}$$
$$\Longleftrightarrow (1 - iH_0^{-1})d\omega' + (1 + iH_0^{-1})d\omega' = 0 \tag{2.6.2}$$

Lemma 2.20. *In the complexified tangent space the equation*

$$[\omega \wedge \omega] = 2 \cdot [\omega' \wedge \omega'']$$

holds.

Proof. For any two left-invariant vector fields X, Y we have

$$[\omega \wedge \omega](X, Y) = [\omega(X), \omega(Y)] - [\omega(Y), \omega(X)] =$$
$$2 \cdot (\omega'(X)\omega'(Y) + \omega'(X)\omega''(Y) + \omega''(X)\omega'(Y) + \omega''(X)\omega''(Y)$$
$$-\omega'(Y)\omega'(X) - \omega'(Y)\omega''(X) - \omega''(Y)\omega'(X) - \omega''(Y)\omega''(X)) =$$
$$2 \cdot ([\omega'(X), \omega'(Y)] + [\omega''(X), \omega''(Y)] + [\omega'(X), \omega''(Y)] + [\omega''(X), \omega'(Y)])$$

which is equal to $2 \cdot [\omega' \wedge \omega''](X, Y)$ by the definition of ω' and ω''. □

We may now insert $d\omega'' = -d\omega' - [\omega' \wedge \omega'']$ and $d\omega' = -d\omega'' - [\omega' \wedge \omega'']$ into (2.6.1) to obtain $2d\omega' = (iH_0 - 1)[\omega' \wedge \omega'']$ and $2d\omega'' = -(iH_0 + 1)[\omega' \wedge \omega'']$. Then an easy computation shows that the λ-dependent form

$$\omega_\lambda = \frac{1}{2}((1+\lambda^{-1})(1+iH_0)\omega' + (1+\lambda)(1-iH_0)\omega'') \tag{2.6.3}$$

satisfies the Maurer-Cartan equation (2.2.2) for $\lambda \in \mathbb{C}^\times$ as well. Gauging with G one obtains

$$\alpha_\lambda := G^{-1}\omega_\lambda G + G^{-1}dG.$$

α_λ satisfies the Maurer Cartan equation as well. Because the Maurer-Cartan equation is an integrability condition, we may, by setting an initial value, integrate (as in remark 2.22) and obtain a corresponding **extended frame**

$$F_\lambda : \mathbb{C} \times \mathbb{C}^\times \longrightarrow Sl_2(\mathbb{C}) \ ; \ dF_\lambda = F_\lambda \alpha_\lambda \ ; \ F_\lambda(0) = \mathbb{1}$$

Since α_λ takes values in su_2 for $\lambda \in \mathbb{S}^1$ we conclude that F_λ takes values in $SU(2)$ if $\lambda \in \mathbb{S}^1$.

2.7 The Sym-Bobenko formula

With the extended frame F_λ we define for $\lambda_0, \lambda_1 \in \mathbb{S}^1$ with $\lambda_0 \neq \lambda_1$ the following map $\tilde{f} : \mathbb{C} \longrightarrow SU(2)$ by the **Sym-Bobenko formula**

$$\tilde{f} = F_{\lambda_1} F_{\lambda_0}^{-1} = F_{\lambda_1} \mathbb{1} F_{\lambda_0}^{-1} \tag{2.7.1}$$

We will call the points $\lambda_0, \lambda_1 \in \mathbb{S}^1$ the **Sym points** of the immersion $\tilde{f}$. For fixed λ_0, λ_1 we set $\Omega = \tilde{f}^{-1} d\tilde{f}$ which we may write as

$$\Omega = AdF_{\lambda_0}(\alpha_{\lambda_1} - \alpha_{\lambda_0}) = F_{\lambda_0}(\alpha_{\lambda_1} - \alpha_{\lambda_0})F_{\lambda_0}^{-1}.$$

Here α_{λ_i} correspond to the matrices α, β introduced above. Using the formula for ω_λ as in (2.6.3) we may split Ω into its $(1,0)$ and $(0,1)$ parts. This gives

$$\Omega' = \frac{1}{2}(\lambda_1^{-1} - \lambda_0^{-1})(1+iH_0)AdF_{\lambda_0}\omega' \quad \text{and} \quad \Omega'' = \frac{1}{2}(\lambda_1 - \lambda_0)(1-iH_0)AdF_{\lambda_0}\omega''.$$

The inner product is the Ad-invariant Killing form and the conformality of ω easily translates into the conformality of Ω:

$$\langle \Omega', \Omega' \rangle = \frac{1}{4}(\lambda_1^{-1} - \lambda_0^{-1})^2(1+iH_0)^2 AdF_{\lambda_0}\langle \omega', \omega' \rangle \tag{2.7.2}$$

and, for $\lambda \in \mathbb{S}^1$ we may write $\lambda_{0,1} = e^{2it_{0,1}}$ and derive for the conformal factor

$$\langle \Omega', \Omega'' \rangle = \frac{1}{2}\sin^2(t_1 - t_0)(1 + H_0^2)v^2 \tag{2.7.3}$$

where v is the conformal factor of the immersion f. We also want to express the mean curvature in terms of Ω. With respect to (2.5.1) we therefore calculate

$$d * \Omega = \frac{i}{4}(1 + H_0^2)(\lambda_0^{-1}\lambda_1 - \lambda_1^{-1}\lambda_0)AdF_{\lambda_0}[\omega' \wedge \omega'']$$

and

$$[\Omega \wedge \Omega] = \frac{1}{2}(1 + H_0^2)(\lambda_1^{-1} - \lambda_0^{-1})(\lambda_1 - \lambda_0)AdF_{\lambda_0}[\omega' \wedge \omega''].$$

The immersion $\tilde{f} : \mathbb{C} \longrightarrow SU(2)$ has constant mean curvature

$$H = i\frac{\lambda_0 + \lambda_1}{\lambda_0 - \lambda_1} \tag{2.7.4}$$

2.8 Conformal immersions and solutions of the sinh-Gordon equation

We now define, for a function $u : \mathbb{C} \longrightarrow \mathbb{R}$ the form α_λ as

$$\alpha_\lambda = \frac{1}{2}\begin{pmatrix} u_z dz - u_{\bar{z}} d\bar{z} & i\lambda^{-1}e^u dz + ie^{-u}d\bar{z} \\ ie^{-u}dz + i\lambda e^u d\bar{z} & -u_z dz + u_{\bar{z}} d\bar{z} \end{pmatrix} \tag{2.8.1}$$

It is traceless and therefore an element of $sl_2(\mathbb{C})$. We still have that α_λ is in $su_2(\mathbb{C})$ for $\lambda \in \mathbb{S}^1$.

Lemma 2.21 (Lax pair and Maurer-Cartan equation). *We have*

$$2d\alpha + [\alpha \wedge \alpha] = 0 \Leftrightarrow \overline{\partial}\alpha' - \partial\alpha'' = [\alpha', \alpha''].$$

Remark 2.22 (The Maurer-Cartan equation is an integrability condition). *For $\alpha = F^{-1}dF$ we may split α into the dz and $d\bar{z}$ parts and write $\alpha = Udz + Vd\bar{z}$. Then the equation above is usually denoted as the equivalence between the Maurer-Cartan equation and a Lax-pair representation. It is usually written as*

$$U_{\bar{z}} - V_z - [U, V] = 0 \Longleftrightarrow d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0$$

The Lax-pair equation is equivalent to the integrability condition $F_{z\bar{z}} = F_{\bar{z}z}$, since we have

$$F_z = FU \Rightarrow F_{z\bar{z}} = F_{\bar{z}}U + FU_{\bar{z}} = FVU + FU_{\bar{z}}$$

$$F_{\bar{z}} = FV \Rightarrow F_{\bar{z}z} = F_z V + F V_z = FUV + FV_z$$

and therefore

$$F_{z\bar{z}} - F_{\bar{z}z} = 0 \Longleftrightarrow VU - UV + U_{\bar{z}} - V_z = 0 \Longleftrightarrow U_{\bar{z}} - V_z - [U, V] = 0.$$

So we may say that if the Lax-pair U, V fulfills the **Lax-equation** $U_{\bar{z}} - V_z - [U, V] = 0$, then $F_{z\bar{z}} = F_{\bar{z}z}$ must hold and so we may consider the differential equation $F_z = FU; F_{\bar{z}} = FV$ to be **exact**. Since $\mathbb{C}$ is connected and simply connected we therefore know that the equation is integrable.

Proof. We evaluate first $[\alpha \wedge \alpha]$ on the two left invariant vectorfields ∂ and $\overline{\partial}$:

$$[\alpha \wedge \alpha](\partial, \overline{\partial}) = [\alpha(\partial), \alpha(\overline{\partial})] - [\alpha(\overline{\partial}), \alpha(\partial)] =$$

$$2 \cdot [\alpha(\partial), \alpha(\overline{\partial})] = 2 \cdot [\alpha^{''}, \alpha^{'}]$$

where we have used $dz(\overline{\partial}) = 0$. Moreover, we have

$$\begin{aligned} 2d\alpha(\partial, \overline{\partial}) &= 2(\partial + \overline{\partial})(\alpha^{'} dz + \alpha^{''} d\bar{z})(\partial, \overline{\partial}) \\ &= 2 \cdot \left(\partial\alpha(\overline{\partial}) - \bar{\partial}\alpha(\partial) + \alpha([\partial, \overline{\partial}])\right) \\ &= 2 \cdot (\partial\alpha'' - \bar{\partial}\alpha^{'}) \end{aligned}$$

□

Theorem 2.23. *The form α_λ fulfills the Maurer-Cartan equation* (2.2.2) *if and only if u is a solution of the sinh-Gordon equation* (2.0.2)*. Then for any solution u of the sinh-Gordon equation and extended frame F_λ, λ_0 and $\lambda_1 \in \mathbb{S}^1, \lambda_0 \neq \lambda_1$ (which is related to u via the integrability condition 2.22), the map defined by the Sym-Bobenko formula* (2.7.1) *is a conformal immersion with constant mean curvature H given by* (2.7.4)*, conformal factor $v = \dfrac{e^u}{\sqrt{H^2+1}}$ and constant Hopf differential Qdz^2 with $Q = \frac{i}{4}(\lambda_1^{-1} - \lambda_0^{-1})$.*

Proof. The proof of this theorem may be found in [7]. For the relationsship betwenn the one-form α_λ and the Maurer-Cartan equation one has to decompose α_λ into its $(1, 0)$ and $(0, 1)$ parts and then calculate the Maurer-Cartan equation by using the above lemma 2.22. □

Proposition 2.24 (Involution of the form α_λ or reality condition in terms of α_λ)**.** *For the form α_λ there holds*

$$\alpha_{\overline{\lambda}^{-1}} = -\overline{\alpha^t_\lambda}. \tag{2.8.2}$$

2.9 Example: the vacuum solution

For the purposes of the numerical computations we want to calculate the frame F explicitly in the case of the vacuum solution. In the vacuum situation we have $u \equiv 0$.The vacuum solution is the only explicitly known solution of the Sinh-Gordon equation which makes it a good starting point of numerical deformation. For $u = 0$ the form α_λ reads

$$\alpha_\lambda = \begin{pmatrix} 0 & \frac{i}{\lambda}dz + id\bar{z} \\ idz + i\lambda d\bar{z} & 0 \end{pmatrix}.$$

If we write the form α_λ as $\alpha^{'} + \alpha^{''} = Udz + Vd\bar{z}$ we have to find a solution of the equation

$$(\partial + \overline{\partial})F = F(Udz + Vd\bar{z}) \equiv dF = F\alpha_\lambda \;\; ; \;\; F(0) = \mathbb{1}.$$

The characteristic polynomial of $\alpha^{'}$ reads $\beta^2 + \frac{1}{\lambda}dz^2 = 0$ so that we have two distinct eigenvalues $\beta^{\alpha^{'}}_{1,2} = \pm\frac{i}{\sqrt{\lambda}}dz$. For $\alpha^{''}$ the characteristic polynomial reads $\beta^2 + \lambda d\bar{z}^2 = 0$. So the distinct eigenvalues read $\beta^{\alpha^{''}}_{1,2} = \pm i\sqrt{\lambda}d\bar{z}$. We assume that B is a change of the basis such that $B\alpha_\lambda B^{-1} = \tilde{\alpha}$ is a diagonal matrix. Then we may write

$$\begin{aligned} \partial F = FU \iff & F^{-1}\partial F = U \iff B^{-1}F^{-1}\partial FB = B^{-1}UB & (2.9.1)\\ \iff & (FB)^{-1}\partial(FB) = BUB^{-1} \iff \tilde{F}^{-1}\partial F = \tilde{U} & (2.9.2) \end{aligned}$$

Now the right hand side is diagonal and therefore we may assume that the left hand side is diagonal too. We rewrite the equations as

$$\frac{d}{dz}(\ln\tilde{F}) = \tilde{U} \quad \text{and} \quad \frac{d}{d\bar{z}}(\ln\tilde{F}) = \tilde{V} \iff d\tilde{F} = (\partial + \overline{\partial})\tilde{F} = \tilde{U} + \tilde{V} \qquad (2.9.3)$$

And, because U and V are constant matrices, we may also conclude

$$\ln\tilde{F} = z\tilde{U} + \bar{z}\tilde{V} + c \Longrightarrow \tilde{F} = \tilde{c}\exp(z\tilde{U} + \bar{z}\tilde{V}).$$

By the initial condition $F(0) = \mathbb{1}$ we get that $\tilde{c} = 1$. Therefore $\tilde{F}$ must be of the form

$$\tilde{F}(z,\bar{z}) = \begin{pmatrix} \exp^{\frac{iz}{\sqrt{\lambda}}} \cdot \exp^{i\bar{z}\sqrt{\lambda}} & 0 \\ 0 & \exp^{-\frac{iz}{\sqrt{\lambda}}} \cdot \exp^{-i\bar{z}\sqrt{\lambda}} \end{pmatrix}.$$

$\tilde{F}$ has determinat 1 so $\tilde{F} \in SL_2(\mathbb{C})$ as it could be expected. In the general case we will not be able to write down the solution expicitely in the way we did it above. The example given above may be used to postulate general features of any solution F_λ to a given form α_λ:

Definition 2.25 (Essential singularity)**.** *Let a be a complex number, assume that f(z) is not defined at a but is analytic in some region U of the complex plane, and that every open neighbourhood of a has non-empty intersection with U. We say a is an essential singularity if neither*

$$\lim_{z\to a} f(z) \quad nor \quad \lim_{z\to a} \frac{1}{f(z)}$$

exists, then a is an essential singularity of both f and $\frac{1}{f}$.

Now, if we take into account that $\tilde{F}$ does depend on λ as well, $\tilde{F} = \tilde{F}(\lambda, z, \bar{z})$, we see that we have essential singularities for $\lambda \longrightarrow 0$ and $\lambda \longrightarrow \infty$.
The **Big Picard theorem** states, that if an analytic function f has an essential singularity at a point a, then on any punctured neighborhood of a, $f(z)$ takes all possible complex values, with at most a single exception, infinitely often. We will assume that the behavior of any solution F is asymptotically the same as for the explicit solution we calculated above.

Remark 2.26. *One can show that the asymptotics of any solution* F_λ *with* α_λ *as in* (2.8.1)*, u solving the sinh-Gordon equation, of the initial value problem* $dF_\lambda = F_\lambda \alpha_\lambda$ *with* $F_\lambda(0) = \mathbb{1}$ *are the same as the asymptotics for the vacuum solution* $\tilde{F}$ *in some sense.*

3 The monodromy and the spectral curve

What we have seen so far is how one solution of the sinh-Gordon equation is in a one-to-one correspondence to a conformal immersion into $\mathbb{S}^3$ with constant mean curvature H and constant Hopf differential Q. We have reached this result by developing the form α to be associated to a CMC immersion into $\mathbb{S}^3$. In our setting the Maurer-Cartan equation was the integrability condition that ensured that we could find a frame $\mathcal{F}$ and finally an immersion f.
In this chapter we will be ultimately interested in periodic solutions of the sinh-Gordon equation. Our main interest will be to understand how the periodicity of u acts on the immersion f. The action of the periodicity of u on the immersion f is basically encoded in the monodromy-operator.

3.1 The monodromy, reality and closing condition

We assume that the form $\alpha_\lambda = F_\lambda^{-1} dF_\lambda$ has period $\tau \neq 0$. That is, if $\tau : \mathbb{C} \to \mathbb{C}; z \mapsto z+\tau$ is a translation, then we assume that α_λ is periodic in $\tau, \tau^*\alpha_\lambda = \alpha_\lambda \circ \tau = \alpha_\lambda$. Periodicity of the form α_λ in τ is equivalent to the assumption that the solution u of the sinh-Gordon equation is periodic with period $\tau \in \mathbb{C}^\times$, $u(z+\tau) = u(z)$.
Given an initial value we know that the equation $\alpha_\lambda = F_\lambda^{-1} dF_\lambda$ is integrable and we may integrate to obtain a frame F_λ. Then we define the monodromy operator as

Definition 3.1 (Monodromy)**.** *The* ***monodromy*** *is the matrix*

$$M_\lambda^\tau = \tau^*(F_\lambda)F_\lambda^{-1} = F_\lambda(z+\tau)F_\lambda^{-1}(z) \Longleftrightarrow F_\lambda(z+\tau) = M_\lambda^\tau F_\lambda(z)$$

The monodromy encodes how the moving frame F_λ varies when traversing a period of u. We demand that the immersion f is periodic as well, $\tau^* f = f$ with $f = F_{\lambda_1} F_{\lambda_0}^{-1}$, and by the Sym-Bobenko formula (2.7.1) we get

$$\tau^* f = \tau^*(F_{\lambda_1} F_{\lambda_0}^{-1}) = \tau^*(F_{\lambda_1})(\tau^* F_{\lambda_0})^{-1} = M_{\lambda_1}^\tau F_{\lambda_1} F_{\lambda_0}^{-1} (M_{\lambda_0}^\tau)^{-1} = M_{\lambda_1}^\tau f (M_{\lambda_0}^\tau)^{-1}. \quad (3.1.1)$$

One can show that the last equation equals f if and only if

$$M_{\lambda_0}^\tau = M_{\lambda_1}^\tau = \pm \mathbb{1}. \quad (3.1.2)$$

We call this the **closing condition**. The monodromy is a map from $\mathbb{C}^* \to SL_2(\mathbb{C}), \lambda \to M_\lambda$ with essential singularities at $\lambda = 0, \infty$, as we stated in remark 2.9. By construction the monodromy takes values in $SU(2)$ for $|\lambda| = 1$. If $|\lambda| = 1$ the condition (3.1.2) is equivalent to the condition that the trace of $M^\tau_{\lambda_0}$ together with $M^\tau_{\lambda_1}$ equals ± 2. This follows from the properties of the matrix-group $SU(2)$.

Proposition 3.2. *The monodromy M^τ_λ does not depend on z.*

Proof.

$$\begin{aligned} dM(z) &= dF(z+\tau)F(z)^{-1} + F(z+\tau)d(F(z)^{-1}) \\ &= F(z+\tau)\alpha(z+\tau)F(z)^{-1} - F(z+\tau)F(z)^{-1}dF(z)F(z)^{-1} \\ &= F(z+\tau)\alpha(z)F(z)^{-1} - F(z+\tau)F(z)^{-1}F(z)\alpha(z)F(z)^{-1} = 0 \end{aligned}$$

□

So far we fixed the basepoint z_0 such that $F_\lambda(z_0) = \mathbb{1}$. By choosing another baseboint z_1 one obtains a z_1-dependent monodromy $M(z_1)$.

Lemma 3.3. *Consider the two fundamental solutions $F_\lambda, G_\lambda \in SL(2,\mathbb{C})$ of*

$$\begin{aligned} dF_\lambda = F_\lambda\alpha_\lambda, &\qquad F_\lambda(z_0) = \mathbb{1} \\ dG_\lambda = G_\lambda\alpha_\lambda, &\qquad G_\lambda(z_1) = \mathbb{1} \end{aligned}$$

for periodic α_λ with period τ. Then the monodromies $M_\lambda(z_0)$ and $M_\lambda(z_1)$ for the frames F_λ and G_λ satisfy the following equation

$$M_\lambda(z_1) = F_\lambda^{-1}(z_1)M_\lambda(z_0)F_\lambda(z_1).$$

Proof. Consider the system

$$dG_\lambda = G_\lambda\alpha_\lambda \text{ with } G_\lambda(z_0) =: G_0.$$

Then one obtains

$$G_\lambda(z) = G_\lambda(z_0)\cdot F_\lambda(z) \quad \forall z$$

since $G_\lambda(z_0)\cdot F_\lambda(z)$ is also a solution of the above system with the same initial value G_0. In particular one has

$$G_\lambda(z_1) = \mathbb{1} = G_\lambda(z_0)\cdot F_\lambda(z_1)$$

and therefore $G_\lambda(z_0) = F_\lambda^{-1}(z_1)$. Since $G_\lambda(z_1) = \mathbb{1}$ we get

$$\begin{aligned} M_\lambda(z_1) &= G_\lambda(z_1+\tau) = G_\lambda(z_0)F_\lambda(z_1+\tau) \\ &= G_\lambda(z_0)M_\lambda(z_0)F_\lambda(z_1) \\ &= F_\lambda^{-1}(z_1)M_\lambda(z_0)F_\lambda(z_1) \end{aligned}$$

and the claim follows. □

If we replace z_1 by the variable z we see that the basepoint-dependent monodromy $M_\lambda(z) = F_\lambda^{-1}(z)M_\lambda(z_0)F_\lambda(z)$ satisfies

$$\begin{aligned} dM_\lambda(z) &= -F_\lambda^{-1}(z)\,(dF_\lambda(z))\,F_\lambda^{-1}(z)M_\lambda(z_0)F_\lambda(z) + F_\lambda^{-1}(z)M_\lambda(z_0)\,(dF_\lambda(z)) \\ &= [M_\lambda(z), \alpha_\lambda(z)]. \end{aligned}$$

Hence the z-dependent monodromy to a given period τ depends on the choice of base point z, but its conjugacy class and hence eigenvalues $\mu_\lambda, \mu_\lambda^{-1}$ does not.

Proposition 3.4 (Reality condition for the monodromy). *The monodromy satisfies the*

$$M(\overline{\lambda^{-1}}) = (\overline{M}^t(\lambda))^{-1} \tag{3.1.3}$$

This result follows from $\alpha_{\overline{\lambda}^{-1}} = -\overline{\alpha_\lambda^t}$. Then, with $dF_\lambda = F_\lambda\alpha_\lambda$ it follows that $F(\overline{\lambda}^{-1}) = (\overline{F}^t(\lambda))^{-1}$ and therefore by the definition of the monodromy 3.1 the result follows. We will denote this property **reality condition**.

Example 3.5 (The monodromy for the vacuum solution). *2.9 The monodromy for the vacuum solution $\tilde{F}(z,\overline{z})$ explained in section 2.9 is given by*

$$M_\lambda^\tau = F_\lambda(z+\tau, \overline{z}+\overline{\tau})F_\lambda^{-1}(z,\overline{z}) = \begin{pmatrix} \exp i(\frac{\tau}{\sqrt{\lambda}} + \overline{\tau}\sqrt{\lambda}) & 0 \\ 0 & \exp -i(\overline{\tau}\sqrt{\lambda} + \frac{\tau}{\sqrt{\lambda}}) \end{pmatrix} \tag{3.1.4}$$

One verifies that proposition 3.4 is fulfilled. The monodromy is independent of the choice of a base-point as was stated in proposition 3.2. As in the assumption about the asymptotics in 2.26 one observes that the eigenvalues μ and $\frac{1}{\mu}$ of M_λ^τ, that are the diagonal entries of the monodromy operator, occur in exponential form.

The eigenfunctions μ and $\frac{1}{\mu}$ of the monodromy have essential singularities at 0 and ∞. Therefore one is often interested in considering the function $\ln(\mu)$ and its differential $d\ln(\mu)$. The holomorphic function $\ln(\mu)$ is multivalued.

3.2 Finite type solutions of the sinh-Gordon equation

For $\lambda \in \mathbb{C}^\times$ the monodromy takes values in the matrix Lie-group $SL_2(\mathbb{C})$ which is a subgroup of $GL_2(\mathbb{C})$.

Proposition 3.6. *If the monodromy is not diagonizable the eigenvalues must be either $+1$ or -1. The opposite is not true.*

Proof. Solving the quadratic equation of the characteristic polynomial gives

$$\mu_{1,2}(\lambda) = \frac{\triangle(\lambda) \pm \sqrt{\triangle(\lambda)^2 - 4}}{2}.$$

where $\triangle(\lambda)$ denotes the trace of the monodromy. If there are no distinct eigenvalues there must hold $\triangle(\lambda) = \pm 2$. So $\mu_{1,2}$ must be either $+1$ or -1. □

Definition 3.7 (Finite type solution)**.** *[7] A periodic solution of the sinh-Gordon equation is of finite type if and only if the monodromy 3.1 fails at only finitely many points* $\lambda \in \mathbb{C}^\times$ *to be diagonalizable.*

If a solution of the sinh-Gordon equation is of finite type, then there are only finitely many points where the monodromy may be represented in a non-trivial Jordan Normal form.

3.3 The spectral curve

In the following we investigate the **characteristic equation** of the monodromy-operator. We may interpret this equation as an equation in two complex parameters since we have the two degrees of freedom λ and the eigenvalue μ. So we define a curve which we call the **spectral curve**.

$$\Gamma = \{(\lambda, \mu) \in \mathbb{C}^\times \times \mathbb{C}^\times \,|\, \det(\mu \mathbb{1} - M(\lambda)) = 0\} \tag{3.3.1}$$

with

$$R(\lambda, \mu) = \det(\mu \mathbb{1} - M(\lambda)) \tag{3.3.2}$$

We have

$$R(\lambda, \mu) = 0 \Longleftrightarrow \det(\mu \mathbb{1} - M(\lambda)) = 0 \Longleftrightarrow \mu_\lambda^2 - \triangle(\lambda)\mu_\lambda + 1 = 0.$$

And therefore

$$\mu_{1,2}(\lambda) = \frac{\triangle(\lambda) \pm \sqrt{\triangle(\lambda)^2 - 4}}{2} = \frac{\triangle(\lambda) \pm \sqrt{(\triangle(\lambda) - 2)(\triangle(\lambda) + 2)}}{2} \tag{3.3.3}$$

Definition 3.8 (Non-singular curve)**.** *We call a point on the curve* $R(\lambda, \mu) = 0$ *non-singular, if*

$$grad_{\mathbb{C}} R|_{R=0} = \left(\frac{\partial R}{\partial \lambda}, \frac{\partial R}{\partial \mu}\right)|_{R(\lambda,\mu)=0} \neq (0, 0) \tag{3.3.4}$$

*So at any singular point the two partial derivatives both vanish. We call such a point a **singularity** or **double point** of the curve.*

For the following we make the assumption:

Remark 3.9 (Assumption)**.** *We assume that* $(\triangle(\lambda)^2 - 4)$ *has no root of order greater than* 2*, that is* $\frac{\partial}{\partial \lambda^k} \triangle \neq 0$ *for* $k > 2$*.*

With the help of this assumption we will be able to characterize the singularities of the spectral curve.

Lemma 3.10 (Characterisation of the singularities and the branch points of Γ)**.** *The curve $\Gamma(\lambda,\mu)$ has its only double points where $\sqrt{\triangle(\lambda)^2-4}$ is zero and in addition the partial derivative of R with respect to λ is zero too. This is the case exactly where the eigenvalues of the monodromy μ and $\frac{1}{\mu}$ are ± 1 and $\triangle' = \frac{\partial}{\partial\lambda}\triangle = 0$.*
If $\triangle' \neq 0$ but there still holds $\sqrt{\triangle(\lambda)^2-4} = 0$ we are given a branch-point of the spectral curve.

Proof. The partial derivatives of R read

$$\frac{\partial R}{\partial\lambda} = \frac{\partial\triangle(\lambda)}{\partial\lambda}\mu = \triangle'\mu \text{ and } \frac{\partial R}{\partial\mu} = 2\mu - \triangle(\lambda) \tag{3.3.5}$$

The second equation gives, together with equation (3.3.3) that the partial derivative with respect to μ is zero, $\frac{\partial R}{\partial\mu} = 0$ if and only if the discriminant is zero, $\sqrt{\triangle(\lambda)^2-4} = 0$. At these points $\triangle(\lambda)$ must be ± 2 and hence the eigenvalues μ_i must be either $+1$ or -1. So the monodromy must be either of the form

$$M_\lambda = \pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \text{ or } M_\lambda = \pm\begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix} \tag{3.3.6}$$

At a singular point the other derivative must also vanish, leading to the condition that $\triangle' = 0$. In this case the expression $(\triangle^2(\lambda) - 4)$ has a double root at that point. In general this case describes roots of $(\triangle^2(\lambda)-4)$ of even order, by the assumption above these are roots of second order of $(\triangle^2(\lambda)-4)$ in our case. The left monodromy matrix belongs to a double point.
The monodromy operator on the right hand side belongs to a branch point. Here $(\triangle^2(\lambda)-4)$ has an odd root, that is $\triangle' \neq 0$. Since we assume that there are only finitely many points where the monodromy fails to be semisimple, we have only finitely many points where $(\triangle^2(\lambda)-4)$ has an odd root. □

Remark 3.11. *The concept of a branch point has a geometric meaning: The surface or curve defined by $R(\lambda,\mu) = 0$ describes a two-sheeted cover of $\mathbb{CP}^1$ if one describes the surface by the parameter $\lambda \in \mathbb{C}$. One may say that over a branch point the two sheets cross. Therefore one may consider the curve to be branched over these points.*
The two sheets correspond to the eigenvalues of the monodromy.

Right now we may distinguish between three cases in the finite type situation for $\lambda \in \mathbb{C}^\times$:

- $M = \pm\mathbb{1}$: This is the case where the closing condition is fulfilled, especially at the **sym-points**. In general we will call such a point a **double point** of the spectral curve.
- The monodromy may be representented in a non-trivial Jordan normalform. If the discriminant $\Delta^2 - 4$ has roots of first order, then the eigenvalues are ± 1 as in the

first case and we call such points **branch-points**. In the finite-type case there are only finitely many of these points. The case when $\Delta^2 - 4$ has roots of second order represents a special case. To any spectral curve there exists a solution so that this point becomes a double point. We will not be interested in this case.

- The monodromy has distinct eigenvalues. Then the monodromy is diagonalizable. This is the case almost everywhere except where one of the above situations occurs.

$\mu_{1,2}(\lambda)$ correspond to the two sheets of the spectral curve with branchpoints at those points where the discriminant $\sqrt{\triangle(\lambda)^2 - 4}$ is zero. The holomorphicity of the functions μ_i with respect to λ is derived by the holomorphicity of the form α_λ and the monodromy $M(\lambda)$. $R(\lambda, \mu)$ is an analytic function of μ and λ in the neighbourhood of any nonsingular point $(\lambda_0, \mu_0) \in \mathbb{C}^\times \times \mathbb{C}^\times$; $R(\lambda_0, \mu_0) = 0$; $\frac{\partial R}{\partial \mu}(\lambda_0, \mu_0) \neq 0$. At any nonsingular point we may differentiate $R(\lambda, \mu)$ to obtain a one form and rewrite the equation without loss of information as

$$dR(\lambda, \mu) = \frac{\partial R}{\partial \lambda} d\lambda + \frac{\partial R}{\partial \mu} d\mu = 0 \Longleftrightarrow \frac{\frac{\partial R}{\partial \lambda}}{\frac{\partial R}{\partial \mu}} = -\frac{d\mu}{d\lambda} \tag{3.3.7}$$

In order to describe the spectral curve by a meromorphic function we introduce two additional parameters κ and p. The appearance of the **quasimomentum** p may be motivated by the asymptotic behaviour of the solutions of the initial value problem $dF_\lambda = F_\lambda \alpha_\lambda$ with $F_\lambda(0) = \mathbb{1}$. The idea to define κ as below is to describe the spectral curve as a meromorphic differential, leaving behind the multivaluedness of $\ln(\mu)$, but depending on both, the parameters μ and λ. Hence we define

$$p := \frac{1}{i} ln(\mu) \ , \ \kappa := \frac{dp}{d \ln \lambda}. \tag{3.3.8}$$

Then, with $\kappa d \ln \lambda = dp$ we calculate (at non-singular points)

$$\kappa = \frac{dp}{d \ln \lambda} = \frac{1}{i} \frac{dln\mu}{d \ln \lambda} = \frac{1}{i} \frac{\lambda}{\mu} \frac{d\mu}{d\lambda} = \frac{i\lambda}{\mu} \frac{\partial R}{\partial \lambda} \Big/ \frac{\partial R}{\partial \mu} \tag{3.3.9}$$

and with

$$\frac{\partial R}{\partial \lambda} = \frac{\partial \triangle(\lambda)}{\partial \lambda} \mu \text{ and } \frac{\partial R}{\partial \mu} = 2\mu - \triangle(\lambda) \tag{3.3.10}$$

and by use of the solutions of

$$R(\lambda, \mu) = 0 \Longleftrightarrow \mu_{1,2} = \frac{\triangle(\lambda) \pm \sqrt{\triangle(\lambda)^2 - 4}}{2} \Longleftrightarrow 2\mu - \triangle(\lambda) = \pm\sqrt{\triangle(\lambda)^2 - 4}. \tag{3.3.11}$$

We may finally write

$$\kappa = \lambda \frac{\partial \triangle(\lambda)}{\partial \lambda} \frac{\sqrt{-1}}{2\mu - \triangle(\lambda)} \tag{3.3.12}$$

The spectral curve in the new parameters κ and λ (without making use of the parameter μ) then reads:

$$Q(\kappa, \lambda) = \kappa^2 + \lambda^2 \left(\frac{\partial \triangle(\lambda)}{\partial \lambda} \right)^2 \frac{1}{\triangle(\lambda)^2 - 4} = 0 \tag{3.3.13}$$

We have to investigate the solutions of

$$\kappa^2 = -\lambda^2 \left(\frac{\partial \triangle(\lambda)}{\partial \lambda} \right)^2 \frac{1}{\triangle(\lambda)^2 - 4} = \left(\frac{\lambda}{i} \right)^2 \left(\frac{\partial \triangle(\lambda)}{\partial \lambda} \right)^2 \frac{1}{\triangle(\lambda)^2 - 4} \tag{3.3.14}$$

On the right hand side of equation 3.3.14 the denominator corresponds to $\frac{\partial R}{\partial \mu}$ whereas the nominator corresponds to $\frac{\partial R}{\partial \lambda}$. Since we consider the case where the solution u is of finite type, that is, there are only finitely many branchpoints, and since we know by remark 3.9 that any branchpoint is not a double point, we may write equation (3.3.13) on $\mathbb{C}^\times \times \mathbb{C}^\times$ in two polynomials $a(\lambda)$ and $b(\lambda)$ such that $\kappa^2 \simeq (\frac{b}{i})^2 \cdot \frac{1}{a \cdot \lambda}$. We will denote the new equation up to now by $\tilde{Y}^*$ where $*$ stands for the lack of the points where $\lambda = \infty, \lambda = 0$.

$$\tilde{Y^*} = \{\kappa^2 - \left(\frac{1}{i} \right)^2 \frac{b(\lambda)^2}{a(\lambda)\lambda} = 0 \mid (\kappa, \lambda) \in \mathbb{C}^\times \times \mathbb{C}^\times\} \tag{3.3.15}$$

Here $b(\lambda)$ describes the zeros of $d\ln(\mu)$ and $a(\lambda)$ the zeros of $d\ln(\lambda)$ which are the branchpoints. If we take $\nu := \frac{b}{i \cdot \kappa}$ as a new parameter, then we may write the spectral curve as

$$Y^* = \{\lambda a(\lambda) = \nu^2 \mid (\nu, \lambda) \in \mathbb{C}^\times \times \mathbb{C}^\times\}. \tag{3.3.16}$$

Notation 3.12. *We will denote the genus of a hyperelliptic Riemann surface by g.*

By 2.26 the eigenvaluefunction $\mu_{1,2}(\lambda)$ has essential singularities at $\lambda = 0$ and $\lambda = \infty$. Therefore the spectral curve must have infinitely many singularities at points with value $\mu = \pm 1$.

We now compactify the spectral curve over $\lambda = 0$ and $\lambda = \infty$. In local parameters this just means that we add 0 and ∞ with the local charts $\lambda \mapsto \frac{1}{\sqrt{\lambda}}$ in $U_\epsilon(\infty)$ and $\lambda \mapsto \sqrt{\lambda}$ in $U_\epsilon(0)$. The **compactified spectral curve** that is a **compact hyperelliptic Riemann surface** we denote by Y. Y is defined on $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Remark 3.13 (Interpretation of $a(\lambda)$ and $b(\lambda)$)**.** *We want to find all tuples $(a(\lambda), b(\lambda))$ that correspond to periodic finite type solutions of the sinh-Gordon equation and describe their spectral curves. The polynomial a defines the hyperelliptic curve and the second polynomial the meromorphic differential $d\ln\mu$ on the spectral curve.*

3.4 Properties of the spectral curve

In this section we shall describe $a(\lambda)$ by a real (that means $a(\lambda)$ satisfies the reality condition 3.4) polynomial of degree $2g$. On the spectral curve there are three involutions which we derive by the behavior of the monodromy-operator M_λ:

Lemma 3.14 (Involutions on Y)**.** *We have the following three involutions on Y:*

- $\sigma(\nu, \lambda) = (-\nu, \lambda)$
- $\eta(\nu, \lambda) = (\overline{\lambda}^{(-(g+1))}\overline{\nu}, \frac{1}{\overline{\lambda}})$
- $\rho(\nu, \lambda) = \eta \circ \sigma(\nu, \lambda)$

σ denotes the hyperelliptic involution. One can derive η by the reality condition for the monodromy-operator, [8].

For later purposes we also write down how these three involutions act on the eigenvalue μ. They act as

$$\sigma^*(\mu) = \frac{1}{\mu} \quad \eta^*(\mu) = \overline{\mu} \quad \rho^*(\mu) = \frac{1}{\overline{\mu}}$$

Corollary 3.15. *For a hyperelliptic Riemann surface of genus g the involution σ has exactly $2g$ fixpoints that correspond to the branchpoints of Y.*

A compact Riemann surface of genus g is topologically a sphere with g handles. For any compact Riemann surface there holds the **Riemann-Hurwitz formula** from which we are able to calculate the genus of the compact Riemann surface Y.
If we donote by λ the projection from Y to $\mathbb{CP}^1$ that is a two sheeted covering in the hyperelliptic case, then we call $\mathbb{CP}^1$ the base space in this setting. Its genus g_0 is 0. We will denote the number of sheets of the covering by N with $N = 2$ in the hyperelliptic case, and we will denote by $B = 2\tilde{g} + 2$ the total number of branch points. The **Riemann-Hurwitz formula** reads

$$2g - 2 = N(2g_0 - 2) + B \tag{3.4.1}$$

In our case it reduces to

$$2g - 2 = 2(0 - 2) + 2\tilde{g} + 2 \tag{3.4.2}$$

So $\tilde{g} = g$ or: The number of branchpoints corresponds to the genus of the compact Riemann surface Y.
In order to describe Y by two polynomials $a(\lambda)$ and $b(\lambda)$, where $a(\lambda)$ corresponds to the hyperelliptic surface Y and $b(\lambda)$ corresponds to the logarithmic differential $d\ln(\mu)$, we state the following:

Lemma 3.16 (Reality- aund positivity condition). *For the two polynomials a and b describing a compact Riemann surface Y and the meromorphic differential* $d\ln(\mu)$ *there must hold the following: If Y is of genus g then* $a(\lambda)$ *must have degree* $2g$ *and* $b(\lambda)$ *must have degree* $g+1$*. Furthermore* $a(\lambda)$ *and* $b(\lambda)$ *transform under the involution* η *as follows:*

$$a \in \mathbb{C}^{2g}[\lambda] : \overline{\eta^* a(\lambda)} = \overline{a(\frac{1}{\overline{\lambda}})} = \lambda^{-2g} a(\lambda) \tag{3.4.3}$$

and

$$b \in \mathbb{C}^{g+1}[\lambda] : \overline{\eta^* b(\lambda)} = \overline{b(\frac{1}{\overline{\lambda}})} = -\lambda^{-(g+1)} b(\lambda) \tag{3.4.4}$$

For $\lambda \in \mathbb{S}^1$ *we have the following* ***negativity condition***

$$\lambda^{-g} a(\lambda) \leq 0 \tag{3.4.5}$$

Proof. The degrees of $a(\lambda)$ and $b(\lambda)$ are consequences of the formula (3.4.1) and formula (3.3.15). The last statement one can calculate by the formula for the involution η 3.14 which reads for $b(\lambda)$ concrete as $\overline{\eta^* b(\lambda)} = \overline{b(\frac{1}{\overline{\lambda}})} \Leftrightarrow \lambda^{g+1}\overline{b(\frac{1}{\overline{\lambda}})} = -b(\lambda)$. A proof for the negativity condition one can find in [8]. By computing $\overline{\eta^*(\overline{\eta^*})}$ on $a(\lambda)$ and $b(\lambda)$ one derives that η is in fact an involution. □

We may interpret the differential $dp = \frac{1}{i} d\ln(\mu)$ in terms of the polynomial $b(\lambda)$ and $\nu(\lambda) = \sqrt{\lambda a(\lambda)}$. We now make use of $b = \sqrt{\kappa^2 a(\lambda)\lambda} = i\kappa \cdot \nu$ and by equation (3.3.9) we also have $\kappa d\ln(\lambda) = \frac{1}{i} d\ln(\mu)$. We calculate

$$dp = \frac{1}{i} d\ln(\mu) = \frac{\kappa\nu}{\nu} d\ln(\lambda) = \frac{1}{i}\frac{b}{\nu}\frac{d\lambda}{\lambda} \tag{3.4.6}$$

We therefore write the differential $d\ln(\mu)$ in terms of $a(\lambda)$ and $b(\lambda)$ as

$$d\ln(\mu) = \frac{b}{\nu}\frac{d\lambda}{\lambda} \tag{3.4.7}$$

One can show that $\ln\mu$ has first order poles at $\lambda = 0$ and $\lambda = \infty$. Thus we conclude that the differential $d\ln(\mu)$ has poles of second order at the points $\lambda = 0$ and $\lambda = \infty$.
The eigenvalue μ transforms under the hyperelliptic involution σ as $\sigma^* d\ln(\mu) = -d\ln(\mu)$. This follows directly by the involutions 3.14 and how they act on the eigenvalue μ. Furthermore we have $\mu(\alpha_i) = \pm 1 \Longleftrightarrow \ln\mu(\alpha_i) \in \pi i\mathbb{Z}$.
By lemma 3.16 and equation (3.4.6) one verifies that adding a double root to $a(\lambda)$ and adding the same simple root to $b(\lambda)$ does not change the differential $d\ln(\mu)$. In the following propositions we will investigate in the precise form of the polynomials $a(\lambda)$ and $b(\lambda)$ if we add certain roots in such a manner that the differential $d\ln\mu$ does not change.
We will denote those roots of $b(\lambda)$ that lie on $\mathbb{S}^1$ and that have no partner under the involution η except themselves by β_{m_i}.

Proposition 3.17 (Properties of the polynomial $b(\lambda)$). *Upon a constant factor $i\phi, \phi \in \mathbb{R}$ and the order of its roots the polynomial $b(\lambda)$ is completely determined by the roots. We think about the monomials that coorespond to those roots that lie on $\mathbb{S}^1$ and have no η-partners except themselves, β_{m_i}, to occur in the "middle" of the polynomial $b(\lambda)$. Then we may index each η-pair of roots by β_i and β_{g+2-i}.*
If we write the polynomial $b(\lambda)$ in the form $b(\lambda) = b_{g+1}\prod_{j=1}^{g+1}(\lambda-\beta_j)$, then the following holds: Any root β_m on $\mathbb{S}^1$ that has no partner under the involution η except itself contributes to the prefactor b_{g+1} by $i\sqrt{\overline{\beta_m}}$. Any pair of roots that is in involution under η, $(\lambda-\beta_j)(1-\lambda\overline{\beta_j}) = -\overline{\beta_j}(\lambda-\beta_j)(\lambda-\frac{1}{\overline{\beta_j}}) = -\overline{\beta_j}(\lambda-\beta_j)(\lambda-\beta_{g+2-j})$ contributes to the prefactor b_{g+1} by $-\overline{\beta_j}$.
Therefore we may write the polynomial $b(\lambda)$ as

$$b(\lambda) = b_{g+1}\prod_{j=1}^{g+1}(\lambda-\beta_j) \text{ with } b_{g+1} = i\phi(\prod i\sqrt{\overline{\beta_{m_j}}})\cdot\prod(-\overline{\beta_j}). \tag{3.4.8}$$

Proof. If $g=0$ then $b(\lambda)$ must have degree 1. Then the first root β_{m_1} must transform into β_{m_1} under the involution η. β_{m_1} must lie on $\mathbb{S}^1$ because of its transformation behavior under η. If $b^0(\lambda)$ is of the form as given in the proposition, then $-\lambda\cdot\overline{\eta^* b(\lambda)} = -\lambda\overline{i^2\phi}\sqrt{\beta_{m_1}}(\frac{1}{\lambda}-\frac{\sqrt{\overline{\beta_{m_1}}}}{\sqrt{\beta_{m_1}}}) = -i^2\phi\sqrt{\overline{\beta_{m_1}}}(\sqrt{\beta_{m_1}}-\lambda) = b^0(\lambda)$ for $\beta_{m_1}\in\mathbb{S}^1$ and $\phi\in\mathbb{R}$. therefore the change of sign of the polynomial $b(\lambda)$ is encoded in the factor i in b_{g+1}. The monomial $b^0(\lambda)$ is uniquely determined by its root β_{m_1} and a constant $0\neq\phi\in\mathbb{R}$. If we write $b^0(\lambda)$ in such a way that the root β_{m_1} occurs as $(\lambda-\beta_{m_1})$, then the prefactor b_{g+1} is given by $i^2\phi\sqrt{\overline{\beta_{m_1}}}$. Any other root β_{m_j} on $\mathbb{S}^1$ that has no partner under the involution η except itself contributes by the factor $i\sqrt{\overline{\beta_{m_j}}}$. The prefactor ensures that the reality condition holds: $\lambda\overline{\eta^* i\sqrt{\overline{\beta_m}}(\lambda-\frac{\sqrt{\beta_m}}{\sqrt{\overline{\beta_m}}})} = i\sqrt{\overline{\beta_m}}(\lambda-\frac{\sqrt{\beta_m}}{\sqrt{\overline{\beta_m}}})$.
In general one derives, by following the ansatz $b^0(\lambda) = \psi(\lambda-\beta_{m_1})$ for an arbitrary prefactor ψ, that there must hold $\frac{\psi}{\overline{\psi}} = \overline{\beta_{m_1}}$.
If one now adds a double root to the polynomial $b(\lambda)$ or any single root on $\mathbb{S}^1$ one has to verify that this is compatible with the reality condition. One has $\lambda^2\overline{\eta^*(\lambda-\beta_i)(1-\lambda\overline{\beta_i})} = \lambda^2(\frac{1}{\lambda}-\overline{\beta_i})(1-\frac{1}{\lambda}\beta_i) = (1-\lambda\overline{\beta_i})(\lambda-\beta_i)$.

□

Remark 3.18 ($b(0) = b_0 = -\overline{b_{g+1}}$). *For $b(\lambda)$ there holds the following symmetry-property: $b_j = -\overline{b_{g+1-j}}\ \forall j = 1\ldots g+1$. Therefore we have $-\overline{b_{g+1}} = b_0$. In standard form $b(\lambda) = \sum_{i=0}^{g+1} b_i\lambda^i$. By the reality condition we have*

$$\overline{b\left(\frac{1}{\overline{\lambda}}\right)} = \sum_{i=0}^{g+1}\overline{b_i}\left(\frac{1}{\lambda}\right)^i = -\lambda^{-(g+1)}\sum_{i=0}^{g+1} b_i\lambda^i \tag{3.4.9}$$

which is equivalent to

$$-\sum_{i=0}^{g+1} \overline{b_i}\lambda^{-i+g+1} = \sum_{i=0}^{g+1} b_i\lambda^i \tag{3.4.10}$$

We then may define j to be $j := -i + g + 1$ so that the equation reads

$$-\sum_{j=0}^{g+1} \overline{b_{g+1-j}}\lambda^j = \sum_{j=0}^{g+1} b_j\lambda^j \tag{3.4.11}$$

and we conclude $b_j = -\overline{b_{g+1-j}}$ $\forall j = 1 \ldots g+1$. Especially we have $b(0) = b_0 = -\overline{b_{g+1}}$. Therefore the highest coefficient of $b(\lambda)$ together with the roots β_i of $b(\lambda)$ determines the polynomial $b(\lambda)$ uniquely up to the order of the roots in a factorization of the polynomial $b(\lambda)$.
We may write $b(\lambda)$ in the form $b(\lambda) = b_{g+1}\prod_{j=1}^{g+1}(\lambda - \beta_j)$ and insert $\lambda = 0$. By doing so we derive the formula $b(0) = b_0 = b_{g+1}(-1)^{g+1}\prod_{j=1}^{g+1}\beta_j$ and we thereby also get the result that $\frac{b_0}{\overline{b_0}} = \prod_{j=1}^{g+1}(-\beta_j)$, that is, the product over all roots of $b(\lambda)$ must be in $\mathbb{S}^1$.

Proposition 3.19 (symmetrie proporties of $a(\lambda)$ and uniqueness of $a(\lambda)$)**.** *We may write the polynomial $a(\lambda)$ uniquely as*

$$a(\lambda) = \tilde{a}_N \prod_{i=1}^{g}(\lambda - \alpha_i)(1 - \lambda\overline{\alpha_i}) \quad \tilde{a}_N = \frac{(-1)^g}{\prod_{i=1}^{g}|\alpha_i|}$$

Proof. By the reality condition for the polynomial $a(\lambda)$ in lemma 3.16, $\lambda^{2g}\overline{a(\frac{1}{\overline{\lambda}})} = a(\lambda)$, one has to ensure that the roots of $a(\lambda)$ are pairwise in involution. By construction of the polynomial $a(\lambda)$ one has $\lambda\overline{\eta^*(\lambda - \alpha_i)} = \lambda \cdot (\frac{1}{\lambda} - \overline{\alpha_i}) = (1 - \lambda\overline{\alpha_i})$. And on the other hand we have $\lambda \cdot \overline{\eta^*(1 - \lambda\overline{\alpha_i})} = \lambda \cdot (1 - \frac{\alpha_i}{\lambda}) = (\lambda - \alpha_i)$. We conclude that if we write the polynomial $a(\lambda)$ as in the proposition, the reality condition will be fulfilled for the factors $(\lambda - \alpha_i)$ and $(1 - \lambda\overline{\alpha_i})$ respectively. Because $\tilde{a}_N \in \mathbb{R}$ the reality condition will be fulfilled for the polynomial $a(\lambda)$ as a whole.
Furthermore it is well known that one obtains uniqueness (of a polynomial) by giving its roots and, in addition, normalizing its highest coefficient. In our case we obtain uniqueness by the claim that the highest coefficient shall have the norm one together with the negativity condition given in proposition 3.17. The highest coefficient is determined by the roots α_i, therefore we obtain uniqueness by dividing the polynomial by the product over the norm of those roots. □

Remark 3.20 (Preparations for numerical computations: $a(\lambda)$)**.** *For the purposes of the later numerical computations we want to write $a(\lambda)$ in the form $a_{2g}\prod_{i=1}^{2g}(\lambda - \alpha_i)$. To reach uniqueness we think about the roots that occur in the polynomial $a(\lambda)$ to be ordered in the following way: For any root α_i its involutionary counterpart shall have* *the index* $2g+2-i$. *A computation gives* $a(\lambda) = \frac{\prod_{i=1}^{g} \overline{\alpha_i}}{\prod_{i=1}^{g} |\alpha_i|} \prod_{i=1}^{2g}(\lambda - \alpha_i)$. *The prefactor* $a_{2g} = \frac{\prod_{i=1}^{g} \overline{\alpha_i}}{\prod_{i=1}^{g} |\alpha_i|} = \prod_{j=1}^{g} \sqrt{\frac{\overline{\alpha_j}}{\alpha_j}}$ *is the product over the first* g *indices of the roots of* $a(\lambda)$.

4 Isoperiodic deformations of the spectral data

4.1 Spectral data

In the following definition we summarize some of the results of the last section. We will denote the roots of $a(\lambda)$ by α_i and the roots of $b(\lambda)$ by β_i. The α_i represent the branchpoints of the spectral curve. Therefore $\mu(\alpha_i) = \pm 1$. To obtain uniqueness we normalize the polynomial $a(\lambda)$ by the condition that the highest coefficient of $a(\lambda)$ has absolute value 1, $|a(0)| = |a_{2g}| = 1$.
We have $\mu(\alpha_i) = \pm 1 \iff \ln \mu(\alpha_i) \in \pi i \mathbb{Z}$ because the exponential map is multivalued, $\exp(2k\pi i) = 1, k \in \mathbb{Z}$ and the eigenvalue μ is a complex exponential.

Definition 4.1 (Spectral data)**.** *Let $a(\lambda)$ be a polynomial of degree $2g$ and let $b(\lambda)$ be a polynomial of degree $g+1$, both fulfilling the reality condition in lemma 3.16. Let λ_0 and $\lambda_1 \in \mathbb{S}^1$, $\lambda_0 \neq \lambda_1$. The spectral data of a CMC cylinder of finite type in $\mathbb{S}^3$ with mean curvature*

$$H = i\frac{\lambda_0 + \lambda_1}{\lambda_0 - \lambda_1} \tag{4.1.1}$$

consists of a tuple $(a(\lambda), b(\lambda), \lambda_0, \lambda_1)$ with the following properties:

1. ***Reality condition*** *The polynomials $a(\lambda)$ and $b(\lambda)$ describing the hyperelliptic Riemann surface Y transform under the involution η as in lemma 3.16. We normalize $a(\lambda)$ by the condition that its highest coefficient a_{2g} has absolute value* 1*.*

2. ***Closing condition*** *The meromorphic differential $d\ln\mu = \frac{b}{\nu}\frac{d\lambda}{\lambda}$ has periods in $2\pi i\mathbb{Z}$*

3. ***Negativity condition*** *$\lambda^{-g}a(\lambda) \leq 0$ for $\lambda \in \mathbb{S}^1$*

For all $g \in \mathbb{N}_0$ we will call the space of equivalence classes of spectral data (a, b) obeying the conditions above the ***moduli space*** *and denote it by $\mathcal{M}_g(a, b)$.*

Remark 4.2 (Assumption)**.** *In general we assume that the polynomials $a(\lambda)$ and $b(\lambda)$ have no common roots. We will pay special attention to the exceptional case.*

We now concentrate on the deformation of the polynomials $a(\lambda)$ and $b(\lambda)$.
We follow the ansatz in [4]. Therefore we introduce deformation parameters t such that $R(\lambda, \mu, t)$ is a onedimensional family of spectral curves. On the Riemann surface Y there exists a regular meromorphic 1-form. Because of the reality condition 3.16 we look for a 1-form ω on Y that fullfills $\overline{\eta^*\omega} = -\omega$. Therefore we modify the ansatz in [4] and choose a logarithmic coordinate in λ too, $\lambda \longmapsto \ln\lambda$ such that the form ω transforms under the involution η in the following way

$$\overline{\eta^* d\ln(\lambda)} = \overline{\left(\frac{d\frac{1}{\bar{\lambda}}}{\frac{1}{\bar{\lambda}}}\right)} = -d\ln(\lambda).$$

The meromorphic 1-form thus reads

$$\omega := \frac{\partial p}{\partial t} d\ln\lambda - \frac{\partial \ln\lambda}{\partial t} dp \qquad (4.1.2)$$

Proposition 4.3. *For the form ω we may choose either p or $\ln(\lambda)$ not to depend on t.*

Proof. The total differential of $R = R(\ln(\lambda), p)$ reads $dR = \frac{\partial R}{\partial \ln(\lambda)} d\ln(\lambda) + \frac{\partial R}{\partial p} dp = 0$. If we assume the functions $\ln(\lambda)$ and p to depend on t as well, we calculate $\frac{\partial R}{\partial t} = \frac{\partial R}{\partial \ln(\lambda)} \frac{\partial \ln(\lambda)}{\partial t} + \frac{\partial R}{\partial p} \frac{\partial p}{\partial t}$. Hence we see that we may write ω by making use of the total differential dR as

$$\omega = -\frac{\partial R}{\partial t} \left(\frac{\partial R}{\partial \ln(\lambda)}\right)^{-1} dp = \frac{\partial R}{\partial t} \left(\frac{\partial R}{\partial p}\right)^{-1} d\ln(\lambda) \qquad (4.1.3)$$

□

If we want to compare the different Riemann surfaces corresponding to different values of t, we may either choose $\ln(\lambda)$ not to depend on t such that p becomes a multivalued function depending on λ and t, or we choose p not to depend on t. Then $\ln(\lambda)$ becomes a multivalued function depending on t and p.
In the sequel we choose the function $\ln(\lambda)$ not to depend on t. Then $\ln\mu$ is a multivalued function dependung on t and λ.

4.2 Isoperiodic deformations of the spectral curve

We want to deform the spectral curve Y and the differential $d\ln(\mu) = idp = \frac{b}{\nu}\frac{d\lambda}{\lambda}$ without leaving the moduli space $\mathcal{M}_g$. So for a given set of polynomials $(a, b) \in \mathcal{M}_g$ we want to flow in the moduli space without leaving it, that is, we want to perserve the conditions stated in the definition of the spectral data in 4.1. We define:

Definition 4.4 (Isoperiodic deformation). *An isoperiodic deformation (of given spectral data $(a,b) \in \mathcal{M}_g$) is a deformation of these data that leaves the periods of the meromorphic differential $d\ln(\mu) = \frac{b}{\nu}\frac{d\lambda}{\lambda} = i \cdot dp$ invariant.*

We consider the following ansatz: We investigate $\partial_t \ln(\mu)$ where t is the deformation parameter we introduced in equation (4.1.2) and we have chosen the function $\ln(\lambda)$ not to depend on t. Then we may write the form ω from equation (4.1.2) as

$$\omega = \dot{p} d\ln\lambda \Longleftrightarrow \dot{p} = \frac{1}{i}\frac{\partial}{\partial t}\ln\mu = \frac{\omega}{d\ln\lambda} \tag{4.2.1}$$

Since the expression on the right hand side is the quotient of two meromorphic one forms, we may express this quotient as a meromorphic function. And because the branchpoints of $\ln\mu$ only differ by elements in $2\pi i\mathbb{Z}$ we get that $\frac{d}{dt}\ln\mu$ is a single valued meromorphic function. Hence we may express the meromorphic function $\frac{\partial}{\partial t}\ln\mu(\lambda,t)$ as the quotient of two polynomials

$$\frac{\partial}{\partial t}\ln\mu(\lambda,t) = \frac{c(\lambda)}{\nu(\lambda)}. \tag{4.2.2}$$

We now consider families $\lambda(t)$ where $\mu(\lambda,t) = \text{const}$. We will denote the partial derivative with respect to t at $t=0$ with a dot, $\dot{p} = \partial_t p|_{t=0}$. We then calculate

$$\frac{d\ln\mu(\lambda(t),t)}{dt} = \frac{1}{\mu(\lambda(t),t)}\frac{\partial\mu(\lambda(t),t)}{\partial\lambda(t)}\dot{\lambda}(t) + \frac{1}{\mu(\lambda(t),t)}\frac{\partial\mu(\lambda(t),t)}{\partial t} = 0 \tag{4.2.3}$$

$$\Longleftrightarrow \frac{1}{\mu(\lambda(t),t)}\frac{d\mu(\lambda(t),t)}{d\lambda(t)}\dot{\lambda}(t) = -\frac{c(\lambda)}{\nu(\lambda)} \tag{4.2.4}$$

$$\Longleftrightarrow \dot{\lambda}(t) = -\frac{c(\lambda)}{b(\lambda)}\lambda(t). \tag{4.2.5}$$

The last expression describes deformations of the roots of the polynomials $a(\lambda)$ and $b(\lambda)$ in the following sense: The roots of $a(\lambda)$ correspond to families $\lambda(t)$ where $\mu(\lambda(t),t) =$ const as well as those roots of $b(\lambda)$ that are also roots of $c(\lambda)$. We consider the exceptions later in this section.
Next we verify that an infinitisimal deformation in the parameter(s) t is an isoperiodic deformation.

Proposition 4.5. *The infinitesimal deformation in the parameter(s) t, $\partial_t \ln(\mu)$ does not change the periods of the differential dp.*

Proof. Let Ψ be a closed cycle on Y, $\Psi : [0,1] \longrightarrow Y$. We then have $\int_\Psi dp = 2\pi\mathbb{Z}$. We show that the partial derivative with respect to t in $t=0$ of $dp(t)$ must be zero, and hence the the deformation $\partial_t \ln(\mu)$ must preserve the periods of dp.

$$\frac{\partial}{\partial t}\int_\Psi dp(t)|_{t=0} = \int_\Psi \frac{\partial}{\partial t} dp(t)|_{t=0} = \int_\Psi d\dot{p}(t) = \dot{p}(\Psi(1)) - \dot{p}(\Psi(0)) = 0 \tag{4.2.6}$$

□

Formula 4.2.3 is only valid at those roots of $b(\lambda)$ that are also roots of $c(\lambda)$. At the critical points we make use of the integrability condition $\partial^2_{t\lambda}\ln(\mu) = \partial^2_{\lambda t}\ln(\mu)$. We therefore compute with $\frac{\partial}{\partial t}\ln(\mu) = i\dot{p} = \frac{c}{\sqrt{\lambda a(\lambda)}}$ and $\frac{\partial}{\partial \lambda}\ln(\mu) = ip^{'} = \frac{b}{\sqrt{\lambda a(\lambda)}\lambda}$

$$i\dot{p}^{'} = \frac{c^{'}\nu - \dfrac{c}{2\sqrt{\lambda a}}(a + a^{'}\lambda)}{\nu} = \frac{c^{'}\lambda a - ca - ca^{'}\lambda}{2\nu^2} \tag{4.2.7}$$

$$i\dot{p^{'}} = \frac{\dot{b}\sqrt{\lambda a}\lambda - \dfrac{b}{\sqrt{\lambda a}\lambda}(0 + \dot{a}\lambda^2)}{\nu\lambda^2} = \frac{2\dot{b}a - b\dot{a}}{2\nu^2} \tag{4.2.8}$$

As result we get the integrability condition, usually denoted as **Whitham equation**

$$-2\dot{b}a + \dot{a}b = -2\lambda ac^{'} + ac + \lambda a^{'}c \tag{4.2.9}$$

In the following propositions we want to summarize some properties that must hold for the polynomials $c(\lambda)$:

Proposition 4.6. *For the polynomial c describing the deformation $\dot{p}$ there holds:*

1. $c \in \mathbb{C}[\lambda]$ *with* $\deg c \leq g + 1$
2. $\overline{\eta^* c} = \lambda^{-(g+1)} c$
3. *if* $\deg c < g + 1$ *then for the period τ there must hold* $\dot{\tau} = 0$

A proof for this statement can be found in [8].

Lemma 4.7 (Deformation polynomials)**.** *In order to deform the spectral data $\mathcal{M}_g(a, b)$ of a finite type solution of the sinh-Gordon equation without changing the periods of p (isoperiodic deformation), we may choose $g + 1$ coordinates of the form*

$$t_k = \Delta(\beta_k) \text{ with } k = 1 \dots g + 1 \tag{4.2.10}$$

Here Δ denotes again the trace of the monodromy M_λ. Each t_k corresponds to a deformation flow of the form $\partial_t \ln(\mu)$. Furthermore there are normalization constants γ_k which may be choosen such that $\partial_{t_k}(t_k = \Delta(\beta_k)) = 1$ and $\partial_{t_k}(t_j) = \delta_{kj}$.
The resulting deformations may be described by polynomials c_k that must vanish at all roots of $b(\lambda)$ except β_k. We find the polynomials $c_k(\lambda)$ and $c_m(\lambda)$ by making use of the ansatz $\partial_t \ln(\mu) = \frac{c}{\nu}$. γ_k and γ_m will denote normalization constants. For the polynomial $c_k(\lambda)$ one may take

$$c_k(\lambda) = \left(\frac{\gamma_k}{\lambda - \beta_k} - \frac{\overline{\gamma_k}\lambda}{1 - \lambda\overline{\beta_k}}\right) b(\lambda)\ ;\ k = 1 \dots g + 1 \tag{4.2.11}$$

and for those roots $\beta_m \in \mathbb{S}^1$ having no "partner" under the involution η the corresponding deformation polynomial c_m reads

$$c_m(\lambda) = \left(\frac{\gamma_m}{\lambda - \beta_m} - \frac{\overline{\gamma_m}\lambda}{1 - \lambda\overline{\beta_m}}\right) b(\lambda) = \left(\frac{\gamma_m}{\lambda - \beta_m} + \frac{\overline{\gamma_m}\lambda\beta_m}{\lambda - \beta_m}\right) b(\lambda). \tag{4.2.12}$$

Proof. To deform the spectral data we use the form $\omega = \dot{p} d\ln(\lambda) - \dot{\ln(\lambda)} dp$ as introduced in (4.1.2). As was shown in proposition 4.5 we may choose either p or $\ln(\lambda)$ not to depend on t. Therefore ω may occur either as $\omega_1 = \dot{p} d\ln(\lambda)$ or $\omega_2 = -\dot{\ln(\lambda)} dp$.
We make the ansatz $\omega = \dot{p}\frac{d\lambda}{\lambda}$ such that $\partial_t \ln(\mu) = \frac{c(\lambda)}{\nu(\lambda)}$. We already know that $\dot{\lambda} = -\frac{c(\lambda)}{b(\lambda)}\lambda$ at those roots of $b(\lambda)$ that are also roots of $c(\lambda)$.
An infinitesimal deformation at the points α_i and β_i thus reads $\dot{\alpha} = -\frac{c(\alpha)}{b(\alpha)}\alpha$ and, if the roots of $b(\lambda)$ are also roots of $c(\lambda)$, $\dot{\beta} = -\frac{c(\beta)}{b(\beta)}\beta$. One may see that $c(\lambda)$ is most simple if it is of the form $c = fb$ with some function f. Since $b(\lambda)$ has $g+1$ roots we make the ansatz to divide for each $k \in \{1 \dots k+1\}$ two roots from the polynomial $b(\lambda)$, where the two roots are choosen such that they are in involution by the involution η. Thereby we have to ensure that the c_k fulfill the reality condition $\overline{\eta^* c} = \lambda^{-(g+1)} c$. We consider the case of the β_{m_i}, β_M later seperately. η acts on the function $\lambda - \beta_k$ for an arbitrary $k \in \{1 \dots g+1\}$, $k \neq m$ as $\overline{\eta^*(\lambda - \beta_k)} = \frac{1}{\lambda} - \overline{\beta_k}$. The latter expression is zero if $\lambda = \frac{1}{\overline{\beta_k}}$.
We make the ansatz $c_k = \frac{b(\lambda)}{\lambda - \beta_k} - \frac{b(\lambda)}{\overline{\eta^*(\lambda - \beta_k)}}$ for $\beta_k \neq \beta_m$. c_k transforms under η as

$$\begin{aligned} &\overline{\eta^*\left(\frac{b}{\lambda - \beta_k} - \frac{b}{\overline{\eta^*(\lambda - \beta_k)}}\right)} = \frac{\overline{\eta^*(b)}}{\overline{\eta^*(\lambda - \beta_k)}} - \frac{\overline{\eta^*(b)}}{\eta^*\eta^*(\lambda - \beta_k)} = \\ &\qquad -\frac{1}{\lambda^{g+1}}\left(\frac{b}{\overline{\eta^*(\lambda - \beta_k)}} - \frac{b}{\lambda - \beta_k}\right) = \frac{1}{\lambda^{g+1}} c_k \end{aligned} \tag{4.2.13}$$

The equation above is still valid if we insert additional parameters γ_k as stated in the lemma. So we may write the c_k as

$$c_k(\lambda) = \left(\frac{\gamma}{\lambda - \beta_k} - \frac{\overline{\gamma}\lambda}{1 - \lambda\overline{\beta_k}}\right) b(\lambda)\ ;\ k = 1 \dots g+1 \tag{4.2.14}$$

In the case where we divide $b(\lambda)$ by β_m we recognize $\beta_m = \frac{1}{\overline{\beta_m}}$. □

In order to prepare later numerical computations we formulate the foregoing results in polynomials that differ slightly from the polynomials introduced in the last lemma.

Remark 4.8 (Preparations for numerical computations: $c_k(\lambda)$)**.** *We are now in the situation to adjust the deformation polynomials $c_k(\lambda)$ and represent them in a unique* *way. If we denote the second summand of the polynomial* $c_k(\lambda)$ *by* c_{k_2} *a short computation gives* $c_{k_2} = -\frac{\overline{\gamma}}{\frac{1}{\lambda}-\overline{\beta}_k} b(\lambda) = \frac{\overline{\gamma}\lambda\beta_{g+2-k}}{\lambda-\beta_{g+2-k}} b(\lambda)$. *We may then write the deformation polynomial uniquely as* $c_k(\lambda) = (\frac{\gamma}{\lambda-\beta_k} + \frac{\overline{\gamma}\lambda\beta_{g+2-k}}{\lambda-\beta_{g+2-k}})b(\lambda)$. *If the root* β *lies on* $\mathbb{S}^1$ *and has no involutionary partner, Then the formula reads* $c_m(\lambda) = (\frac{\gamma}{\lambda-\beta_m} + \frac{\overline{\gamma}\lambda\beta_m}{\lambda-\beta_m})b(\lambda)$.

Proposition 4.9 (The normalization constants)**.** *In this proposition we will give formulas for the normalization constants* γ_k *and* γ_{m_i}.

Proof. To derive the normalization constants we claim $\partial_{t_k}(t_k = \Delta(\beta_k)) = 1$. The β_k are evaluated at $t = 0$. We may write the trace of the Monodromy Δ as $\Delta = \mu + \frac{1}{\mu} = 2\cosh(\ln(\mu))$. And therefore

$$\dot{\Delta} = 2\sinh(\ln(\mu))\dot{\ln(\mu)} = \left(\mu - \frac{1}{\mu}\right)\dot{\ln(\mu)} = \left(\mu - \frac{1}{\mu}\right)\frac{c}{\nu} \tag{4.2.15}$$

The normalization condition then reads

$$\frac{\partial}{\partial_{t_k}}(\Delta(\beta_k)) = 2\sinh(\ln(\mu(\beta_k)))\frac{c(\beta_k)}{\nu(\beta_k)} = 1 \tag{4.2.16}$$

$b'(\beta_k)$ equals the polynomial $b(\beta_k)$ divided by the factor $(\lambda - \beta_k)$. We again consider the case of γ_k where $g+1$ is odd seperately. Where $\beta_k \neq \beta_m$ the γ_k are of the form

$$\gamma_k = \frac{\nu(\beta_k)}{2\sinh(\ln(\mu(\beta_k)))b'(\beta_k)} \tag{4.2.17}$$

and for any "partner" of one β_k this partner must be of the form $\frac{1}{\overline{\beta}_k}$. We will denote this partner $\frac{1}{\overline{\beta}_k}$ by β_{g+2-k}. It is a root of the denominator of c_2, the second summand of the deformation polynomial $c(\lambda)$. In this case the normalization constant reads

$$\overline{\gamma}_{g+2-k} = \frac{\nu(\beta_{g+2-k})}{2\sinh(\ln(\mu(\beta_{g+2-k})))b'(\beta_{g+2-k})\beta_{g+2-k}^2} \tag{4.2.18}$$

In order to derive normalization constants for those deformation flows that correspond to the β_{m_i} and β_M we derive the equation

$$\gamma_m + \overline{\gamma_m}\beta_m^2 = \frac{\nu(\beta_m)}{2\sinh(\ln(\mu(\beta_m)))b'(\beta_m)} \tag{4.2.19}$$

where $\beta_m = \frac{1}{\overline{\beta}_m}$. □

The roots β_{m_i} lying on $\mathbb{S}^1$ are not determined uniquely by the relation given above. We will determine them with the help of the following

Proposition 4.10. *Under a Möbius rotation of the form* $\lambda \mapsto i\phi \cdot \lambda$ *with* $\phi \in \mathbb{R}$ *the corresponding Riemann surfaces described by the spectral data* $(a, b) \in \mathcal{M}_g(a, b)$ *are biholomorphic equivalent. Hence we may only consider spectral data* $(a, b) \in \mathcal{M}_g(a, b)$ *up to a Möbius rotation in the given form.*
The corresponding deformation polynomials are of the form $c_{\text{möb}} = i\phi b$*, with* $\phi \in \mathbb{R}$.

Proof. We have to ensure that under a Möbius rotation which we denote by $m : \lambda \mapsto m(\lambda)$ the two distinguished points 0 and ∞ of Y the function λ are preserved under m. Therefore m must be of the form $m : \lambda \mapsto k \cdot \lambda$ with a complex constant k. We furthermore have to ensure that m is compatible with the involutions defined on $\mathcal{M}_g$, i.e. with η; $m \circ \eta \equiv \eta \circ m$. This leads to the condition that $|k| = 1$, and thus $m(\lambda)$ is given by $m : \lambda \mapsto \exp(i\phi t)\lambda$ with $\alpha \in \mathbb{R}$. In the following calculation we set $\lambda(t) = \exp(i\phi t)\lambda$ with $\phi \in \mathbb{R}$ such that $\dot{\lambda} = i\phi\lambda$. By the chain rule we have

$$\frac{\partial}{\partial t}\bigg|_{t=0}\Big(\ln \mu(\lambda(t))\Big) = \frac{\partial}{\partial \lambda}\Big(\ln \mu\Big)\dot{\lambda} = \frac{\partial \ln \mu}{\partial \lambda} i\phi\lambda \overset{!}{=} \frac{c}{\nu}.$$

By making use of $\frac{\partial \ln \mu}{\partial \lambda} = \frac{b}{\lambda\nu}$ we get, that the deformation polynomial corresponding to a Möbiustransformation $m(\lambda)$ must be of the form $c_{\text{möb}} = i\phi b$. □

We are now in the situation to give a representation of $c_m(\lambda)$ that depends only on one real parameter ϕ.

Proposition 4.11. *Up to a Möbius rotation the deformation polynomial* $c_m(\lambda)$ *is given by* $\gamma_m \cdot \frac{1+\beta_m^3\lambda}{\lambda-\beta_m} b(\lambda)$ *. For the normalization constant* γ_m *there holds*

$$\gamma_m = \frac{\nu(\beta_m)}{2\sinh(\ln \mu(\beta_m))(1+\beta_m^4)b'(\beta_m)}.$$

Proof. We chose the following coordinates for $\mathbb{C}$: $\overline{\beta_m}$ and $i\beta_m$. In these coordinates we have $\gamma_m = \phi_1 i\beta_m + \phi_2\overline{\beta_m}$ and therefore the previous relation reads $\gamma_m + \overline{\gamma_m}\lambda\beta_m = (\phi_{(1)} i(\beta_m - \lambda) + \phi_{(2)}(\overline{\beta_m} + \beta_m^2\lambda)b(\lambda))$; $\phi_{(i)} \in \mathbb{R}$. The corresponding deformation polynomial then reads

$$c_m(\lambda) = \frac{-i\phi_{(1)}(\lambda - \beta_m) + \phi_{(2)}(\overline{\beta_m} + \beta_m^2\lambda)}{\lambda - \beta_m} b(\lambda).$$

We may omit the term involving $\phi_{(1)}$ because it is a Möbius transformation. Therefore γ_m only has one degree of freedom which is encoded in the real constant $\phi_{(2)}$ and we may write γ_m as $\gamma_m = \phi_{(2)}\overline{\beta_m}$. By making use of this last equation one derives the formula for the normalization constant γ_m. □

Remark 4.12. *In the previous proposition we used that $\phi_i \in \mathbb{R}$. Especially when we inserted $\gamma_m = \phi_{(2)}\overline{\beta_m}$ into the equation of the normalization constant to derive the new statement about γ_m we heavily used that $\phi_{(2)} \in \mathbb{R}$. We want to show that this is indeed true.*
For $\lambda \in \mathbb{S}^1$ the Monodromy M_λ is in SU_2. Hence it is conjugate to a diagonal matrix with entries $\mu_1 = e^{i\phi}$, $\mu_2 = e^{-i\phi}$, $\phi \in \mathbb{R}$. Therefore $d\ln\mu \in i\mathbb{R}$ for $\lambda \in \mathbb{S}^1$. We also have that $\frac{d\lambda}{\lambda} \in i\mathbb{R}$ for $\lambda \in \mathbb{S}^1$. So we may conclude that the term $\frac{b(\lambda)}{\nu}$ occuring in the equation $d\ln\mu = \frac{b(\lambda)}{\nu}\frac{d\lambda}{\lambda}$ must be in $i\mathbb{R}$, $\frac{b(\lambda)}{\nu} \in \mathbb{R}$.
We also have that, for $\lambda \in \mathbb{S}^1$, $\mu(\lambda) = e^{\pm i\phi}$, the expression $\mu - \frac{1}{\mu} = \mu - \overline{\mu}$ must be in $i\mathbb{R}$. And because $\dot{\triangle} = 1 \in \mathbb{R}$ we follow by making use of the equation $\dot{\triangle} = 2\sinh(\ln\mu)\dot{\ln}\,\mu = (\mu - \overline{\mu})\dot{\ln}\,\mu = (\mu - \overline{\mu})\frac{c(\lambda)}{\nu(\lambda)}$ that $\frac{c(\lambda)}{\nu(\lambda)} \in i\mathbb{R}$.
We now consider the equation $i\mathbb{R} \ni \frac{c(\lambda)}{\nu(\lambda)} = (\gamma + \overline{\gamma}\beta_m^2)\frac{b'(\beta_m)}{\nu(\beta_m)} = (\overline{\beta_m}\gamma + \overline{\gamma}\beta_m)\frac{\beta_m b'(\beta_m)}{\nu(\beta_m)}$. For $(\overline{\beta_m}\gamma + \overline{\gamma_m}\beta_m) \in \mathbb{R}$ we conclude that $\frac{\beta_m b'(\beta_m)}{\nu(\beta_m)} \in i\mathbb{R}$.
We finally consider the equation $\gamma_m + \overline{\gamma_m}\beta_m^2 = \beta_m(\overline{\beta_m}\gamma_m + \overline{\gamma_m}\beta_m) = \frac{\nu(\beta_m)}{\sinh(\ln\mu(\beta_m))(1+\beta_m^4)b'(\beta_m)} = \frac{\nu(\beta_m)}{\sinh(\ln\mu(\beta_m))\beta_m^2(\overline{\beta_m}^2 + \beta_m^2)b'(\beta_m)}$. By the previous thougts this equation can only be true if γ_m is of the form $\phi\overline{\beta_m}$ with $\phi \in \mathbb{R}$.

In order to prepare the numerical computations we notice that for each normalization constant γ we have a root in the nominator and the denominator in some initial state, that is, at those points where the α_i, α_{i+2} equal one β_i. These are the branchpoints of the spectral curve and they will be the initial state when deforming the spectral data. To be prepared for the numerical computations we state the following proposition:

Proposition 4.13 (The normalization constants at the double points of the spectral curve)**.** *At the initial state of deformation we have at each double point $\alpha_i = \alpha_{i+1} = \beta_j$ of the spectral curve that* $\ln\mu(\beta_j) = 0$. *Therefore there also holds* $\sinh(\ln\mu(\beta_j)) = 0$ *and $\nu(\lambda) = \sqrt{\lambda a(\lambda)} = 0$ at $\lambda = \alpha_{i,i+1}$. β_1 denotes the root of $b(\lambda)$ that does not occur in $a(\lambda)$ in the initial state. A computation gives*

$$\lim_{\lambda \to \beta_j} \frac{\nu(\lambda)}{2\sinh\ln(\mu(\lambda))} = \frac{\beta_j^2 a_{2g}\prod_{i=1\,;\,i\neq j}^{g}(\beta_j - \beta_i)}{2b_{g+1}(\beta_j - \beta_1)} \tag{4.2.20}$$

Proof. We write the polynomial $a(\lambda)$ and $b(\lambda)$ as $a(\lambda) = a_{2g}\prod_{i=1}^{2g}(\lambda - \alpha_i) = a_{2g}\prod_{j=1}^{g}(\lambda - \beta_j)^2$ and $b(\lambda) = b_{g+1}\prod_{i=1}^{g+1}(\lambda - \beta_i)$. At the initial state all double roots of $a(\lambda)$ correspond to one root of $b(\lambda)$. The only exception is β_1, the root that occurs also in $b^0(\lambda)$. So at any branch point of the spectral curve $\alpha_i = \alpha_{i+1} = \beta_j$ we consider the limes $\lambda \to \beta_j =$

$\alpha_i = \alpha_{i+1}$. We make use of $\ln(\mu)' = \frac{b(\lambda)}{\nu(\lambda)\lambda}$.

$$\begin{aligned}
\lim_{\lambda\to\beta_j} \quad \frac{\nu(\lambda)}{2\sinh\ln(\mu(\lambda))} &= \lim_{\lambda\to\beta_j} \frac{\sqrt{\lambda a(\lambda)}}{2\sinh\ln(\mu(\lambda))} = \lim_{\lambda\to\beta_j} \frac{\sqrt{\lambda\cdot a_{2g}\prod_{i=1}^{g}(\lambda-\beta_i)}}{2\sinh\ln(\mu(\lambda))} \\
= \quad & \lim_{\lambda\to\beta_j} \sqrt{a_{2g}}\frac{\sqrt{\lambda}\prod_{i=1}^{g}(\lambda-\beta_i)}{2\sinh\ln(\mu(\lambda))} =^{\text{l'H.}} \lim_{\lambda\to\beta_j} \frac{\frac{\sqrt{a_{2g}}}{2\sqrt{\lambda}}\prod_{i=1}^{g}(\lambda-\beta_i)+\sqrt{\lambda}\sum_{i=1}^{g}\frac{\sqrt{a(\lambda)}}{\lambda-\beta_i}}{2\cosh\ln(\mu(\lambda))\ln'\mu(\lambda)} \\
= \quad & \lim_{\lambda\to\beta_j} \frac{\sqrt{\lambda a(\lambda)}}{(\lambda-\beta_j)2\ln'\mu(\lambda)} = \lim_{\lambda\to\beta_j} \frac{\lambda\sqrt{\lambda a(\lambda)}\sqrt{a(\lambda)\lambda}}{(\lambda-\beta_j)2b(\lambda)} = \lim_{\lambda\to\beta_j} \frac{\lambda^2\sqrt{a_{2g}}\sqrt{a(\lambda)}}{2b_{g+1}(\lambda-\beta_1)(\lambda-\beta_j)} \\
= \quad & \frac{\beta_j^2 a_{2g}\prod_{i=1\,;\,i\neq j}^{g}(\beta_j-\beta_i)}{2b_{g+1}(\beta_j-\beta_1)}
\end{aligned}$$

If one inserts this result into the formulas for the γ_i this leads to further simplification because one may cancel down the product in the nominator. □

4.3 Deformation ODEs

We are intereseted in deformations of the spectral data $\mathcal{M}_g(a,b)$ at the zeroes of a and b, namely of the points α_i and β_i. By equation (4.2.3) these deformations of the roots are of the form $\dot\alpha_i = -\frac{c}{b}\alpha_i$ and $\dot\beta_i = -\frac{c}{b}\beta_i$. We will furthermore develop a deformation equation for the coefficient $b(0) = b_0 = -\overline{b}_{g+1}$ of $b(\lambda)$. The dot denotes the derivative with respect to one $t_i; i \in \{1\ldots g+1\}$, evaluated at $t=0$ as before.
If we insert one β_j into (4.2.3) we get in general $\frac{\partial}{\partial_{t_k}}\beta_j = \lim_{\lambda\to\beta_j} -\frac{c_k(\lambda)}{b(\lambda)}\lambda$. We have to distinguish between roots of $b(\lambda)$ that have a partner under the involution η and those roots β_{m_i} of $b(\lambda)$ that transform under the involution η into themselves.
By the formulas for the $c_k(\lambda)$, $b(\lambda)$ and $a(\lambda)$ the equations for those roots that occur in η-pairs and whose denominator of $c_k(\lambda)$ does not vanish read

$$\begin{aligned}
\left.\frac{\partial}{\partial_{t_k}}\beta_j\right|_{t=0} &= \lim_{\lambda\to\beta_j} -\frac{\left(\frac{\gamma_k}{\lambda-\beta_k}-\frac{\overline{\gamma_k}\lambda}{1-\lambda\overline{\beta_k}}\right)b(\lambda)}{b(\lambda)}\lambda = -\left(\frac{\gamma_k}{\beta_j-\beta_k}-\frac{\overline{\gamma_k}\beta_j}{1-\beta_j\overline{\beta_k}}\right)\beta_j \\
&= -\left(\frac{\gamma_k}{\beta_j-\beta_k}+\frac{\overline{\gamma_k}\beta_j\beta_{g+2-k}}{\beta_j-\beta_{g+2-k}}\right)\beta_j \text{ for } j\neq k, j\neq g+2-k.
\end{aligned}$$

If, on the other hand, a root $\beta_m \in \mathbb{S}^1$ has no partner under the involution η we make use of the relation $\beta_m = \frac{1}{\overline{\beta_m}}$ and derive

$$\begin{aligned}\frac{\partial}{\partial_{t_m}}\beta_j\Big|_{t=0} &= \lim_{\lambda\to\beta_j} -\frac{\left(\frac{\gamma_m}{\lambda-\beta_m}-\frac{\overline{\gamma_m}\lambda}{1-\lambda\overline{\beta_m}}\right)b(\lambda)}{b(\lambda)}\lambda \\ &= -\left(\frac{\gamma_m}{\beta_j-\beta_m}+\frac{\overline{\gamma_m}\beta_j\beta_m}{\beta_j-\beta_m}\right)\beta_j \text{ for } j\neq m.\end{aligned}$$

One verifies that the first deformation equation is singular at the two points β_k and $\frac{1}{\overline{\beta_k}} = \beta_{g+2-k}$ with respect to the derivative $\frac{\partial}{\partial_{t_k}}$. We therefore make use of the Whitham equation (4.2.9) to get an expression at these two points with respect to each of the derivatives $\frac{\partial}{\partial t_k}|_{t=0}\beta_k$ and $\frac{\partial}{\partial t_k}|_{t=0}\beta_{g+2-k}$ for $k \in \{1\ldots g+1\}$.

The denominator of the second deformation equation is zero in case of $\beta_j = \beta_m = \sqrt{\frac{\beta_m}{\overline{\beta_m}}}$. In order to derive well-defined equations in these cases we calculate

$$\begin{aligned}\dot{b}(\lambda) &= \partial_t(b_{g+1}\prod_{i=1}^{g+1}(\lambda-\beta_i)) = \dot{b}_{g+1}\prod_{i=1}^{g+1}(\lambda-\beta_i) - b_{g+1}\sum_{i=1}^{g+1}\dot{\beta}_i\prod_{l=1;l\neq i}(\lambda-\beta_l) \quad (4.3.1)\\ &= \dot{b}_{g+1}\prod_{i=1}^{g+1}(\lambda-\beta_i)+\sum_{i=1}^{g+1}-\dot{\beta}_i\frac{b}{\lambda-\beta_i} \quad (4.3.2)\end{aligned}$$

and

$$b'(\lambda) = \sum_{j=1}^{g+1} b_{g+1}\prod_{j=1;i\neq j}^{g+1}(\lambda-\beta_i) = \sum_{j=1}^{g+1}\frac{b}{\lambda-\beta_j}. \quad (4.3.3)$$

If we evaluate b' at β_k one gets

$$b'(\beta_k) = b_{g+1}\prod_{i=1;i\neq k}^{g+1}(\beta_k-\beta_i). \quad (4.3.4)$$

We now insert β_k into (4.3.1) and obtain by making use of the fact that the β_i are the zeros of b

$$-\dot{b}(\beta_k) = \dot{\beta}_k b'(\beta_k). \quad (4.3.5)$$

In order to find better expressions for the derivative $\frac{\partial}{\partial_{t_k}}\beta_k|_{t_k=0}$ we now split the polynomial c into two parts c^1 and c^2 corresponding to the sum in the lemma 4.7, since only one part of the polynomial c is singular at one β_k, that is, for $\beta_k \neq \beta_m$ we set

$$c_{k_1}(\lambda) = \frac{\gamma_k}{\lambda-\beta_k}b(\lambda) \quad c_{k_2}(\lambda) = \frac{-\overline{\gamma_k}}{1-\lambda\overline{\beta_k}}b(\lambda) \Rightarrow \frac{\partial}{\partial t_k}\beta_j\Big|_{t=0} = \lim_{\lambda\to\beta_j} -\frac{c_{k_1}(\lambda)+c_{k_2}(\lambda)}{b(\lambda)}\lambda.$$

Let us first assume that β_k is a root of the denominator of c_{k_1}. In the following we will make use of the Whitham equation to express the summand of $c_k(\lambda)$ which has a root in the the denominator. By the Whitham equation one has

$$2a(\beta_k)\dot{\beta}_k b^{'}(\beta_k) = -2\beta_k a(\beta_k)c^{'}(\beta_k) + a(\beta_k)c(\beta_k) + \beta_k a^{'}(\beta_k) \tag{4.3.6}$$

$$\Longleftrightarrow \dot{\beta}_k = \frac{1}{2b^{'}(\beta_k)}\left(-2\beta_k c^{'}(\beta_k) + c(\beta_k) + \beta_k c(\beta_k)\frac{a^{'}(\beta_k)}{a(\beta_k)}\right). \tag{4.3.7}$$

After that, we will outline the results in a lemma where we take also in account the summand of $c_k(\lambda)$ that has no root in the denominator at β_k.

Proposition 4.14. *If β_k is a root of the denominator of $c_{k_1}(\lambda)$, and if we denote the first part of the equation $\dot{\beta}_k = \lim_{\lambda\to\beta_k} -\frac{c_{k_1}(\lambda)}{b(\lambda)}\lambda - \frac{c_{k_2}(\lambda)}{b(\lambda)}\lambda$ by $\dot{\beta}_k^{(1)}$ ($= \lim_{\lambda\to\beta_k} -\frac{c_{k_1}(\lambda)}{b(\lambda)}\lambda$), then we may write the $\dot{\beta}_k^{(1)}$ as*

$$\dot{\beta}_k^{(1)} = \frac{\gamma_k}{2}\left(-2\beta_k\sum_{j\neq k}^{g+1}\frac{1}{\beta_k-\beta_j} + 1 + \beta_k\sum_{j=1}^{2g}\frac{1}{\beta_k-\alpha_j}\right) \tag{4.3.8}$$

Proof. We have to do several steps to derive the result. First we evaluate $\lim_{\lambda\to\beta_k} c_k^{'}$. By the definition of c_k one has

$$\lim_{\lambda\to\beta_k} c_k^{'} = \lim_{\lambda\to\beta_k}\frac{b^{'}(\lambda)\gamma_k(\lambda-\beta_k) - b(\lambda)\gamma_k}{(\lambda-\beta_k)^2}. \tag{4.3.9}$$

By the rule of l'Hospital we obtain

$$\lim_{\lambda\to\beta_k} c_k^{'} = \lim_{\lambda\to\beta_k}\gamma_k\frac{b^{''}(\lambda)(\lambda-\beta_k) + b^{'}(\lambda) - b^{'}(\lambda)}{2(\lambda-\beta_k)} = \frac{b^{''}(\beta_k)}{2}\gamma_k. \tag{4.3.10}$$

We also have the formula $c_k(\beta_k) = b^{'}(\beta_k)\gamma_k$ which is derived by inserting the expression of $b^{'}(\beta_k)$ into the equation $c_k = \frac{b\gamma_k}{\lambda-\beta_k}$. By inserting these results into (4.3.6) we derive

$$\dot{\beta}_k^{(1)} = \frac{\gamma_k}{2}\left(\frac{-\beta_k b^{''}(\beta_k)}{b^{'}(\beta_k)} + 1 + \beta_k\frac{a^{'}(\beta_k)}{a(\beta_k)}\right). \tag{4.3.11}$$

We now want to find a better expression for $\frac{b^{''}(\beta_k)}{b^{'}(\beta_k)}$. We make use of the formula for $b^{'}(\lambda)$ and calculate

$$b^{''}(\lambda) = \sum_{j=1}^{g+1}\left(\frac{b(\lambda)}{\lambda-\beta_j}\right)^{'} = \sum_{j=1}^{g+1}\frac{b^{'}(\lambda)(\lambda-\beta_j) - b(\lambda)}{(\lambda-\beta_j)^2}. \tag{4.3.12}$$

Inserting β_k leads to

$$\begin{aligned} b^{''}(\beta_k) &= \sum_{j=1;j\neq k}^{g+1} \frac{b^{'}(\beta_k)}{\beta_k-\beta_j} + \lim_{\lambda\to\beta_k}\left(\frac{b'(\lambda)}{\lambda-\beta_k} - \frac{b(\lambda)}{(\lambda-\beta_k)^2}\right) \\ &= \sum_{j=1;j\neq k}^{g+1} \frac{b^{'}(\beta_k)}{\beta_k-\beta_j} + \lim_{\lambda\to\beta_k}\frac{b'(\lambda)(\lambda-\beta_k)-b(\lambda)}{(\lambda-\beta_k)^2} \\ &\overset{l'H.}{=} \sum_{j=1;j\neq k}^{g+1} \frac{b^{'}(\beta_k)}{\beta_k-\beta_j} + \frac{b^{''}(\beta_k)}{2} \Leftrightarrow b^{''}(\beta_k) = 2\sum_{j=1,j\neq k}^{g+1}\frac{b^{'}(\beta_k)}{\beta_k-\beta_j}. \end{aligned}$$

We have found the following expression

$$\frac{b^{''}(\beta_k)}{b^{'}(\beta_k)} = \sum_{j=1,j\neq k}^{g+1}\frac{2}{\beta_k-\beta_j} \tag{4.3.13}$$

We finally investigate $\frac{a^{'}(\beta_k)}{a(\beta_k)}$. By making use of the formulas for $a(\lambda)$ and $a^{'}(\lambda)$ we calculate $\frac{a^{'}(\lambda)}{a(\lambda)} = \sum_{j=1}^{2g}\frac{1}{\lambda-\alpha_j}$. Inserting $\lambda = \beta_k$ gives $\frac{a^{'}(\beta_k)}{a(\beta_k)} = \sum_{j=1}^{2g}\frac{1}{\beta_k-\alpha_j}$. This finishes the proof. □

If β_k is a root of the denominator of c_{k_1} then β_{g+2-k} must be a root of the denominator of c_{k_2}.

Proposition 4.15. *If β_{g+2-k} is a root of the denominator of $c_{k_2} = \frac{-\overline{\gamma_k}\lambda}{1-\overline{\beta_k}\lambda}b(\lambda)$, then we may write the singular term of $\dot{\beta}_{g+2-k} = \frac{\partial}{\partial t_k}|_{t_k=0}\beta_{g+2-k}$ in (4.3.6), which we denote by $\dot{\beta}^{(2)}_{g+2-k}$, in the non-singular form*

$$\dot{\beta}^{(2)}_{g+2-k} = \frac{\beta^2_{g+2-k}\overline{\gamma_k}}{2}\left(-\beta_{g+2-k}\sum_{j=1;j\neq g+2-k}^{g+1}\frac{2}{\beta_{g+2-k}-\beta_j} - 1 + \beta_{g+2-k}\sum_{j=1}^{2g}\frac{1}{\beta_{g+2-k}-\alpha_j}\right)$$

Proof. We have to do similar calculations as in the proposition above. We first calculate by making use of the rule of Hospital in the third step

$$\begin{aligned} \lim_{\lambda\to\beta_{g+2-k}} c^{'}_{k_2} &= \lim_{\lambda\to\beta_{g+2-k}} \frac{(-b(\lambda)^{'}\overline{\gamma_k}\lambda - b(\lambda)\overline{\gamma_k})(1-\lambda\overline{\beta_k}) - (\overline{\beta_k}b(\lambda)\overline{\gamma_k}\lambda)}{(1-\lambda\overline{\beta_k})^2} \\ &= \lim_{\lambda\to\beta_{g+2-k}} \frac{\overline{-\gamma_k}(b^{''}(\lambda)\lambda + 2b^{'}(\lambda))(1-\lambda\overline{\beta_k})}{-2(1-\lambda\overline{\beta_k})\overline{\beta_k}} \\ &= \lim_{\lambda\to\beta_{g+2-k}} \frac{\overline{\gamma_k}}{2\overline{\beta_k}}(b^{''}(\lambda)\lambda + 2b^{'}(\lambda)) \\ &= \frac{\overline{\gamma_k}}{2\overline{\beta_k}}(b^{''}(\beta_{g+2-k})\beta_{g+2-k} + 2b^{'}(\beta_{g+2-k})) \end{aligned}$$

for which the limes exists. We now insert the result into equation (4.3.6). As in the proposition above we want to express $c_{k_2}(\lambda)$ in terms of $b'(\lambda)$. By making use of $\overline{\beta_k} = \frac{1}{\beta_{g+2-k}}$ and the formula for $b'(\lambda)$ we derive

$$\frac{-\overline{\gamma_k}\lambda}{1-\lambda\overline{\beta_k}} = \frac{-\overline{\gamma_k}}{\frac{1}{\lambda}-\frac{1}{\beta_{g+2-k}}} = \frac{\overline{\gamma_k}\lambda\beta_{g+2-k}}{\lambda-\beta_{g+2-k}} \Longrightarrow \quad c_{k_2}(\lambda) = \overline{\gamma_k}\beta_{g+2-k}\lambda b'(\lambda)$$

and inserting β_{g+2-k} gives $c_{k_2}(\beta_{g+2-k}) = \overline{\gamma_k}b'(\beta_{g+2-k})\beta^2_{g+2-k}$. We now calculate

$$\begin{aligned}
\dot{\beta}^{(2)}_{g+2-k} &= \frac{1}{2b'(\beta_{g+2-k})}\Bigg(-2\beta_{g+2-k}c'_{k_2}(\beta_{g+2-k}) + c_{k_2}(\beta_{g+2-k}) + \\
&\quad +\beta_{g+2-k}c_{k_2}(\beta_{g+2-k})\frac{a'(\beta_{g+2-k})}{a(\beta_{g+2-k})}\Bigg) \\
&= \frac{\overline{\gamma_k}}{2}\Bigg(-\beta^3_{g+2-k}\frac{b''(\beta_{g+2-k})}{b'(\beta_{g+2-k})} - 2\beta^2_{g+2-k} + \beta^2_{g+2-k} + \beta^3_{g+2-k}\frac{a'(\beta_{g+2-k})}{a(\beta_{g+2-k})}\Bigg) \\
&= \frac{\beta^2_{g+2-k}\overline{\gamma_k}}{2}\left(-\beta_{g+2-k}\frac{b''(\beta_{g+2-k})}{b'(\beta_{g+2-k})} - 1 + \beta_{g+2-k}\frac{a'(\beta_{g+2-k})}{a(\beta_{g+2-k})}\right)
\end{aligned}$$

In the last proposition we have derived formulas for the expressions $\frac{b''(\beta_{g+2-k})}{b'(\beta_{g+2-k})}$ and $\frac{a'(\beta_{g+2-k})}{a(\beta_{g+2-k})}$. By inserting them we derive the stated result in the proposition. □

Finally we have to consider the case where β_m is a root of the denominator of $c_m(\lambda)$ in the case where $\beta_m \in \mathbb{S}^1$ has no η-partner.

Proposition 4.16. *If $\beta_m \in \mathbb{S}^1$ is a root of the denominator of $c_m(\lambda) = \left(\frac{\gamma_m}{\lambda-\beta_m} - \frac{\overline{\gamma_m}\lambda}{1-\lambda\overline{\beta_m}}\right)b(\lambda)$, then we may write the expression for $\dot{\beta}_m$ in the Witham equation (4.3.6) as the sum of the expressions of the previous two propositions. We may express $\dot{\beta}_m$ in the non-singular form*

$$\dot{\beta}_m = \frac{\partial}{\partial\beta_m}\beta_m\bigg|_{t=0} = \dot{\beta}^{(1)}_m + \dot{\beta}^{(2)}_m \tag{4.3.14}$$

$$= \left(\frac{\gamma_m + \beta_m^2\overline{\gamma}_m}{2}\right)\left(\sum_{j=1;j\neq m}^{g+1}\frac{-2\beta_m}{\beta_m-\beta_j} - 1 + \beta_m\sum_{j=1}^{2g}\frac{1}{\beta_m-\alpha_j}\right) + \gamma_m \tag{4.3.15}$$

Proof. We may add up the expressions we have found in the previous propositions. This gives the result. □

If one assumes that the leading coefficient of $a(\lambda)$, a_{2g}, is constant under the deformation flow, $\dot{a}_{2g} = 0$, one can calculate the deformation equation for the leading coefficient b_{g+1} of $b(\lambda)$. This assumption would also fix the Möbius rotation one has in the normalization constant γ_m in proposition 4.11.

Proposition 4.17. *If we assume that $\dot{a}_{2g} = 0$, then the following differential equation describes the flow of the leading coefficient $b_{g+1} = -\overline{b(0)}$ under an isoperiodic deformation:*

$$\frac{\partial}{\partial t_k} b_{g+1}\Big|_{t=0} = b_{g+1}\left(\frac{1}{2}\sum_{i=1}^{2g}\frac{\dot{\alpha}_i}{\alpha_i} + \frac{\gamma_k}{2\beta_k} - \sum_{i=1}^{g+1}\frac{\dot{\beta}_i}{\beta_i}\right) \tag{4.3.16}$$

Proof. We use the notation $b(\lambda) = b_{g+1}\prod(\lambda - \beta_i)$ and $a(\lambda) = a_{2g}\prod(\lambda - \alpha_j)$. We have

$$\begin{aligned}
\dot{b}(\lambda) &= \dot{b}_{g+1}\prod(\lambda - \beta_j) + b_{g+1}\sum_i(-\dot{\beta}_i)\prod_{j\neq i}(\lambda - \beta_j) \\
&= \dot{b}_{g+1}\prod(\lambda - \beta_j) + b_{g+1}\sum_i\frac{c(\beta_i)\beta_i}{b(\beta_i)}\prod_{j\neq i}(\lambda - \beta_j)
\end{aligned}$$

By the Whitham equation one gets, after inserting 0, that

$$\dot{b(0)} = \frac{\dot{a(0)}b(0) - a(0)c(0)}{2a(0)}$$

If we insert this formula into the formula for $\dot{b}(\lambda)$ and insert 0 we derive:

$$\begin{aligned}
\dot{b}_{g+1} &= \frac{1}{\prod(-\beta_j)}\left(\frac{\dot{a(0)}b(0) - a(0)c(0)}{2a(0)} - b_{g+1}\sum_i\frac{c(\beta_i)\beta_i}{b(\beta_i)}\prod_{j\neq i}(-\beta_j)\right) \\
&= \frac{b_{g+1}}{b(0)}\left(\frac{\dot{a(0)}b(0) - a(0)c(0)}{2a(0)} + b(0)\sum_i\frac{c(\beta_i)}{b(\beta_i)}\right) \\
&= b_{g+1}\left(\frac{\dot{a}(0)}{2a(0)} - \frac{c(0)}{2b(0)} + \sum_i\frac{c(\beta_i)}{b(\beta_i)}\right) \\
&= b_{g+1}\left(\frac{1}{2}\sum_i\frac{\dot{\alpha}_i}{\alpha_i} - \frac{c(0)}{2b(0)} - \sum_i\frac{\dot{\beta}_i}{\beta_i}\right)
\end{aligned}$$

Calculating the last expression with respect to one derivative $\frac{\partial}{\partial t_k}$, i.e. making use of $c_k(0) = -\frac{\gamma_k}{\beta_k}b(0)$ gives the result stated in the proposition. □

We finally summarize the achieved results.

Lemma 4.18. *We may describe isoperiodic deformations of given spectral data* $(a, b) \in \mathcal{M}_g$ *without leaving the moduli space* $\mathcal{M}_g$ *by deforming the roots of* a *and* b *and the highest coefficient of* $b(\lambda)$ *by means of the differential equations given below. We sort those roots of the polynomial* $b(\lambda)$*, that have an* η*-partner, by means of* $\beta_k = \frac{1}{\overline{\beta_{g+2-k}}}$*. These deformation equations are given by*

$$\begin{aligned}
\frac{\partial}{\partial t_k}\alpha_j\bigg|_{t=0} &= -\left(\frac{\gamma_k}{\alpha_j - \beta_k} + \frac{\overline{\gamma_k}\alpha_j\beta_{g+2-k}}{\alpha_j - \beta_{g+2-k}}\right)\alpha_j \\
\frac{\partial}{\partial t_k}\beta_j\bigg|_{t=0} &= -\left(\frac{\gamma_k}{\beta_j - \beta_k} + \frac{\overline{\gamma_k}\beta_j\beta_{g+2-k}}{\beta_j - \beta_{g+2-k}}\right)\beta_j \text{ for } j \neq k \text{ ; } j \neq g+2-k.
\end{aligned}$$

For the roots of $b(\lambda) \in \mathbb{S}^1$ *we have*

$$\frac{\partial}{\partial_{t_m}}\beta_j\bigg|_{t=0} = -\left(\frac{\gamma_m}{\beta_j - \beta_m} + \frac{\overline{\gamma_m}\beta_j\beta_m}{\beta_j - \beta_m}\right)\beta_j \text{ for } j \neq m.$$

And for any η*-pair of roots* β_k *and* β_{g+2-k} *of* $b(\lambda)$ *the equations for the derivatives with respect to* t_k *and* t_{g+2-k} *at* β_k*,* β_{g+2-k} *read*

$$\begin{aligned}
\frac{\partial}{\partial t_k}\beta_k\bigg|_{t=0} &= \dot{\beta}_k^{(1)} - \lim_{\lambda\to\beta_k}\frac{c_{k_2}(\lambda)}{b(\lambda)}\lambda = \dot{\beta}_k^{(1)} - \frac{\overline{\gamma_k}\beta_k\beta_{g+2-k}}{\beta_k - \beta_{g+2-k}}\beta_k \\
\frac{\partial}{\partial t_k}\beta_{g+2-k}\bigg|_{t=0} &= \dot{\beta}_{g+2-k}^{(2)} - \lim_{\lambda\to\beta_{g+2-k}}\frac{c_{k_1}(\lambda)}{b(\lambda)}\lambda = \dot{\beta}_{g+2-k}^{(2)} - \frac{\gamma_k}{\beta_{g+2-k} - \beta_k}\beta_{g+2-k}.
\end{aligned}$$

Furthermore we have, for those roots $\beta_{m_i} \in \mathbb{S}^1$ *that have no partner under the involution* η *except themselves, the following differential equation for the derivative at* β_m *in the direction* t_m*:*

$$\frac{\partial}{\partial t_m}\beta_m\bigg|_{t=0} = \beta_m^{(1)} + \beta_m^{(2)}$$

5 Numerical computations

Our starting point for computing isoperiodic deformations of the spectral curve will be the monodromy of the vacuum solution we computed in 2.9. From there we want to compute the polynomials $a(\lambda)$ and $b(\lambda)$. By equation (3.4.6) and the monodromy of the vacuum solution 3.5 we have

$$d\ln\mu = \Big(-\frac{\tau}{2}\lambda^{-\frac{3}{2}} + \frac{\overline{\tau}}{2}\lambda^{-\frac{1}{2}}\Big)d\lambda = \frac{\frac{1}{2}(-\tau+\overline{\tau}\lambda)}{\sqrt{\lambda}}\frac{d\lambda}{\lambda} \stackrel{!}{=} \frac{b}{\nu}\frac{d\lambda}{\lambda} \tag{5.0.1}$$

And by the definition of the spectral curve we also have $\nu^2 = \lambda a(\lambda)$. When looking for polynomials $a(\lambda)$ and $b(\lambda)$ to deform, we have to ensure that these starting polynomials fulfill the following conditions:

1. The reality conditions for the polynomials $a(\lambda)$ and $b(\lambda)$ have to be fulfilled, by lemma 3.16 they read $\lambda^{2g}\overline{a(\frac{1}{\overline{\lambda}})} = a(\lambda)$ and $\lambda^{g+1}\overline{b(\frac{1}{\overline{\lambda}})} = -b(\lambda)$.

2. The normalization conditions have to be fulfilled. By the definition of the spectral data in 4.1 the highest coefficient of $a(\lambda)$ must be in $\mathbb{S}^1$. For the polynomial $b(\lambda)$ there must hold $b_0 = -\overline{b_{g+1}}$ for the highest and the lowest coefficients of $b(\lambda)$.

With equation (5.0.1) we make the ansatz $\nu = \sqrt{\lambda}$ we get $\nu^2 = \lambda a(\lambda)$ and therefore $a(\lambda) = 1$. So the highest coefficient has absolute value $|a_0| = 1$ as claimed. If we take $b^0(\lambda)$ where 0 indicates the genus $g = 0$ to be $\tilde{b}^0(\lambda) = \frac{1}{2}(\overline{\tau}\lambda - \tau)$, then one verifies that the conditions we claim for the polynomial $\tilde{b}(\lambda)$ are fulfilled too. So the first root β_M is $\frac{\tau}{\overline{\tau}}$. Applying the normalization properties of the proposition 3.17 we conclude

Proposition 5.1. *For the monodromy of the vacuum solution 2.9 the polynomials $a(\lambda)$ and $b(\lambda)$ read*

$$a^0(\lambda) = 1 \qquad b^0(\lambda) = b_{g+1}\prod_{j=1}^{1}(\lambda - \beta_M) = \frac{1}{2}\sqrt{\frac{\overline{\tau}}{\tau}}(\lambda - \frac{\tau}{\overline{\tau}}) \tag{5.0.2}$$

One verifies that $b(0) = -\frac{1}{2}\sqrt{\frac{\tau}{\overline{\tau}}} = -\overline{b_{g+1}}$ and that the reality condition is satisfied was well for the polynomial $b(\lambda)$.

5.1 From $g = 0$ to $g = 1$

At $\lambda = \alpha_1 \in \mathbb{S}^1$ with $\mu = \pm 1$ we may add a double root to the polynomial $a^0(\lambda) = 1$ and a simple root to the polynomial $b^0(\lambda)$. We may then deform the double root of $a(\lambda)$ into two branchpoints. We already know that $\beta_1 = \frac{\tau}{\overline{\tau}}$. To deform the roots of $a(\lambda)$ and $b(\lambda)$ we use the deformation polynomials we derived in the last chapter to deform the spectral data $(a^1, b^1) \in \mathcal{M}_1$.
The double root $\alpha_{1,2}, \alpha_1 = \alpha_2 \in \mathbb{S}^1$ we add to the polynomial hence must satisfy $\mu(\alpha_{1,2}) = \pm 1$ and $|\alpha_{1,2}| = 1$. The first condition gives

$$\begin{aligned}\mu(\alpha_{1,2}) &= \exp(i(\frac{\tau}{\sqrt{\alpha_{1,2}}} + \overline{\tau}\sqrt{\alpha_{1,2}})) = \pm 1 \\ &\iff (\frac{\tau}{\sqrt{\alpha_{1,2}}} + \overline{\tau}\sqrt{\alpha_{1,2}}) = \pi n,\ n \in \mathbb{Z}\end{aligned}$$

We assume that $n = 0$. Hence we conclude that the root $\alpha_{1,2}$ we add must fulfill $\alpha_{1,2} = -\frac{\tau}{\overline{\tau}}$. There are several ways to add this root to the polynomial $a^0(\lambda)$. We follow the normalization given in the proposition 3.17 straight forward. For $\alpha_{1,2} = \beta_2 \in \mathbb{S}^1$ and β_2 has no η-partner, we have

$$a^1(\lambda) = \frac{\overline{\alpha_1}}{|\alpha_1|}\left(\lambda - \left(\frac{-\tau}{\overline{\tau}}\right)\right)^2 = \frac{-\overline{\tau}}{\tau}\left(\lambda - \left(\frac{-\tau}{\overline{\tau}}\right)\right)^2 \tag{5.1.1}$$

$$b^1(\lambda) = \frac{i}{2}\sqrt{\frac{\overline{\tau}}{\tau}}\sqrt{\frac{-\overline{\tau}}{\tau}}\left(\lambda - \frac{\tau}{\overline{\tau}}\right)\left(\lambda - \left(-\frac{\tau}{\overline{\tau}}\right)\right) \tag{5.1.2}$$

One verifies that the reality conditions for both, $a(\lambda)$ and $b(\lambda)$ are fulfilled. Furthermore one has $b_0^1 = b^1(0) = \frac{1}{2}\frac{\tau}{\overline{\tau}} = -\overline{b_{g+1}}$.

Remark 5.2 (The right choice of the deformation equations)**.** *The two roots of $b^1(\lambda)$ are not in involution, or, to say it more precisely, they are only in involution with themselves. We have, for $\beta_i \in \mathbb{S}^1$ and $\beta_1 \neq \beta_2$ that $\eta^*\overline{(\lambda - \beta_i)} = (\frac{1}{\lambda} - \overline{\beta_i})$, which zero for the same $\lambda = \beta_i$ because $\beta_i = \frac{1}{\overline{\beta_i}}$ if $\beta_i \in \mathbb{S}^1$. Therefore one has to use for both β_i the deformation polynomial $c_m(\lambda)$ for those roots β that lie on $\mathbb{S}^1$ and that are only in involution with themselves.*

We also want to show how to calculate the normalization constants in this case. We have to distinguish between the two roots $\beta_1 = \frac{\tau}{\overline{\tau}}$ and $\beta_2 = -\frac{\tau}{\overline{\tau}}$. We have $\mu(\beta_1) = \exp(i \cdot (\frac{\tau}{\sqrt{\beta_1}} + \overline{\tau}\sqrt{\beta_1})) = \exp(2i|\tau|)$. Therefore $\ln\mu(\beta_1) = 2i|\tau| \mod 2\pi i\mathbb{Z}$.
For $\beta_2 = \alpha_{1,2}$ we have $\ln\mu(\beta_2) = 0$ because the α's are branchpoints of the spectral curve. Since $\sinh(0) = 0$ we have to consider the expression $\frac{\sqrt{\nu(\beta_2)}}{\sinh\ln(\mu(\beta_2))}$. In proposition

4.13 we already calculated that $\lim_{\lambda \to \beta_j} \dfrac{\nu(\lambda)}{2 \sinh \ln(\mu(\lambda))} = \dfrac{\beta_j a_{2g} \prod_{i=1\,;\, i \neq j}^{g} (\beta_j - \beta_i)}{2 b_{g+1}(\beta_j - \beta_1)}$. We formulate the following proposition about the normalization constants in the case where the genus $g = 1$.

Proposition 5.3. *If we switch from $g = 0$ to $g = 1$, then the initial normalization constants γ_1 and γ_2 corresponding to $\beta_1 = \frac{\tau}{\bar{\tau}}$ and $\beta_2 = -\frac{\tau}{\bar{\tau}}$ are of the following form:*

$$\gamma_1 = \frac{\nu(\beta_1)}{2 \sinh(2i|\tau|)(1 + \beta_1)^4 b'(\beta_1)} \;;\; |\tau| \neq k\pi \; k \in \mathbb{Z}$$

$$\gamma_2 = \frac{\beta_2^2 a_{2g}}{2 b_{g+1}(\beta_2 - \beta_1)(1 + \beta_2)^4 b'(\beta_2)}$$

By the periodicity of sinh *we have omitted the periods in $\ln \mu(\beta_1) = 2i\|\tau\| \mod 2\pi i\mathbb{Z}$. Since $\beta_1 \in \mathbb{S}^1$ and β_1 is not in involution with β_2 we have used the formulas for the for the deformation constants 4.11 that correspond to c_m.*

If one considers the inital normalization constant γ_1 one recognizes that $|\tau|$ may not be $k\pi; k \in \mathbb{Z}$. This restriction of the initial state is the general situation: Since the root β_1, that occurs already in the polynomial $b^0(\lambda)$, one always has the restriction that $\ln \mu(\beta_1)$ may not be in $\pi i\mathbb{Z}$. $\ln \mu \in \pi i\mathbb{Z}$ if and only if $\mu(\beta_1) = \pm 1$. Since we did not add a double root α at this point, β_1 is a singularity of the spectral curve. For a discussion of this problem one may refer [5]. In our numerical computations we will take care about the initial state.

5.2 From $g = n$ to $g = n + 2$

In contrast to the case when we add just one root to $b(\lambda)$ (and one double root to $a(\lambda)$) the case where we add two roots to $b(\lambda)$ is different: The conditions for the roots $\alpha_{1,2}, \alpha_{3,4}$ read $|\alpha_i| \neq 1$ and $\mu(\alpha_i) = \pm 1$. By the reality condition we want the α_i fulfill $\alpha_1 = \frac{1}{\bar{\alpha}_2}, \alpha_3 = \frac{1}{\bar{\alpha}_4}$. Therefore we have to solve for $k \in \mathbb{Z}$

$$\frac{\tau}{\sqrt{\alpha}} - \bar{\tau}\sqrt{\alpha} = \pi k \Longleftrightarrow \sqrt{\alpha}_{\{1,2\},\{3,4\}} = \frac{-k\pi \pm \sqrt{(k\pi)^2 - 4\bar{\tau}\tau}}{2\bar{\tau}} \quad k \in \mathbb{Z}; \quad k \neq 0$$

The condition that $k \neq 0$ is derived by the last section since we know that $|\alpha_i| \neq 1$ if $k \neq 0$. We have $\sqrt{\alpha}_{1,2} \cdot \sqrt{\alpha}_{3,4} = \frac{\tau}{\bar{\tau}} \in \mathbb{S}^1$ for all $k \in \mathbb{Z}$. We then also have $\alpha_1 \cdot \alpha_3 \in \mathbb{S}^1$ because the elements of $\mathbb{S}^1$ form a group, such that we have $\frac{\tau}{\bar{\tau}} \in \mathbb{S}^1 \Longrightarrow \frac{\tau}{\bar{\tau}} \cdot \frac{\tau}{\bar{\tau}} \in \mathbb{S}^1$. More precisely we have $|\sqrt{\alpha}_1 \cdot \sqrt{\alpha}_3| = 1 \Longrightarrow |\sqrt{\alpha}_1| = \frac{1}{|\sqrt{\alpha}_3|}$. One may compute directly that the following proposition holds:

Proposition 5.4. *For the roots $\sqrt{\alpha}_i$ there must hold $\sqrt{\alpha}_{1,2} = \frac{1}{\sqrt{\bar{\alpha}}_{3,4}}$. The same relation must be true for the α_i themselves.*

If we start from a polynomial corresponding to an arbitrary genus g then we again have to ensure that the normalization conditions and the reality conditions are fulfilled. First we notice that the minus occuring in $b_0 = -\overline{b_{g+1}}$ is ensured by the polynomial $b^0(\lambda)$. The reality condition for the polynomial $b(\lambda)$ will be preserved if we add the two additional roots that are in involution. We do so by following (3.4.8) and proposition 3.20. We have to take care how the prefactor changes when adding a double root.

$$\begin{aligned} a^{2g+4}(\lambda) &= a^{2g}(\lambda)\frac{\overline{\alpha_{g+1}}^2}{|\overline{\alpha_{g+1}}|^2}(\lambda-\alpha_{g+1})^2(\lambda-\alpha_{g+2})^2 \\ b^{g+2}(\lambda) &= b^g(\lambda)\overline{\beta_{g+2}\beta_{g+3}}(\lambda-\beta_{g+2})(\lambda-\beta_{g+3}) \end{aligned}$$

By the reality condition we may replace $|\overline{\alpha_{g+1}}|$ by $\frac{1}{|\alpha_{g+2}|}$.

Remark 5.5. *At the doublepoints α_i one has $\mu(\alpha_i) = \pm 1$. If one considers the limits $\lim_{k\to+\infty}(\alpha_i)(k), i = 1, 2$ of the α_i then the limit points equal 0 and ∞.*

As in the case where we jumped form genus $g = 0$ to $g = 1$ we have the problem that two of the three normalization constants have a root in the nominator and denominator when evaluated at the β_1 and β_2 (the two roots of the polynomial $b(\lambda)$ that correspond to the two double-points in the polynomial $a(\lambda)$). At these points we have $\ln\mu(\beta = \alpha) = 0$. We have to consider the expression $\frac{\sqrt{\nu(\beta)}}{\sinh\ln(\mu(\beta))}$ we calculated for this special case in proposition 4.13.
If we switch from $g = 0$ to $g = 2$ and add two roots β_1 and β_3 (the former β_1 will get the new β_2) that are in involution to the polynomial $b(\lambda)$, then α_1, α_4 correspond to β_1 and α_2, α_3 correspond to β_3. In this case the normalization constants read

$$\begin{aligned} \gamma_1 &= \frac{\beta_2 a_{2g}(\beta_1-\beta_3)}{2b_{g+1}(\beta_1-\beta_2)b'(\beta_1)} \\ \overline{\gamma_3} &= \overline{\gamma_{(g+2-k)}} = \frac{\beta_3 a_{2g}(\beta_3-\beta_1)}{2b_{g+1}(\beta_3-\beta_2)b'(\beta_3)\beta_3^2} \end{aligned}$$

5.3 Facing singular initial conditions

If one tries to plug in the so far derived roots in the deformation polynomials, one immediately realizes that they are producing singular initial values since we want to start with α_i and $\alpha_{i+1} = \beta_i$. That is, in the two cases we stated above we always add the same root to the polynomials $a(\lambda)$ and $b(\lambda)$.

But if one considers the polynomial expression for the differential $d\ln\mu = \frac{b}{\nu}\frac{d\lambda}{\lambda}$ with $\nu^2 = \lambda a(\lambda)$ one verifies that expanding the polynomials $a(\lambda)$ and $b(\lambda)$ in the way we stated above, gives a removable singularity of the differential $d\ln\mu$. So we may say that adding one or two double roots to the polynomial $a(\lambda)$ and the simple roots to the polynomial $b(\lambda)$ does not change the values of the differential $d\ln\mu$. The problem of the singular initial values must correspond to the coodinates we have choosen to deform the spectral data, and the problem appears to be unnatural in some sense. This relation ist discussed in [5].

5.3.1 First ansatz

A first ansatz to solve the problem of singular initial values and to make the deformation equations usable would be to expand the roots we derived by power series. We then would insert these series in the deformation ODE and solve them for $t = 0$. If we insert one root on $\mathbb{S}^1$ the Taylor series read

$$\alpha_{i,j}(t) = \alpha_{i,j}(0) + \dot{\alpha}_{i,j}(0)t + O(t^2) \quad \beta_i(t) = \beta_i(0) + \dot{\beta}_i(0)t + O(t^2) \tag{5.3.1}$$

Unfortunately a calculation shows that inserting this series into the deformation polynomial does not solve the problem of singular initial value conditions. We therefore change our ansatz by making a coordinate transformation of the deformation parameters t_k by means of

$$t_k \to s_k^2 \Rightarrow s_k = \pm\sqrt{t_k}; \quad dt_k = 2s_k ds_k \Rightarrow \frac{d}{dt_k} = \frac{1}{2s_k}\frac{d}{ds_k} \tag{5.3.2}$$

We now write down the deformation polynomials in the $g = 1$ case. If we denote the Taylor-expansion (in s) of $\alpha_i = \alpha_j$ and $\beta_{1,2}$ by $T_i(\alpha)$ and $T_i(\beta), i = 1,2$ then, under the (quite natural) assumption that $\dot{\alpha} - \dot{\beta} \neq 0$, by the rule of l'Hospital and by omitting the terms involving $O(t^2)$, the deformation equation for $\dot{\alpha}_i$ reads

$$\frac{\partial}{\partial s_1 := s}T(\alpha)\bigg|_{t=0} = \lim_{s\to 0} -2s\left(\frac{\gamma}{T(\alpha) - T(\beta)} + \frac{\overline{\gamma}T(\overline{\alpha})}{T(\overline{\alpha}) - T(\overline{\beta})}\right)T(\alpha) \tag{5.3.3}$$

$$= \lim_{s\to 0}\frac{-2\gamma\alpha - 4s\gamma\dot{\alpha}}{\dot{\alpha} - \dot{\beta}} + \frac{-2\overline{\gamma}\alpha - 4s\overline{\gamma}\dot{\alpha}}{\dot{\alpha} - \dot{\beta}} \tag{5.3.4}$$

And the limes is given by

$$\dot{\alpha}_{1,2} = \left(\frac{-2\gamma_1\alpha(0)}{\dot{\alpha}(0) - \dot{\beta}(0)} + \frac{-2\overline{\gamma_1}\alpha(0)}{\overline{\dot{\alpha}(0)} - \overline{\dot{\beta}(0)}}\right) \tag{5.3.5}$$

There are two possibilities of inserting small values of the parameter s, so we have the possibility to move the α_i away from each other. Since we have used the deformation polynomials from the last chapter, we can be sure that the conditions for the polynomials $a(\lambda)$ and $b(\lambda)$, especially the reality condition, are preserved when inserting $\pm\epsilon$ into s. We do the same for the β_i. We therefore use the deformation equation for β_m since we jump from $g = 0$ to $g = 1$. We again use the transformation $t_k \to s_k^2$. Then, in the new coordinates, the equation for β reads

$$\frac{\partial}{\partial s_1 := s} T(\beta_1)\Big|_{t=0} = \dot{\beta} = \tag{5.3.6}$$

$$\frac{-2s\sqrt{T(\beta_1)}i}{2}\left(\left(1 - T(\beta_1)\sum_{j=1}^{2}\frac{1}{T(\beta_1) - T(\alpha_j)}\right)\left(\gamma_1 T(\beta_1) - \overline{\gamma_1}\right) + 2\overline{\gamma_1}\right) \tag{5.3.7}$$

Here we already used that $\sum_{j=1;j\neq m}\frac{2}{\beta_m - \beta_j} = 0$ in this case.

5.3.2 Second ansatz

In order to solve the problem of singular initial value conditions we may also concentrate on the roots of $a(\lambda)$. These roots occur in pairs. If we take one root of each pair and move it infinitesimally, we obtain the other root by ensuring that the reality condition $\alpha_{i+1} = \eta(\alpha_i) = \frac{1}{\overline{\alpha_i}}$ is fulfilled.
We first consider the case $a^1(\lambda) = a_{2g}(\lambda - \alpha_1)(\lambda - \alpha_2)$ with $\alpha_1 = \alpha_2 \in \mathbb{S}^1$. One possible deformation of the root α_1 is to scale it by a small factor $(1 + \epsilon), \epsilon \in \mathbb{R}$, and thereby scaling it away orthogonally from $\mathbb{S}^1$. Thereby we derive

$$\begin{aligned} a^1(\lambda) &= a_{2g}(\lambda - \alpha_1)(\lambda - \alpha_1) \longrightarrow \tilde{a^1}(\lambda) = a_{2g}\Big(\lambda - \alpha_1 \cdot (1+\epsilon)\Big)\left(\lambda - \frac{1}{\overline{\alpha_1 \cdot (1+\epsilon)}}\right) \\ \tilde{\alpha}_1 &= \alpha_1 \cdot (1+\epsilon) \text{ and } \tilde{\alpha}_2 = \frac{1}{\overline{\alpha_1 \cdot (1+\epsilon)}} \end{aligned}$$

Remark 5.6 (Things we may want to check when we deform the spectral data)**.** *When deforming the spectral data we may want to check that the deformation flows are commuting. And one should control wether the results differ to the expectations one has. Another question is how much better the numerical result gets by performing smaller steps (because the numerical computation is a kind of discretisation - we can recalculate the normalization constants in every step. So it is interesting how different the results get when performing either one big step or ten small steps.)*

We will now follow the second ansatz.

5.4 Computation of the deformation flow for $g = 1$

When switching from genus $g = 0$ to $g = 1$ both, the polynomial $b(\lambda)$ and the polynomial $a(\lambda)$ have degree 2. Because the roots of $b(\lambda)$ are only in involution with themselves, we expect that, under any deformation flow, they rest on $\mathbb{S}^1$.
To the two roots of $b(\lambda)$ there correspond two deformation flows. We should expect that the deformation flow $\frac{\partial}{\partial t_1}$ corresponding to β_1 does not open the double point. It should be the same flow as in the genus $g = 0$ case, where the only possible deformation should lead to a motion on $\mathbb{S}^1$. The other deformation flow, $\frac{\partial}{\partial t_2}$, should, contrary, lead to open the double point.
For the computations we used the software mathematica.
There occurs one problem affecting the normalisation constants: In propositions 4.9 we derived formulas for these constants. Because they depend on the roots α_i, β_i one would expect that they also change under the deformation flow. But at the starting point of deformation where $\alpha_i = \alpha_{i+1}$ we know that the term $\frac{\nu(\lambda)}{\sinh(\ln \mu(\lambda))}$ is indefinite because the nominator and denominator are both zero at such a point. Therefore we calculated the limes in proposition 4.13. But the limes only gives the deformation constants in the initial state and hence does not give the function we are looking for. Hence, if we want the normalization constants to be functions depending on the deformation parameters t_i, we would have to handle expressions of the form

$$\text{normkonst}(1)(\text{t\_}) = \frac{\sqrt{\text{beta}(1)(t)\text{pola}(\text{beta}(1)(t), t)}}{2(\text{beta}(1)(t)+1)^4\left(\frac{\partial \text{polb}(s,t)}{\partial s} /.\, s \to \text{beta}(1)(t)\right)\sinh\left(\int_{\text{alpha}(1)(t)}^{\text{beta}(1)(t)} \frac{\text{polb}(s,t)}{s\sqrt{\text{spola}(s,t)}}\, ds\right)}.$$

Here the parameters of the integral are the functions we derive by solving the ODEs in theorem 4.18. Since it is very difficult to handle both, solving the ODEs and calculating the integral continuous, and since we are ultimately interestet in the qualitative behavier of the roots under the deformation flow, we avoid this difficulty by setting the normalization constants to be true constants rather than functions.
Because of proposition 4.11 we set $\gamma_i = \overline{\beta_i}(0)$. Then the initial state is implemented as follows:

$$\begin{aligned}
\text{eps} &= 0.001; \text{genus} = 1; \text{tau} = 0.5 + 1.4i;\\
\text{alphat0}(2) &= \frac{1}{\text{alphat0}(1)^*}; \text{lcb0} = \frac{1}{2} i \sqrt{(-\text{betat0}(1))^* \text{betat0}(2)^*};\\
\text{betat0}(1) &= \frac{\text{tau}}{\text{tau}^*}; \text{betat0}(2) = -\frac{\text{tau}}{\text{tau}^*};\\
\text{alphat0}(1) &= -\frac{(\text{eps}+1)\text{tau}}{\text{tau}^*}; \text{lca} = \frac{\text{alphat0}(1)^*}{\|\text{alphat0}(1)\|};\\
\text{alphas} &= \text{Table}[\text{alpha}(i), \{i, 2\text{genus}\}]; (\text{alphast0} = \text{Table}[\text{alphat0}(i), \{i, 2\text{genus}\}];\\
\text{betas} &= \text{Table}[\text{beta}(i), \{i, \text{genus}+1\}]; \text{betast0} = \text{Table}[\text{betat0}(i), \{i, \text{genus}+1\}];\\
\text{normkonsts} &= \text{Table}[\text{normkonst}(i), \{i, 2(\text{genus}+1)\}];
\end{aligned}$$

In these terms the polynomials $a(\lambda)$ and $b(\lambda)$ read

$$\begin{aligned} Polb[s_] &= lcb0 * Product[s - betat0[i], i, 1, 2]; \\ pola[s_] &= lca * Product[s - alphat0[i], i, 1, 2];. \end{aligned}$$

The ODEs are implemented as matrices. For the roots α_i they read:

$$\begin{aligned} \text{cms} &= \text{Table}\left[\text{cm}(i)(\text{s\_}) = \frac{\text{normkonsts}[[i]]}{s - \text{beta}(i)(t)} - \frac{s\text{normkonsts}[[i]]^*}{1 - s(\text{beta}(i)(t))^*}, \{i,1,2\}\right]; \\ \text{dglas} &= \text{Table}\left[\text{alpha}(i)'(t) = \text{alpha}(i)(t)(-\text{cm}(j)(\text{alpha}(i)(t))), \{i,1,2\}, \{j,1,2\}\right]; \end{aligned}$$

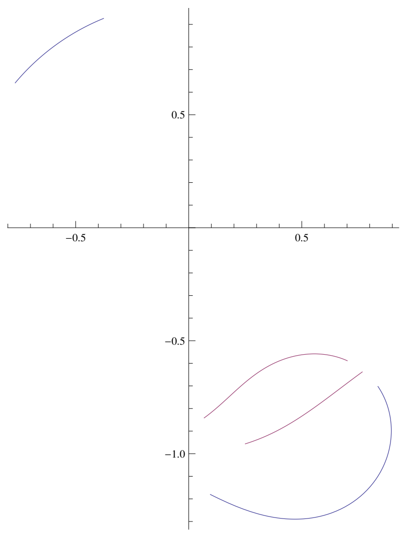
These ODEs are then solved numerically. One has two directions t_1 and t_2 that correspond to the two columns of the matrices for the ODEs. We therefore initialize

$$\text{fluss} = 2; \text{t1} = -0.01; \text{t2} = -0.5;$$

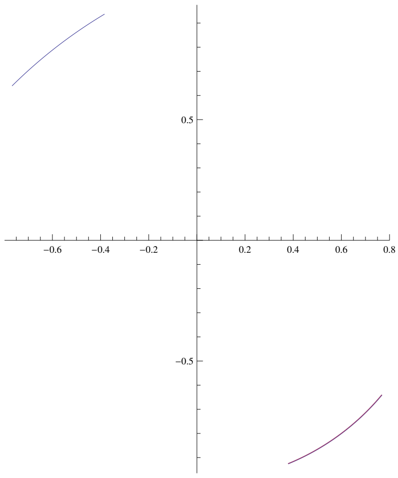
where $fluss$ corresponds to the direction. Next we join the tables that contain the ODEs together with the initial conditions. We solve the system with the command NDsolve.

$$\begin{aligned} \text{tableba} &= \text{Join}[\text{Table}[\text{beta}(i), \{i,1,2\}], \text{Table}[\text{alpha}(i), \{i,1,2\}]]; \\ \text{step1a} &= \text{Join}[\text{Table}[\text{dglas}[[i]][[\text{fluss}]], \{i,1,2\}], \text{Table}[\text{alpha}(i)(0) = \text{alphast0}[[i]], \{i,1,2\}]]; \\ \text{step1b} &= \text{Join}[\text{Table}[\text{simpledglbs}[[i]][[\text{fluss}]], \{i,1,2\}], \text{Table}[\text{dglas}[[i]][[\text{fluss}]], \{i,1,2\}], \\ &\quad \text{Table}[\text{alpha}(i)(0) = \text{alphast0}[[i]], \{i,1,2\}], \text{Table}[\text{beta}(i)(0) = \text{betat0}(i), \{i,1,2\}]]; \\ \text{solb} &= \text{NDSolve}[\text{step1b}, \text{tableba}, \{t, \text{t1}, \text{t2}\}]; \end{aligned}$$

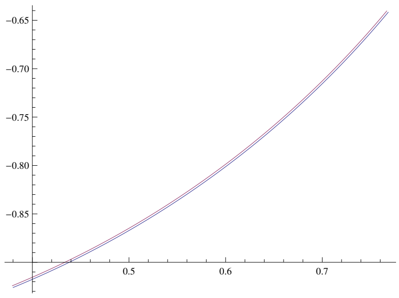
One may then plot the solution $solb$ corresponding to one direction t_i. t_2 opens up the double point. Up to a numerical error one observes that the roots β_i rest on the unit



circle whereas the α_i open up but keep in involution. This is the behavior we expected in the preliminary considerations. Contrary the solution in the first direction should correspond to a rotation of the roots of $a(\lambda)$. Below is the plot for the solution in the first direction:



the two roots $\alpha_{1,2}$ rotate around the point of origin, each with constant distance to $\mathbb{S}^1$:



This is, again, the behavior we expected in the preliminary thougts of this section.

6 Conclusion and outlook

The main results of this diploma thesis are summarized in lemma 4.18. In this lemma the deformation equations to deform the spectral data of a CMC cylinder of finite type isoperiodically are presented in a accessible way for numerical computations.
In the previous chapter the numerical computation is discussed explicitely. One problem that remains is how to make functions out of the normalization constants that are suited for numerical computations. We also took a closer look on the simple case when one raises the genus of the spectral curve from 0 to 1. After we set up assumptions on the behavior of the roots of the two polynomials $a(\lambda)$ and $b(\lambda)$ we explicitly calculated the deformation flows of these roots in the directions t_1 and t_2. The plot of the solution of the ODEs confirmed the assumptions of the preliminary thoughts.
It would be of big interest to do further computations that raise the genus of the spectral curve to a much higher degree. Thereby one should control the involutions for the polynomials $a(\lambda)$ and $b(\lambda)$. One could investigate the commutativity of the deformation flows.
Another interesting problem would be to use the inverse scattering method to find new solutions of the sinh-Gordon equation and corresponding CMC immersions. Then one could investigate in the relationship beween the deformation of the spectral data and the corresponding solution of the sinh-Gordon equation.

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