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Diploma Thesis

Compact surfaces of constant mean curvature in the 3-sphere

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Introduction

The study of minimal surfaces has a long and rich history dating from the experiments of the Belgian physicist J. Plateau who showed that by the laws of surface tension the soap film formed by dipping a wire form in a soap solution represented a surface which was stable with respect to area. Mathematically this wire can be described by a polygon Γ and thus by investigating the so-called *Plateau Problem* one is interested in the existence of a *minimal surface* $\mathcal{M}_\Gamma$ that has Γ as boundary [1].

The goal of this diploma thesis is to investigate a family of minimal surfaces Σ_g going back to Lawson [24, 26]. As Lawson constructs those surfaces using solutions to the Plateau Problem for a given polygon Γ_g in S^3 that is patched together by geodesics in S^3 we will be dealing with this problem and the resulting "initial surface" $\mathcal{M}_{\Gamma_g}$ first. Since those geodesics are serving as boundary for the "initial surface" it will be convenient to reflect some conditions posed upon that polygon. Lawson uses the symmetries encoded in the "initial surface" to construct the whole surface by a reflection principle explained in [24, 26].

Thus in order to understand the surface one has to understand the symmetry group arising from reflections across the boundary arcs of the geodesic polygon in S^3. Since every 2-dimensional orientable Riemannian manifold can be considered as a Riemann surface one might search for another description of Σ_g considered as a compact Riemann surface. As all the Σ_g are hyperelliptic one can realize them as 2-sheeted cover of $\mathbb{CP}^1$ or equivalently one can construct a function $w : \Sigma_g \to \mathbb{CP}^1$ such that

$$w^2 = P(z) = \prod_{j=1}^{2g+2} (z - e_j),$$

[1] A minimal surface is a surface with a mean curvature of zero.

with P being a polynomial of degree $2g+2$. Equipped with this realization of the compact Riemann surface Σ_g one can find the quadratic Hopf differential Q that together with the mean curvature H and the conformal factor u uniquely describes an immersion into S^3.

We shall for this purpose investigate quadratic differentials and this investigation will lead to a triangulation of the surfaces Σ_g. It will turn out that the zeros of the Hopf differential will play an important role in the following part of the thesis. This part will be devoted to the study of conformal immersions via the concept of moving frames. For globally constant Hopf differential one has to consider tori and those have been studied very intensively by Bobenko (see [3], [4] and [5]) applying methods from integrable systems theory. In fact, Bobenko gave explicit formulas for CMC tori for the spaceforms $\mathbb{R}^3, S^3$ and $\mathbb{H}^3$ in terms of theta-functions. These functions are described in terms of the so-called *spectral curve*. The notion of the spectral curve arises if one considers the eigenvalue-curve of the *monodromy* that is explained below.

The situation changes for higher genus $g \geq 2$ as one has to deal with discrete symmetry groups now. It is not clear which object is associated to those surfaces. However the considerations at the end of this work suggest that one might find answers when merging the concept of the spectral curves and the monodromy around distinguished points.

We now give a short overview of the content of the various chapters.

In the second chapter we are going through some notational conventions as well as the basic concepts of differential geometry such as the first and second fundamental form or equivalently the three quantities u, Q and H, that is the conformal factor u, the Hopf differential Q and the mean curvature H. Since the surface Σ_g is compact this chapter also deals with compact Riemann surfaces and the notion of the genus g (a topological invariant) and describes the Riemann-Roch theorem in terms of divisors and sheafs. We also give a short introduction into Lie group theory and reduce our attention to the Lie group $SU(2) \simeq S^3$. Finally the concept of moving frames is elucidated at the end of this chapter. In particular the relationship between solutions F to the *Lax pair*

$$F_z = FU, \quad F_{\bar{z}} = FV$$

with the compatibility condition $U_{\bar{z}} - V_z - [U, V] = 0$ and solutions u to the Gauss and Codazzi equations

$$2u_{z\bar{z}} + 2e^{2u}(1+H^2) - \frac{1}{2}Q\overline{Q}e^{-2u} = 0, \quad Q_{\bar{z}} = 2H_z e^{2u}$$

for given Q and $H \equiv \text{const}$ are highlighted.

The third chapter starts with introducing tools needed to construct the surface Σ_g, that is the reflection principle, followed by the construction procedure itself. The focus lies on describing Σ_g as an algebraic curve and that is done by investigating the symmetry group first. In this context it will be necessary to recall some facts about hyperelliptic Riemann surfaces.

In the fourth chapter the Hopf differential Qdz^2 is investigated and therefore a short introduction into the theory of quadratic differentials is given. Moreover the notion of a (horizontal) trajectory for a quadratic differential $\varphi(z)dz^2$ is discussed, i.e. a curve γ parameterized on an open interval (a,b) of the real axis with $\varphi(\gamma(t))\left(\frac{d\gamma(t)}{dt}\right)^2 > 0$ for every $t \in (a,b)$. This will lead to a canonical triangulation of Σ_g. We shall particularly focus on the significance of the zeros of the Hopf differential, as they will play an important role in the rest of this work.

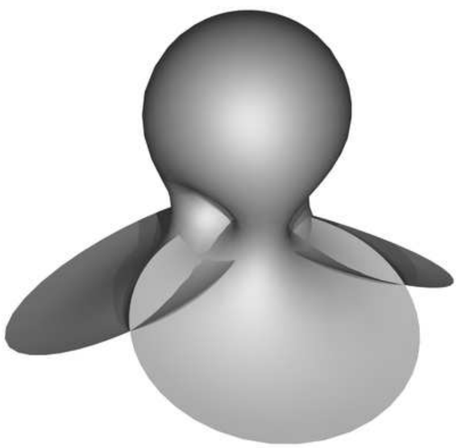

Figure 1.1: A Smyth surface in S^3. The image is taken from [32].

Finally the fifth chapter deals with Lax pairs for S^3 and λ-dependent *extended frames* F_λ, where λ is called the spectral parameter. We are interested in the object associated to Σ_g and therefore shall investigate the effect of coordinate transformations on the conformal factor u and the Hopf differential Q as well as on the extended frame F_λ. We will also have to recall some facts from the theory of covering surfaces. Introducing the monodromy M_λ as an operator that describes the change of the frame as one traverses a loop δ around a given point via $F_\lambda(\delta) = M_\lambda^\delta F_\lambda$, several properties and the behavior of M_λ at distinguished points are studied.

There is also a sixth chapter that contains the most important results of this thesis and gives an outlook on possible interesting further research.

There are many people I am indebted to. First I would like to thank Martin U. Schmidt for providing me with the topic of this work, for regular fruitful discussions and his steady encouragement. I also would like to thank my parents for supporting me, without them all this would not have been possible. Speaking of my family I also want to thank my brother Marcel (for his occasional late visits) and my best friend Rainer Jäkel, on whom I can always count. Special thanks also go to my workmate Vania Neugebauer (for bearing with me for several months) and to Jörg Zentgraf for proofreading this work. Finally I would like to thank Anna for her love and for always standing by my side.

Chapter 2

Preliminaries

In order to understand the construction procedure for the surfaces we need to look over some notational conventions and recall essential facts from differential geometry, the theory of Riemann Surfaces, Lie groups, and the concept of moving frames.

2.1 Differential geometric preliminaries

Let M be a C^∞ Riemannian manifold where at each point $p \in M$ the metric in the tangent space $T_p(M)$ at p is denoted by the bracket $\langle \cdot, \cdot \rangle$. Let $\mathcal{X}_M$ denote the space of C^∞ vector fields on M.

Definition 2.1. A **connection** on M is a rule which assigns to each $X \in \mathcal{X}_M$ a linear map $\nabla_X : \mathcal{X}_M \to \mathcal{X}_M$ such that for all $X, Y, Z \in \mathcal{X}_M$ and all $f, g \in C^\infty(M)$ we have

1. $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$,
2. $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$.

By "The Fundamental Theorem of Riemannian Geometry" there exists a unique connection on M, called the **Riemannian connection**, which satisfies the further conditions:

3. $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$,
4. $\nabla_X Y - \nabla_Y X = [X, Y]$.

The third condition states that the Riemannian connection is *metric*, the fourth that it is *torsion-free*.

Let $\overline{M}$ be a Riemannian $\overline{m}$-manifold and $M \subset \overline{M}$ a topologically embedded submanifold of dimension m. Denote the metric on $\overline{M}$ by $\langle \cdot, \cdot \rangle$ and the associated Riemannian connection by $\overline{\nabla}$. For any $p \in M \subset \overline{M}$ we have an orthogonal splitting

$$T_p(\overline{M}) = T_p(M) \oplus T_p^{\perp}(M)$$

into the tangent and normal spaces of M at p respectively. With respect to this splitting we decompose any vector $X \in T_p(M)$ as

$$X = X^{\top} + X^{\perp}.$$

The unique Riemannian connection ∇ of M can then be given as follows. Denote by $\mathcal{X}_p$ the set of tangent vector fields of M each of which is defined in some neighborhood of p on M. Then for $X, X \in \mathcal{X}_p$,

$$\nabla_X Y = (\overline{\nabla}_X Y)^{\top}.$$

Definition 2.2. The local normal vector field at p

$$B_{X,Y} = (\overline{\nabla}_X Y)^{\perp}$$

represents a C^{∞}-section of $T^*(M) \otimes T^*(M) \otimes T^{\perp}(M)$ and is called the **second fundamental form** of the submanifold M.

Remark 2.3. At each point $p \in M$, $B_{X,Y}|_p$ represents a symmetric bilinear map of $T_p(M)$ into $T_p^{\perp}(M)$:

$$(\overline{\nabla}_X Y)^{\perp} = (\overline{\nabla}_Y X + [X,Y])^{\perp} = (\overline{\nabla}_Y X)^{\perp}$$

and thus

$$B_{X,Y} = B_{Y,X}.$$

Definition 2.4. For each p

$$H_p = \mathrm{tr}(B_p)$$

is a smooth field of normal vectors on M called the **mean curvature vector field**.

Remark 2.5. Locally H has the form

$$H_p = \sum_{k=1}^{m} (\nabla_{X_k} X_k)^{\perp}$$

for pointwise orthonormal vector fields $X_1, \ldots, X_m$. An immersion $f : M \to \overline{M}$ is called minimal if $\mathrm{tr}(B) \equiv 0$.

We now turn our attention to the case where $\overline{M} = S^3 := \{x \in \mathbb{R}^4 \mid |x| = 1\}$ is the 3-sphere and M is an arbitrary Riemann surface.

Definition 2.6. A Riemann surface is a pair (M, Σ), consisting of a connected 2-dimensional manifold M with a complex structure Σ, that is an equivalence class of biholomorphic equivalent collections of charts, that cover M.

We will be considering 2-dimensional submanifold of S^3, i.e. conformal immersions of the form

$$f : M \longrightarrow S^3.$$

The function f will always be considered as $\mathbb{R}^4$-valued with $|f|^2 = 1$.

Now let $z = x + iy$ be a local complex coordinate on M and define

$$\partial := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

and $\bar{\partial}$ respectively by

$$\bar{\partial} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

It will be convenient to reformulate the above considerations in a slightly different language.

Definition 2.7. Let $f : M \to S^3$ be an immersion where S^3 is equipped with the metric defined by restricting the metric $h = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$ of $\mathbb{R}^4$ to the 3-dimensional tangent spaces of S^3. The induced metric $g : T_pM \times T_pM \to \mathbb{R}$ is defined by

$$g(v, w) = h(df(v), df(w)) = \langle df(v), df(w)\rangle, \quad v, w \in T_pM, \quad p \in M$$

and is called the **first fundamental form**. Both g and ds^2 are commonly used notations.

Since (x, y) is a coordinate for M and f is an immersion, a basis for T_pM can be chosen as

$$f_x = \left(\frac{\partial f}{\partial x}\right)_p, \; f_y = \left(\frac{\partial f}{\partial y}\right)_p,$$

then the metric $g = ds^2$ is represented by the matrix

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \langle f_x, f_x\rangle & \langle f_x, f_y\rangle \\ \langle f_y, f_x\rangle & \langle f_y, f_y\rangle \end{pmatrix}.$$

In case of a conformal immersion f there exists a function $u : M \to \mathbb{R}$, called the conformal factor such that

$$ds^2 = 4e^{2u}dz d\bar{z} = 4e^{2u}(dx^2 + dy^2).$$

Remark 2.8. $f : M \to S^3$ is an immersion $\Leftrightarrow$ g has positive determinant.

With the unit normal vector to the surface $f(M)$ defined as

$$N = \frac{f_x \times f_y}{|f_x \times f_y|}, \quad |f_x \times f_y|^2 = \langle f_x \times f_y, f_x \times f_y \rangle,$$

one sees that N is perpendicular to the tangent plane T_pM at $f(p)$ for every point $f(p)$ and we have the following equivalent

Definition 2.9. The symmetric bilinear map $b : T_pM \times T_pM \to \mathbb{R}$ defined by

$$b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = - \begin{pmatrix} \langle N_x, f_x \rangle & \langle N_y, f_x \rangle \\ \langle N_x, f_y \rangle & \langle N_y, f_y \rangle \end{pmatrix} = \begin{pmatrix} \langle N, f_{xx} \rangle & \langle N, f_{xy} \rangle \\ \langle N, f_{yx} \rangle & \langle N, f_{yy} \rangle \end{pmatrix}$$

is called the **second fundamental form**.

The second fundamental form can also be written in terms of symmetric 2-differentials as

$$b = b_{11}dx^2 + b_{12}dxdy + b_{21}dydx + b_{22}dy^2.$$

Converting to the complex coordinate $z = x + iy$ one obtains

$$b = Qdz^2 + Hdzd\bar{z} + \overline{Q}d\bar{z}^2,$$

where Q is the complex-valued function

$$Q := \frac{1}{4}(b_{11} - b_{22} - ib_{12} - ib_{21})$$

and H is the real valued function

$$H := \frac{1}{2}(b_{11} + b_{22}).$$

Definition 2.10. The symmetric 2-differential Qdz^2 is called the **Hopf differential** of the immersion f.

Definition 2.11. The linear map $S : T_pM \to T_pM$ with

$$S := g^{-1}b.$$

is called the **shape operator** of the immersion f.

The eigenvalues k_1, k_2 and corresponding eigenvectors of the shape operator $g^{-1}b$ are the *principal curvatures* and *principal curvature directions* of the surface $f(M)$ at $f(p)$. We can now define the Gauss and mean curvature using the language introduced above.

Definition 2.12. The determinant and half-trace of the shape operator $S = g^{-1}b$ of $f : M \to S^3$ are the **Gauss curvature** K and the **mean curvature** H, respectively. The immersion f is CMC (i.e. of constant mean curvature) if H is constant, and is minimal if H is identically zero.

Remark 2.13. We note that the above K is not the same as the intrinsic curvature.

In the following lemma we state an equivalent minimality-condition that will be useful later on (see [7]).

Lemma 2.14. The minimal mapping f satisfies the equation

$$\partial\bar{\partial} f = -2e^{2u} f.$$

Proof. Since the immersion f of M in S^3 is locally minimal the curvature vector of f as an immersion in $\mathbb{R}^4$

$$\frac{1}{2e^{2u}}\partial\bar{\partial} f$$

is everywhere orthogonal to S^3, i.e. proportional to f. Hence we get

$$\partial\bar{\partial} f = \lambda f$$

for some complex valued function λ. From the fact that $|f|^2 = 1$ we obtain

$$\begin{aligned} 0 &= \frac{1}{2}\partial\bar{\partial}(1) = \frac{1}{2}\partial\bar{\partial}\langle f, f\rangle = \frac{1}{2}(\partial(\langle\bar{\partial} f, f\rangle + \langle f, \bar{\partial} f\rangle)) \\ &= \frac{1}{2}(\langle\partial\bar{\partial} f, f\rangle + \langle\bar{\partial} f, \partial f\rangle + \langle\partial f, \bar{\partial} f\rangle + \langle f, \partial\bar{\partial} f\rangle) \\ &= \langle\partial\bar{\partial} f, f\rangle + \langle\partial f, \bar{\partial} f\rangle = \langle\lambda f, f\rangle + 2e^{2u} \\ &= \lambda\langle f, f\rangle + 2e^{2u} = \lambda + 2e^{2u}, \end{aligned}$$

and $\lambda = -2e^{2u}$, as asserted. □

We will now compute the (intrinsic) Gauss curvature K and make the observation, that it can be expressed in terms of the conformal factor u.

Lemma 2.15. The Gauss curvature K for a conformal immersed surface with isothermal coordinates $(z, \bar{z})$, such that $ds^2 = 4e^{2u}|dz|^2$, where $u = u(z, \bar{z})$ is a given function, is given by

$$K = -e^{-2u}\Delta u,$$

where Δ is the Laplacian.

Proof. The assertion follows from a straight-forward calculation using the "Theorema Egregium" and the fact that the surface is immersed conformally such that for $F = e^{2u}$:

$$\begin{aligned} 4F^4K &= F(4\partial F\bar{\partial}F) + 2F^2(-2\partial\bar{\partial}F) = 16F^3\partial u\bar{\partial}u + 2F^2(-2\partial(2e^{2u}\bar{\partial}u)) \\ &= 16F^3\partial u\bar{\partial}u - 16F^3\partial u\bar{\partial}u - 16F^3\partial\bar{\partial}u, \end{aligned}$$

from which we get $K = -\frac{4}{F}\partial\bar{\partial}u = -e^{-2u}\Delta u$. □

The quantities u, Q and H obey the well-known Gauss and Codazzi equations stated below:

$$2u_{z\bar{z}} + 2e^{2u}(1 + H^2) - \frac{1}{2}Q\overline{Q}e^{-2u} = 0, \;\; Q_{\bar{z}} = 2H_z e^{2u}.$$

For a minimal immersion f we have

$$\frac{1}{2}\mathrm{tr}(S) = 8e^{-2u}H = 4e^{-2u}(b_{11} + b_{22}) \equiv 0$$

and therefore $b_{11} = -b_{22}$. For the Hopf differential Q one now obtains (since $b_{12} = b_{21}$)

$$Q\overline{Q} = \frac{1}{4}(b_{11} - ib_{12})(b_{11} + ib_{12}) = \frac{1}{4}((b_{12})^2 - b_{11}b_{22}).$$

Since $H = 0$ the Gauss equation can be restated as follows

$$\begin{aligned} Q\overline{Q} &= 4e^{2u}u_{z\bar{z}} + 4e^{4u} = 4e^{4u}(e^{-2u}u_{z\bar{z}} + 1) \\ &= 4e^{4u}(1 - K) \end{aligned}$$

with K being the Gauss curvature. The above curvature equation will help to understand the meaning of the zeros of the Hopf differential.

Lemma 2.16. If the immersion f is minimal, then the quadratic Hopf differential $\omega = Qdz^2$ is holomorphic on M, where

$$Q = \frac{1}{2}(b_{11} - ib_{12}) = -\frac{i}{2e^{2u}}f \wedge \partial f \wedge \bar{\partial}f \wedge \partial^2 f.$$

Proof. Since f is conformal and $\langle f, f\rangle = 1$ we have

$$\begin{aligned} \langle \partial f, \partial f\rangle &= \langle \bar{\partial} f, \bar{\partial} f\rangle = 0, \\ \langle \partial^2 f, \partial f\rangle &= \langle \bar{\partial}^2 f, \bar{\partial} f\rangle = 0, \\ \langle \partial f, \bar{\partial} f\rangle &= 2e^{2u}. \end{aligned}$$

ω is holomorphic if Q^2 is holomorphic since $\bar{\partial}Q^2 = 2Q\bar{\partial}Q$. Evaluating $Q^2 = \left(\frac{1}{ie^{2u}} f \wedge \partial f \wedge \bar{\partial} f \wedge \partial^2 f\right)^2$ gives

$$\begin{aligned} Q^2 &= -\frac{1}{4e^{4u}} \det \begin{pmatrix} \langle f, f\rangle & \langle f, \partial f\rangle & \langle f, \bar{\partial} f\rangle & \langle f, \partial^2 f\rangle \\ \langle \partial f, f\rangle & \langle \partial f, \partial f\rangle & \langle \partial f, \bar{\partial} f\rangle & \langle \partial f, \partial^2 f\rangle \\ \langle \bar{\partial} f, f\rangle & \langle \bar{\partial} f, \partial f\rangle & \langle \bar{\partial} f, \bar{\partial} f\rangle & \langle \bar{\partial} f, \partial^2 f\rangle \\ \langle \partial^2 f, f\rangle & \langle \partial^2 f, \partial f\rangle & \langle \partial^2 f, \bar{\partial} f\rangle & \langle \partial^2 f, \partial^2 f\rangle \end{pmatrix} \\ &= \langle \partial^2 f, \partial^2 f\rangle. \end{aligned}$$

By applying $\bar{\partial}$ and the equivalent minimality-condition stated in lemma 2.14 we obtain

$$\bar{\partial}\langle \partial^2 f, \partial^2 f\rangle = 2\langle \partial(\partial\bar{\partial} f), \partial^2 f\rangle = -4\langle \partial(e^{2u} f), \partial^2 f\rangle = 0$$

and thus that ω is holomorphic. □

The next result illustrates the relationship between the holomorphic quadratic Hopf differential Qdz^2 and the (intrinsic) Gauss curvature K.

Lemma 2.17. The Gauss curvature K of a minimal surface in S^3 satisfies $K \leq 1$, with equality precisely at the isolated zeros of the holomorphic Hopf differential Qdz^2.

Proof. Since f is minimal we may consider

$$\begin{aligned} 0 \ &\leq \ |Q|^2 = \frac{1}{4}(b_{11} - ib_{12})(b_{11} + ib_{12}) \\ &= \ \frac{1}{4}((b_{12})^2 - b_{11}b_{22}) = 4e^{4u}(1-K), \end{aligned}$$

where we have made use of the fact that $b_{22} = -b_{11}$ and the Gauss curvature equation. From the last equation the lemma follows immediately. □

The Hopf differential Qdz^2 will be of central importance to us. Besides the fact that the investigated surface will be CMC if and only if Q is holomorphic, the Hopf differential can also be used to determine the umbilic points of a surface.

Definition 2.18. Let M be a 2-dimensional manifold. The **umbilic points** of an immersion $f : M \to S^3$ are the points where the two principal curvatures are equal.

Proposition 2.19. If M is a Riemann surface and $f : M \to S^3$ is a conformal immersion, then $p \in M$ is an umbilic point if and only if $Q = 0$.

Proof. The shape operator corresponding to the conformal immersion f is

$$S = g^{-1}b = \frac{1}{4e^{2u}} \begin{pmatrix} H + Q + \bar{Q} & i(Q - \bar{Q}) \\ i(Q - \bar{Q}) & H - Q - \bar{Q} \end{pmatrix}$$

with respect to the basis f_x and f_y of each tangent space of $f(M)$. The two principal curvatures are then the two eigenvalues of this self-adjoint operator, i.e. solutions of

$$\begin{aligned} 4e^{2u}\det(S - k \cdot \mathbb{1}) \ &= \ (H + Q + \bar{Q} - k)(H - Q - \bar{Q} - k) + (Q - \bar{Q})^2 \\ &= \ ((H - k) + (Q + \bar{Q}))((H - k) - (Q + \bar{Q})) - (Q - \bar{Q})^2 \\ &= \ (H - k)^2 - (Q + \bar{Q})^2 - (Q - \bar{Q})^2 \\ &= \ (H - k)^2 - 4|Q|^2 = 0, \end{aligned}$$

and thus one obtains

$$k_1 = H + 2|Q|, \quad k_2 = H - 2|Q|.$$

Finally one gets $k_1 = k_2 \Leftrightarrow |Q| = 0 \Leftrightarrow Q = 0$ and the result follows. □

2.2 Compact Riemann Surfaces

In this section we will focus on Divisors and the Riemann-Roch Theorem that will be powerful tools in the following chapters for analyzing the constructed surfaces. Most results and terminology are taken from [9] and [11].

Definition 2.20. Let M be a Riemann surface. A **divisor** on M is a mapping

$$D : M \to \mathbb{Z}$$

such that for every compact subset $K \subset M$ there are only finitely many points $x \in K$ with $D(x) \neq 0$. With respect to addition the set of all divisors on M is an abelian group, denoted by $\mathrm{Div}(M)$.
For $D, D' \in \mathrm{Div}(M)$ we set $D \leq D'$ if $D(x) \leq D'(x)$ for every $x \in M$.

For a compact Riemann surface M of genus $g \geq 0$ let $\mathcal{K}(M)$ denote the field of meromorphic functions on M. Now suppose that N is an open subset of M. For a meromorphic function $f \in \mathcal{K}(N)$ and $a \in N$ define

$$\mathrm{ord}_a(f) := \begin{cases} 0, & \text{if } f \text{ is holomorphic and non-zero at } a, \\ k, & \text{if } f \text{ has a zero of order } k \text{ at } a, \\ -k, & \text{if } f \text{ has a pole of order } k \text{ at } a, \\ \infty, & \text{if } f \text{ is identically zero in a neighborhood of } a. \end{cases}$$

Thus for any meromorphic function $f \in \mathcal{K}(M)\backslash\{0\}$, the mapping $x \mapsto \mathrm{ord}_x(f)$ is a divisor on M. It is called the *divisor of f* and will be denoted by (f).
The function f is said to be a *multiple* of the divisor D if $(f) \geq D$. Then f is holomorphic if and only if $(f) \geq 0$.

For a meromorphic 1-form ω one can define its order at a point $a \in N$ as follows. Choose a coordinate neighborhood (U, z) of a. Then on $U \cap N$ one has $\omega = f dz$, where f is a meromorphic function. Set $\mathrm{ord}_a(\omega) = \mathrm{ord}_a(f)$. Again the mapping $x \mapsto \mathrm{ord}_x(\omega)$ is a divisor on M, denoted by (ω).

A divisor $D \in Div(M)$ is called a *principal divisor* if there exists a function $f \in \mathcal{K}(M)\backslash\{0\}$ such that $D = (f)$. Two divisors $D, D' \in Div(M)$ are said to be *equivalent* if their difference $D - D'$ is principal divisor.
A *canonical divisor* is the divisor (ω) of a meromorphic 1-form ω.

Definition 2.21. For a compact Riemann surface M the mapping

$$\deg : \mathrm{Div}(M) \to \mathbb{Z}$$

is called the **degree** (of the divisor D), whereas

$$\deg D := \sum_{x \in M} D(x).$$

The mapping $\deg : \mathrm{Div}(M) \to \mathbb{Z}$ is a group homomorphism and $\deg(f) = 0$ for any principal divisor (f) on a compact Riemann surface since a meromorphic function has as many zeros as poles.

Before we can state the Riemann-Roch Theorem we have to introduce the notion of a sheaf and its corresponding cohomology.

Definition 2.22. Suppose M is a topological space and $\mathcal{I}$ is the system of open sets in M. A **presheaf** of abelian groups on M is a pair $(\mathcal{F}, \rho)$ consisting of

1. a family $\mathcal{F} = (\mathcal{F}(U))_{U \in \mathcal{I}}$ of abelian groups,
2. a family $\rho = (\rho_V^U)_{U,V \in \mathcal{I}, V \subset U}$ of group homomorphisms (called *restriction homomorphisms*)

$$\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V), \text{ where } V \text{ is open in } U,$$

with the following properties:

$$\begin{aligned} \rho_U^U &= id_{\mathcal{F}(U)} \text{ for every } U \in \mathcal{I}, \\ \rho_W^V \circ \rho_V^U &= \rho_W^U \text{ for } W \subset V \subset U. \end{aligned}$$

Instead of $\rho_V^U(f)$ for $f \in \mathcal{F}(U)$ one writes $f|V$. We can now define a sheaf.

Definition 2.23. A presheaf $\mathcal{F}$ on a topological space M is called a **sheaf** if for every open set $U \subset M$ and every family of open subsets $U_i \subset U, i \in I$, with $U = \bigcup_{i \in I} U_i$, the following conditions are satisfied:

(S1) If $f, g \in \mathcal{F}(U)$ are elements such that $f|U_i = g|U_i$ for every $i \in I$, then $f = g$.

(S2) Given elements $f_i \in \mathcal{F}(U_i), i \in I$, obeying

$$f_i|U_i \cap U_j = f_j|U_i \cap U_j \text{ for all } i, j \in I,$$

then there exists $f \in \mathcal{F}(U)$ such that $f|U_i = f_i$ for every $i \in I$.

(S1) and (S2) are called the *Sheaf Axioms*.

Definition 2.24. Let M be a topological space and $\mathcal{F}$ a sheaf of abelian groups on M. Let $\mathcal{U}$ be an open covering of M, i.e. a family $\mathcal{U} = (U_i)_{i\in I}$ of open subsets of M such that $\bigcup_{i\in I} U_i = M$. For $q = 0, 1, 2, \ldots$ define the **qth cochain group** of $\mathcal{F}$, with respect to $\mathcal{U}$, as

$$C^q(\mathcal{U}, \mathcal{F}) := \prod_{(i_0,\ldots,i_q)\in I^{q+1}} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_q}).$$

The elements of $C^q(\mathcal{U}, \mathcal{F})$ are called q-cochains.

Now define *coboundary operators*

$$\delta : C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F})$$
$$\delta : C^1(\mathcal{U}, \mathcal{F}) \to C^2(\mathcal{U}, \mathcal{F})$$

as follows:

1. For $(f_i)_{i\in I} \in C^0(\mathcal{U}, \mathcal{F})$ let $\delta((f_i)_{i\in I}) = (g_{ij})_{i,j\in I}$ where
$$g_{ij} := f_j - f_i \in \mathcal{F}(U_i \cap U_j).$$
2. For $(f_{ij})_{i,j\in I} \in C^1(\mathcal{U}, \mathcal{F})$ let $\delta((f_{ij})_{i,j\in I}) = (g_{ijk})$ where
$$g_{ijk} := f_{jk} - f_{ik} + f_{ij} \in \mathcal{F}(U_i \cap U_j \cap U_k).$$

These coboundary operators are group homomorphisms, so we can define

Definition 2.25. Let

$$\begin{aligned} Z^1(\mathcal{U}, \mathcal{F}) &:= Ker(C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^2(\mathcal{U}, \mathcal{F})), \\ B^1(\mathcal{U}, \mathcal{F}) &:= Im(C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathcal{F})). \end{aligned}$$

The elements of $Z^1(\mathcal{U}, \mathcal{F})$ are called **1-cocycles** and those of $B^1(\mathcal{U}, \mathcal{F})$ are called **1-coboundaries**.

Definition 2.26. The quotient group

$$H^1(\mathcal{U}, \mathcal{F}) := Z^1(\mathcal{U}, \mathcal{F})/B^1(\mathcal{U}, \mathcal{F})$$

is called the **1st cohomology group** with coefficients in $\mathcal{F}$ with respect to the covering $\mathcal{U}$.

An open covering $\mathcal{B} = (V_k)_{k\in K}$ is finer with respect to the covering $\mathcal{U} = (U_i)_{i\in I}$, denoted by $\mathcal{B} < \mathcal{U}$, if every V_k is contained in at least one U_i. Thus there is a mapping $\tau : K \to I$ such that

$$V_k \subset U_{\tau(k)} \text{ for every } k \in K.$$

We can now define a mapping

$$t^{\mathcal{U}}_{\mathcal{B}} : Z^1(\mathcal{U}, \mathcal{F}) \to Z^1(\mathcal{B}, \mathcal{F})$$

in the following way. For $(f_{ij}) \in Z^1(\mathcal{U}, \mathcal{F})$ let $t^{\mathcal{U}}_{\mathcal{B}}((f_{ij})) = (g_{kl})$ where

$$g_{kl} := f_{\tau(k),\tau(l)}|V_k \cap V_l \text{ for every } k, l \in K.$$

This mapping induces a homomorphism of the cohomology groups (also denoted by $t^{\mathcal{U}}_{\mathcal{B}}$) and we are finally ready to define $H^1(M, \mathcal{F})$.

Definition 2.27. Given three open coverings such that $\mathcal{W} < \mathcal{B} < \mathcal{U}$, one has

$$t^{\mathcal{B}}_{\mathcal{W}} \circ t^{\mathcal{U}}_{\mathcal{B}} = t^{\mathcal{U}}_{\mathcal{W}}.$$

Now define the following equivalence relation $\sim$ on the disjoint union of the $H^1(\mathcal{U}, \mathcal{F})$, where $\mathcal{U}$ runs through all open coverings of M, for two cohomology classes $\xi \in H^1(\mathcal{U}, \mathcal{F})$, $\eta \in H^1(\mathcal{U}', \mathcal{F})$ by

$$\begin{aligned} \xi \sim \eta \quad :\Leftrightarrow \quad & \exists \text{ open covering } \mathcal{B} \text{ with } \mathcal{B} < \mathcal{U} \text{ and} \\ & \mathcal{B} < \mathcal{U}' \text{ such that } t^{\mathcal{U}}_{\mathcal{B}}(\xi) = t^{\mathcal{U}'}_{\mathcal{B}}(\eta). \end{aligned}$$

The set of equivalence classes is called the **1st cohomology group** of M with coefficients in the sheaf $\mathcal{F}$:

$$H^1(M, \mathcal{F}) = \left(\bigcup_{\mathcal{U}} H^1(\mathcal{U}, \mathcal{F}) \right) / \sim .$$

Now suppose D is a divisor on the Riemann surface M. For any open set $U \subset M$ define $\mathcal{O}_D(U)$ to be the set of all meromorphic functions on U which are multiples of the divisor $-D$, i.e.

$$\mathcal{O}_D(U) := \{ f \in \mathcal{K}(U) \mid \mathrm{ord}_x(f) \geq -D(x) \text{ for every } x \in U \}.$$

Together with the natural restriction mappings $\mathcal{O}_D$ is a sheaf. In the special case of the zero divisor $D = 0$ one has $\mathcal{O}_0 = \mathcal{O}$.

We will recall the definition of the genus of a compact Riemann surface before we state the theorem that is central in the theory of compact Riemann surfaces.

Definition 2.28. For a compact Riemann surface M,

$$g := \dim H^1(M, \mathcal{O})$$

is called the **genus** of M.

Theorem 2.29 (The Riemann-Roch Theorem)**.** *Suppose D is a divisor on a compact Riemann surface M of genus g. Then $H^0(M, \mathcal{O}_D)$ and $H^1(M, \mathcal{O}_D)$ are finite dimensional vector spaces and*

$$\dim H^0(M, \mathcal{O}_D) - \dim H^1(M, \mathcal{O}_D) = 1 - g + \deg D.$$

Proof. See [11, Thm. 16.9]. □

Definition 2.30. The positive integer

$$i(D) := \dim H^1(M, \mathcal{O}_D)$$

is called the **index of speciality** of the divisor D.

We may reformulate the Riemann-Roch Theorem in the following form

$$\dim H^0(M, \mathcal{O}_D) = 1 - g + \deg D + i(D).$$

We will now state the Serre Duality Theorem that allows a simpler interpretation of the cohomology groups $H^1(M, \mathcal{O}_D)$ in terms of differential forms.

For this purpose let M be a compact Riemann surface. For any divisor $D \in \mathrm{Div}(M)$ we denote by Ω_D the sheaf of meromorphic 1-forms which are multiples of $-D$. Thus for any open set $U \subset M$ the set $\Omega_D(U)$ consists of all differential forms ω such that $\mathrm{ord}_x(\omega) \geq -D(x)$ for every $x \in U$.

Theorem 2.31 (The Duality Theorem of Serre)**.** *Any divisor D on a compact Riemann surface M induces an isomorphism*

$$H^0(M, \Omega_{-D}) \simeq H^1(M, \mathcal{O}_D)^*.$$

Proof. See [11, Thm. 17.9]. □

Remark 2.32. From the Serre Duality Theorem one immediately obtains

$$\dim H^1(M, \mathcal{O}_D) = \dim H^0(M, \Omega_{-D}).$$

In particular for $D = 0$ one has

$$g = \dim H^1(M, \mathcal{O}) = \dim H^0(M, \Omega).$$

Thus the genus of a compact Riemann surface M is equal to the maximum number of linearly independent holomorphic 1-forms on M. One can now formulate the Riemann-Roch Theorem as follows:

$$\dim H^0(M, \mathcal{O}_{-D}) - \dim H^0(M, \Omega_D) = 1 - g - \deg D.$$

Theorem 2.33. *The divisor of a non-vanishing meromorphic 1-form ω on a compact Riemann surface of genus g satisfies*

$$\deg(\omega) = 2g - 2.$$

Proof. See [11, Thm. 17.12]. □

Proposition 2.34. For $g \geq 1$ the quadratic Hopf differential Qdz^2 defined in section 2.1 has exactly $4g - 4$ zeros to multiplicity.

Proof. Since $\omega = Qdz^2$ is holomorphic we have $D := (\omega) \geq 0$ and therefore

$$\sharp \text{ zeros } = \deg(\omega) = 2(2g-2) = 4g - 4.$$

□

2.3 Lie groups

In order to understand the concept of moving frames and the following considerations, one has to recall some basic facts about Lie groups.

Definition 2.35. Let G be a Lie group and define left- and right-multiplication by an element $g \in G$ via

$$\begin{aligned} L_g : G \to G, &\qquad h \mapsto gh \\ R_g : G \to G, &\qquad h \mapsto hg \end{aligned}$$

A vector field $X : G \to TG$ is called **invariant**, if

$$d_h L_g(X(h)) = X(gh) \text{ for all } g \in G.$$

With the above definition one immediately sees that left-invariant vector fields are uniquely determined through their values at the identity, since

$$X(g) = d_1 L_g X(1).$$

Denoting the set of left-invariant vector fields by $\Gamma_L(G)$ one obtains the following vector space isomorphism

$$\begin{aligned} \Gamma_L(G) &\cong T_1 G \\ X &\mapsto X(1), \end{aligned}$$

with inverse map given by $T_1 G \ni v_1 \mapsto X \in \Gamma_L(G),\ X(g) := d_1 L_g(v_1)$.

Definition 2.36. The **Lie algebra** $\mathfrak{g}$ associated to a Lie group G is the tangent space of G at the identity 1, i.e. $\mathfrak{g} = T_1 G$. Furthermore, there is a bracket operation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined as

$$[X,Y](f) = X(Y(f)) - Y(X(f)), \quad X, Y \in \mathfrak{g}, \quad f : G \to \mathbb{R} \text{ smooth}.$$

Thus the left-invariant vector fields, equipped with the commutator $[\cdot,\cdot]$ correspond to $\mathfrak{g}$. Moreover, the tangent bundle of a Lie group is trivial:

$$\begin{array}{rcl} TG & \cong & G \times \mathfrak{g} \\ v_g & \mapsto & (g, d_g L_g^{-1}(v_g)), \end{array}$$

where the inverse map of this isomorphism is given by $(g, v_1) \mapsto d_1 L_g(v)$. We may now define the Maurer-Cartan form.

Definition 2.37. The (left) Maurer-Cartan form is the $\mathfrak{g}$-valued 1-form $g \mapsto \theta_g$ with

$$\begin{array}{rcl} \theta_g : TG & \to & \mathfrak{g} \\ v_g & \mapsto & d_g L_g^{-1}(v_g). \end{array}$$

This is often written $\theta = g^{-1} dg$.

Proposition 2.38. The Maurer-Cartan form satisfies the following equation

$$2d\theta + [\theta \wedge \theta] = 0.$$

It is called structure equation or Maurer-Cartan equation.

Proof. First we note that

$$d\theta = d(g^{-1}) \wedge dg.$$

To compute $d(g^{-1})$, consider the identity e and note that it is the product of two non-constant functions:

$$0 = d(e) = d(g^{-1}g) = d(g^{-1})g + g^{-1}dg.$$

So, $d(g^{-1}) = -g^{-1}(dg)g^{-1}$ and thus

$$d\theta = -g^{-1}(dg)g^{-1} \wedge dg = -(g^{-1}dg) \wedge (g^{-1}dg) = -\theta \wedge \theta =: -\frac{1}{2}[\theta \wedge \theta].$$

□

We state the following proposition that will be useful later on.

Proposition 2.39. For a map $f : M \to G$, the pullback $\alpha := f^\star \theta$ also satisfies the Maurer-Cartan equation, i.e.

$$2d\alpha + [\alpha \wedge \alpha] = 0.$$

Proof. A short calculation yields

$$\begin{array}{rcl} 2d\alpha + [\alpha \wedge \alpha] & = & 2d(f^\star\theta) + [f^\star\theta \wedge f^\star\theta] = 2f^\star d\theta + f^\star[\theta \wedge \theta] \\ & = & f^\star(2d\theta + [\theta \wedge \theta]) = 0. \end{array}$$

□

It will be convenient to identify S^3 with a certain Lie group, namely the group $SU(2)$,

$$\begin{aligned} SU(2) &= \{A \in M_{2\times 2}(\mathbb{C}) \mid \det(A) = 1,\ \bar{A}^t = A^{-1}\} \\ &= \left\{ \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \,\middle|\, |z|^2 + |w|^2 = 1 \right\} \cong S^3, \end{aligned}$$

The corresponding Lie algebra is denoted by $\mathfrak{su}(2)$ and a direct computation shows that

$$\mathfrak{su}(2) = \left\{ -\frac{i}{2} \begin{pmatrix} -x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_3 \end{pmatrix} \,\middle|\, x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

2.4 The concept of moving frames

The fundamental theorem of surface theory states that there exists an immersion $f : M \to S^3$ with first fundamental form g and second fundamental form b if and only if g and b satisfy a pair of equations called the Gauss and Codazzi equations. Furthermore we know that f is uniquely determined by g and b up to rigid motions.
The 1-form formulations for g and b are

$$g = 4e^{2u} dz d\bar{z}, \quad b = Q dz^2 + H dz d\bar{z} + \overline{Q} d\bar{z}^2.$$

The symmetric 2-form Qdz^2 is the Hopf differential as defined before. In the conformal situation, the Gauss and Codazzi equations can be written in terms of the functions u, H and Q.

Definition 2.40. Let M be a smooth manifold of dimension n. A **frame** is an n-tuple $(X_1, \ldots, X_n)$ of vector fields such that $X_1(p), \ldots, X_n(p)$ is an ordered basis of T_pM at every $p \in M$.

If one considers a conformal immersion

$$f : M \to S^3 \subset \mathbb{R}^4$$

of a Riemann surface with complex coordinate z, then one has $\langle f, f \rangle = 1$ and

$$\langle f_z, N \rangle = \langle f_{\bar{z}}, N \rangle = \langle f, N \rangle = 0, \quad \langle N, N \rangle = 1,$$

where N is a unit normal of f in S^3. Then the frame $\mathcal{F} = (f, f_z, f_{\bar{z}}, N)$ satisfies the following conditions.

Proposition 2.41. The frame $\mathcal{F} = (f, f_z, f_{\bar{z}}, N)$ satisfies

$$\mathcal{F}_z = \mathcal{F}\mathcal{U}, \quad \mathcal{F}_{\bar{z}} = \mathcal{F}\mathcal{V},$$

where

$$\mathcal{U} = \begin{pmatrix} 0 & 0 & -2e^{2u} & 0 \\ 1 & 2u_z & 0 & -H \\ 0 & 0 & 0 & -\frac{1}{2}Qe^{-2u} \\ 0 & Q & 2He^{2u} & 0 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} 0 & -2e^{2u} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\overline{Q}e^{-2u} \\ 1 & 0 & 2u_{\bar{z}} & -H \\ 0 & 2He^{2u} & \overline{Q} & 0 \end{pmatrix}.$$

Proof. Since the immersion is conformal we have

$$\langle f_z, N\rangle = \langle f_{\bar{z}}, N\rangle = \langle f, N\rangle = \langle f_z, f_z\rangle = \langle f_{\bar{z}}, f_{\bar{z}}\rangle = 0, \quad \langle N, N\rangle = 1$$

and $\langle f, f\rangle = 1$. In addition one has $\langle f_z, f_{\bar{z}}\rangle = 2e^{2u}$. Therefore $\mathcal{F} = (f, f_z, f_{\bar{z}}, N)$ indeed is a framing and after normalization one obtains

$$\begin{aligned}
f_z &= \langle f_z, f\rangle \frac{f}{2e^{2u}} + \langle f_z, f_{\bar{z}}\rangle \frac{f_z}{2e^{2u}} + \langle f_z, f_z\rangle \frac{f_{\bar{z}}}{2e^{2u}} + \langle f_z, N\rangle N \\
f_{\bar{z}} &= \langle f_{\bar{z}}, f\rangle \frac{f}{2e^{2u}} + \langle f_{\bar{z}}, f_{\bar{z}}\rangle \frac{f_z}{2e^{2u}} + \langle f_{\bar{z}}, f_z\rangle \frac{f_{\bar{z}}}{2e^{2u}} + \langle f_{\bar{z}\bar{z}}, N\rangle N \\
f_{zz} &= \langle f_{zz}, f\rangle \frac{f}{2e^{2u}} + \langle f_{zz}, f_{\bar{z}}\rangle \frac{f_z}{2e^{2u}} + \langle f_{zz}, f_z\rangle \frac{f_{\bar{z}}}{2e^{2u}} + \langle f_{zz}, N\rangle N \\
f_{z\bar{z}} &= \langle f_{z\bar{z}}, f\rangle \frac{f}{2e^{2u}} + \langle f_{z\bar{z}}, f_{\bar{z}}\rangle \frac{f_z}{2e^{2u}} + \langle f_{z\bar{z}}, f_z\rangle \frac{f_{\bar{z}}}{2e^{2u}} + \langle f_{z\bar{z}}, N\rangle N \\
f_{\bar{z}\bar{z}} &= \langle f_{\bar{z}\bar{z}}, f\rangle \frac{f}{2e^{2u}} + \langle f_{\bar{z}\bar{z}}, f_{\bar{z}}\rangle \frac{f_z}{2e^{2u}} + \langle f_{\bar{z}\bar{z}}, f_z\rangle \frac{f_{\bar{z}}}{2e^{2u}} + \langle f_{\bar{z}\bar{z}}, N\rangle N \\
N_z &= \langle N_z, f\rangle \frac{f}{2e^{2u}} + \langle N_z, f_{\bar{z}}\rangle \frac{f_z}{2e^{2u}} + \langle N_z, f_z\rangle \frac{f_{\bar{z}}}{2e^{2u}} + \langle N_z, N\rangle N \\
N_{\bar{z}} &= \langle N_{\bar{z}}, f\rangle \frac{f}{2e^{2u}} + \langle N_{\bar{z}}, f_{\bar{z}}\rangle \frac{f_z}{2e^{2u}} + \langle N_{\bar{z}}, f_z\rangle \frac{f_{\bar{z}}}{2e^{2u}} + \langle N_{\bar{z}}, N\rangle N
\end{aligned}$$

Recall that the Hopf differential Q and the mean curvature H are defined by

$$Q = \langle f_{zz}, N\rangle, \quad 2He^{2u} = \langle f_{z\bar{z}}, N\rangle.$$

Differentiating the equation

$$\langle f_z, f_{\bar{z}}\rangle = 2e^{2u}$$

one obtains

$$\langle f_{zz}, f_{\bar{z}}\rangle = 2e^{2u}u_z.$$

Equipped with all these equations one can directly check, that the matrices $\mathcal{U}, \mathcal{V}$ are of the form stated above. $\square$

Theorem 2.42. *Let $D \subset \mathbb{C}^2$ be an open simply connected set containing $(0,0)$. For $\mathcal{U},\mathcal{V} : D \to \mathfrak{sl}(n,\mathbb{C})$ there exists a solution $F = F(z,\bar{z}) : D \to SL(n,\mathbb{C})$ of the Lax Pair*

$$F_z = F\mathcal{U}, \quad F_{\bar{z}} = F\mathcal{V}$$

for any initial condition $F(0,0) \in SL(n,\mathbb{C})$ if and only if

$$\mathcal{U}_{\bar{z}} - \mathcal{V}_z + [\mathcal{V},\mathcal{U}] = 0 \quad \textit{with } [\mathcal{U},\mathcal{V}] = \mathcal{U}\mathcal{V} - \mathcal{V}\mathcal{U}.$$

Proof. Assume there exists an invertible solution $F(z,\bar{z})$. Since $F_{z\bar{z}} = F_{\bar{z}z}$ one obtains

$$0 = F_{z\bar{z}} - F_{\bar{z}z} = F\mathcal{U}_{\bar{z}} - F\mathcal{V}_z + F_{\bar{z}}\mathcal{U} - F_z\mathcal{V}$$

and therefore

$$0 = F\mathcal{U}_{\bar{z}} - F\mathcal{V}_z + F\mathcal{V}\mathcal{U} - F\mathcal{U}\mathcal{V}.$$

Thus

$$\mathcal{U}_{\bar{z}} - \mathcal{V}_z + [\mathcal{V},\mathcal{U}] = 0$$

must hold.

Now suppose that $\mathcal{U}_{\bar{z}} - \mathcal{V}_z + [\mathcal{V},\mathcal{U}] = 0$ holds. Reworking this into the coordinates (x,y) we get

$$U_x + iU_y - V_x + iV_y - 2[U,V] = 0.$$

Then we can solve the ordinary differential equation

$$(F(x,0))_x = F(x,0)(\mathcal{U}+\mathcal{V})(x,0)$$

with initial condition $F(0,0)$. For each fixed x_0 it remains to solve

$$(F(x_0,y))_y = F(x_0,y)i(\mathcal{U}-\mathcal{V})(x_0,y)$$

with initial condition $F(x_0,0)$. Hence $F(x,y)$ is defined and we have
$F_y = Fi(\mathcal{U}-\mathcal{V})$ for all x,y.
Since

$$(F_x - F(\mathcal{U}+\mathcal{V}))(x,y) = 0$$

if $y = 0$ and $F_{xy} = F_{yx}$, we have

$$\begin{aligned}
(F_x - F(\mathcal{U}+\mathcal{V}))_y &= F_{xy} - F_y(\mathcal{U}+\mathcal{V}) - F(\mathcal{U}_y+\mathcal{V}_y) \\
&= (Fi(\mathcal{U}-\mathcal{V}))_x - F_y(\mathcal{U}+\mathcal{V}) - F(\mathcal{U}_y+\mathcal{V}_y) \\
&= F_x i(\mathcal{U}-\mathcal{V}) + Fi(\mathcal{U}_x-\mathcal{V}_x) - F_y(\mathcal{U}+\mathcal{V}) - F(\mathcal{U}_y+\mathcal{V}_y) \\
&= F_x i(\mathcal{U}-\mathcal{V}) + Fi(2[U,V]) - F_y(\mathcal{U}+\mathcal{V}) \\
&= F_x i(\mathcal{U}-\mathcal{V}) + Fi(2[U,V]) - Fi(\mathcal{U}-\mathcal{V})(\mathcal{U}+\mathcal{V}) \\
&= (F_x - F(\mathcal{U}+\mathcal{V}))i(\mathcal{U}-\mathcal{V}).
\end{aligned}$$

Set $G = F_x - F(\mathcal{U} + \mathcal{V})$. G is a solution of $G_y = Gi(\mathcal{U} - \mathcal{V})$ with initial condition 0. By the uniqueness of the solution G is zero and therefore $F_x - F(\mathcal{U} + \mathcal{V}) \equiv 0$. Hence F is a solution to the Lax pair, since

$$F_z = \frac{1}{2}(F_x - iF_y) = F\mathcal{U}, \quad F_{\bar{z}} = \frac{1}{2}(F_x + iF_y) = F\mathcal{V}.$$

Considering

$$\det(F) \cdot \operatorname{tr}(F^{-1}F_z) = (\det(F))_z \text{ and } \det(F) \cdot \operatorname{tr}(F^{-1}F_w) = (\det(F))_w$$

with $\mathcal{U}, \mathcal{V} \in \mathfrak{sl}(n, \mathbb{C})$ we have

$$(\det(F))_z = (\det(F))_w = 0$$

and it follows $\det(F) = 1$ because $\det(F(0,0)) = 1$. □

The matrices $\mathcal{U}, \mathcal{V}$ obey the compatibility condition

$$\mathcal{U}_{\bar{z}} - \mathcal{V}_z - [\mathcal{U}, \mathcal{V}] = 0,$$

which implies that

$$2u_{z\bar{z}} + 2e^{2u}(1 + H^2) - \frac{1}{2}Q\overline{Q}e^{-2u} = 0, \quad Q_{\bar{z}} = 2H_z e^{2u}.$$

These are precisely the Gauss equation and Codazzi equation, respectively. Following this ansatz makes it possible to apply methods from integrable systems theory, provided that the Hopf differential is constant.

We want to take another point of view and will treat the Gauss and Codazzi equations as a zero-curvature condition.

For this purpose we identify the 3-sphere $S^3 \subset \mathbb{R}^4$ with

$$S^3 \cong (SU(2) \times SU(2))/D,$$

where D is the diagonal in $SU(2) \times SU(2)$. The Lie algebra of the matrix Lie group $SU(2)$ is $\mathfrak{su}(2)$, equipped with the commutator $[\cdot, \cdot]$.
The Maurer-Cartan form $\theta : TSU(2) \to \mathfrak{su}(2)$ satisfies the Maurer-Cartan equation

$$2d\theta + [\theta \wedge \theta] = 0.$$

For a map $F : \mathbb{R}^2 \to SU(2)$, the pullback $\alpha = F^*\theta$ also satisfies the above equation and conversely, every solution $\alpha \in \Omega^1(\mathbb{R}^2, \mathfrak{su}(2))$ of the above equation integrates to a smooth map $F : \mathbb{R}^2 \to SU(2)$ with $\alpha = F^*\theta$.
Setting $\alpha = Udz + Vd\bar{z}$ one obtains the Gauss and Codazzi equations from the Maurer-Cartan equation for $U, V \in SU(2)$. Later we will see how to rework our Lax pair $\mathcal{U}, \mathcal{V}$ into the $SU(2)$-setting.

If one thinks of θ as a connection form, $d\theta + \theta \wedge \theta = d\theta + \frac{1}{2}[\theta \wedge \theta]$ is the corresponding curvature form. Thus the Maurer-Cartan equation is a *zero curvature condition.*

Chapter 3

Construction of Lawson's surfaces

We now want to study a construction procedure proposed by H. Blaine Lawson that generates for every non-negative integer g a minimal embedding of a compact orientable surface of genus g into S^3.
Before stating this procedure we first introduce what Lawson calls the "reflection principle".

3.1 The reflection principle

Let γ be the geodesic in S^3 given by $x_3 = x_4 = 0$, and let $\mathbf{S}$ be the great 2-sphere given by $x_4 = 0$.

Definition 3.1. Define the *geodesic reflection across* γ via the map $r_\gamma : S^3 \to S^3$ where

$$r_\gamma(x_1, x_2, x_3, x_4) = (x_1, x_2, -x_3, -x_4)$$

and analogously *geodesic reflection across* $\mathbf{S}$ via the map $r_{\mathbf{S}} : S^3 \to S^3$ where

$$r_{\mathbf{S}}(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, -x_4).$$

Remark 3.2.

1. These maps can be interpreted as sending a point p to its "opposite" point on a geodesic through p which meets γ (or $\mathbf{S}$) orthogonally.

2. It is clear that geodesic reflection across an arbitrary geodesic ι is obtained by conjugation with a rotation ϕ that maps ι into γ:

$$\iota = \phi^{-1} \circ \gamma \circ \phi.$$

For purposes that will become clear later we state the two following propositions due to Lawson.

Proposition 3.3. Let $f : M \to \mathbb{R}^n$ be an immersion and let H be the mean curvature vector field of f. Then

$$\Delta f = H,$$

where $\Delta f = (\Delta f_1, \ldots, \Delta f_n)$.

Proof. Let $p \in M$ and choose pointwise orthonormal vector fields $E_1, \ldots, E_m$ on M. Then we have

$$E_i E_i f = \overline{\nabla}_{E_i} E_i$$

with $\overline{\nabla}$ being the euclidean connection. Thus we have

$$\begin{aligned}
\Delta f &= \sum_{i=1}^{m} \nabla^2_{E_i,E_i} f = \sum_{i=1}^{m} (\nabla_{E_i}\nabla_{E_i} f - \nabla_{(\nabla_{E_i} E_i)} f) \\
&= \sum_{i=1}^{m} (E_i E_i f - \nabla_{E_i} E_i f) = \sum_{i=1}^{m} (\overline{\nabla}_{E_i} E_i - \nabla_{E_i} E_i) \\
&= \sum_{i=1}^{m} (\overline{\nabla}_{E_i} E_i)^{\perp} = H.
\end{aligned}$$

□

Proposition 3.4. Let M be a Riemannian m-manifold and let $f : M \to S^n \subset \mathbb{R}^{n+1}$ be an almost conformal minimal immersion, i.e. a mapping that fails to be an immersion at isolated points. Then

$$\Delta f = -\langle \nabla f, \nabla f \rangle f.$$

Proof. First we observe that for the mean curvature vector fields H^* in S^n and H in $\mathbb{R}^{n+1}$ we have

$$H^* = \sum_{i=1}^{m} (\nabla^*_{E_i} E_i)^{\perp} = \sum_{i=1}^{m} ((\overline{\nabla}_{E_i} E_i)^{\top})^{\perp} = \sum_{i=1}^{m} ((\overline{\nabla}_{E_i} E_i)^{\perp})^{\top} = (H)^{\top},$$

where $\nabla^*, \overline{\nabla}$ are the connections on S^n and $\mathbb{R}^{n+1}$, respectively. Since f is an almost conformal minimal immersion, $\Delta f(p)$ is parallel to the normal to S^n almost everywhere, i.e.

$$\Delta f = \lambda f, \quad \lambda \in C^{\infty}(M).$$

Since $\langle f, f \rangle = |f|^2 = 1$ one obtains

$$\begin{aligned}
0 &= \frac{1}{2}\Delta |f|^2 = \frac{1}{2}\sum_{i=1}^{m} E_i E_i \langle f, f \rangle = \sum_{i=1}^{m} E_i(\langle f, E_i f \rangle) \\
&= \sum_{i=1}^{m} (\langle f, E_i E_i f \rangle + \langle E_i f, E_i f \rangle) = \langle f, \Delta f \rangle + \langle \nabla f, \nabla f \rangle \\
&= \lambda \langle f, f \rangle + \langle \nabla f, \nabla f \rangle = \lambda + \langle \nabla f, \nabla f \rangle
\end{aligned}$$

and thus $\lambda = -\langle \nabla f, \nabla f \rangle$. □

Proposition 3.5. Let M be a minimal surface with C^2-boundary ∂M. If ∂M contains a geodesic arc γ (in S^3), M can be continued as an analytic minimal surface across each non-trivial component of $\partial M \cap \gamma$ by geodesic reflection.

Proof. Let w.l.o.g. γ be given by $x_3 = x_4 = 0$ and choose p in the interior of $\partial M \cap \gamma$. Since ∂M is of class C^2 it is of class C^1_1, i.e. the immersion is μ-Hölder continuous with $\mu = 1$. We can now apply [28, Thm. 9.3.1] to obtain a conformal map

$$\hat{f} : \overline{\mathbb{D}} \to S^3 \subset \mathbb{R}^4.$$

$\hat{f}$ is a regular representation of M (and of ∂M) in a neighborhood of p with $\overline{\mathbb{D}} := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Due to [28, Thm. 9.3.1] we may also set

$$f(0, 0) = p,\ f_3(x, 0) = f_4(x, 0) = 0.$$

Via the Riemann mapping theorem we can transform $\hat{f}$ to $f : \overline{\mathbb{D}}^+ \to S^3$ defined on the upper half disk $\overline{\mathbb{D}}^+ := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1, y \geq 0\}$. Since f is minimal it follows from proposition 3.4 that

$$\Delta f = -\langle \nabla f, \nabla f \rangle f$$

over $\overline{\mathbb{D}}^+$ (Δ is the Laplace-Beltrami operator for the induced metric). Now we extend f to the entire unit disk by setting (for the lower half disk)

$$f_k(x, y) = (-1)^{[k/3]} f_k(x, -y) \text{ for } k = 1, \ldots, 4.$$

For each k we have $f_k \in C(\overline{\mathbb{D}})$. The only interesting points here are those determined by $y = 0$. If we apply the minimal surface equation from proposition 3.4 we obtain (for $k = 3, 4$)

$$\begin{aligned} \left(\frac{\partial^2}{\partial x^2} f_k + \frac{\partial^2}{\partial y^2} f_k \right)(x, 0) &= \left(-\frac{\partial^2}{\partial x^2} f_k - \frac{\partial^2}{\partial y^2} f_k \right)(x, 0) \\ &\Leftrightarrow\ 2 \cdot \Delta f_k(x, 0) = 0, \end{aligned}$$

hence we see that the second partial derivatives with respect to x and y agree on $y = 0$ since $f_k(x, 0) = 0$ for $k = 3, 4$. Thus the first derivatives $\frac{\partial}{\partial x} f_k$ and $\frac{\partial}{\partial y} f_k$ coincide on $y = 0$ for $k = 3, 4$ as well. In particular we have in this situation

$$\frac{\partial}{\partial y} f_3 = \frac{\partial}{\partial y} f_4 = 0.$$

The second partial mixed derivatives obviously agree on $y = 0$ and so we have $f_k \in C^2(\overline{\mathbb{D}})$ for $k = 3, 4$.
We also have $\frac{\partial}{\partial x} f_k$, $\frac{\partial^2}{\partial x^2} f_k$ and (by the chain-rule) $\frac{\partial^2}{\partial y^2} f_k \in C(\overline{\mathbb{D}})$ for $k = 1, 2$. We will now show that $\frac{\partial}{\partial y} f_1 = \frac{\partial}{\partial y} f_2 = 0$ on $y = 0$. Since $|f|^2 \equiv 1$ we have

$$\langle f, \frac{\partial}{\partial x} f \rangle = f_1 \cdot \frac{\partial}{\partial x} f_1 + f_2 \cdot \frac{\partial}{\partial x} f_2 = 0 = \langle f, \frac{\partial}{\partial y} f \rangle = f_1 \cdot \frac{\partial}{\partial y} f_1 + f_2 \cdot \frac{\partial}{\partial y} f_2$$

on $y = 0$. Furthermore we have

$$\left(\frac{\partial}{\partial x} f_1\right)^2 + \left(\frac{\partial}{\partial x} f_2\right)^2 > 0$$

on $y = 0$ and thus applying the derivative with respect to x to the equation above yields $\frac{\partial}{\partial y} f_k = \frac{\partial^2}{\partial x \partial y} f_k = 0$ on $y = 0$ for $k = 1, 2$. It follows that f satisfies the minimality condition stated in lemma 2.14 and therefore by [26, Lemma 1.1] it is analytic in $\overline{\mathbb{D}}$. □

3.2 The construction procedure

We now want to apply the methods developed above in order to construct complete, non-singular minimal surfaces in S^3. Roughly speaking the whole procedure boils down to the solution of the Plateau Problem (see below) for a given polygon Γ serving as boundary for the desired surface. By choosing a special polygon Γ obeying some conditions we can extend the resulting surface by geodesic reflection to obtain a family of compact orientable surfaces of genus g in S^3. We shall start with some terminology.

Definition 3.6. For two distinct geodesics γ and δ which meet in S^3, let $S(\gamma, \delta)$ be the unique 2-sphere containing $\gamma \cup \delta$.
A subset $X \subset S^3$ is **bounded** by $S(\gamma, \delta)$ if X is contained in one of the two closed hemispheres determined by $S(\gamma, \delta)$.

Definition 3.7. By a **geodesic polygon** in S^3 we mean a polygon whose edges are geodesics $\gamma_0, \gamma_1, \dots, \gamma_n = \gamma_0$ having vertices $v_0, v_1, \dots, v_n = v_0$ such that for each $1 \leq i \leq n$, γ_i meets γ_{i-1} in v_i at an angle of the form $\frac{\pi}{k_i+1}$ where k_i is a positive integer.
For each i we denote by N_i the geodesic perpendicular to $S(\gamma_{i-1}, \gamma_i)$ at v_i.

Definition 3.8. A geodesic polygon Γ is called **proper** if for each i, it is bounded either by $S(\gamma_{i-1}, N_i)$ or by $S(\gamma_i, N_i)$.

Before stating the necessary conditions for the special choice of the geodesic polygon Γ we have to make another definition.

Definition 3.9. Define the **convex hull** of Γ by

$$\mathcal{C}(\Gamma) = \bigcap \{H \mid H \text{ is a closed hemisphere containing } \Gamma\}.$$

If $\Gamma \subset \partial \mathcal{C}(\Gamma)$, Γ is called **convex**. Now set

$$\begin{aligned} \mathcal{S}_\Gamma \quad = \quad & \{S \mid S \text{ is a geodesic 2-sphere in } S^3 \text{ such that} \\ & S \cap \Gamma \text{ has at least four components}\}. \end{aligned}$$

The polygon Γ is now assumed to be a proper, convex curve satisfying the following:

(A) Γ lies in an open hemisphere of S^3.

(B) For each $p \in \mathcal{C}(\Gamma)^o$ there is a geodesic 2-sphere S_p containing p such that $S_p \notin \mathcal{S}_\Gamma$.

(C) Whenever one of the pair $S(\gamma_{i-1}, N_i)$, $S(\gamma_i, N_i)$ fails to bound Γ, we have $k_i = 1$.

(D) There exists a continuous map $\pi : \mathcal{C}(\Gamma) \to \overline{\mathbb{D}}$ which is differentiable in $\mathcal{C}(\Gamma)^o$ and carries Γ monotonically onto $\partial\overline{\mathbb{D}}$ such that for each $S \in \mathcal{S}_\Gamma$ the differential of the map $\pi|S \cap \mathcal{C}(\Gamma)^o$ is everywhere of rank 2.

Definition 3.10. Let

$$X_\Gamma = \{f : \overline{\mathbb{D}} \to S^3 \mid f \text{ is piecewise } C^1 \text{ and } f|\partial\overline{\mathbb{D}} \text{ is a monotone parametrization of } \Gamma\}$$

and define the **area function** $A : X_\Gamma \to \mathbb{R}^+ \cup \{\infty\}$ by the following integral:

$$A(f) = \iint_{\overline{\mathbb{D}}} |f_x \wedge f_y| dxdy$$

where $|f_x \wedge f_y|^2 = |f_x|^2|f_y|^2 - \langle f_x, f_y\rangle^2$.

Definition 3.11. Define the **Dirichlet integral** by

$$D(f) = \iint_{\overline{\mathbb{D}}} \left(|f_x|^2 + |f_y|^2\right) dxdy.$$

Remark 3.12.

1. We observe that

$$|f_x \wedge f_y|^2 = |f_x|^2|f_y|^2 - \langle f_x, f_y\rangle^2 \leq |f_x|^2|f_y|^2 \leq \frac{1}{4}\left(|f_x|^2 + |f_y|^2\right)^2$$

   where equality holds if and only if $|f_x| = |f_y|$ and $\langle f_x, f_y\rangle = 0$. Thus we have

$$A(f) \leq \frac{1}{2}D(f)$$

   where equality holds if and only if f is conformal almost everywhere in $\overline{\mathbb{D}}$.

2. In case of a conformal immersion f we have due to the isoperimetric inequality for minimal surfaces (see [8, pp. 129-131])

$$D(f) = 2A(f) \leq \frac{1}{2\pi}l(\Gamma)^2$$

   where $l(\Gamma)$ denotes the length of Γ. Thus the Dirichlet integral $D(f)$ always exists for such f.

The **Plateau Problem** for given Γ is now to find a $f \in X_\Gamma$ such that $A(f) = \mathcal{A}_\Gamma$ where

$$\mathcal{A}_\Gamma = \inf_{f \in X_\Gamma} A(f).$$

We want to state a theorem due to Morrey (see [28]) without proof that guarantees the existence of a solution to the Plateau Problem for Γ.

Theorem 3.13 (Morrey)**.** *Let Γ be a geodesic polygon as constructed above. Then there exists a continuous immersion*

$$f : \overline{\mathbb{D}} \to S^3$$

that is analytic and almost conformal in $\overline{\mathbb{D}}^o$ and minimizes the Dirichlet and area integral among all maps in $C(\overline{\mathbb{D}}, S^3) \cap H_2^1(\overline{\mathbb{D}}, S^3)$ which represent Γ on $\partial\overline{\mathbb{D}}$, i.e. $f(\partial\overline{\mathbb{D}}) = \Gamma$.

Now the *construction procedure* splits up into the following parts:

1. Let $f : \overline{\mathbb{D}} \to S^3$ represent Morrey's solution to the Plateau Problem for Γ and set $\mathcal{M}_\Gamma = f(\overline{\mathbb{D}})$.

2. The surface $\mathcal{M}_\Gamma$ can be analytically continued as a non-singular minimal surface across each of its boundary arcs $\gamma_0, \ldots, \gamma_n$ by geodesic reflection.

3. If G_Γ denotes the subgroup of $O(4)$ generated by the reflections across the boundary arcs then

$$M_\Gamma = \bigcup_{g \in G_\Gamma} g(\mathcal{M}_\Gamma)$$

   is a complete, non-singular submanifold. If G_Γ is finite, M_Γ is compact.

Theorem 3.14 (Lawson)**.** *The immersion $f : \overline{\mathbb{D}} \to S^3$ from the above theorem fulfills*

$$f((\overline{\mathbb{D}})^o) \subset \mathcal{C}(\Gamma)^o.$$

Proof. This follows immediately from [25, Thm. 2] and $f(\partial\overline{\mathbb{D}}) = \Gamma$. □

Theorem 3.15 (Lawson)**.** *The immersion $f : \overline{\mathbb{D}} \to S^3$ is non-singular and one-to-one in $(\overline{\mathbb{D}})^o = \mathbb{D}$ and thus f conformally embeds $(\overline{\mathbb{D}})^o = \mathbb{D}$ into $\mathcal{C}(\Gamma)^o \subset S^3$.*

Proof. Due to the previous theorem we have $f(\mathbb{D}) \subset \mathcal{C}(\Gamma)^o$. Take a point $p \in \mathbb{D}$, then $f(p) \in \mathcal{C}(\Gamma)^o$. Condition B implies that there exists a geodesic 2-sphere S_p with $f(p) \in S_p$ such that $S_p \notin \mathcal{S}_\Gamma$, i.e. $S_p \cap \Gamma$ has at most three components. We may now apply [25, Thm. 4 (a)] to deduce that $\mathrm{rank}(df|_p) = 2$, thus f is non-singular in $\mathbb{D}$.

The second part of the assertion is gained by combining condition D and Theorem 4 in [25]: Since we are considering an immerson into S^3 we have to set $n = 3$ in [25, Thm. 4 (d)] and thus there exists at least one geodesic hypersphere S, containing $f(p)$, such that $\Gamma \cap S$ has at least $2\frac{n+1}{2} = 4$ components, i.e. $S \in \mathcal{S}_\Gamma$.
Now suppose there exists some $p \in \mathbb{D}$ such that $\pi(f(p)) \neq p$ and therefore $\mathrm{rank}(d\pi|_p) < 2$. Due to condition (D) there is no $S \in \mathcal{S}_\Gamma$ with $f(p) \in S$. But this clearly contradicts the above statement and we see that f is one-to-one in $\mathbb{D}$. □

We need to check that f is analytic on $\partial\overline{\mathbb{D}}$ except at isolated points. The fact that f is one-to-one is based on the finiteness of $D(f)$ as the following lemma shows.

Lemma 3.16. The immersion $f : \overline{\mathbb{D}} \to S^3$ is one-to-one on $\partial\overline{\mathbb{D}}$ and analytic at each point of the boundary which is mapped to the interior of an analytic sub-arc of Γ.

Proof. Consider a point p on $\partial\overline{\mathbb{D}}$ and the Dirichlet integral over a "wedge" Δ_ε in the interior given in polar coordinates r, θ about p such that $r \leq 2\varepsilon$ for a small ε.

$$M(2\varepsilon) := D(f, \Delta) = 2 \cdot A(f, \Delta) = \int_{r \leq 2\varepsilon} \int_\theta \left(|f_r|^2 r + \frac{1}{r} |f_\theta|^2 \right) dr d\theta.$$

This integral represents twice the area of the image of this region; hence it exists and $\lim_{\varepsilon \to 0} M(2\varepsilon) = 0$. Furthermore for $r \leq \rho < 2\varepsilon$ we have

$$\iint_{\Delta_\rho} \left(|f_r|^2 r + \frac{1}{r} |f_\theta|^2 \right) dr d\theta \leq M(2\varepsilon).$$

Now set $s = r \cdot \theta$ with $ds = r d\theta$. The last inequality then becomes

$$\begin{aligned} M(2\varepsilon) &= \iint_{\Delta_\varepsilon} \left(|f_r|^2 + |f_s|^2 \right) dr ds \geq \int_0^{2\varepsilon} \int_0^{2\pi r} |f_s|^2 ds dr \\ &\geq \int_\varepsilon^{2\varepsilon} \int_0^{2\pi r} |f_s|^2 ds dr = \int_\varepsilon^{2\varepsilon} p(r) dr \end{aligned}$$

where $p(r) = \int_0^{2\pi r} |f_s|^2 ds$. By the mean value theorem, there exists a value $\rho = \rho_0$ in the range $\varepsilon \leq \rho \leq 2\varepsilon$ such that

$$\int_\varepsilon^{2\varepsilon} p(r) dr = p(\rho) \int_\varepsilon^{2\varepsilon} dr = p(\rho) \cdot \varepsilon = \varepsilon \int_0^{2\pi\rho} |f_s|^2 ds \leq M(2\varepsilon)$$

and therefore $\int_0^{2\pi\rho} |f_s|^2 ds \leq \frac{M(2\varepsilon)}{\varepsilon}$. Now we consider a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ with $\lim_{n \to \infty} \varepsilon_n = 0$. To this sequence corresponds a sequence of radii ρ_n

tending to zero, for which the above inequality is satisfied. We now want to estimate the length l_n along an arc given by a fixed ρ_n.

Applying Schwarz' inequality we get ($\rho_n \leq 2\epsilon$)

$$l_n^2 = \left(\int_0^{2\pi\rho_n} \sqrt{|f_s|^2 ds}\right)^2 \leq 2\pi\rho_n \frac{M(2\varepsilon)}{\varepsilon} \leq 4\pi M(2\varepsilon).$$

Hence l_n tends to zero as $\rho_n \to 0$, and therefore the distance between the endpoints of an arc $\gamma \subset \Gamma$ tends to zero. Thus the diameter of this arc tends to zero and the first assertion is proved.

The second claim follows from [13, Thm. 9.3]. □

We have seen that f is analytic on $\partial\overline{\mathbb{D}}$ away from the points corresponding to the vertices of Γ. We now want to investigate the behaviour of f at these points. For a fixed i, $1 \leq i \leq n$ let

$$f^{-1}[\gamma_i] =: \delta_i \subset \partial\overline{\mathbb{D}}$$

be the pre-image of an arc $\gamma_i \subset \Gamma$. By a conformal mapping we may carry f into the upper half disk such that δ_i corresponds to the arc $y = 0$. Since f is analytic and one-to-one on δ_i proposition 3.5 shows that reflection across δ_i defines an analytic continuation of f throughout all of $\overline{\mathbb{D}}$.

Lemma 3.17. For a fixed i there are no points in the interior of δ_i where $|\nabla f| = 0$.

Proof. Fix a point in the interior of δ_i, say $p = (x, 0) \in \delta_i^o$, and choose a small disk $B_\varepsilon(p) \subset \mathbb{D}$ around p. Since Γ is convex for each i there is a geodesic 2-sphere $S \supset \gamma_i$ which divides S^3 into hemispheres H^+ and H^- such that $f(\overline{\mathbb{D}}^+) \subset H^+$.
Applying [25, Thm. 2] yields

$$f(int(\overline{\mathbb{D}}^+)) \subset int(H^+) \text{ and } f(int(\overline{\mathbb{D}}^-)) \subset int(H^-).$$

Thus $f(\partial B_\varepsilon(p)) \cap S$ consists of exactly two points. Restricting f to $B_\varepsilon(p)$ one gets a surface

$$f|_{B_\varepsilon(p)} : B_\varepsilon(p) \to S^3$$

and therefore applying [25, Thm. 4 (a)] shows that $|\nabla f| \neq 0$ at $p = (x, 0)$. □

If we set $\mathcal{M}_\Gamma = f(\overline{\mathbb{D}})$ for a solution $f : \overline{\mathbb{D}} \to S^3$ to the Plateau Problem for Γ we can now smoothly continue the surface $\mathcal{M}_\Gamma$ across each of its boundary arcs by geodesic reflection. Reflecting this surface successively $2k_i + 2$ times around a vertex v_i (with two arcs γ_i and γ_{i-1} intersecting at an angle of

$\frac{\pi}{k_i+1}$) yields the original surface $\mathcal{M}_\Gamma$. Hence we get a smooth surface near v_i that may be singular at v_i itself.

We will now see that the resulting surface $\mathcal{M}^*$ is smooth and non-singular at v_i.

Lemma 3.18 (Lawson)**.** The surface $\mathcal{M}^*$ is smooth at v_i.

Proof. We only give a sketch of the proof of this lemma. First we observe that if we reflect $\mathcal{M}_\Gamma$ k_i-times at v_i we get a surface $\mathcal{M}^+$ which is bounded near v_i by an unbroken geodesic arc γ. Since S^3 is a compact Riemannian manifold it is geodesically complete and therefore γ is just the continuation of the arc γ_i across v_i.
Reflection of $\mathcal{M}^+$ thus yields the total surface $\mathcal{M}^*$. Let $f^+ : \overline{\mathbb{D}}^+ \to S^3$ be a conformal, smooth parametrization of $\mathcal{M}^+$ with

$$f^+\left([-1,1] \times 0\right) = \gamma \text{ and } f^+(0,0) = v_i.$$

Due to our previous considerations f^+ is smooth on $[-1,1]$ except possibly at $(0,0)$ and we may extend f^+ to a map $f^* : \overline{\mathbb{D}} \to S^3$ by reflection across γ. For any domain $D \subset \mathcal{M}^*$ we have the rough inequality

$$A(D) \leq \frac{2k_i+2}{4\pi} l(\partial D)^2$$

resulting from the isoperimetric inequality for $\mathcal{M}_\Gamma$. Let C be a system of local coordinates for S^3 obtained by stereographic projection from the point $-v_i$. The metric in these coordinates has the form

$$ds^2 = \frac{4}{(1+|X|^2)^2}|dX|^2,$$

where $X = (X_1, X_2, X_3)$ and $|X|$ denotes the euclidean norm. The Dirichlet integral for any S^3-valued function Φ defined in a plane domain D represented in such coordinates is

$$D(\Phi, D) = \iint_D \frac{4}{(1+|\Phi|^2)^2}|\nabla\Phi|^2 dxdy.$$

For $f^*(\overline{\mathbb{D}}) = \mathcal{M}^*$ we can find constants K and μ (see [13, §4]), independent of r and R, with $0 < \mu < 1$ such that

(1) f^* is μ-Hölder continuous in $\mathbb{D}$

(2) For any $p \in \mathbb{D}$ and any r, R with $0 < r \leq R$ one has

$$D(f^*, B_r(p)) \leq K \left(\frac{r}{R}\right)^\mu D(f^*, B_R(p)),$$

where $B_r(p) = \{z \in \overline{\mathbb{D}} \mid |p - z| < r\}$.
Over domains in $\overline{\mathbb{D}}$ which parameterize domains on $\mathcal{M}_\Gamma$ or one of its images, f^* minimizes the Dirichlet integral D. Since f^* is analytic except possibly at $(0,0)$, it represents a weak solution to the Euler-equation

$$\frac{\partial}{\partial x}\left(\frac{f_x^*}{(1+|f^*|^2)^2}\right) + \frac{\partial}{\partial y}\left(\frac{f_y^*}{(1+|f^*|^2)^2}\right) + 2\frac{|\nabla f_x^*|^2}{(1+|f^*|^2)^3} = 0.$$

However, this system satisfies the condition (1.10.8”) in [28, p.33] and combining (1) and (2) one can conclude that f^* is analytic at $(0,0)$. □

Lemma 3.19 (Lawson)**.** We have $|\nabla f^*(0,0)| \neq 0$ and thus the non-singularity at v_i.

Proof. Choose a small disk $B_\varepsilon(0,0) \subset \overline{\mathbb{D}}$ such that

$$f^*\left(B_\varepsilon(0,0)\right) \subset H \text{ for an open hemisphere } H.$$

By [25, Thm. 3], we know that $|\nabla f^*(0,0)| \neq 0$ implies that for every geodesic 2-sphere S containing v_i the pre-image $(f^*|\partial B_\varepsilon(0,0))^{-1}(S)$ has at least four components. In order to prove the lemma we have to find S for which this set has only 2 points.

Suppose Γ is bounded by both $S(\gamma_{i-1}, N_i)$ and $S(\gamma_i, N_i)$. Then Γ lies in a region L that is determined by these hyperspheres. Moreover, $\mathcal{M}_\Gamma^o \subset L^o$. Observe now that there is a tessellation of S^3 by $2k_i + 2$ regions congruent to L each of which meets N_i. When $\mathcal{M}_\Gamma$ is reflected at v_i each distinct image lies in a different one of these regions (with its interior in the interior of the region). The surface $\mathcal{M}^*$ meets the interfaces of the regions in great circles which are parameterized one-to-one. It follows that if $S = S(\gamma_i, N_i)$, then $(f^*|\partial B_\varepsilon(0,0))^{-1}(S)$ consists of exactly two points.

Suppose on the other hand, that $S_0 := S(\gamma_i, N_i)$ bounds Γ and $k_i = 1$. Since Γ is convex, there exists a geodesic 2-sphere $S_1 \supset \gamma_{i-1}$ which also bounds Γ. S_0 and S_1 are perpendicular and separate S^3 into four disjoint, congruent domains. It is not difficult to see that the interiors of each of the four images of $\mathcal{M}_\Gamma$ reflected at v_i lie in different domains and that $f^*(B_\varepsilon(0,0))$ meets $S_0 \cup S_1$ in great circular arcs. It follows that S_0 has precisely two pre-images in $\partial B_\varepsilon(0,0)$ and the lemma is proved. □

We have seen that reflection across the boundary arcs of $\mathcal{M}_\Gamma$ produces a complete, non-singular submanifold in S^3, that will be denoted by M_Γ. Thus we obtain the following result.

Lemma 3.20. Let G_Γ denote the subgroup of $O(4)$ generated by the reflections $\{r_{\gamma_k}\}_{k=1,\ldots,n}$ corresponding to the boundary arcs $\gamma_1, \ldots, \gamma_n$. Then

$$M_\Gamma = \bigcup_{g \in G_\Gamma} g[\mathcal{M}_\Gamma],$$

and in particular M_Γ is compact if and only if G_Γ is finite.

Proof. For the first part, observe that one has a smooth action

$$G_\Gamma \times S^3 \to S^3,$$

where all elements $r_{\gamma_i} \in G_\Gamma$ are isometries and thus

$$M_\Gamma = G_\Gamma \cdot \mathcal{M}_\Gamma = \bigcup_{g \in G_\Gamma} g[\mathcal{M}_\Gamma].$$

For the second part of the claim we note that $\mathcal{M}_\Gamma$ is compact as an image of a compact set $\overline{\mathbb{D}}$ under the continuous mapping f. Since every $g \in G_\Gamma$ is an isometry it is continuous and thus $g[\mathcal{M}_\Gamma]$ is compact for every $g \in G_\Gamma$.

Let G_Γ be finite, then

$$M_\Gamma = \bigcup_{g \in G_\Gamma} g[\mathcal{M}_\Gamma]$$

is compact as union of finitely many compact sets.

On the other hand let M_Γ be compact and set

$$H_\Gamma = \{g \in G_\Gamma \mid g[\mathcal{M}_\Gamma] = \mathcal{M}_\Gamma\}.$$

Since H_Γ leaves $\mathcal{M}_\Gamma$ invariant it is a subgroup of the group of symmetries of Γ and thus H_Γ is finite. Each coset of H_Γ in G_Γ corresponds to a distinct image of $\mathcal{M}_\Gamma$ under G_Γ. These distinct images may intersect but do not coincide. Since M_Γ is compact we can calculate the volume of M_Γ in the following way

$$\mathrm{vol}(M_\Gamma) = \mathrm{vol}(\bigcup_{g \in G_\Gamma / H_\Gamma} g[\mathcal{M}_\Gamma]) = \frac{\mathrm{ord}(G_\Gamma)}{\mathrm{ord}(H_\Gamma)} \cdot \mathrm{vol}(\mathcal{M}_\Gamma).$$

From this equation we can deduce the finiteness of G_Γ. □

Summing up the preceding results we have the following.

Theorem 3.21 (Lawson)**.** *To each proper convex polygon Γ in S^3 having vertex angles of the type $\frac{\pi}{k+1}$, where k is a positive integer which depends on the vertex, and satisfying conditions (A), (B) and (C) there exists a complete, non-singular minimal submanifold $M_\Gamma \subset S^3$ with $\Gamma \subset M_\Gamma$. The surface M_Γ is compact if and only if the group G_Γ generated by reflections across the geodesic sub-arcs of Γ is finite. If Γ further satisfies condition (D), then the fundamental region $\mathcal{M}_\Gamma$, which has boundary Γ and generates M_Γ under G_Γ, has no self-intersections.*

3.3 The surfaces Σ_g

We now want to construct a family of surfaces proposed by Lawson, namely Σ_g and realize them as a 2-sheeted cover of $\mathbb{CP}^1$. Before we can do this we shall recall some facts about hyperelliptic Riemann surfaces.

Definition 3.22. Let M be a compact Riemann surface of genus $g \geq 1$. A point $p \in M$ is called **Weierstrass point**, if for a basis $\omega_1, \ldots, \omega_g$ of $\Omega(M)$ and a coordinate neighborhood (U, z) of p (with $\omega_k = f_k dz$ on U), the Wronskian determinant

$$W_z(\omega_1, \ldots, \omega_g) := W(f_1, \ldots, f_g) := \det \begin{pmatrix} f_1 & f_2 & \cdots & f_g \\ f_1' & f_2' & \cdots & f_g' \\ \vdots & \vdots & & \vdots \\ f_1^{(g-1)} & f_2^{(g-1)} & \cdots & f_g^{(g-1)} \end{pmatrix}$$

has a zero at p. The order of this zero is called the weight of the Weierstrass point.

Definition 3.23. A compact Riemann surface of genus $g \geq 1$ that admits a 2-sheeted covering of the Riemann sphere $f : M \to \mathbb{CP}^1$ is called **hyperelliptic**.

Remark 3.24.

1. f is non-constant with 2 poles and each ramification point has branch number 1. Moreover we have the relation

$$\sharp\{\text{branchpoints}\} = 2g + 2$$

   where g shall denote the genus of M.

2. Every surface of genus $g \leq 2$ is hyperelliptic.

Theorem 3.25 (Farkas, Kra). *Let M be a hyperelliptic surface of genus $g \geq 2$ and $f : M \to \mathbb{CP}^1$ the corresponding covering of $\mathbb{CP}^1$. Then the branch points of f are precisely the Weierstrass points of M and f is unique up to Möbiustransformations.*
The hyperelliptic surfaces of genus $g \geq 2$ are the only ones with precisely $2g + 2$ Weierstrass points.

Proof. See [9, III.7.3]. □

Definition 3.26. Let M be a Riemann surface. By $Aut(M)$ we denote the group of conformal automorphisms of M. Let $H \subset Aut(M)$ be a finite subgroup. For $p \in M$ set

$$H_p := \{h \in H \mid h(p) = p\}$$

Thus we may consider the natural projection

$$\pi : M \to M/H$$

and the Riemann-Hurwitz Theorem allows us to compute the genus of M/H in terms of the genus of M and the branch points of π (the fixed points of elements of H). We now have the following result.

Proposition 3.27 (Farkas, Kra)**.** Let M be a compact Riemann surface of genus g. Then M is hyperelliptic if and only if there exists a conformal involution $J \in Aut(M)$ (with $J^2 = 1$) on M that fixes $2g + 2$ points.

Proof. See [9, III.7.9]. □

Some immediate consequences of this result will be stated in the following remark.

Remark 3.28.

1. The projection $\pi : M \to M/\langle J \rangle$ is a 2-sheeted covering such that $M/\langle J \rangle$ has genus 0, thus M has a meromorphic function of degree 2.
2. If $g \geq 2$, then the fixed points of the hyperelliptic involution are the Weierstrass points.
3. If $g \geq 2$, then J is the unique involution with $2g + 2$ fixed points.
4. The hyperelliptic involution J on a (hyperelliptic) surface M of genus $g \geq 2$ is in the center of $Aut(M)$.

We now state some other useful results for calculating elements of $Aut(M)$.

Proposition 3.29. Let M be a hyperelliptic Riemann surface of genus $g \geq 2$. Let $T \in Aut(M)$ with $T \notin \langle J \rangle$, where J is the hyperelliptic involution. Then T has at most four fixed points.

Proof. See [9, III.7.11]. □

Corollary 3.30. If T fixes a Weierstrass point, then T has at most 2 other fixed points.

Proposition 3.31 (Farkas, Kra)**.** Let M be a compact Riemann surface of genus g. If $1 \neq T \in Aut(M)$, then T has at most $2g + 2$ fixed points.

Proof. See [9, V.1.1]. □

Proposition 3.32. Let $g \geq 2$ and let $W(M)$ be the set of Weierstrass points of M. Then for $T \in Aut(M)$ we have

$$T(W(M)) = W(M).$$

Corollary 3.33. There is a group homomorphism

$$\lambda : Aut(M) \to Perm(W(M)),$$

where $Perm(W(M))$ denotes the permutation group of the Weierstrass points on M. Furthermore, λ is injective unless M is hyperelliptic, in which case $Ker(\lambda) = \langle J \rangle$.

The following proposition will help us to obtain the anti-holomorphic involutions of the hyperelliptic curve Σ_g.

Proposition 3.34. Let M be a hyperelliptic Riemann surface of genus $g \geq 2$ and let T be an anti-conformal involution on M. Then T induces an anti-holomorphic involution on $\mathbb{CP}^1$ that leaves the fixed point set of T invariant.

Proof. Let $f : M \to \mathbb{CP}^1$ be a function with two poles on M. For T as above, $T \circ J \circ T^{-1}$ is a conformal involution with at least $2g+2$ fixed points. Thus one obtains

$$T \circ J \circ T^{-1} = J$$

and one gets an anti-conformal and hence anti-holomorphic involution A on $\mathbb{CP}^1$ with $f \circ T = A \circ f$. If now p is a fixed point of T, then

$$f(p) = f(T(p)) = A(f(p)),$$

and $f(p)$ is a fixed point of A. □

Via the identification

$$S^3 = \{(z,w) \in \mathbb{C} \times \mathbb{C} \mid |z|^2 + |w|^2 = 1\}$$

one may consider a set of coordinates (X_1, X_2, X_3) as a coordinate system for S^3 obtained by stereographic projection from the point $(z,w) = (0,-1)$. The resulting coordinates (X_1, X_2, X_3) have the form

$$(X_1, X_2, X_3) = \left(\frac{x_1}{1+x_3}, \frac{x_2}{1+x_3}, \frac{x_4}{1+x_3}\right)$$

where $(x_1, x_2, x_3, x_4) \in S^3 \backslash \{(0,0,-1,0)\}$.

In the construction procedure one has to choose 2 "sorts" of points acting as vertices of the geodesic polygon Γ. Lawson restricts to two distinguished great circles $C_1 = X_3$-axis and $C_2 := \{(X_1, X_2, 0) \mid X_1^2 + X_2^2 = 1\}$. The condition for C_2 reads

$$\frac{x_1^2}{(1+x_3)^2} + \frac{x_2^2}{(1+x_3)^2} = 1 \Leftrightarrow x_1^2 + x_2^2 = (1+x_3)^2,$$

and thus setting $x_3 = 0$ one obtains

$$C_2 \sim \{(z, 0) \in \mathbb{C}^2 \mid |z| = |x_1 + ix_2| = x_1^2 + x_2^2 = 1\}.$$

A similar consideration yields

$$C_1 \sim \{(0, w) \in \mathbb{C}^2 \mid |w| = 1\}.$$

In order to construct surfaces Σ_g of genus g we have to proceed in the following manner:

- Pick $P_1, P_2 \in C_1$ and $Q_1, Q_2 \in C_2$ such that $d(P_1, P_2) = \arccos\langle P_1, P_2\rangle = \frac{\pi}{2}$ and $d(Q_1, Q_2) = \frac{\pi}{g+1}$.
  We ***choose*** $P_1 = (0, 0, 1, 0),\ P_2 = (0, 0, 0, 1),\ Q_1 = (1, 0, 0, 0),\ Q_2 = (\frac{1}{2}, \frac{1}{2}\sqrt{3}, 0, 0)$.
- Define Γ_g to be the polygon $P_1Q_1P_2Q_2$.

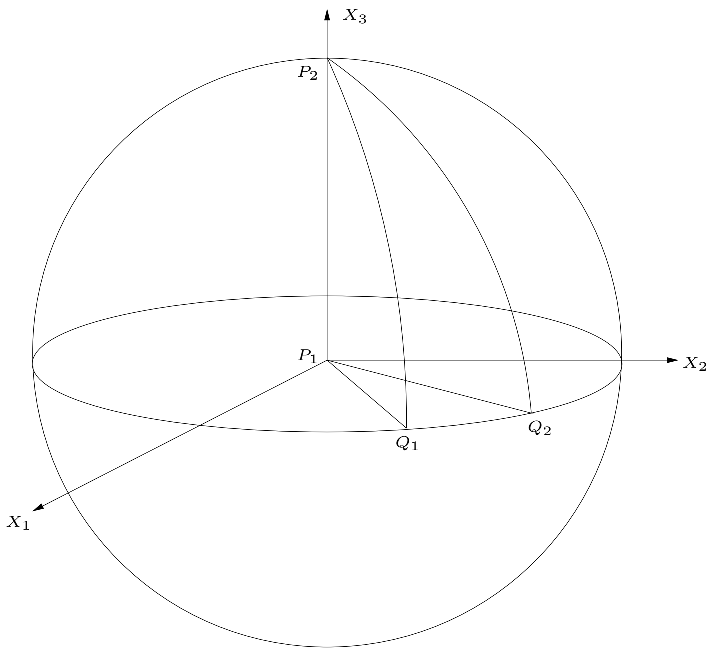

Figure 3.1: The polygon Γ_g.

Before being able to proceed one has to make sure that conditions (A) to (D) for Γ_g are satisfied.

Proposition 3.35. The polygon Γ_g is proper, convex and satisfies the conditions (A) to (D).

Proof. From the above construction one immediately obtains that Γ_g is proper and convex (see above figure 3.1). The construction also implies that Γ_g lies in an open hemisphere of S^3 and thus condition (A) is fulfilled. The picture also indicates the correctness of condition (C).

It remains to verify conditions (B) and (D):
For condition (B) we have to find for each $p \in \mathcal{C}(\Gamma_g)^o$ a geodesic 2-sphere S_p with $p \in S_p$ and $S_p \notin \mathcal{S}_{\Gamma_g}$, i.e. $S_p \cap \Gamma_g$ has at most 3 components. For this purpose one simply considers the family of great spheres passing through C_1, i.e. the family of planes passing through the X_3-axis in figure 3.1. For these spheres the number of components of $S_p \cap \Gamma_g$ is at most 2.
Condition (D) is gained by first rotating Γ_g in a way so that the X_3-axis becomes the center line of symmetry with (see figure 3.2)

$$Q_1, Q_2 \in \{(X_1, X_2, X_3) \mid X_3 = 0\}$$

and

$$P_1, P_2 \in \{(0, X_2, X_3) \mid X_2^2 + X_3^2 = 1,\ X_3 > 0\}.$$

Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ denote the orthogonal projection onto the (X_1, X_2)-plane. Then $\pi|_{\mathcal{C}(\Gamma_g)^o}$ is a map with the properties necessary for condition (D). □

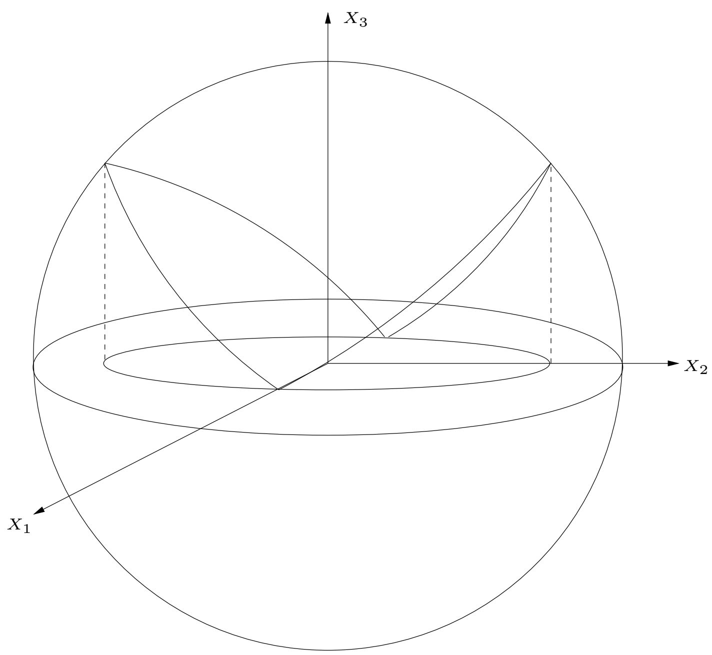

Figure 3.2: Rotated Γ_g.

Lemma 3.36. For the polygon Γ_2 the group of symmetries G_{Γ_2} induced by reflection across the boundary has the group structure of D_6 (i.e. the dihedral group of order 12).

Proof. The proof splits up into the following parts:

- Find the geodesic arcs γ_{ij} running from P_i to Q_j with $i, j \in \{1, 2\}$.
- Write down the reflections across those arcs as mappings $r_{\gamma_{ij}} : S^3 \to S^3$.
- Identify the group structure.

First we observe that γ_{ij} is a geodesic in S^3 if and only if $\ddot{\gamma}_{ij}$ is normal to S^3. This means that $\ddot{\gamma}_{ij}$ and γ_{ij} should be proportional as vectors in $\mathbb{R}^4$. Great circles

$$\gamma(t) = a\cos(\alpha t) + b\sin(\alpha t),$$

where $a, b \in \mathbb{R}^4$, $|a| = |b| = 1$ and $a \perp b$ clearly have this property. Furthermore, since $\gamma(0) = a \in S^3$ and $\dot{\gamma}(0) = \alpha b \in T_aS^3$, we see that we have a geodesic for each initial value problem.

From this one immediately obtains the geodesics

$$\begin{aligned}
\gamma_{11}(t) &= (\sin(t), 0, \cos(t), 0),\\
\gamma_{12}(t) &= (\frac{1}{2}\sin(t), \frac{1}{2}\sqrt{3}\sin(t), \cos(t), 0),\\
\gamma_{21}(t) &= (\sin(t), 0, 0, \cos(t)),\\
\gamma_{22}(t) &= (\frac{1}{2}\sin(t), \frac{1}{2}\sqrt{3}\sin(t), 0, \cos(t)).
\end{aligned}$$

γ_{11} and γ_{21} are geodesics with $x_2 = x_4 = 0$ and $x_2 = x_3 = 0$ respectively. Thus we obtain

$$\begin{aligned}
r_{\gamma_{11}}(x_1, x_2, x_3, x_4) &= (x_1, -x_2, x_3, -x_4),\\
r_{\gamma_{21}}(x_1, x_2, x_3, x_4) &= (x_1, -x_2, -x_3, x_4).
\end{aligned}$$

The reflections across γ_{12} and γ_{22} are obtained via conjugation by a certain rotation ϕ (see remark 3.2):

$$\begin{aligned}
r_{\gamma_{12}} &= \phi^{-1} \circ r_{\gamma_{11}} \circ \phi\\
&= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 & 0\\ \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & 0\\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}\\
&= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & 0\\ \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1 \end{pmatrix}
\end{aligned}$$

and thus one gets (with a similar calculation for $r_{\gamma_{22}}$)

$$\begin{aligned} r_{\gamma_{12}}(x_1,x_2,x_3,x_4) &= (-\frac{1}{2}x_1+\frac{1}{2}\sqrt{3}x_2, \frac{1}{2}\sqrt{3}x_1+\frac{1}{2}x_2, x_3, -x_4), \\ r_{\gamma_{22}}(x_1,x_2,x_3,x_4) &= (-\frac{1}{2}x_1+\frac{1}{2}\sqrt{3}x_2, \frac{1}{2}\sqrt{3}x_1+\frac{1}{2}x_2, -x_3, x_4). \end{aligned}$$

Recall that a group G is said to decompose as a semi-direct product of subgroups G_1 and G_2 (written $G = G_1 \rtimes G_2$), if

1. The subgroup G_1 is normal;
2. $G_1G_2 = G$;
3. $G_1 \cap G_2 = \{e\}$.

In this case there is an isomorphism

$$G_1 \rtimes G_2 \to G, \ \ (g_1, g_2) \mapsto g_1g_2,$$

where the direct product of G_1 and G_2 is equipped with the group operation

$$(g_1,g_2)\circ(h_1,h_2) = (h_2^{-1}g_1h_2h_1, g_2h_2).$$

First we observe that

$$\begin{pmatrix} A & \\ & -\mathbb{1} \end{pmatrix} := r_{\gamma_{11}} \circ r_{\gamma_{22}} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & 0 \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

where for brevity we set $A := \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}$ and $\mathbb{1} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Obviously we have $A^3 = \mathbb{1}$ and thus we obtain

$$\begin{aligned} G_1 &:= \langle \begin{pmatrix} A & \\ & -\mathbb{1} \end{pmatrix} \rangle = \left\{ \begin{pmatrix} A & \\ & -\mathbb{1} \end{pmatrix}, \begin{pmatrix} A^2 & \\ & \mathbb{1} \end{pmatrix}, \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix}, \begin{pmatrix} A & \\ & \mathbb{1} \end{pmatrix}, \begin{pmatrix} A^2 & \\ & -\mathbb{1} \end{pmatrix}, \begin{pmatrix} \mathbb{1} & \\ & \mathbb{1} \end{pmatrix} \right\} \\ &\simeq \mathbb{Z}_3 \times \mathbb{Z}_2 \simeq \mathbb{Z}_6. \end{aligned}$$

We have to check that G_1 is a normal subgroup of G_{Γ_2} and therefore

$$r_{\gamma_{ij}} G_1 r_{\gamma_{ij}}^{-1} \subset G_1 \ \ \forall i,j \in \{1,2\}.$$

This follows immediately from (the only interesting parts are the entries encoded in A and A^2; all the $r_{\gamma_{ij}}$ are involutions)

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} &= \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix} = A^2, \\ \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix} A \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix} &= \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix} = A^2. \end{aligned}$$

Setting

$$G_2 := \langle r_{\gamma_{21}} \rangle \simeq \mathbb{Z}_2$$

one obtains

$$\begin{aligned} r_{\gamma_{11}} &= \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix} \circ r_{\gamma_{21}} \\ r_{\gamma_{12}} &= \begin{pmatrix} A^2 & \\ & -\mathbb{1} \end{pmatrix} \circ r_{\gamma_{21}} \\ r_{\gamma_{22}} &= \begin{pmatrix} A^2 & \\ & \mathbb{1} \end{pmatrix} \circ r_{\gamma_{21}} \end{aligned}$$

and thus one gets all generators of G_{Γ_2}. From $G_1 \cap G_2 = \{e\}$ follows

$$G_{\Gamma_2} = G_1 \rtimes G_2 \simeq D_6,$$

where D_6 denotes the dihedral group of order 12. □

We now will restrict to the $g = 2$ case with the following distinguished points

$$\begin{aligned} P_1 &= (0,0,1,0), & P_3 &= (0,0,-1,0), \\ P_2 &= (0,0,0,1), & P_4 &= (0,0,0,-1), \\ Q_1 &= (1,0,0,0), & Q_4 &= (-1,0,0,0), \\ Q_2 &= (\frac{1}{2}, \frac{1}{2}\sqrt{3}, 0, 0), & Q_5 &= (-\frac{1}{2}, -\frac{1}{2}\sqrt{3}, 0, 0), \\ Q_3 &= (-\frac{1}{2}, \frac{1}{2}\sqrt{3}, 0, 0), & Q_6 &= (\frac{1}{2}, -\frac{1}{2}\sqrt{3}, 0, 0). \end{aligned}$$

The points P_i and Q_j are chosen equally spaced and for C_{ij} denoting the great circle containing P_i and Q_j one may consider the 1-skeleton of the geodesic triangulation

$$Sk_2 = C_1 \cup C_2 \cup \left(\bigcup_{i,j} C_{ij} \right).$$

Proposition 3.37. The group G_{Γ_2} leaves Sk_2 invariant.

Proof. Consider the generators of G_{Γ_2}. Since each element is an isometry and leaves the set of $\{P_i\}$ and $\{Q_j\}$ invariant, the C_{ij} are left invariant. A short glance at the generators also reveals the invariance of C_1 and C_2. □

Corollary 3.38. The triangulation of Σ_2 consists of 12 faces, 24 edges and 10 vertices and thus Σ_2 is of genus $g = 2$.

Proof. Consider the action of G_{Γ_2} on the "edges" P_iQ_j and the corresponding faces. We shall abbreviate P_iQ_j by ij and start with the geodesic polygon given by

$$P_1Q_1P_2Q_2, \text{ i.e. } (11, 21, 22, 12).$$

The action of G_{Γ_2} yields

Number	Face	Number	Face
1	(11,21,22,12)	7	(31,21,26,36)
2	(15,25,26,16)	8	(33,23,22,32)
3	(13,23,24,14)	9	(35,25,24,34)
4	(11,41,46,16)	10	(31,41,42,32)
5	(13,43,42,12)	11	(35,45,46,36)
6	(15,45,44,14)	12	(33,43,44,34)

The triangulation via such "geodesic" rectangles consists of 10 vertices (V) (all the P_i, Q_j), 24 edges (E) and 12 faces (F). Summing up one has via the Gauss-Bonnet-formula

$$\chi(\Sigma_2) = V - E + F = 10 - 24 + 12 = 2 - 2g,$$

and thus $g = 2$. □

Conclusion 3.39. The surface Σ_g is a compact orientable surface of genus g embedded in S^3 with $G_{\Gamma_g} = D_{2g+2}$.

Proof. This is just a generalization of the preceding results: Since G_{Γ_g} leaves the corresponding 1-skeleton Sk_g invariant it is finite and thus Σ_g compact. Finally one obtains

$$G_{\Gamma_g} = \mathbb{Z}_{2g+2} \rtimes \mathbb{Z}_2 \simeq D_{2g+2}$$

and by a similar counting argument

$$\chi(\Sigma_g) = V - E + F = (2g + 2 + 4) - 4(2g + 2) + 2(2g + 2) = 2 - 2g.$$

Thus we get a compact, embedded (hence orientable) surface of genus g. □

Since *every orientable surface can be considered as a Riemann surface* or complex algebraic curve one therefore gets

Corollary 3.40. For every genus g the surface Σ_g is hyperelliptic.

Proof. Consider the conformal involution (see notation introduced above) given by

$$J := \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix}.$$

J fixes the $2g + 2$ points $Q_1, \ldots, Q_{2g+2}$, since

$$Q_j \in C_2 \sim \{(z, 0) \in \mathbb{C}^2 \mid |z| = 1\}.$$

The claim now follows from proposition 3.27. □

Proposition 3.41. The subgroup $\mathbb{Z}^{g+1} \subset G_{\Gamma_g}$, generated by $\begin{pmatrix} A \\ \mathbb{1} \end{pmatrix}$, that is

$$\mathbb{Z}^{g+1} = \langle \begin{pmatrix} A \\ \mathbb{1} \end{pmatrix} \rangle,$$

fixes the distinguished points $\{P_1, \ldots, P_4\}$.

Proof. This follows immediately from $P_i \in C_1 \sim \{(0, w) \in \mathbb{C}^2 \mid |w| = 1\}$. □

Considering Σ_g as a Riemann surface it makes sense to investigate the automorphism group $Aut(\Sigma_g)$ in order to get an algebraic equation that represents that surface, i.e.

$$w^2 = P(z) = \prod_{j=1}^{2g+2} (z - e_j),$$

with P being of order $2g + 2$ and $e_1, \ldots, e_{2g+2}$ the images of the Weierstrass points on Σ_g. We will call these points Weierstrass points as well.

Now if $T \in Aut(\Sigma_g)$ is an automorphism, $T \circ J \circ T^{-1}$ has at least $2g + 2$ fixed points and therefore

$$T \circ J \circ T^{-1} = J.$$

Thus the hyperelliptic involution J commutes with T and therefore any automorphism projects to a Möbius transformation A of the Riemann sphere $\mathbb{CP}^1 = \Sigma_g / \langle J \rangle$. Each Möbius transformation A maps the Weierstrass points onto themselves and therefore the reduced automorphism group $Aut(\Sigma_g)/\langle J \rangle$, that is isomorphic to a certain subgroup of $Perm(W(\Sigma_g))$, can be used to determine the algebraic equation.

The following results are stated for the genus $g = 2$ case, but can immediately generalized to arbitrary genus g.

Lemma 3.42. The reduced automorphism group of $Aut(\Sigma_2)$ contains D_6, i.e.

$$D_6 \subset Aut(\Sigma_2)/\langle J \rangle.$$

Proof. We have already examined the isometries resulting from reflections across the geodesic boundary arcs of $P_1Q_1P_2Q_2$ that map the "fundamental" polygon $P_1Q_1P_2Q_2$ into isometric copies. Since the isometries I resulting from a composition of two reflections belong to the group of conformal automorphisms of Σ_2 (as they are orientation-preserving) we will treat those isometries as holomorphic $T \in Aut(\Sigma_2)$. It is clear that other symmetries of the surface can be found if one considers the symmetries of the polygon

$P_1Q_1P_2Q_2$. In this case one has 2 symmetries arising from plane-reflections interchanging Q_1 and Q_2 or P_1 and P_2 respectively.

The latter shall be denoted by $\eta : S^3 \to S^3$ with

$$\eta(x_1, x_2, x_3, x_4) = (x_1, x_2, x_4, x_3).$$

In order to get the first reflection $\sigma : S^3 \to S^3$ one has to conjugate the reflection

$$\mathrm{diag}(1, -1, 1, 1),$$

where $\mathrm{diag}(a_1, \ldots, a_n)$ denotes a $n \times n$-matrix with entries $a_1, \ldots, a_n$ on the diagonal, with a rotation about $\frac{\pi}{6}$. Summing up one has to calculate

$$\begin{pmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}\sqrt{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix},$$

and thus one arrives at

$$\sigma(x_1, x_2, x_3, x_4) = (\frac{1}{2}x_1 + \frac{1}{2}\sqrt{3}x_2, \frac{1}{2}\sqrt{3}x_1 - \frac{1}{2}x_2, x_3, x_4).$$

A composition of σ with $r_{\gamma_{12}}$ yields

$$\begin{aligned} G_1 := r_{\gamma_{12}} \circ \sigma &= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & 0 \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & 0 \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 & 0 \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \end{aligned}$$

with $B = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}$ being of order 6, i.e. $B^6 = \mathbb{1}$. From the above considerations we know that $Q_1, \ldots, Q_6$ are the Weierstrass points and it is clear that G_1 acts like a rotation around $\frac{2\pi}{6}$ on Q_j. Since $r_{\gamma_{21}}$ fixes $\{Q_1, Q_4\}$ and we also have

$$Q_2 \xrightarrow{r_{\gamma_{21}}} Q_6, \quad Q_3 \xrightarrow{r_{\gamma_{21}}} Q_5,$$

we may set $G_2 := r_{\gamma_{21}} \circ \eta$. In a similar way one can now check that

$$(\langle G_1 \rangle \rtimes \langle G_2 \rangle) / \langle J \rangle = D_6.$$

□

Remark 3.43. Note that for G_1 one has $\det(G_1) = -1$ and therefore $G_1 \notin SO(4)$. This will be crucial when determining the Hopf differential Q of Σ_g.

Conclusion 3.44. Switching over to arbitrary g one sees that the reduced automorphism group of $Aut(\Sigma_g)$ contains D_{2g+2}, i.e.

$$D_{2g+2} \subset Aut(\Sigma_g)/\langle J \rangle.$$

Lemma 3.45. The group of Möbius transformations splits up into 2 classes, namely those elements with 1 respectively 2 fixed points.

Proof. Let $f \in$ Möb be an arbitrary Möbius transformation, i.e.

$$f(z) = \frac{az+b}{cz+d}$$

We solve the fixed point equation $f(\gamma) = \gamma$. For $c \neq 0$, we obtain two roots:

$$\gamma_{1,2} = \frac{(a-d) \pm \sqrt{(a-d)^2 + 4bc}}{2c} = \frac{(a-d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2c}.$$

When c = 0, one of the fixed points is at infinity; the other one is given by

$$\gamma = -\frac{b}{a-d}.$$

The transformation will be a simple transformation composed of translations, rotations, and dilations:

$$z \mapsto \alpha z + \beta.$$

If c = 0 and a = d, then "both" fixed points are at infinity (we have one fixed point), and the Möbius transformation corresponds to a pure translation:

$$z \mapsto z + \beta.$$

□

Theorem 3.46. *For genus $g = 2$ the Lawson surface Σ_2 is of the form*

$$w^2 = z^6 - 1.$$

Proof. With $Q_1 = e_1, \ldots, Q_6 = e_6$ as Weierstrass points we see that we have an action

$$D_6 \times W(\Sigma_2) \to W(\Sigma_2).$$

$W(\Sigma_2)$ is the disjoint union of the orbits under $\mathbb{Z}_6 \subset D_6$. For each $g \in \mathbb{Z}_6$ let $W(\Sigma_2)^g$ denote the set of elements in $W(\Sigma_2)$ that are fixed by g. Burnside's lemma asserts the following formula for the number of orbits, denoted $|W(\Sigma_2)/\mathbb{Z}_6|$:

$$|W(\Sigma_2)/\mathbb{Z}_6| = \frac{1}{|\mathbb{Z}_6|} \sum_{g \in \mathbb{Z}_6} |W(\Sigma_2)^g| = \frac{1}{6} |W(\Sigma_2)^{\mathrm{id}}| = \frac{1}{6} \cdot 6 = 1.$$

Thus the 6 Weierstrass points lie on one orbit under a cyclic group action with generator T. Since T maps the set of Weierstrass points onto itself, the order of T is the length of a cycle of S_6, i.e. 6 and one gets $T(x) = e^{\frac{2\pi i}{6}}x$. The resulting algebraic equation has the form

$$w^2 = z^6 - a,$$

with $a \in \mathbb{C}^*$. Let $\xi \in \mathbb{C}$ solve the equation $z^6 = a$, i.e. $\xi^6 = a$. Via the transformation

$$\widetilde{w} = \frac{1}{\xi^3}w, \quad \widetilde{z} = \frac{1}{\xi}z$$

one obtains

$$\widetilde{w}^2 = \frac{1}{\xi^6}w^2 = \frac{1}{a}w^2 = \frac{1}{a}(z^6 - a) = (\frac{1}{\xi}z)^6 - 1 = \widetilde{z}^6 - 1$$

and we have a biholomorphic equivalence between those surfaces. □

Conclusion 3.47. The Lawson surface Σ_g is of the form

$$w^2 = z^{2g+2} - 1.$$

Corollary 3.48. The automorphism group $Aut(\Sigma_2)$ is of order 24 and the reduced automorphism group, i.e. $Aut(\Sigma_2)/\langle J\rangle$ with J being the hyperelliptic involution, is D_6.

Proof. Every element $T \in Aut(\Sigma_2)/\langle J\rangle$ corresponds to an $\sigma \in Perm(W(\Sigma_2))$ that is realized via a Möbius transformation $A \in$ Möb. By the 3-point-formula we know that every Möbius transformation that fixes more than 2 (Weierstrass) points must be the identity.

Thus we have to distinguish 3 cases:

1. One Weierstrass point is fixed by A:
   Since A must leave $W(\Sigma_2)$ invariant it must leave the circle $|z| = 1$ invariant. If one considers the generators of $A \in$ Möb one sees that this condition is fulfilled by the rotations and inversions. But the group generated by those 2 elements clearly fixes 2 or 0 Weierstrass points and thus this case can't be realized via a Möbius transformation.

2. No Weierstrass point is fixed by A:
   Following the above argument we see that this can be realized by the 5 rotations around $\frac{2\pi k}{6}$ $(k = 1, \ldots, 5)$ and the 3 inversions resulting from conjugating $f : z \mapsto \frac{1}{z}$ with rotations around $\frac{\pi}{6}$, $\frac{3\pi}{6}$ and $\frac{5\pi}{6}$. This amounts to 8 possible transformations for Σ_2.

3. Two Weierstrass points are fixed by A:
   We start with the inversion $f : z \mapsto \frac{1}{z}$ and rotate around $\frac{k\pi}{3}$ ($k = 1, \ldots, 3$) and obtain 3 automorphisms that fix 2 Weierstrass points.

These 11 Möbius transformations together with the identity map have the group structure of D_6 (since this is one way to define the group D_6; it can be regarded as the rotational and reflection symmetries of a regualar hexagon), so that there are no other holomorphic automorphisms. Summing up we see that

$$\varphi : Aut(\Sigma_2) \to D_6 \subset Perm(W(\Sigma_2))$$

is a surjective group homomorphism and thus ($ker\varphi = \langle J \rangle$)

$$Aut(\Sigma_2)/\langle J \rangle \simeq D_6.$$

Now Lagrange's theorem yields (since $\text{ord}(Aut(\Sigma_2)/\langle J \rangle) = \text{ord}(D_6) = 12$)

$$\text{ord}(Aut(\Sigma_2)) = \text{ord}(\langle J \rangle) \cdot \text{ord}(Aut(\Sigma_2)/\langle J \rangle) = 2 \cdot 12 = 24.$$

□

Corollary 3.49. The surface Σ_2 is uniquely determined by $w^2 = z^6 - 1$.

Conclusion 3.50. The surface Σ_g is uniquely determined by $w^2 = z^{2g+2} - 1$.

Remark 3.51.

1. The holomorphic automorphisms that represent $\mathbb{Z}^6 \subset D_6 \simeq Aut(\Sigma_2)/\langle J \rangle$ are $(z, w) \mapsto (\xi z, w)$ with ξ a 6-th root of unity, $\xi^6 = 1$.
2. Comparing the fixed point sets, one observes the same behaviour as in the constructed model of Lawson.

Let $Aut^-(\Sigma_2/\langle J \rangle)$ denote the anti-holomorphic, i.e. the anticonformal, involutions on $\Sigma_2/\langle J \rangle$.

Proposition 3.52. The antiholomorphic involutions of $\Sigma_2/\langle J \rangle$ are

$$Aut^-(\Sigma_2/\langle J \rangle) = \{A^{-1} \circ \rho \circ A \mid A \in Aut(\Sigma_2/\langle J \rangle)\},$$

with $\rho : \mathbb{CP}^1 \to \mathbb{CP}^1, \rho(z) = \bar{z}$.

Proof. First we make the following observation: For a anti-conformal involution $R \in Aut^-(\Sigma_2)$, one has

$$f \circ R = A^{-1} \circ \rho \circ A \circ f,$$

i.e. R projects to anti-holomorphic involution on $\mathbb{CP}^1$. Since every anti-holomorphic involution of $\mathbb{CP}^1$ is conjugated to $\rho(z) = \bar{z}$ and ρ is an involution of Σ_2:

$$P(\rho(z)) = \bar{z}^6 - 1 = \overline{(z^6 - 1)} = \overline{w^2} = \bar{w}^2,$$

with $(z, w) \mapsto (\bar{z}, \bar{w})$ the corresponding mapping, the proposition follows.

□

Chapter 4

A triangulation for Σ_g

We are now focusing on quadratic differentials and consider the Hopf differential Qdz^2 of Σ_g. It will turn out that the Qdz^2 can be used to obtain a natural triangulation of Σ_g.
We mainly follow the terminology introduced by Strebel in [33].

4.1 Quadratic differentials on Σ_g

Definition 4.1. Let M be a Riemann surface with a given conformal structure $\{(U_\nu, \phi_\nu)\}$. A holomorphic **quadratic differential** φ on M is a set of holomorphic function elements φ_ν in the local parameters $z_\nu = \phi_\nu(P)$ for which the transformation law

$$\varphi_\nu(z_\nu)dz_\nu^2 = \varphi_\mu(z_\mu)dz_\mu^2 \text{ with } dz_\mu = \frac{dz_\mu}{dz_\nu}dz_\nu,$$

holds whenever z_μ and z_ν are local parameters that correspond to the same point P of M.

If we keep a certain parameter fixed, we will call it z instead of z_ν and write $\omega = \varphi(z)dz^2$ instead of $\varphi_\nu(z_\nu)dz_\nu^2$.

Since the value of a quadratic differential ω at a point $P \in M$ depends on the local parameter near P, it only makes sense to consider the zeros and poles of ω as distinguished points.

Definition 4.2. The **critical points** of a quadratic differential ω are its zeros and poles. Poles of order one and the zeros are called *finite critical points*. All other points of M are called *regular points* of ω. A holomorphic point is either a regular point or a zero.

We want to determine the Hopf differential $Q(z)dz^2$ of the surface $f : \Sigma_g \to S^3$. This differential must be compatible with the coordinate changes induced by the $SO(4)$-automorphisms of Σ_g as the following lemma shows.

Lemma 4.3. Coordinate changes induced by an $O(4)$-automorphism A of a surface $f : M \to f(M) \subset S^3$ leave the Hopf differential $Q(z)dz^2$ invariant if and only if $A \in SO(4)$.

Proof. First we recall that the normalized $\{f, f_x, f_y, N\}$ form an orthonormal basis for the tangent space T_pM with $N = f \times f_x \times f_y$. We know that

$$f_{zz} = \langle f_{zz}, f\rangle \frac{f}{2e^{2u}} + \langle f_{zz}, f_{\bar{z}}\rangle \frac{f_z}{2e^{2u}} + \langle f_{zz}, f_z\rangle \frac{f_{\bar{z}}}{2e^{2u}} + \langle f_{zz}, N\rangle N$$

with complex coordinates z and $\bar{z}$. The cross product is defined via

$$\frac{1}{2e^{2u}}\langle w, f \times f_z \times f_{\bar{z}}\rangle = \frac{1}{2e^{2u}} \det(f, f_z, f_{\bar{z}}, w) \ \ \forall w \in T_pM$$

Since f_{zz} can be expressed in terms of $f, f_z, f_{\bar{z}}$ and N we have the following equation

$$\begin{aligned}
Q &= \langle f_{zz}, N\rangle = \frac{1}{2e^{2u}}\langle f_{zz}, f \times f_z \times f_{\bar{z}}\rangle \\
&= \frac{1}{4e^{4u}}\langle\langle f_{zz}, f\rangle f + \langle f_{zz}, f_{\bar{z}}\rangle f_z + \langle f_{zz}, f_z\rangle f_{\bar{z}} + \langle f_{zz}, N\rangle N, f \times f_z \times f_{\bar{z}}\rangle \\
&= \frac{1}{4e^{4u}} \det(f, f_z, f_{\bar{z}}, \langle f_{zz}, f\rangle f + \langle f_{zz}, f_{\bar{z}}\rangle f_z + \langle f_{zz}, f_z\rangle f_{\bar{z}} + \langle f_{zz}, N\rangle N) \\
&= \frac{1}{4e^{4u}} \det \begin{pmatrix} \langle f, f\rangle & \langle f, f_z\rangle & \langle f, f_{\bar{z}}\rangle & \langle f, f_{zz}\rangle \\ \langle f_z, f\rangle & \langle f_z, f_z\rangle & \langle f_z, f_{\bar{z}}\rangle & \langle f_z, f_{zz}\rangle \\ \langle f_{\bar{z}}, f\rangle & \langle f_{\bar{z}}, f_z\rangle & \langle f_{\bar{z}}, f_{\bar{z}}\rangle & \langle f_{\bar{z}}, f_{zz}\rangle \\ \langle f_{zz}, f\rangle & \langle f_{zz}, f_z\rangle & \langle f_{zz}, f_{\bar{z}}\rangle & \langle f_{zz}, f_{zz}\rangle \end{pmatrix}.
\end{aligned}$$

For an automorphism $A \in O(4)$ one has $\langle Av, Aw\rangle = \langle v, w\rangle$ and therefore the quantities

$$\langle f_{zz}, f\rangle,\ \langle f_{zz}, f_{\bar{z}}\rangle,\ \langle f_{zz}, f_z\rangle,\ \langle f_{zz}, N\rangle$$

are left invariant for $\widetilde{f(z, \bar{z})} := Af(z, \bar{z})$. The transformed Hopf differential $\widetilde{Q}$ then obeys

$$\begin{aligned}
\widetilde{Q} &= \langle (Af)_{zz}, \widetilde{N}\rangle = \langle Af_{zz}, \widetilde{N}\rangle = \det(Af, Af_z, Af_{\bar{z}}, Af_{zz}) \\
&= \det(A) \cdot \det(f, f_z, f_{\bar{z}}, f_{zz}) \\
&= Q \iff A \in SO(4).
\end{aligned}$$

□

Now we determine a basis for the quadratic differentials on Σ_g, and then check which can be considered as suitable candidate. The following lemmas and observations are taken from [9, III.5.2] and [9, III.7.5].

Lemma 4.4. Let M be a compact Riemann surface of genus $g \geq 2$. The dimension of the space of holomorphic 2-differentials on M is

$$\dim \mathcal{H}^2(M) = 3g - 3.$$

Lemma 4.5. The g differentials

$$\frac{z^j dz}{w}, \quad j = 0, \ldots, g-1$$

form a basis for the abelian differentials of the first kind on a hyperelliptic Riemann surface of the form $w^2 = P(z)$.

Lemma 4.6. On a hyperelliptic surface of genus $g \geq 2$ the products of the holomorphic abelian differentials (taken 2 at a time) form a $(2g-1)$-dimensional subspace of the $(3g-3)$-dimensional space of all holomorphic quadratic differentials. The basis of $\mathcal{H}^2(M)$ then consists of

$$\frac{z^j dz^2}{w^2}, \quad j = 0, \ldots, 2g-2$$

and for $g > 2$ one has to add to the above list the $(3g-3)-(2g-1) = (g-2)$ differentials

$$\frac{z^j dz^2}{w}, \quad j = 0, \ldots, g-3.$$

Theorem 4.7. *The Hopf differential of the surface Σ_g is of the form*

$$Q = a\frac{z^{g-1}dz^2}{w^2}$$

and the zeros are precisely the points lying over zero and infinity, i.e.

$$\{zeros\ of\ Q\} = z^{-1}(\infty) \cup z^{-1}(0).$$

Moreover for each zero F one has ord$(F) = g-1$.

Proof. We have already seen that Q must be invariant under coordinate changes induced by an $SO(4)$-isometry. In our case we have to check whether one of the candidates transforms correctly under

$$\theta_g : z \mapsto \exp\left(\frac{2\pi i}{g+1}\right) z.$$

With $\tilde{z} = e^{\frac{2\pi i}{g+1}} z$ one sees immediately that none of the candidates of the form $\frac{z^j dz^2}{w}$, $j = 0, \ldots, g-3$, is left invariant under this change of coordinates (since $j \leq g-3$!). If one considers an element of the basis of the form

$$\frac{z^k dz^2}{w^2}, \quad k = 0, \ldots, 2g-2,$$

one obtains

$$\begin{aligned} Q(\tilde{z})d\tilde{z}^2 = Q(z)dz^2 \quad &\Longleftrightarrow \quad z^k e^{\frac{2\pi k i}{g+1}} \left(\frac{d\tilde{z}}{dz}\right)^2 \frac{dz^2}{w^2} = z^k \frac{dz^2}{w^2} \\ &\Longleftrightarrow \quad z^k e^{\frac{2\pi k i}{g+1}} e^{\frac{4\pi i}{g+1}} \frac{dz^2}{w^2} = z^k \frac{dz^2}{w^2} \\ &\Longleftrightarrow \quad e^{\frac{2\pi (k+2) i}{g+1}} = 1 \\ &\Longleftrightarrow \quad k = g-1. \end{aligned}$$

Without loss of generality we have the following divisor for z

$$(z) = \frac{F_3 F_4}{F_1 F_2},$$

where $\{F_1, F_2\}$ and $\{F_3, F_4\}$ are the points lying over infinity and zero respectively, i.e. $z^{-1}(\infty) = \{F_1, F_2\}$ and $z^{-1}(0) = \{F_3, F_4\}$. Since

$$(dz) = \frac{E_1 \cdots E_{2g+2}}{F_1^2 F_2^2}$$

and

$$(w) = \frac{E_1 \cdots E_{2g+2}}{F_1^{g+1} F_2^{g+1}}$$

we see that

$$\begin{aligned} \left(\frac{z^j dz^2}{w^2}\right) &= \frac{F_3^j F_4^j}{F_1^j F_2^j} \cdot \frac{E_1^2 \cdots E_{2g+2}^2}{F_1^4 F_2^4} \cdot \frac{F_1^{2g+2} F_2^{2g+2}}{E_1^2 \cdots E_{2g+2}^2} \\ &= F_3^j F_4^j F_1^{2g+2-j-4} F_2^{2g+2-j-4} \\ &= F_3^j F_4^j F_1^{2g-2-j} F_2^{2g-2-j} \end{aligned}$$

and therefore with $j = g-1$ one obtains 4 zeros $F_1, \ldots, F_4$ for Q with $\operatorname{ord}(F_i) = g-1$ for all i. □

Corollary 4.8. The Hopf differential of the surface Σ_2 is of the form

$$Q = a\frac{z dz^2}{w^2}.$$

with a total number of 4 zeros lying over zero and infinity each of order 1.

4.2 The local parameter $w = \Phi(z)$

For a differential an important quantity is its integral as a function on the underlying surface M. In order to get an invariant integral in the case of a quadratic differential φ, one has to pass to a linear differential, by taking the square root, then integrate:

$$w = \Phi(z) = \int \sqrt{\varphi(z)}dz.$$

The above statement is made more precise in the following theorem.

Theorem 4.9 (Strebel)**.** *In a neighborhood of every regular point P of φ we can introduce a local parameter w, in terms of which the representation of φ is identically equal to one. The parameter is given by the integral*

$$w = \Phi(z) = \int \sqrt{\varphi(z)}dz.$$

It is uniquely determined up to a transformation $w \mapsto \pm w + const$ and is called the distinguished or natural parameter near P.

Proof. Let $w = \Phi(z)$ be the function defined above. Every regular point P of φ has a neighborhood in which a single valued branch of this function can be chosen (by integrating one of the two single valued branches of $\sqrt{\varphi(z)}$). For any two determinations $\Phi_1(z)$ and $\Phi_2(z)$ near the same regular point we obviously have

$$\Phi_2(z) = \pm\Phi_1(z) + \text{const}.$$

Choose a small neighborhood U of P, such that U is mapped homeomorphically by a branch of Φ onto an open set V in the w-plane. Let w now be the conformal parameter in U, the differential dw then becomes

$$dw = \Phi'(z)dz = \sqrt{\varphi(z)}dz,$$

and therefore squaring gives

$$dw^2 = \varphi(z)dz^2.$$

In terms of this parameter the quadratic differential has the representation $\equiv 1$. If $\tilde{w}$ is another parameter near P with this property, we have (since $dw^2 = d\tilde{w}^2$)

$$\tilde{w} = \pm w + \text{const}.$$

□

In the next paragraph we will be dealing with so-called trajectories of a quadratic differential. In order to have another representation for these trajectories we already introduce some terminology related to the distinguished parameter $w = \Phi(z)$.

Definition 4.10. Let φ be a meromorphic quadratic differential on an arbitrary Riemann surface M. A φ-**disk** is a region which is mapped homeomorphically onto a disk in the complex plane by a branch of Φ.

If γ is a rectifiable curve in a disk U_0 around a regular point P_0, one can calculate the length of γ by means of the differential $dw = \sqrt{\varphi(z)}dz$ in terms of local parameter z on M.

Definition 4.11. The differential $|dw| = |\varphi(z)|^{\frac{1}{2}}|dz|$ is called the **length element** of the φ-metric, the metric associated to the quadratic differential φ. The length of a curve γ in this metric is denoted by $|\gamma|_\varphi$ and is defined as

$$|\gamma|_\varphi = \int_{\gamma'} |dw| = \int_\gamma |\varphi(z)|^{\frac{1}{2}}|dz|,$$

where $\gamma' = \Phi[\gamma]$. $|\gamma|_\varphi$ is called the φ-**length** of γ.

4.3 Trajectory structure of the Hopf differential

The local theory of trajectories makes use of special parameters in terms of which the differential has a simple representation.

Definition 4.12. Let M be a Riemann surface and $\omega = \varphi(z)dz^2$ a holomorphic quadratic differential on M. A curve

$$\gamma : (a,b) \ni t \mapsto \gamma(t) = z \in M,$$

parameterized on an open interval (a,b) of a real axis is called **horizontal trajectory** of ω if

$$\varphi(\gamma(t))\left(\frac{d\gamma(t)}{dt}\right)^2 > 0 \text{ for every } t \in (a,b)$$

and one speaks of a **vertical trajectory** if

$$\varphi(\gamma(t))\left(\frac{d\gamma(t)}{dt}\right)^2 < 0 \text{ for every } t \in (a,b).$$

We now have the following theorem.

Theorem 4.13 (Strebel)**.** *Let ω be a holomorphic quadratic differential on an arbitrary Riemann surface M. Then through every regular point of ω there exists a uniquely determined trajectory.*
In particular, two trajectories never have a common point, unless they coincide.

Remark 4.14. Since the vertical trajectories of ω are the horizontal trajectories of $-\omega$ it is sufficient to consider the horizontal trajectories of quadratic differentials. However, in general the geodesics are composed of straight arcs of different inclinations, with vertices being zeros or first-order poles, as we shall see later.

Before computing the trajectories for Qdz^2 we introduce the notion of a trajectory ray.

Definition 4.15. Let $\gamma : [u_1, u_2] \to M$,

$$u \mapsto \gamma(u) = P, \quad -\infty \le u_1 < u < u_2 \le +\infty$$

be a non closed trajectory in its natural parametrization. Then, for $u_0 \in (u_1, u_2)$, the restriction of γ to one of the subintervals $(u_1, u_0]$, $[u_0, u_2)$ is called a **trajectory ray** with initial point $\gamma(u_0) = P_0$.
Rays will usually be denoted by the symbols γ^- and γ^+ respectively.

Proposition 4.16. The horizontal and vertical leaves of a quadratic differential of the form $\omega = z^m dz^2$ near a zero of order m are

$$\alpha_k : (0, \infty) \ni t \mapsto t \cdot \exp\left(\frac{2\pi i k}{m+2}\right) \in \mathbb{C}, \quad k = 0, \ldots, m+1,$$

respectively,

$$\beta_k : (0, \infty) \ni t \mapsto t \cdot \exp\left(\frac{\pi i + 2\pi i k}{m+2}\right) \in \mathbb{C}, \quad k = 0, \ldots, m+1.$$

Proof. By making the ansatz $\gamma' = \gamma$, one obtains

$$(\gamma(t))^m (\gamma'(t))^2 = (\gamma'(t))^{m+2} = \pm 1$$

and therefore

$$\gamma'(t) = \sqrt[m+2]{\pm 1}.$$

Integration then yields the result. □

Straightforward calculations carried out in [33, pp. 27-29] yield the following theorems due to Strebel.

Theorem 4.17 (Strebel)**.** *In the neighborhood of any finite critical point P of order m we can introduce a local parameter ξ with $P \leftrightarrow \xi = 0$, in terms of which the quadratic differential has the representation*

$$\varphi(\xi) d\xi^2 = \left(\frac{m+2}{2}\right)^2 \xi^m d\xi^2.$$

Corollary 4.18. The horizontal, respectively vertical, trajectories near the zeros of the Hopf differential $Qd\xi^2$ of Σ_g in terms of the local parameter ξ are of the form

$$\alpha_k: \quad (0,\infty) \ni t \mapsto t \cdot \exp\left(\frac{2\pi i k}{g+1}\right) \in \mathbb{C}, \quad k = 0, \dots, g,$$
$$\beta_k: \quad (0,\infty) \ni t \mapsto t \cdot \exp\left(\frac{\pi i + 2\pi i k}{g+1}\right) \in \mathbb{C}, \quad k = 0, \dots, g.$$

Theorem 4.19 (Strebel)**.** *Let P be a finite critical point of order m and the above ξ chosen as local parameter. Subdivide the disk determined by $|\xi| < \rho$ for some suitable $\rho > 0$ by the radii*

$$\arg \xi = \frac{2\pi k}{m+2}, \quad k = 0, \dots, m+1$$

into $m+2$ sectors. Map each of the sectors onto a half circle in the upper or lower half plane by means of the function $w = \xi^{\frac{m+2}{2}}$. The trajectory arcs are the lines which are mapped into the horizontals. In particular, the distinguished radii are the critical trajectory arcs ending at (or emerging from) P.

4.4 The limit set of a trajectory ray

In the following we will represent trajectories by Φ^{-1} and therefore state the following lemma.

Lemma 4.20. Let φ be a meromorphic quadratic differential on an arbitrary Riemann surface M. The trajectory α of φ through a regular point P_0 can be represented by a branch of the analytic mapping Φ^{-1}.

Remark 4.21. Since Φ^{-1} is defined in a neighborhood of α, it also describes the relation between α and the neighboring trajectories.

Proof. Consider a regular point P_0 of φ and let U_0 be the maximal φ-disk with center P_0. We fix a branch Φ_0 of Φ in U_0 with $\Phi_0(P_0) = 0$. Pick a point $u_1 \in V_0 = \Phi(U_0)$ on the real axis. The point $P_1 = \Phi_0^{-1}(u_1)$ is a regular point of φ. Let U_1 be the maximal φ-disk around P_1. We choose the branch Φ_1 of Φ in U_1 such that

$$\Phi_1|_{U_0 \cap U_1} = \Phi_0|_{U_0 \cap U_1}.$$

Then Φ_1^{-1} is the analytic continuation of Φ_0^{-1}. Picking a point u_2 on the real axis in the disk $V_1 = \Phi_1(U_1)$ one continues as above and therefore gets a finite chain

$$C = V_0 \cup V_1 \cup \ldots \cup V_k$$

of disks in the w-plane with centers on the real axis and a mapping of C into the surface M, which is locally one-to-one and conformal. Let D be the union of all these chains. If $w \in D$, we have $w \in C$ for some chain C. We define $\Phi^{-1}(w)$ as the value at w of the analytic continuation of Φ_0^{-1} along C. As the intersection of two chains is connected and contains v_0, Φ^{-1} is uniquely defined in D. Now let

$$\Delta = D \cap \mathbb{R}.$$

For $u \in \Delta$, $\Phi^{-1}(u)$ defines the trajectory α through P_0 in the natural parametrization. This is a consequence of the following consideration:

If $u \in \Delta$ one has $u \in V$ for some disk $V = \Phi(U)$, i.e. V is the image by a branch of $\int \sqrt{\varphi(z)}dz$ of a φ-disk. Therefore

$$0 < du^2 = \varphi(z(u))dz^2.$$

Let I be a closed subinterval of α which contains P_0. Then it can be covered by finitely many φ-disks with centers on I with U_0 being among them. Choosing the proper branch Φ_0 in U_0 one sees that I corresponds to a subinterval of Δ. Thus α is maximal. □

Let φ be a holomorphic quadratic differential on a Riemann surface M and let $\Phi^{-1}(\Delta) = \alpha$ be the trajectory through the point P_0, $\Phi^{-1}(0) = P_0$, $\Delta = (u_{-\infty}, u_\infty)$. The length $a = |\alpha|$ of α in the φ-metric is

$$a = \int_\alpha |\varphi(z)|^{\frac{1}{2}} |dz| = \int_\Delta |du| = u_\infty - u_{-\infty}.$$

With this parametrization by u we get two half open subintervals corresponding to the 2 trajectory rays that are emerging from P_0, i.e.

$$\alpha^+ = \Phi^{-1}([0, u_\infty)) \quad \text{and } \alpha^- = \Phi^{-1}([0, u_{-\infty}))$$

and the orientation is supposed to be chosen in a way such that P_0 is the initial point of either of them.
For two arbitrarily chosen values $u_1 < u_2$ on Δ there exists a number $b > 0$ such that Φ^{-1} is a homeomorphism of the rectangle

$$u_1 \le u \le u_2, \;\; 0 \le v \le b \;\; (\text{respectively} - b \le v \le 0)$$

into the surface M. The image S is called a horizontal rectangle.

Definition 4.22. Let M be a Riemann surface and $\Phi^{-1}([0, u_\infty))$ be a trajectory ray. Then

$$A^+ = \lim_{u \to u_\infty} \overline{\Phi^{-1}([u, u_\infty))}$$

is the **limit set** of the trajectory ray. It is the set of all points $P \in M$, for which there exists a sequence of numbers $(u_n)_{n \in \mathbb{N}}$ with $\lim_n u_n = u_\infty$ such that $\lim_n P_n = \lim_n \Phi^{-1}(u_n) = P$.

A trajectory α has two limit sets A^+ and A^-, according to its two rays α^+ and α^-. The next lemma investigates the limit set of a trajectory ray given that A^+ contains a regular or critical point of the quadratic differential $\varphi(z)dz^2$.

Lemma 4.23. Let $\varphi(z)dz^2$ be a holomorphic quadratic differential. Then for the trajectory ray α^+ one has:

(a) Let $P \in A^+$ be a regular point. Then $u_\infty = \infty$ and for the trajectory γ through P one has $\gamma \subset A^+$.

(b) Let $P \in A^+$ be a finite critical point (i.e. a zero, since $\varphi(z)dz^2$ is holomorphic). Then $u_\infty < \infty$ if α^+ tends to P (i.e. is a critical ray) with $A^+ = \{P\}$. Otherwise $u_\infty = \infty$ and A^+ contains at least two neighboring rays ending at P.

Proof.

(a) Let $P \in A^+$ be a regular point. On the trajectory γ through P we choose an arbitrary point Q. Now consider the closed subinterval I of γ with endpoints P and Q of φ-length a, i.e.

$$I = [P, Q], \quad |I| = a.$$

Then there exists an open horizontal rectangle S which contains I on its middle line. Let $(u_n)_{n \in \mathbb{N}}$ be a sequence with

$$\lim_n u_n = u_\infty, \quad \lim_n P_n = \lim_n \Phi^{-1}(u_n) = P.$$

For $P_n \in S$, the trajectory α can be continued through S. Therefore $u_\infty = \infty$ since α contains infinitely many disjoint subintervals of length a. Moreover

$$\lim_n (u_n + a) = \lim_n u_n = \infty \quad \text{and} \quad \lim_n Q_n := \lim_n \Phi^{-1}(u_n + a) = Q.$$

Thus $Q \in A^+$ and as $Q \in \gamma$ was arbitrary, $\gamma \subset A^+$.

(b) Let now $P \in A^+$ be a finite critical point, i.e. a zero of order m, with

$$\lim_n u_n = u_\infty, \quad \lim_n P_n = \lim_n \Phi^{-1}(u_n) = P.$$

From theorem 4.19 we know that there are finitely many trajectory rays ending at P that subdivide the neighborhood of P into $(m+2)$ sectors with equal angles at P. For each sector consider the upper part of the horizontal rectangle given by a subinterval I with φ-length $|I| = 2a$ and P as midpoint (see figure 4.1).
The interior of the union of these rectangles forms a neighborhood U

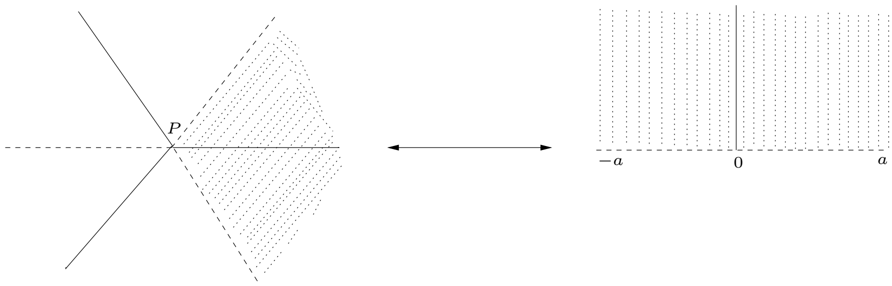

Figure 4.1: Horizontal rectangle.

of P. If α^+ is one of the trajectory rays ending at P one has $u_\infty < \infty$ and $A^+ = \{P\}$.
In the other case there are infinitely many points P_n in at least one of the sectors S of U and the horizontal (!) intervals through these points extend through S. Since these intervals correspond to subintervals of α^+ we get $u_\infty = \infty$ and with

$$\lim_n (u_n \pm a) = \infty \text{ and } P_n' = \Phi^{-1}(u_n + a),\ P_n'' = \Phi^{-1}(u_n - a)$$

one gets two points P', P'' on ∂S as accumulation points of $(P_n')_{n\in\mathbb{N}}$ and $(P_n'')_{n\in\mathbb{N}}$ respectively. Thus A^+ contains at least two neighboring rays ending at P.

□

Remark 4.24. If the initial point P_0 of α^+ is contained in A^+, then $\alpha \subset A^+$. Therefore the closure $\overline{\alpha} \subset A^+$, as A^+ is closed. On the other hand one obviously has $A^+ \subset \overline{\alpha}$ and thus

$$\overline{\alpha} = A^+ = A.$$

A trajectory ray α^+ with $P_0 \in A^+$ is called **recurrent**.

4.5 A canonical triangulation for Σ_g

The goal of this section is to construct a canonical triangulation for the surfaces Σ_g that is induced by the trajectory structure of the Hopf differential. We will see that the critical rays will play an important role, since they connect the zeros lying over 0 and ∞. First we will have to introduce the notion of a maximal rectangle, but first define a triangulation for a submanifold.

Definition 4.25. Let N be a smooth n-manifold and suppose $M \subset N$ is a compact, oriented, embedded m-dimensional submanifold. A **smooth triangulation** of M is a smooth m-cycle $c = \sum_i \sigma_i$ in N with the following properties:

- Each $\sigma_i : \Delta_p \to M$ is a smooth orientation-preserving embedding.
- If $i \neq j$, then $\sigma_i(\mathrm{Int}\Delta_p) \cap \sigma_j(\mathrm{Int}\Delta_p) = \emptyset$.
- $M = \bigcup_i \sigma_i(\Delta_p)$.

Definition 4.26. Let α^+ be a trajectory ray of finite φ-length $|\alpha^+| = a$ with $\alpha^+ = \Phi^{-1}([0,a])$. For every $u \in [0,a]$ there is a maximal half-open interval $[0, v(u))$ on which Φ^{-1} is defined and one-to-one. Let

$$\overline{v} = \min_{0 \leq u \leq a} v(u).$$

The image of the rectangle $0 \leq u \leq a,\ 0 \leq v < \overline{v}$ under the mapping Φ^{-1}, i.e.

$$R := \Phi^{-1}([0,a] \times [0, \overline{v})),$$

is called the **maximal rectangle** R associated to the critical trajectory ray α^+.

We now have the following important theorem due to Strebel.

Theorem 4.27. *Two maximal rectangles are either disjoint or else identical.*

Proof. Let R_1 and R_2 be two maximal rectangles. There is a one-to-one conformal mapping Φ_1^{-1} of some $S_1 : 0 \leq v < \overline{v}_1$ onto R_1 and a Φ_2^{-1} of a domain $S_2 : 0 \leq v < \overline{v}_2$ onto R_2. Let $P_0 \in R_1 \cap R_2$ and denote by $p_i \in S_i$ the points with $\Phi_1^{-1}(p_1) = P_0 = \Phi_2^{-1}(p_2)$. In a neighborhood U of P_0 the two function elements $w_i = \Phi_i(P)$ satisfy

$$\Phi_2(P) = \pm\Phi_1(P) + \mathrm{const}$$

and by translation we can achieve that

$$\Phi_1(P_0) = \Phi_2(P_0) = 0 \text{ and } w = \Phi_2(P) = \Phi_1(P) \ \ \forall P \in U.$$

But then

$$\Phi_1^{-1}(w) = \Phi_2^{-1}(w) \ \ \forall w \in U_1 = \Phi_1(U).$$

Because of the maximality of S_1 and S_2 the two domains of definition of the mappings Φ_1^{-1} and Φ_2^{-1} must be the same and hence $R_1 = R_2$. □

Before describing the triangulation for Σ_g we give a short list of the possible trajectories on a compact Riemann surface (see also [17],[18]):

1. Closed trajectories.
2. Non-closed trajectories:

(a) Critical trajectories: at least one ray, say γ^+, of γ tends to a finite critical point. The other ray either tends to another finite critical point (possibly the same), to an infinite critical point or else it is recurrent. The length $|\gamma|_\varphi < \infty$ if and only if γ^+ and γ^- tend to finite critical points. There are only finitely many critical trajectories.

(b) Trajectories both rays of which tend to infinite critical points.

(c) Spirals, i.e. trajectories both rays of which are recurrent.

Since these are the only possibilities and we know that the maximal rectangles are either disjoint or they coincide we have

Theorem 4.28. *The Riemann surface Σ_g minus the critical points and the critical (horizontal and vertical) trajectories of $Q(z)dz^2 = a\frac{z^{g-1}}{w^2}dz^2$ is subdivided into maximal rectangles and one obtains a canonical triangulation for Σ_g with 4 vertices (V), $4g+4$ edges (E) and $2g+2$ faces (F).*

Proof. As a hyperelliptic Riemann surface is specified by its branch points and combinatorial data that describes which sheets are joined by which branch points and branch cuts, we shall describe this situation for Σ_g first. The combinatorial data is called the *gluing rules.*

Take two copies of the Riemann sphere $\mathbb{CP}^1$ and label them sheet I and sheet II. On each sheet for each $k = 1, \ldots, g+1$ we draw a "cut" joining the branch points e_{2k-1} to e_{2k}. Each "cut" is considered to have two banks; an N-bank and an S-bank. A concrete model for Σ_g is then obtained by joining every S-bank on sheet I to an N-bank of the corresponding "cut" on sheet II, and then joining the corresponding S-bank on sheet II to the N-bank of the corresponding "cut" on sheet I.

The Hopf differential of Σ_g is of the form

$$Q(z)dz^2 = a\frac{z^{g-1}dz^2}{w^2}.$$

The horizontal and vertical trajectories in terms of a local parameter ξ around a zero of Q are of the form

$$\alpha_k : \qquad (0,\infty) \ni t \mapsto t \cdot \exp\left(\frac{2\pi i k}{g+1}\right) \in \mathbb{C}, \quad k = 0, \ldots, g,$$

$$\beta_k : \qquad (0,\infty) \ni t \mapsto t \cdot \exp\left(\frac{\pi i + 2\pi i k}{g+1}\right) \in \mathbb{C}, \quad k = 0, \ldots, g.$$

Without loss of generality and due to symmetry reasons we can assume that the horizontal trajectories pass through the cuts and thus change the sheet

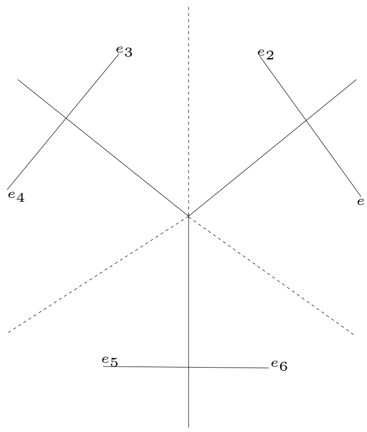

Figure 4.2: Horizontal trajectories passing through cuts for $g = 2$.

on their way to infinity whereas the vertical trajectories stay on the same sheet (see the simplified figure 4.2). In order not to get confused we will call the zeros of Q lying over zero 0^+ and 0^-, and those lying over infinity ∞^+, ∞^-.

Now consider a horizontal critical trajectory ray α^+ joining 0^+ and ∞^-. It is of finite Q-length and therefore one may consider the maximal rectangle R associated to it. For $w = \Phi(\xi)$ one gets

$$w = \int \xi^{\frac{g-1}{2}} d\xi = c\xi^{\frac{g+1}{2}}$$

and therefore

$$\Phi^{-1}(w) = \xi = c' w^{\frac{2}{g+1}}.$$

We are interested in the maximal rectangle adjacent to the left of α^+. From the above considerations we see that the vertical border line of the maximal rectangle R corresponds to the vertical trajectory ray β^+ emanating from 0^+ (see figure 4.3). Since this (critical) ray hits the zero at infinity ∞^+ we get $\overline{v} < \infty$ and therefore a maximal rectangle of finite area. Now consider the two corresponding distinguished rays ending at ∞^+ and ∞^-. In the case of the horizontal critical ray the left-adjacent maximal rectangle is bounded by a horizontal critical ray. This ray passes through a cut defined by the gluing rules on its way to the zeros $0^+, 0^-$ and therefore hits 0^-. Again one gets a finite-area rectangle. The left-adjacent maximal rectangle of this horizontal ray is in turn bounded by a vertical critical ray, and as it is critical one arrives at ∞^- since the sheet stays the same.

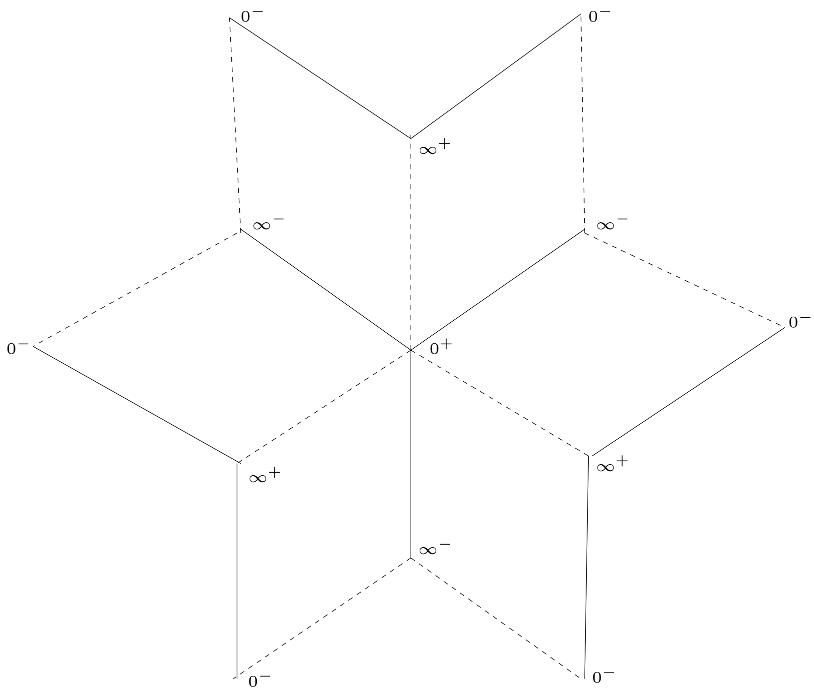

Figure 4.3: Triangulation for $g = 2$.

All the constructed maximal rectangles have borders in common and therefore coincide. Repeating the above procedure for the $2g + 2$ horizontal and critical trajectories one obtains a triangulation for Σ_g with 4 distinguished vertices (V) $0^+, 0^-, \infty^+, \infty^-$, $2g+2$ faces (F) $R_1, \ldots, R_{2g+2}$ and $4g+4$ edges (E) such that the Gauss-Bonnet formula

$$V + F - E = 4 + (2g + 2) - (4g + 4) = 2 - 2g$$

holds. This concludes the proof. □

Chapter 5

The moduli problem for the genus $g \geq 2$ case

In this chapter we consider how the Lax pairs and methods from integrable systems theory can be used to study surfaces in S^3 and investigate the nature of spectral curves corresponding to the higher genus surfaces with $g \geq 2$ and non-constant Hopf differential.

5.1 The Lax pair for CMC surfaces in S^3

It will be convenient to introduce a spectral parameter λ in order to obtain a whole family of CMC surfaces.

Lemma 5.1. Introducing a spectral parameter $\lambda \in S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and setting $Q_\lambda = \lambda Q$ leaves the Gauss-Codazzi equations invariant and thus one obtains a family of CMC surfaces with $H \equiv$ const and conformal factor u.

Proof. This follows directly if one considers the Gauss-Codazzi equations, i.e.

$$\begin{aligned} 2u_{z\bar{z}} + 2e^{2u}(1+H^2) - \frac{1}{2}Q_\lambda \overline{Q}_\lambda e^{-2u} &= 2u_{z\bar{z}} + 2e^{2u}(1+H^2) - \frac{1}{2}\lambda\bar{\lambda}Q\overline{Q}e^{-2u} \\ &= 2u_{z\bar{z}} + 2e^{2u}(1+H^2) - \frac{1}{2}Q\overline{Q}e^{-2u} \\ &= 0. \end{aligned}$$

The claim now follows from the fundamental theorem of surface theory. □

We now want to rework the 4×4 Lax pair into a 2×2 Lax pair and therefore state the following

Lemma 5.2. The double cover of $SO(4)$ is $SU(2) \times SU(2)$ via the action

$$X \mapsto FXG^{-1}$$

and the Lax pair $\mathcal{U}, \mathcal{V}$ is transformed to

$$U = \frac{1}{2}\begin{pmatrix} -u_z & e^{-u}Q \\ -2(H-i)e^u & u_z \end{pmatrix}, \quad V = \frac{1}{2}\begin{pmatrix} u_{\bar z} & 2(H+i)e^u \\ -e^{-u}\overline{Q} & -u_{\bar z} \end{pmatrix},$$

$$\widetilde{U} = \frac{1}{2}\begin{pmatrix} -u_z & e^{-u}Q \\ -2(H+i)e^u & u_z \end{pmatrix}, \quad \widetilde{V} = \frac{1}{2}\begin{pmatrix} u_{\bar z} & 2(H-i)e^u \\ -e^{-u}\overline{Q} & -u_{\bar z} \end{pmatrix}.$$

Proof. Consider the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With the 2×2 identity matrix $\mathbb{1}$ the Pauli matrices can be used to form a basis $\{\mathbb{1}, i\sigma_1, i\sigma_2, i\sigma_3\}$ for the ring of quaternions, since we have the quaternionic relations

$$\begin{aligned} (i\sigma_1)(i\sigma_2) &= (i\sigma_3) = -(i\sigma_2)(i\sigma_1), \\ (i\sigma_2)(i\sigma_3) &= (i\sigma_1) = -(i\sigma_3)(i\sigma_2), \\ (i\sigma_2)(i\sigma_1) &= (i\sigma_2) = -(i\sigma_1)(i\sigma_3) \end{aligned}$$

and $(i\sigma_j)^2 = -\mathbb{1}$ for $j = 1, 2, 3$.

With the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ we first note that for any 2×2-matrix X one has

$$X = \sigma_2 \overline{X} \sigma_2 \Longrightarrow X = \begin{pmatrix} a & b \\ -\bar b & \bar a \end{pmatrix}.$$

Such a matrix X represents a point in $\mathbb{R}^4$ via $X \leftrightarrow (a_1, b_2, b_1, a_2) \in \mathbb{R}^4$, where $a = a_1 + ia_2$, $b = b_1 + ib_2$. So we may consider $\mathbb{R}^4$ to be the set of matrices X satisfying the above equation. The 3-sphere S^3 is then the set of those X in $\mathbb{R}^4$ such that $|a|^2 + |b|^2 = 1$, i.e. S^3 is identified with $SU(2)$. For such an $X \in \mathbb{R}^4$ we see that

$$X \mapsto F \cdot X \cdot G^{-1}$$

represents a general rotation of S^3, where

$$F = \begin{pmatrix} c & d \\ -\bar d & \bar c \end{pmatrix}, \quad G^{-1} = \begin{pmatrix} e & f \\ -\bar f & \bar e \end{pmatrix} \in SU(2).$$

To see this, writing the point $X = (a_1, b_2, b_1, a_2) \in \mathbb{R}^4$ in vector form, this map $X \mapsto F \cdot X \cdot G^{-1}$ translates into $X \mapsto R \cdot X$ in the vector formulation for $\mathbb{R}^4$, where

$$R = \begin{pmatrix} Re(ce - \bar d f) & -Im(de + \bar c f) & -Re(de + \bar c f) & Im(\bar d f - ce) \\ Im(cf - \bar d e) & Re(\bar c e + df) & -Im(\bar c e + df) & Re(cf - \bar d e) \\ Re(\bar d e + cf) & Im(\bar c e - df) & Re(\bar c e - df) & -Im(\bar d e + cf) \\ Im(ce + \bar d f & Re(de - \bar c f) & Im(\bar c f - de) & Re(ce + \bar d f) \end{pmatrix}.$$

A short check now yields that $R \in SO(4)$.

Let $F_1 = F_1(z,\bar{z},\lambda),\ F_2 = F_2(z,\bar{z},\lambda) \in SU(2)$ be the matrices that rotate $i\sigma_1, i\sigma_2$ and $i\sigma_3$ to the 2×2-matrix forms of e_1, e_2 and N respectively, i.e.

$$e_1 = F_1\,(i\sigma_1)\,F_2^{-1},\quad e_2 = F_1\,(i\sigma_2)\,F_2^{-1}\quad N = F_1\,(i\sigma_3)\,F_2^{-1}.$$

We now define

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} := F_1^{-1}(F_1)_z, \qquad V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} := F_1^{-1}(F_1)_{\bar{z}}$$
$$\widetilde{U} = \begin{pmatrix} \widetilde{U}_{11} & \widetilde{U}_{12} \\ \widetilde{U}_{21} & \widetilde{U}_{22} \end{pmatrix} := F_2^{-1}(F_2)_z, \qquad \widetilde{V} = \begin{pmatrix} \widetilde{V}_{11} & \widetilde{V}_{12} \\ \widetilde{V}_{21} & \widetilde{V}_{22} \end{pmatrix} := F_2^{-1}(F_2)_{\bar{z}}$$

and can then compute U and V in terms of the conformal factor u, the mean curvature H and the Hopf differential Q. Making use of

$$e_1 = \frac{f_x}{|f_x|} = \frac{f_x}{2e^u} = F_1 \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} F_2^{-1}, \quad e_2 = \frac{f_y}{|f_y|} = \frac{f_y}{2e^u} = F_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} F_2^{-1}$$

we get

$$f_z = 2ie^u F_1 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} F_2^{-1}, \quad f_{\bar{z}} = 2ie^u F_1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} F_2^{-1}.$$

The entries of the matrices U and V are now obtained in the following way:

Differentiating $f_{\bar{z}}$ with respect to z leads to

$$\begin{aligned} f_{\bar{z}z} &= u_z f_{\bar{z}} + 2ie^u \left((F_1)_z \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} F_2^{-1} + F_1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} (F_2)_z^{-1} \right) \\ &= u_z f_{\bar{z}} + 2ie^u \left(F_1 U \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} F_2^{-1} + F_1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \widetilde{U}^{-1} F_2^{-1} \right) \\ &= u_z f_{\bar{z}} + 2ie^u \left(F_1 \begin{pmatrix} -\widetilde{U}_{21} & U_{11} + \widetilde{U}_{11} \\ 0 & U_{21} \end{pmatrix} F_2^{-1} \right). \end{aligned}$$

We now differentiate f_z with respect to $\bar{z}$:

$$\begin{aligned} f_{z\bar{z}} &= u_{\bar{z}} f_z + 2ie^u \left((F_1)_{\bar{z}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} F_2^{-1} + F_1 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} (F_2)_{\bar{z}}^{-1} \right) \\ &= u_{\bar{z}} f_z + 2ie^u \left(F_1 V \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} F_2^{-1} + F_1 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \widetilde{V}^{-1} F_2^{-1} \right) \\ &= u_{\bar{z}} f_z + 2ie^u \left(F_1 \begin{pmatrix} V_{12} & 0 \\ V_{22} + \widetilde{V}_{22} & -\widetilde{V}_{12} \end{pmatrix} F_2^{-1} \right). \end{aligned}$$

Since $f_{z\bar{z}} = f_{\bar{z}z}$ we therefore obtain

$$u_{\bar{z}} f_z + 2ie^u \left(F_1 \begin{pmatrix} V_{12} & 0 \\ V_{22} + \widetilde{V}_{22} & -\widetilde{V}_{12} \end{pmatrix} F_2^{-1} \right) = u_z f_{\bar{z}} + 2ie^u \left(F_1 \begin{pmatrix} -\widetilde{U}_{21} & U_{11} + \widetilde{U}_{11} \\ 0 & U_{21} \end{pmatrix} F_2^{-1} \right)$$

and thus

$$u_{\bar{z}} f_z - u_z f_{\bar{z}} = 2ie^u \left(F_1 \begin{pmatrix} -\widetilde{U}_{21} & U_{11} + \widetilde{U}_{11} \\ 0 & U_{21} \end{pmatrix} F_2^{-1} - F_1 \begin{pmatrix} V_{12} & 0 \\ V_{22} + \widetilde{V}_{22} & -\widetilde{V}_{12} \end{pmatrix} F_2^{-1} \right),$$

implying

$$u_{\bar{z}} f_z - u_z f_{\bar{z}} = 2ie^u \left(F_1 \begin{pmatrix} -\widetilde{U}_{21} - V_{12} & U_{11} + \widetilde{U}_{11} \\ -V_{22} - \widetilde{V}_{22} & U_{21} + \widetilde{V}_{12} \end{pmatrix} F_2^{-1} \right).$$

Writing out the left part of the above equation yields

$$\begin{aligned} u_{\bar{z}} f_z - u_z f_{\bar{z}} &= 2ie^u \left(F_1 \begin{pmatrix} 0 & -u_z \\ u_{\bar{z}} & 0 \end{pmatrix} F_2^{-1} \right) \\ &= 2ie^u \left(F_1 \begin{pmatrix} -\widetilde{U}_{21} - V_{12} & U_{11} + \widetilde{U}_{11} \\ -V_{22} - \widetilde{V}_{22} & U_{21} + \widetilde{V}_{12} \end{pmatrix} F_2^{-1} \right). \end{aligned}$$

Hence we get

$$U_{11} + \widetilde{U}_{11} + u_z = 0, \quad V_{22} + \widetilde{V}_{22} + u_{\bar{z}} = 0, \quad U_{21} = -\widetilde{V}_{12}, \quad \widetilde{U}_{21} = -V_{12}.$$

Computing f_{zz} yields

$$\begin{aligned} f_{zz} &= u_z f_z + 2ie^u \left((F_1)_z \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} F_2^{-1} + F_1 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} (F_2)_z^{-1} \right) \\ &= u_z f_z + 2ie^u \left(F_1 U \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} F_2^{-1} + F_1 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \widetilde{U}^{-1} F_2^{-1} \right) \\ &= u_z f_z + 2ie^u \left(F_1 \begin{pmatrix} U_{12} & 0 \\ U_{22} + \widetilde{U}_{22} & -\widetilde{U}_{12} \end{pmatrix} F_2^{-1} \right). \end{aligned}$$

We know that $f_{zz} = 2u_z f_z + QN$ and with $N = F_1 (i\sigma_3) F_2^{-1}$ therefore obtain

$$2u_z f_z + QN = u_z f_z + 2ie^u \left(F_1 \begin{pmatrix} U_{12} & 0 \\ U_{22} + \widetilde{U}_{22} & -\widetilde{U}_{12} \end{pmatrix} F_2^{-1} \right),$$

thus

$$\begin{aligned} 2ie^u \left(F_1 \begin{pmatrix} U_{12} & 0 \\ U_{22} + \widetilde{U}_{22} & -\widetilde{U}_{12} \end{pmatrix} F_2^{-1} \right) &= u_z f_z + QN \\ &= 2ie^u F_1 \begin{pmatrix} \frac{1}{2} e^{-u} Q & 0 \\ u_z & -\frac{1}{2} e^{-u} Q \end{pmatrix} F_2^{-1}. \end{aligned}$$

This gives

$$U_{12} = \widetilde{U}_{12} = \frac{1}{2} e^{-u} Q, \quad U_{22} + \widetilde{U}_{22} - u_z = 0.$$

Differentiating $f_{\bar z}$ with respect to $\bar z$ shows $V_{11} + \widetilde{V}_{11} - u_{\bar z} = 0$. We now look at

$$\begin{aligned} f_{\bar z z} &= u_z f_{\bar z} + 2ie^u \left(F_1 \begin{pmatrix} -\widetilde{U}_{21} & U_{11} + \widetilde{U}_1 1 \\ 0 & U_{21} \end{pmatrix} F_2^{-1} \right) \\ &= 2ie^u \left(F_1 \begin{pmatrix} -\widetilde{U}_{21} & 0 \\ 0 & U_{21} \end{pmatrix} F_2^{-1} \right). \end{aligned}$$

With $N = F_1 \,(i\sigma_3)\, F_2^{-1}$ and $f_{\bar z z} = -2e^{2u} f + 2He^{2u} N$ we get

$$\begin{aligned} f_{\bar z z} &= -2e^{2u} f + 2He^{2u} N \\ &= 2ie^u F_1 \begin{pmatrix} ie^u & 0 \\ 0 & ie^u \end{pmatrix} F_2^{-1} + 2ie^u F_1 \begin{pmatrix} He^u & 0 \\ 0 & -He^u \end{pmatrix} F_2^{-1} \end{aligned}$$

and thus $U_{21} = -(H-i)e^u$, $\widetilde{U}_{21} = -(H+i)e^u$. Considering $f_{\bar z \bar z}$ one obtains $V_{21} = \widetilde{V}_{21} = -e^{-u}\overline{Q}$ and summing up the Lax pairs in terms of 2×2-matrices are of the form

$$U = \frac{1}{2} \begin{pmatrix} -u_z & e^{-u} Q \\ -2(H-i)e^u & u_z \end{pmatrix}, \quad V = \frac{1}{2} \begin{pmatrix} u_{\bar z} & 2(H+i)e^u \\ -e^{-u}\overline{Q} & -u_{\bar z} \end{pmatrix},$$

$$\widetilde{U} = \frac{1}{2} \begin{pmatrix} -u_z & e^{-u} Q \\ -2(H+i)e^u & u_z \end{pmatrix}, \quad \widetilde{V} = \frac{1}{2} \begin{pmatrix} u_{\bar z} & 2(H-i)e^u \\ -e^{-u}\overline{Q} & -u_{\bar z} \end{pmatrix}.$$

□

Now we can give a formula for CMC surfaces in S^3 described by 2 solutions of the above Lax pair. For technical reasons the following considerations will be mostly stated in the language of $\mathfrak{su}(2)$-valued 1-forms $\alpha_\lambda = U_\lambda dz + V_\lambda d\bar z$.

For $\omega \in \Omega^1(\mathbb{R}^2, \mathfrak{sl}(2,\mathbb{C}))$ we perform a splitting into the $(1,0)$-part ω' and the $(0,1)$-part ω'', i.e.

$$\omega = \omega' + \omega'',$$

according to the decomposition of the complexified tangent bundle $T\mathbb{C} \simeq T\mathbb{R}^2$ with $d = \partial + \bar\partial$. Setting the $*$-operator on $\Omega^1(\mathbb{R}^2, \mathfrak{sl}(2,\mathbb{C}))$ to

$$*\omega = -i\omega' + i\omega''$$

one may prove the following lemma.

Lemma 5.3. Let $f : \mathbb{R}^2 \to S^3$ be a conformal immersion and $\omega = f^{-1} df$. The mean curvature H is given by

$$2d * \omega = H[\omega \wedge \omega].$$

Proof. Let $U \subset \mathbb{R}^2$ be open and simply connected with coordinate $z : U \to \mathbb{C}$ and set $df' = f_z dz,\ df'' = f_{\bar z} d\bar z$. Since f is conformal one has

$$\langle f_z, f_z\rangle = \langle f_{\bar z}, f_{\bar z}\rangle = 0, \quad \langle f_z, f_{\bar z}\rangle = 2v^2$$

with a function $v \in C^\infty(U, \mathbb{R}\backslash\{0\})$. From the left invariance we obtain

$$\langle \omega', \omega'\rangle = \langle df', df'\rangle = 0,$$

i.e. conformality is equivalent to $\langle \omega', \omega'\rangle = 0$. For two smooth maps $F, G : U \to SU(2)$ that transform the basis $\{\mathbb{1}, \epsilon_-, \epsilon_+, \epsilon\}$ into a frame $\{f, f_z, f_{\bar z}, N\}$, where

$$\epsilon_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \epsilon_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

the previous lemma implies

$$f = FG^{-1}, \quad df = 2ivF(\epsilon_- dz + \epsilon_+ d\bar z)G^{-1}, \quad N = F\epsilon G^{-1}.$$

Thus $\alpha := Udz + Vd\bar z = F^{-1}dF$ and $\beta := \widetilde{U}dz + \widetilde{V}d\bar z = G^{-1}dG$ are of the form

$$\begin{aligned} \alpha &= \left(-v(H-i)dz - \frac{1}{2}v^{-1}\overline{Q}d\bar z\right)\epsilon_- + \left(\frac{1}{2}v^{-1}Qdz + v(H+i)d\bar z\right)\epsilon_+ \\ &\quad + \left(\frac{1}{2}v_z dz - \frac{1}{2}v_{\bar z}d\bar z\right) i\epsilon \end{aligned}$$

and

$$\begin{aligned} \beta &= \left(-v(H+i)dz - \frac{1}{2}v^{-1}\overline{Q}d\bar z\right)\epsilon_- + \left(\frac{1}{2}v^{-1}Qdz + v(H-i)d\bar z\right)\epsilon_+ \\ &\quad + \left(\frac{1}{2}v_z dz - \frac{1}{2}v_{\bar z}d\bar z\right) i\epsilon. \end{aligned}$$

With $0 = d(GG^{-1}) = (dG)G^{-1} + G(dG^{-1})$ one calculates

$$\begin{aligned} \omega &= f^{-1}df = (FG^{-1})^{-1}d(FG^{-1}) = GF^{-1}d(FG^{-1}) \\ &= GF^{-1}(dF)G^{-1} + GF^{-1}F(dG^{-1}) = GF^{-1}(dF)G^{-1} + G(dG^{-1}) \\ &= GF^{-1}(dF)G^{-1} - (dG)G^{-1} = G(F^{-1}dF - G^{-1}dG)G^{-1} \\ &= G(\alpha - \beta)G^{-1}. \end{aligned}$$

A computation now yields

$$d * \omega = 4iv^2 HG\epsilon G^{-1} dz \wedge d\bar z.$$

Furthermore we have

$$[\omega \wedge \omega] = 8iv^2 G\epsilon G^{-1} dz \wedge d\bar z$$

and thus

$$2d * \omega = H[\omega \wedge \omega].$$

□

Introducing a spectral parameter λ we make α λ-dependent and are in the position to express the mean curvature of the immersed surface in terms of λ.

Theorem 5.4. *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and define*

$$\alpha_\lambda = \frac{1}{2}\begin{pmatrix} u_z dz - u_{\bar z} d\bar z & i\lambda^{-1} e^u dz + i\overline{Q} e^{-u} d\bar z \\ iQe^{-u} dz + i\lambda e^u d\bar z & -u_z dz + u_{\bar z} d\bar z \end{pmatrix}.$$

Then $2d\alpha_\lambda + [\alpha_\lambda \wedge \alpha_\lambda] = 0$ if and only if Q is holomorphic and u is a solution of the reduced Gauss equation

$$2u_{z\bar z} + \frac{1}{2}(e^{2u} - Q\overline{Q}e^{-2u}) = 0, \quad Q_{\bar z} = 0.$$

For any solution u of the above equation and corresponding extended frame F_λ, and $\lambda_0, \lambda_1 \in S^1, \lambda_0 \neq \lambda_1$, i.e. $\lambda_k = e^{it_k}$ the map defined by the Sym-Bobenko-formula

$$f = F_{\lambda_1} F_{\lambda_0}^{-1}$$

is a conformal immersion with constant mean curvature

$$H = i\frac{\lambda_0 + \lambda_1}{\lambda_0 - \lambda_1} = \cot(t_0 - t_1),$$

conformal factor $v = e^u/\sqrt{H^2+1}$, and Hopf differential $\widetilde{Q}dz^2$ with $\widetilde{Q} = \frac{i}{4}(\lambda_1^{-1} - \lambda_0^{-1})Q$.

Proof. We decompose α_λ into the $(1,0)$- and $(0,1)$-parts $\alpha_\lambda = \alpha'_\lambda dz + \alpha''_\lambda d\bar z$ and get

$$\begin{aligned} \bar\partial \alpha'_\lambda &= \frac{1}{2}\begin{pmatrix} u_{z\bar z} & i\lambda^{-1} u_{\bar z} e^u \\ -iu_{\bar z} e^{-u} Q + ie^{-u} Q_{\bar z} & -u_{z\bar z} \end{pmatrix}, \\ \partial \alpha''_\lambda &= \frac{1}{2}\begin{pmatrix} -u_{z\bar z} & -iu_z e^{-u}\overline{Q} + ie^{-u}\overline{Q}_z \\ i\lambda u_z e^u & u_{z\bar z} \end{pmatrix}, \\ [\alpha'_\lambda, \alpha''_\lambda] &= \frac{1}{4}\begin{pmatrix} -e^{2u} + Q\overline{Q}e^{-2u} & 2iu_{\bar z}\lambda^{-1}e^u + 2iu_z e^{-u}\overline{Q} \\ -2i\lambda u_z e^u - 2iu_{\bar z} Q e^{-u} & e^{2u} - Q\overline{Q}e^{-2u} \end{pmatrix}. \end{aligned}$$

Since $2d\alpha_\lambda + [\alpha_\lambda \wedge \alpha_\lambda] = 0$ is equivalent to $\bar\partial\alpha'_\lambda - \partial\alpha''_\lambda = [\alpha'_\lambda, \alpha''_\lambda]$ we see that u must fulfill the reduced Gauss equation and $Q_{\bar z} = 0$.

Now let u be a solution of the above equation and consider for $\lambda_0, \lambda_1 \in S^1$, $\lambda_0 \neq \lambda_1$ the map $f = F_{\lambda_1} F_{\lambda_0}^{-1}$ defined by the Sym-Bobenko-formula. Setting $\omega = f^{-1}df = F_{\lambda_0}(\alpha_{\lambda_1} - \alpha_{\lambda_0})F_{\lambda_0}^{-1}$ one has

$$\begin{aligned} f^{-1}\partial f &= F_{\lambda_0} F_{\lambda_1}^{-1}\left((\partial F_{\lambda_1})F_{\lambda_0}^{-1} + F_{\lambda_1}(\partial F_{\lambda_0}^{-1})\right) \\ &= F_{\lambda_0} F_{\lambda_1}^{-1}\left(F_{\lambda_1}\alpha'_{\lambda_1} F_{\lambda_0}^{-1} - F_{\lambda_1} F_{\lambda_0}^{-1}(\partial F_{\lambda_0})F_{\lambda_0}^{-1}\right) \\ &= F_{\lambda_0}\left(\alpha'_{\lambda_1} - \alpha'_{\lambda_0}\right)F_{\lambda_0}^{-1} \end{aligned}$$

and therefore

$$f^{-1}\partial f = \frac{1}{2} i e^u (\lambda_1^{-1} - \lambda_0^{-1}) F_{\lambda_0} \epsilon_+ F_{\lambda_0}^{-1}.$$

A similar calculation reveals $f^{-1}\bar\partial f = -\frac{1}{2} i e^u (\lambda_1 - \lambda_0) F_{\lambda_0} \epsilon_- F_{\lambda_0}^{-1}$ and it is clear that $\langle f^{-1}\partial f, f^{-1}\partial f\rangle = \langle f^{-1}\bar\partial f, f^{-1}\bar\partial f\rangle = 0$. For the conformal factor one has to calculate

$$v^2 = 2\langle f^{-1}\partial f, f^{-1}\bar\partial f\rangle = \frac{1}{4} e^{2u} (\lambda_1^{-1} - \lambda_0^{-1})(\lambda_1 - \lambda_0).$$

For $\widetilde\omega = f^{-1} df = F_{\lambda_0}(\alpha_{\lambda_1} - \alpha_{\lambda_0}) F_{\lambda_0}^{-1}$ one has the splitting

$$\widetilde\omega = \frac{1}{2} i F_{\lambda_0} \left((\lambda_1^{-1} - \lambda_0^{-1})\omega' + (\lambda_1 - \lambda_0)\omega'' \right) F_{\lambda_0}^{-1},$$

where $\omega = \omega' + \omega''$ belongs to α (without the added spectral parameter λ). Another calculation shows

$$d * \widetilde\omega = \frac{1}{4} i (\lambda_1 \lambda_0^{-1} - \lambda_0 \lambda_1^{-1}) F_{\lambda_0} [\omega' \wedge \omega''] F_{\lambda_0}^{-1}$$

and

$$[\widetilde\omega, \widetilde\omega] = \frac{1}{2} (1 - \lambda_1 \lambda_0^{-1})(1 - \lambda_0 \lambda_1^{-1}) F_{\lambda_0} [\omega' \wedge \omega''] F_{\lambda_0}^{-1},$$

and thus $H = i\frac{\lambda_0 + \lambda_1}{\lambda_0 - \lambda_1}$ is the mean curvature for f. From this formula we obtain

$$(H^2 + 1)(\lambda_1^{-1} - \lambda_0^{-1})(\lambda_1 - \lambda_0) = 4$$

and thus $v^2 = e^{2u}/(H^2 + 1)$. Finally we want to find the Hopf differential and consider the normal $N = F_{\lambda_1} \epsilon F_{\lambda_0}^{-1}$. Similar to the above calculations one obtains $\partial N = F_{\lambda_1}(\alpha'_{\lambda_1}\epsilon - \epsilon\alpha'_{\lambda_0}) F_{\lambda_0}^{-1}$ with

$$\alpha'_{\lambda_1}\epsilon - \epsilon\alpha'_{\lambda_0} = \begin{pmatrix} 0 & \frac{1}{2} e^u (\lambda_1^{-1} + \lambda_0^{-1}) \\ -Q e^{-u} & 0 \end{pmatrix}.$$

Thus one has

$$\widetilde Q = -\langle \partial\partial f, N\rangle = \langle \partial f, \partial N\rangle = \langle F_{\lambda_1}^{-1} \partial f F_{\lambda_0}, F_{\lambda_1}^{-1} \partial N F_{\lambda_0}\rangle = \frac{i}{4} (\lambda_1^{-1} - \lambda_0^{-1}) Q,$$

and the claim is proved. □

5.2 Covering spaces for Σ_g and transformations of the frame F_λ

We can draw the following conclusions from the Gauss and Codazzi equations: Away from umbilic points one can choose local coordinates w for Σ_g so that the Hopf differential Q is identically 1, $Q \equiv 1$. Such coordinates correspond to $w = \Phi(\xi)$ away from the zeros of the Hopf differential as we have already seen. In our case we have

$$w = \Phi(\xi) = \int \xi^{\frac{g-1}{2}} d\xi = \frac{2}{g+1}\xi^{\frac{g+1}{2}}.$$

Now one can find covering spaces that correspond to the above coordinate transformation for Σ_g as we shall see below. In the more general setting you have to deal with a generalization of manifolds, the so-called orbifolds (see [2]) which are introduced later.

Definition 5.5. Let $f : M \to N$ be a non-constant holomorphic mapping of degree n (i.e. $f^{-1}(Q)$ has cardinality n for all almost all $Q \in N$) between the compact Riemann surfaces M, N of genus g and γ, respectively. Define the **total branching number** B of f by

$$B = \sum_{P \in M} b_f(P),$$

where $b_f(P)$ is the branch number of f at P.

Theorem 5.6 (Riemann-Hurwitz Relation)**.** *With the above notation we have*

$$2g - 2 = 2n(\gamma - 1) + B.$$

Proof. See [9, I.2.7]. □

Lemma 5.7. The hyperelliptic Riemann surface Y of genus $g' = 2g + 1$ given by

$$Y : \widetilde{w}^2 = y^{4g+4} - 1$$

is a 2-fold cover of $\Sigma_g : w^2 = z^{2g+2} - 1$ with branch points precisely at the zeros of the Hopf Differential Q.

Proof. Let z and y be local coordinates on Σ_g and Y respectively. The map

$$\begin{aligned} f : \quad & Y \to \Sigma_g \\ & y \mapsto f(y) = y^2 = z \end{aligned}$$

is of degree 2 and induces a meromorphic function $\widetilde{w}$ on Y via

$$\widetilde{w}(y) = w(f(y)) = \sqrt{y^{4g+4} - 1}.$$

Since we have $4g + 4$ branch points for Y the genus is $g' = 2g + 1$ and thus the Riemann-Hurwitz relation yields

$$B = 2g' - 2 - 2n(g - 1) = 2(2g + 1) - 2 - 4(g - 1) = 4.$$

Lifting the Hopf differential $Q(z)dz^2$ of Σ_g to Y yields

$$\frac{\widetilde{Q}(y)dy^2}{w^2} = \frac{Q(f(y))(f'(y))^2dy^2}{w^2} = \frac{4y^{2g}dy^2}{w^2}$$

by the transformation rule for quadratic differentials. If for a $P \in Y$, y vanishes at P, then z vanishes at $f(P)$, i.e.

$$z = y^2.$$

This observation together with the transformation rule gives

$$\mathrm{ord}_P\widetilde{Q} = (b_f(P) + 1)\mathrm{ord}_{f(P)}Q + 2b_f(P).$$

The four zeros of Q lying over zero and infinity $F_1, \ldots, F_4$ are of order $(g-1)$ and therefore one has $\mathrm{ord}_{\widetilde{F}_i}\widetilde{Q} = 2g$ for the corresponding zeros of $\widetilde{Q}$, i.e.

$$\begin{aligned} 2g &= \mathrm{ord}_{\widetilde{F}_i}\widetilde{Q} = (b_f(\widetilde{F}_i) + 1)\mathrm{ord}_{f(\widetilde{F}_i)}Q + 2b_f(\widetilde{F}_i) \\ &= (b_f(\widetilde{F}_i) + 1)(g-1) + 2b_f(\widetilde{F}_i) \\ &\Leftrightarrow b_f(\widetilde{F}_i) = 1 \ \forall i. \end{aligned}$$

Since $B = 4$ we see that these four zeros are the branching points of $f : Y \to \Sigma_g$. □

Lemma 5.8. The immersion $f : \Sigma_g \to S^3$ can be lifted to an almost conformal immersion $\widetilde{f} : Y \to S^3$.

Proof. Away from the branch points of the covering $\widetilde{f}$ is a smooth immersion since it is a composition of smooth immersions. This only fails at the four branch points - a finite set of points. Hence the claim follows. □

The coordinate $w = \Phi(\xi) = \frac{2}{g+1}\xi^{\frac{g+1}{2}}$ now indicates that one might regard Y as a $(g+1)$-fold cover of some other Riemann surface. On this surface the corresponding Hopf differential $\hat{Q}$ is identically 1, $\hat{Q} \equiv 1$. We now first want to investigate the possibilities of "going down" from one surface to another. It turns out that this can usually be achieved if one has a finite group acting smoothly and effectively.

Definition 5.9. An n-dimensional smooth **orbifold** $\mathcal{O}$ is a paracompact Hausdorff topological space together with a collection $\{(\widehat{U}_i, G_i, f_i, U_i)\}$ where

(i) $\{U_i\}$ is an open cover of $\mathcal{O}$;

(ii) $\widehat{U}_i$ is a smooth connected n-manifold;

(iii) G_i is a finite group acting on $\widehat{U}_i$ smoothly and effectively;

(iv) $f_i : \widehat{U}_i \to U_i$ is a continuous map which induces a homeomorphism from $\widehat{U}_i/G_i$ to U_i;

(v) (Compatibility condition) If $y \in \widehat{U}_i$ and $y' \in \widehat{U}_j$ satisfy $f_i(y) = f_j(y')$, then there is a diffeomorphism f of a neighborhood V_y of y to a neighborhood of y' with $f(y) = y'$ such that $f_j \circ f = f_i$.

Definition 5.10. For any point $x \in \mathcal{O}$, orbifold chart $(\widehat{U}_i, G_i, f_i, U_i)$ and $y \in f^{-1}(x)$, the **stabilizer** or **isotropy group** Γ_x of x is

$$\Gamma_x = \{g \in G_i \mid g(y) = y\}.$$

A point with non-trivial stabilizer is called a **singular point**. A singular point $x \in \mathcal{O}$ whose stabilizer Γ_x consists only of orientation-preserving conformal diffeomorphisms is called a **cone point**.

The following lemma due to [2] can be applied in a more general setting and is interesting in its own right, since it allows the "smoothing" of an orbifold.

Lemma 5.11. Let $\mathcal{O}$ be a two-dimensional conformal orbifold such that all its singular points are cone points. Then $\mathcal{O}$ can be given the (unique) structure of a smooth conformal surface $\mathcal{O}_s$ such that the identity map $\mathcal{O} \to \mathcal{O}_s$ is smooth and conformal on $\mathcal{O}\backslash\{\text{cone points}\}$.

Proof. Γ_x is cyclic. Suppose it has order p. By the Riemann mapping theorem we have a uniformizing map, i.e. a conformal diffeomorphism $f : (\widehat{U}, 0) \to (\mathbb{D}, 0)$ to the open unit disk $\mathbb{D}$. Since a conformal diffeomorphism of $\mathbb{D}$ which preserves the origin must be a rotation, it follows that f is equivariant with respect to the action of Γ_x on $\widehat{U}$ and the action of $\mathbb{Z}_p$ on $\mathbb{D}$ generated by rotation through $2\pi/p$. The map f factors to a homeomorphism from $U = \widehat{U}/\Gamma_x$ to the "cone" $\mathbb{D}/\mathbb{Z}_p$, which is smooth and conformal away from x.
Now given such a cone point x this homeomorphism can be composed with the homeomorphism given by

$$\mathbb{D}/\mathbb{Z}_p \to \mathbb{D}, \quad y \mapsto y^p$$

and thus defines a conformal structure on U. It is clear that this endows $\mathcal{O}$ with a well-defined smooth conformal structure which agrees with the conformal structure on $\mathcal{O}\backslash\{\text{cone points}\}$. □

Remark 5.12. If $\mathcal{O}_s$ is endowed with the above conformal structure, each orbifold chart $f : \widehat{U}_i \to U_i \subset \mathcal{O}_s$ is smooth and conformal with a branch point of order p at each point in the inverse image of a cone point of cone angle $\frac{2\pi}{p}$.

In the present situation we have again a covering with 4 distinguished points:

Lemma 5.13. The hyperelliptic Riemann surface Y of genus $g' = 2g + 1$ given by

$$Y : \widetilde{w}^2 = y^{4g+4} - 1$$

is a $(g + 1)$-fold cover of the surface $X : w^2 = x^4 - 1$ of genus $\gamma = 1$ with branch points precisely at the distinguished points that correspond to the zeros of the Hopf Differential Q of Σ_g.

Proof. The proof of lemma 5.7 carries over to this situation. It is clear that the genus of X is 1. □

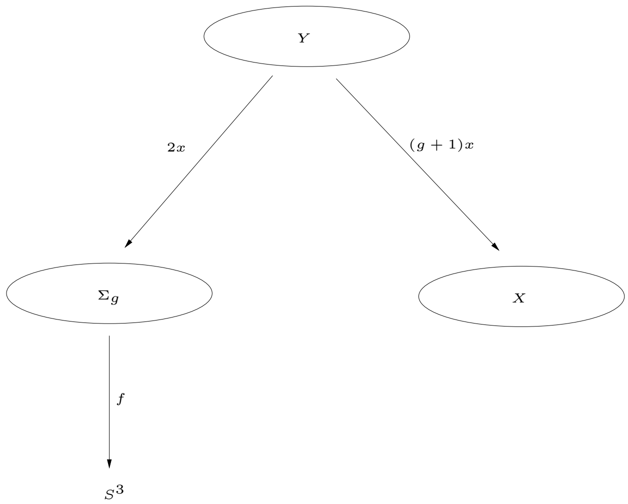

Figure 5.1: Covering spaces induced by Σ_g.

Lemma 5.14. The conformal factor $\widetilde{u}$ of the surface Y fulfills

$$\widetilde{u}(\theta_g(y), \overline{\theta_g(y)}) = \widetilde{u}(y, \overline{y}).$$

Proof. Let y be a conformal coordinate around a zero of Q. Since the map $\theta_g : y \mapsto \exp\left(\frac{2\pi i}{2g+2}\right) \cdot y$ leaves the metric on Y invariant, we have for $\xi_g = \exp\left(\frac{2\pi i}{2g+2}\right)$

$$e^{2\widetilde{u}(\theta_g(y), \overline{\theta_g(y)})} \xi_g \bar{\xi}_g dy d\overline{y} = e^{2\widetilde{u}(\theta_g(y), \overline{\theta_g(y)})} dy d\overline{y} = e^{2\widetilde{u}(y, \overline{y})} dy d\overline{y}$$

and therefore the result follows. □

Lemma 5.15. The constant quadratic differential (the Hopf differential) $\widehat{Q}$ of X can be lifted to the Hopf differential $\widetilde{Q}$ of Y.

Proof. This follows immediately from the definition of a lift of a quadratic differential to the covering surface Y, since setting $\frac{2}{g+1}y^{g+1} = x$ yields

$$\frac{\widehat{Q}dx^2}{w^2} = \frac{1dx^2}{w^2} = \frac{4y^{2g}dy^2}{w^2} = \frac{\widetilde{Q}dy^2}{w^2}.$$

□

Lemma 5.16. Coordinate changes of the form $z \mapsto w = w(z)$ leave the Gauss and Codazzi equations invariant away from the zeros of $w(z)$.

Proof. The Gauss and Codazzi equations for S^3 are given by

$$2u_{z\bar{z}} + 2e^{2u}(1 + H^2) - \frac{1}{2}Q\overline{Q}e^{-2u} = 0, \quad Q_{\bar{z}} = 2H_z e^{2u}.$$

Since $H \equiv \text{const}$ we only need to consider the first equation and investigate the transformation of the corresponding terms resulting from the coordinate change. First we observe that from the equation

$$e^{2\widetilde{u}(w,\bar{w})}dwd\bar{w} = e^{2u(z,\bar{z})}dzd\bar{z}$$

we get

$$e^{2u(z,\bar{z})} = e^{2\widetilde{u}(w,\bar{w})}\left(\frac{dw}{dz}\right)\overline{\left(\frac{dw}{dz}\right)}$$

and therefore

$$u(z,\bar{z}) = \widetilde{u}(w,\bar{w}) + \ln\left(\left|\frac{dw}{dz}\right|\right).$$

Differentiation yields

$$\begin{aligned}
2u(z,\bar{z})_{z\bar{z}} &= \left|\frac{dw}{dz}\right|^2 2\widetilde{u}(w,\bar{w})_{w\bar{w}} + \ln\left(\frac{dw}{dz}\right)_{z\bar{z}} + \ln\left(\frac{d\bar{w}}{d\bar{z}}\right)_{z\bar{z}} \\
&= \left|\frac{dw}{dz}\right|^2 2\widetilde{u}(w,\bar{w})_{w\bar{w}} + \left(\frac{1}{\left(\frac{dw}{dz}\right)} \cdot \frac{d^2w}{dz^2}\right)_{\bar{z}} + \left(\frac{1}{\left(\frac{d\bar{w}}{d\bar{z}}\right)} \cdot \frac{d^2\bar{w}}{d\bar{z}dz}\right)_{\bar{z}} \\
&= \left|\frac{dw}{dz}\right|^2 2\widetilde{u}(w,\bar{w})_{w\bar{w}} + \left(-\frac{1}{\left(\frac{dw}{dz}\right)^2} \cdot \frac{d^2w}{dzd\bar{z}} \cdot \frac{d^2w}{dzdz} + \frac{1}{\left(\frac{dw}{dz}\right)} \cdot \frac{d^3w}{dz^2d\bar{z}}\right) + 0 \\
&= \left|\frac{dw}{dz}\right|^2 2\widetilde{u}(w,\bar{w})_{w\bar{w}},
\end{aligned}$$

since w is holomorphic and $\bar{w}$ anti-holomorphic, respectively. Now consider the quadratic Hopf differential and its transformation rule for coordinate changes, namely

$$Qdz^2 = \widetilde{Q}dw^2 \Longleftrightarrow Q = \widetilde{Q}\left(\frac{dw}{dz}\right)^2.$$

Under a coordinate change of the above form the Gauss and Codazzi equations are transformed into

$$\left|\frac{dw}{dz}\right|^2 2\widetilde{u}_{w\bar{w}} + \left|\frac{dw}{dz}\right|^2 2e^{2\widetilde{u}}(1+H^2) - \frac{1}{2}\left(\frac{dw}{dz}\right)^2 \overline{\left(\frac{dw}{dz}\right)^2} \widetilde{Q}\overline{\widetilde{Q}} \frac{1}{\left(\frac{dw}{dz}\right)\overline{\left(\frac{dw}{dz}\right)}} e^{-2\widetilde{u}}$$

and therefore one obtains

$$\left|\frac{dw}{dz}\right|^2 \left(2\widetilde{u}_{w\bar{w}} + 2e^{2\widetilde{u}}(1+H^2) - \frac{1}{2}\widetilde{Q}\overline{\widetilde{Q}}e^{-2\widetilde{u}}\right) = 0.$$

From the above considerations we see that the Gauss and Codazzi equations are left invariant away from the zeros of $w(z)$ and behave singular at these points. □

Remark 5.17. Since

$$u(z,\bar{z}) = \widetilde{u}(w,\bar{w}) + \ln\left(\left|\frac{dw}{dz}\right|\right).$$

we see that the conformal factor u has a singularity at the zeros of $w(z)$.

It is possible to define the quantities $\widehat{u}$, $\widehat{Q}$ and H on the surface X and one obtains the following

Theorem 5.18. *Via the $(g+1)$-fold cover $p_2 : Y \to X$ with $y \mapsto y^{g+1} =: x$, the quantities $\widehat{u}$, $\widehat{Q}$ and H on the surface $X : w^2 = x^4 - 1$ are well-defined. Furthermore $\widehat{u}$ solves the* sinh-*Gordon equation*

$$2\widehat{u}_{x\bar{x}} + \sinh(2\widehat{u}) = 0.$$

Proof. Considering the above construction one sees that it is possible to define global coordinates via $d\widehat{x} = \frac{dx}{w}$ so that the Hopf differential $\widehat{Q}$ in these coordinates is identically 1, i.e. $\widehat{Q} \equiv 1$.
Moreover lemma 5.14 ensures that the conformal factor $\widehat{u}$ is well-defined on X as well, since it is compatible with the group action induced by the conformal diffeomorphism $\theta_g : y \mapsto \exp\left(\frac{2\pi i}{2g+2}\right) \cdot y$. Since $\widehat{Q} \equiv 1$ the Gauss equation may be reduced to

$$2\widehat{u}_{x\bar{x}} + \frac{1}{2}\left(e^{2\widehat{u}} + e^{-2\widehat{u}}\right) = 2\widehat{u}_{x\bar{x}} + \sinh(2\widehat{u}) = 0$$

and we obtain a doubly periodic solution to the sinh-Gordon equation with singularities appearing precisely at the distinguished points corresponding to the zeros of Qdz^2. □

We will see that a frame F_λ on Σ_g corresponds to a frame $\widehat{F}_\lambda$ on X with a special behavior around the 4 distinguished points corresponding to the zeros of Q on Σ_g.

Theorem 5.19. *A coordinate transform of the above form leaves the frame F_λ respectively the $\mathfrak{su}(2)$-valued 1-form α_λ invariant.*

Proof. Considering

$$\alpha_\lambda = \frac{1}{2}\begin{pmatrix} u_z dz - u_{\bar z} d\bar z & i\lambda^{-1} e^u dz + i\overline{Q} e^{-u} d\bar z \\ iQe^{-u} dz + i\lambda e^u d\bar z & -u_z dz + u_{\bar z} d\bar z \end{pmatrix}$$

one has to investigate the transformations of the quantities appearing in this matrix. According to the splitting $\alpha_\lambda = \alpha'_\lambda + \alpha''_\lambda$ this leads to

$$(e^u)' = (e^{\widetilde{u}})' \left(\frac{dw}{dz}\right)$$

and therefore applying ln and differentiation yields

$$u_z = \left(\frac{dw}{dz}\right) \widetilde{u}_w$$

as well as

$$u_{\bar z} = \left(\frac{d\bar w}{d\bar z}\right) \widetilde{u}_{\bar w}.$$

With the transformation rule for quadratic differentials in mind we see that $\alpha_\lambda = U_\lambda dz + V_\lambda d\bar z$ becomes

$$\widehat{\alpha}_\lambda = \widehat{U}_\lambda dw + \widehat{V}_\lambda d\bar w,$$

where

$$U_\lambda = \left(\frac{dw}{dz}\right) \widehat{U}_\lambda, \;\; V_\lambda = \left(\frac{d\bar w}{d\bar z}\right) \widehat{V}_\lambda.$$

Considering

$$\begin{aligned} F_\lambda &= (F_\lambda)_z U_\lambda^{-1} = (\widehat{F}_\lambda)_w \left(\frac{dw}{dz}\right) U_\lambda^{-1} = (\widehat{F}_\lambda)_w \left(\frac{dw}{dz}\right) \frac{1}{\left(\frac{dw}{dz}\right)} \widehat{U}_\lambda^{-1} \\ &= \widehat{F}_\lambda, \end{aligned}$$

and the analogue for V yields the invariance of the frame F_λ and concludes the proof. □

5.3 Closing conditions and monodromy of the moving frame

When one starts with a CMC H conformal immersion f into S^3 defined on a simply-connected domain D, and then extends f to a conformal CMC immersion $\widehat{f}$ on a larger non-simply-connected domain $\widehat{D}$, the extension $\widehat{f}$ will be unique. However, it is not necessarily true that $\widehat{f}$ is well-defined on $\widehat{D}$. The extended immersion $\widehat{f}$ being well-defined on $\widehat{D}$ is equivalent to $\widehat{f}$ being well-defined on every closed loop δ in $\widehat{D}$, i.e. $\delta : [0,1] \to \widehat{D}$ and τ is the Deck transformation associated to δ on the universal cover $\widetilde{D}$ of $\widehat{D}$.

Definition 5.20. A **Deck transformation** of a cover $p : C \to X$ is a homeomorphism $f : C \to C$ such that $p \circ f = p$. The set of all Deck transformations of p forms a group under composition, the **Deck transformation group** $\mathrm{Deck}(C/X)$. Every Deck transformation permutes the elements of each fiber and defines a group action of the Deck transformation group on each fiber.

Definition 5.21. Let δ be a closed loop and τ the associated Deck transformation on the universal cover. We call M_δ for which $F(\tau(z), \overline{\tau(z)}, \lambda) = M_\tau \cdot F(z, \bar{z}, \lambda)$ the **monodromy** of F.

Remark 5.22. The monodromy $\mathbb{C}^* \to SL(2, \mathbb{C})$, $\lambda \mapsto M_\lambda$ is a holomorphic map with essential singularities at $\lambda = 0$ and $\lambda = \infty$ and by construction takes values in $SU(2)$ for $|\lambda| = 1$.

We want to derive some properties of the monodromy and start with the so-called "closing conditions". These are necessary and sufficient conditions for f to be well-defined on the loop δ, i.e.

$$f(\tau(z), \overline{\tau(z)}) = f(z, \bar{z}).$$

Theorem 5.23. *Let F be an unitary frame with mondoromy $M_\tau(\lambda)$. Let $f(z, \bar{z}) = F_{\lambda_1} F_{\lambda_0}^{-1}$. Then the closing condition is $M_\tau(\lambda_0) = M_\tau(\lambda_1) = \pm \mathbb{1}$.*

Proof. The claim follows directly from the Sym-Bobenko formula:

$$\begin{aligned}
f(\tau(z), \overline{\tau(z)}) &= F(\tau(z), \overline{\tau(z)}, \lambda_1) F^{-1}(\tau(z), \overline{\tau(z)}, \lambda_0) \\
&= M_\tau(\lambda_1) F(z, \bar{z}, \lambda_1) \left(M_\tau(\lambda_0) F(z, \bar{z}, \lambda_0) \right)^{-1} \\
&= M_\tau(\lambda_1) F(z, \bar{z}, \lambda_1) F^{-1}(z, \bar{z}, \lambda_0) M_\tau^{-1}(\lambda_0) \\
&= M_\tau(\lambda_1) f(z, \bar{z}) M_\tau^{-1}(\lambda_0) = f(z, \bar{z}) \\
&\Leftrightarrow M_\tau(\lambda_0) = M_\tau(\lambda_1) = \pm \mathbb{1}.
\end{aligned}$$

□

Proposition 5.24. The monodromy satisfies

$$M(\bar{\lambda}^{-1}) = (\overline{M}^t(\lambda))^{-1}.$$

Proof. We have to show that $\alpha_{\bar{\lambda}^{-1}} = (\overline{\alpha_\lambda}^t)^{-1}$ holds. Inserting $\bar{\lambda}^{-1}$ into α_λ one gets

$$\alpha_{\bar{\lambda}^{-1}} = \frac{1}{2} \begin{pmatrix} u_z dz - u_{\bar{z}} d\bar{z} & i\bar{\lambda} e^u dz + i\overline{Q} e^{-u} d\bar{z} \\ iQe^{-u} dz + i\bar{\lambda}^{-1} e^u d\bar{z} & -u_z dz + u_{\bar{z}} d\bar{z} \end{pmatrix}.$$

On the other hand one has

$$\begin{aligned}
\overline{\alpha}_\lambda &= \frac{1}{2}\begin{pmatrix} u_{\bar z}d\bar z - u_z dz & -i\bar\lambda^{-1}e^u d\bar z - iQe^{-u}dz \\ -i\overline{Q}e^{-u}d\bar z - i\bar\lambda e^u dz & -u_{\bar z}d\bar z + u_z dz \end{pmatrix}, \\
\overline{\alpha}^t_\lambda &= \frac{1}{2}\begin{pmatrix} u_{\bar z}d\bar z - u_z dz & -i\overline{Q}e^{-u}d\bar z - i\bar\lambda e^u dz \\ -i\bar\lambda^{-1}e^u d\bar z - iQe^{-u}dz & -u_{\bar z}d\bar z + u_z dz \end{pmatrix}, \\
(\overline{\alpha}^t_\lambda)^{-1} &= \frac{1}{2}\begin{pmatrix} u_z dz - u_{\bar z}d\bar z & i\bar\lambda e^u dz + i\overline{Q}e^{-u}d\bar z \\ iQe^{-u}dz + i\bar\lambda^{-1}e^u d\bar z & -u_z dz + u_{\bar z}d\bar z \end{pmatrix}.
\end{aligned}$$

Since $dF_\lambda = F_\lambda\alpha_\lambda$ we have

$$F_{\bar\lambda^{-1}} = (\overline{F_\lambda}^t)^{-1}$$

and hence the result follows from the definition of the monodromy. □

Proposition 5.25. For the Pauli matrix σ_2 one has

(i) $\sigma_2 M(\lambda)\sigma_2 = (M(\lambda)^t)^{-1}$,

(ii) $\sigma_2\overline{M(\bar\lambda^{-1})}\sigma_2 = M(\lambda)$.

Proof.

(i) Again we are considering α_λ and note that from the previous proposition we can deduce

$$(\alpha^t_\lambda)^{-1} = \frac{1}{2}\begin{pmatrix} u_{\bar z}d\bar z - u_z dz & -i\lambda e^u d\bar z - iQe^{-u}dz \\ -i\overline{Q}e^{-u}d\bar z - i\lambda^{-1}e^u dz & -u_{\bar z}d\bar z + u_z dz \end{pmatrix}.$$

Computing $\sigma_2 M(\lambda)\sigma_2$ yields

$$\begin{aligned}
\sigma_2 M(\lambda)\sigma_2 &= \sigma_2\frac{1}{2}\begin{pmatrix} -\lambda^{-1}e^u dz - \overline{Q}e^{-u}d\bar z & -iu_z dz + iu_{\bar z}d\bar z \\ -iu_z dz + iu_{\bar z}d\bar z & Qe^{-u}dz + \lambda e^u d\bar z \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} u_{\bar z}d\bar z - u_z dz & -i\lambda e^u d\bar z - iQe^{-u}dz \\ -i\overline{Q}e^{-u}d\bar z - i\lambda^{-1}e^u dz & -u_{\bar z}d\bar z + u_z dz \end{pmatrix}
\end{aligned}$$

and the first claim is proved.

(ii) Note that (ii) is equivalent to $\overline{M(\bar\lambda^{-1})} = \sigma_2 M(\lambda)\sigma_2$ and that one has

$$\overline{(\alpha_{\bar\lambda^{-1}})} = \frac{1}{2}\begin{pmatrix} u_{\bar z}d\bar z - u_z dz & -i\lambda e^u d\bar z - iQe^{-u}dz \\ -i\overline{Q}e^{-u}d\bar z - i\lambda^{-1}e^u dz & -u_{\bar z}d\bar z + u_z dz \end{pmatrix}.$$

Hence the result follows.

□

We now return to the rotations in S^3 via the double cover

$$X \mapsto FXG^{-1} \quad \longleftrightarrow \quad X \mapsto RX$$

and consider the special case of $F = G$, that is

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_1^2 - c_2^2 - d_1^2 + d_2^2 & 2(c_1c_2 + d_1d_2) & -2c_1d_1 + 2c_2d_2 \\ 0 & -2c_1c_2 + 2d_1d_2 & c_1^2 - c_2^2 + d_1^2 - d_2^2 & 2(c_2d_1 + c_1d_2) \\ 0 & 2(c_1d_1 + c_2d_2) & 2c_2d_1 - 2c_1d_2 & c_1^2 + c_2^2 - d_1^2 - d_2^2 \end{pmatrix}.$$

The construction procedure of the Lawson surface Σ_g yields a reflection θ_g that acts like a rotation around $\frac{2\pi}{g+1}$. For $g = 2$ the corresponding mapping is

$$\theta_2 = \begin{pmatrix} A & \\ & \mathbb{1} \end{pmatrix} := \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 & 0 \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Going through the construction procedure of Σ_g one sees immediately that interchanging the roles of the P_i's and Q_j's one obtains the same surface, since one just assigns another north pole in S^3. Applying this consideration to the present case one gets

$$\theta_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ 0 & 0 & -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix},$$

and we may solve the above equations to obtain a $A \in SU(2)$ such that

$$AfA^{-1} \quad \longleftrightarrow \quad R \cdot f = \theta_2 \cdot f.$$

Thus we get the following equations

$$\begin{aligned} \text{(I)}\ c_1^1 - c_2^2 - d_1^2 + d_2^2 &= 1, \\ \text{(II)}\ c_1^2 - c_2^2 + d_1^2 - d_2^2 &= -\frac{1}{2}, \\ \text{(III)}\ c_1^2 + c_2^2 - d_1^2 - d_2^2 &= -\frac{1}{2}, \end{aligned}$$

as well as

$$\begin{aligned} \text{(IV)}\ c_1c_2 = -d_1d_2, &\qquad c_1c_2 = d_1d_2, \\ \text{(V)}\ c_1d_1 = -c_2d_2, &\qquad c_1d_1 = c_2d_2. \end{aligned}$$

Equation (II) and (III) give $c_2^2 = d_1^2$ and inserting that into equation (I) one gets $c_1^2 - 2c_2^2 + d_2^2 = 1$. Moreover equation (IV) yields $2d_1d_2 = 0 \Leftrightarrow d_1 = 0 \vee d_2 = 0$. Setting

$$c_1 := -\frac{1}{2}, \;\; c_2 = d_1 := 0, \;\; d_2 := -\frac{\sqrt{3}}{2}$$

one may check that equations (I)-(V) are fulfilled. Furthermore one has

$$\begin{aligned} 2(c_1 d_1 + c_1 d_2) &= \frac{1}{2}\sqrt{3} \\ 2c_2 d_1 - 2c_1 d_2 &= -\frac{1}{2}\sqrt{3} \end{aligned}$$

and thus A is of the form

$$A = \begin{pmatrix} -\frac{1}{2} & -i\frac{\sqrt{3}}{2} \\ -i\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

Summing up and considering the general genus g case one has the following

Theorem 5.26. *The immersion $f : \Sigma_g \to S^3$ is equivariant with respect to the diffeomorphism θ_g, i.e.*

$$f(\theta_g(z), \overline{\theta_g(z)}) = Af(z, \bar{z})A^{-1}, \quad A^{g+1} = \mathbb{1}.$$

Proof. The group generated by (A, A) under the adjoint group action $(F, G) : X \mapsto FXG^{-1}$ in $SU(2)$ is cyclic and of order $(g+1)$, since $A^{g+1} = \mathbb{1}$. Thus

$$G_2 = \langle (A, A) \rangle \simeq \mathbb{Z}^{g+1}.$$

For $G_1 = \langle \theta_g \rangle \simeq \mathbb{Z}^{g+1}$ one therefore has

$$f(g(z), \overline{g(z)}) = A_g f(z, \bar{z}) A_g^{-1} \quad \forall g \in \mathbb{Z}^{g+1}$$

and the theorem is proved. □

Corollary 5.27. The almost conformal immersion $g = f \circ p_1 : Y \to S^3$ is equivariant with respect to the lifted diffeomorphism $\widetilde{\theta}_g$,

$$g(\widetilde{\theta}_g(y), \overline{\widetilde{\theta}_g(y)}) = Ag(y, \bar{y})A^{-1}, \quad A^{g+1} = \mathbb{1}.$$

Proof. Since $p_1 : Y \to \Sigma_g$ has branch points at the zeros of $Q(z)dz^2$ on Σ_g, the symmetry induced by θ_g can be lifted to Y. □

Theorem 5.28. *For the diffeomorphism θ_g the extended frame F_λ obeys the following transformation rules*

$$F_{\lambda_0} \circ \theta_g = AF_{\lambda_0}A^{-1}, \quad F_{\lambda_1} \circ \theta_g = AF_{\lambda_1}A^{-1}, \quad \lambda_0, \lambda_1 \in S^1.$$

Proof. We have already seen that we can find $\lambda_0, \lambda_1 \in S^1$ such that f can be written as

$$f = F_{\lambda_1} F_{\lambda_0}^{-1},$$

and therefore we get (since $f(\theta_g(z), \overline{\theta_g(z)}) = Af(z, \bar{z})A^{-1}$)

$$F_{\lambda_0} \circ \theta_g = AF_{\lambda_0}B^{-1}, \quad F_{\lambda_1} \circ \theta_g = AF_{\lambda_1}B^{-1}.$$

Note that the point $\widetilde{P}_1 = (1,0,0,0)$ corresponds to $\mathbb{1} \in SU(2)$ and therefore the point z_0 on Σ_g can be chosen such that $z_0 := 0$ corresponds to $\widetilde{P}_1$ in a neighborhood of z_0. Since z_0 is a zero of the Hopf differential it is a fixed point of the diffeomorphism θ_g. In the above construction we have precisely made use of that fact. The frames for f are obtained by integrating the equation

$$dF_\lambda = F_\lambda \alpha_\lambda$$

with initial condition $F_\lambda(z_0) = \mathbb{1}$. Considering the initial condition for $\lambda = \lambda_{0,1}$ one sees

$$\begin{aligned} F_\lambda(0) &= F_\lambda(\theta_g(0)) = AF_\lambda(0)B^{-1} = \mathbb{1} \\ &\Leftrightarrow B = A. \end{aligned}$$

Rotating λ_0 and λ_1 while keeping the angle between them fixed, one obtains the same result for the whole family associated to Σ_g. □

Recall that a smooth solution u of the Gauss and Codazzi equations must also exist at the zeros of the Hopf differential. Applying the transformation rules for the extended frames and the preceding considerations now yields the following

Theorem 5.29. *Considering the $(g+1)$-fold cover $p_2 : Y \to X$ one obtains a monodromy M_λ around the 4 distinguished points on X with*

$$M_\lambda^{g+1} = \mathbb{1}.$$

Proof. Due to theorem 5.4 we see that for all $\lambda \in \mathbb{C}^*$, u is a solution of the reduced Gauss equation if and only if the zero-curvature condition for α_λ, that is

$$2d\alpha_\lambda + [\alpha_\lambda \wedge \alpha_\lambda] = 0,$$

is fulfilled. But the zero-curvature condition is an integrability condition and thus we can integrate to obtain a corresponding extended frame F_λ. We know that such a smooth solution must also exist at the zeros of the Hopf differential. Therefore there is no monodromy on Σ_g.

Now consider a loop $\widehat{\delta}$ on X. Transforming the frame on Σ_g to a frame $\widehat{F}_\lambda$ on X we have already seen that the zero-curvature condition is no longer valid at the distinguished points corresponding to the zeros of the Hopf differential on Σ_g. Thus one obtains a monodromy

$$M_\lambda(\tau) = \tau^*(\widehat{F}_\lambda)\widehat{F}_\lambda^{-1}$$

for the corresponding Deck transformation τ. From the theory of covering spaces we know that going around the loop $\widehat{\delta}$ for $(g+1)$ times results in one loop $\widetilde{\delta}$ on the covering surface Y and this in turn corresponds to a 2-fold loop δ on the surface Σ_g. Combining this consideration with the invariance of the frame $\widehat{F}_\lambda$ under coordinate changes induced by the covering spaces one obtains a monodromy on X with

$$M_\lambda^{g+1} = \mathbb{1} \ \ \forall\, \lambda \in \mathbb{C}^*.$$

□

5.4 The spectral curve for Σ_g

We have already introduced the notion of monodromy and will now construct a hyperelliptic Riemann surface from a solution $\widehat{u}$ of the sinh-Gordon equation. This surface is the so-called *spectral curve.*

Although the monodromy depends on the choice of the base point z_0 the conjugacy classes and hence the eigenvalues do not. Given a conformal immersion of a torus, i.e. $f : \mathbb{R}^2/\Gamma \to SU(2)$, with lattice

$$\Gamma = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z},$$

and corresponding extended frame F_λ one can consider the monodromies $M_1(\lambda),\ M_2(\lambda)$ of F_λ with respect to ω_1 and ω_2. Denoting the corresponding eigenvalues with $\mu_1,\ \mu_2$ one can make the following definition.

Definition 5.30. The **spectral curve** of a conformally immersed torus in S^3 is the hyperelliptic curve given by

$$\Sigma_f = \{(\lambda, \mu_1, \mu_2) \,|\, \det(\mu_1\mathbb{1} - M_1(\lambda)) = \det(\mu_2\mathbb{1} - M_2(\lambda)) = 0\}.$$

The following theorem yields a description of CMC tori in terms of spectral curves.

Theorem 5.31. *Let Y be a hyperelliptic Riemann surface with branch points over $\lambda = 0$ (y^+) and $\lambda = \infty$ (y^-). Then Y is the spectral curve of an immersed CMC torus in S^3 if and only if the following four conditions hold:*

(i) Besides the hyperelliptic involution σ, the surface Y has two further anti-holomorphic involutions η and $\rho = \eta \circ \sigma = \sigma \circ \eta$, such that η has no fixed points and $\eta(y^+) = y^-$.

(ii) There exist two non-zero holomorphic functions $\mu_1,\ \mu_2$ on $Y\backslash\{y^+, y^-\}$ such that for $i = 1, 2$

$$\sigma^*\mu_i = \mu_i^{-1},\ \eta^*\bar{\mu}_i = \mu_i,\ \rho^*\bar{\mu}_i = \mu_i^{-1}.$$

(iii) The forms $d\ln\mu_i$ are meromorphic differentials of the second kind with double poles at $y^\pm$. The singular parts at y^+ respectively y^- of these two differentials are linearly independent.

(iv) *There are four fixed points* $y_1,\ y_2 = \sigma(y_1),\ y_3,\ y_4 = \sigma(y_3)$ *of* ρ*, such that the functions* μ_1 *and* μ_2 *are either* 1 *or* -1 *there.*

Proof. We only give a short sketch of the proof and consider the "only if"-part. Thus we have to verify the four conditions stated above (for details see [29]).

(i) From proposition 5.24 and 5.25 one obtains the existence of the three involutions. Obviously $\rho = \eta \circ \sigma = \sigma \circ \eta$ holds as well as $\eta(y^+) = y^-$. To complete the proof we have to check that η has no fixed points: If ν is an eigenvector of M_λ then $\bar{\nu}$ is an eigenvector of $\overline{M}_{\bar{\lambda}^{-1}}$ since

$$\begin{aligned} \det(\mu\mathbb{1} - M_\lambda) &= \det(\mu\mathbb{1} - \sigma_2\overline{M}_{\bar{\lambda}^{-1}}\sigma_2) \\ &= \det(\sigma_2(\mu\mathbb{1} - \overline{M}_{\bar{\lambda}^{-1}})\sigma_2) \\ &= \det(\mu\mathbb{1} - \overline{M}_{\bar{\lambda}^{-1}}) = 0. \end{aligned}$$

We further have

$$\overline{M}_{\bar{\lambda}^{-1}}\bar{\nu} = \overline{M_{\bar{\lambda}^{-1}}\nu} = \overline{\bar{\mu}\nu} = \mu\bar{\nu}.$$

With $\overline{M}_{\bar{\lambda}^{-1}} = \sigma_2 M_\lambda \sigma_2$ we get

$$\begin{aligned} &\overline{M}_{\bar{\lambda}^{-1}}\bar{\nu} = \mu\bar{\nu} \\ \Leftrightarrow\ &\sigma_2 M_\lambda \sigma_2 \bar{\nu} = \mu\bar{\nu} \\ \Leftrightarrow\ &M_\lambda \sigma_2 \bar{\nu} = \mu\sigma_2\bar{\nu} \end{aligned}$$

and therefore $\sigma_2\bar{\nu}$ is an eigenvector of M_λ. If η would have fixed points, the eigenvectors of M_λ would linearly depend on each other, i.e. $\sigma_2\bar{\nu} = \gamma\nu$. But this would imply

$$-\bar{\nu} = \overline{\sigma_2}\sigma_2\bar{\nu} = \gamma\overline{\sigma_2}\nu = \gamma\overline{(\sigma_2\bar{\nu})} = \gamma\bar{\gamma}\bar{\nu}$$

and therefore $\gamma\bar{\gamma} = -1$, which is a contradiction. Hence the eigenvectors are linearly independent and η has no fixed points.

(ii) With the help of proposition 5.24 and 5.25 we compute

$$\begin{aligned} \overline{P(\bar{\lambda}^{-1}, \bar{\mu})} &= \overline{\det(\bar{\mu}\mathbb{1} - M(\bar{\lambda}^{-1}))} = \det(\mu\mathbb{1} - \overline{M}(\bar{\lambda}^{-1})) \\ &= \det(\mu\mathbb{1} - \sigma_2 M(\lambda)\sigma_2) = P(\mu, \lambda), \\ P(\lambda, \mu^{-1}) &= \det(\mu^{-1}\mathbb{1} - M(\lambda)) = \det(\mu^{-1}\mathbb{1} - \sigma_2(M^t(\lambda))^{-1}\sigma_2) \\ &= \det(\mu^{-1}\mathbb{1} - (M^t(\lambda))^{-1}) = \det(\mu\mathbb{1} - M^t(\lambda)) \\ &= P(\lambda, \mu), \\ \overline{P(\bar{\lambda}^{-1}, \bar{\mu}^{-1})} &= \overline{\det(\bar{\mu}^{-1}\mathbb{1} - M(\bar{\lambda}^{-1}))} = \det(\mu^{-1}\mathbb{1} - \overline{M}(\bar{\lambda}^{-1})) \\ &= \det(\mu^{-1}\mathbb{1} - (M^t(\lambda))^{-1}) = P(\lambda, \mu). \end{aligned}$$

(iii) To prove the claim one has to introduce the *Baker-Akhiezer function* and this is done in [3]. Following the arguments explained in [29] yields the claim.

(iv) This follows directly from the closing conditions, since $\rho^*\bar{\lambda} = \lambda^{-1} = \bar{\lambda}$ if and only if $|\lambda| = 1$. Moreover $\sigma^*\lambda = \lambda$ and with $M_{\lambda_0} = M_{\lambda_1} = \pm\mathbb{1}$ the result follows.

□

Considering the Riemann surface X one obtains the following

Conclusion 5.32. The monodromy M_λ on X has eigenvalues that are $(g+1)$-roots of unity. Hence the associated spectral curve is trivial.

Chapter 6

Conclusions and outlook

In this chapter we summarize the results of this work, especially those which are new. We also give some remarks on other interesting questions that are beyond the scope of this thesis.

First we introduced the construction procedure for the surfaces Σ_g according to [24] and [25] that strongly relies on the *reflection principle*, i.e. reflection across a geodesic γ in S^3. The surfaces themselves are patched together by isometric copies of the same "initial surface" $\mathcal{M}_{\Gamma_g}$ which is a solution to the famous Plateau Problem. In the present case the boundary is a geodesic polygon Γ_g, i.e. a polygon in S^3 that is composed of geodesic arcs γ_i. In the following we gave an outline of the techniques applied by Lawson (like the conditions posed upon the geodesic polygon Γ_g) to ensure that reflection across these boundary arcs produces a complete, non-singular submanifold in S^3. Moreover we proved that the subgroup of $O(4)$ generated by the reflections across the boundary arcs (denoted by G_{Γ_g}) is D_{2g+2}, i.e.

$$G_{\Gamma_g} = \mathbb{Z}_{2g+2} \rtimes \mathbb{Z}_2 \simeq D_{2g+2},$$

where D_n denotes the dihedral group of order $2n$. It was also shown that for each g the surface Σ_g is a hyperelliptic Riemann surface with reduced automorphism group D_{2g+2} and therefore

$$\Sigma_g : w^2 = z^{2g+2} - 1$$

describes Σ_g as an algebraic curve.

In the following we were dealing with the Hopf differential Qdz^2 of Σ_g and proved that up to phase-scaling one has

$$Qdz^2 = a\frac{z^{g-1}dz^2}{w^2},$$

where we have made use of the fact that Qdz^2 stays invariant under certain coordinate changes induced by G_{Γ_g}. We also determined the zeros of Q

and discovered their significance for the following part. Introducing a local parameter $w = \Phi(z)$ and the techniques necessary in order to deal with trajectories of quadratic differentials (see [33]), we could prove a canonical triangulation for Σ_g with a certain form of trajectory rays as edges and the zeros of Qdz^2 as vertices.

Finally we focused on the moduli problem for the genus $g \geq 2$ case and therefore investigated the nature of the object associated to Σ_g. We showed that the surface Σ_g is equivariant with respect to a subgroup of diffeomorphisms induced by the reflection across the boundary arcs. Considering the monodromy for an extended moving frame F_λ on a torus, one can introduce the notion of the *spectral curve*. We proved that starting with the surface Σ_g one has to consider doubly periodic solutions $\widehat{u}$ to the sinh-Gordon equation with singularities at distinguished points on a torus X, namely the points corresponding to the zeros of the Hopf differential Q of Σ_g. This was achieved by introducing coverings $p : Y \to X$ that induce coordinate transformations of the form $z \mapsto w = w(z)$. The extended frame F_λ is invariant under these coordinate transformations and Σ_g is covered by a hyperelliptic Riemann surface Y that in turn covers the torus X. Considering the extended frame $\widehat{F}_\lambda$ on X we proved that one obtains a monodromy around the distinguished points that satisfies

$$M_\lambda^{g+1} = \mathbb{1},$$

i.e. the corresponding spectral curve is trivial. Thus it might be possible to merge the knowledge gained from the study of spectral curves and the fact that $M_\lambda^{g+1} = \mathbb{1}$ at the distinguished points in order to get a description for CMC-surfaces of higher genus g.

Many open questions are related to the above results. For example one could investigate how the triangulation transforms if one passes to the torus X. Knowing the symmetry group, one may pose symmetry-conditions that must be fulfilled by solutions $\widehat{u}$ of the sinh-Gordon equation. On a torus the two periods induced by the lattice $\Gamma = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ commute, but in the present situation this is not the case. Therefore one has to investigate the behaviour of periods that result from a loop around a distinguished point.

For future research it also might be of interest to consider the asymptotic analysis of the monodromy, that is the behavior of M_λ around $\lambda = 0$ and $\lambda = \infty$, where M_λ has essential singularities.
For this purpose one has to express the monodromy in terms of polar coordinates. Then traversing a loop is equivalent to adding a period p. This periodicity corresponds to a real translation and therefore one may solve

$$dF_\lambda = F_\lambda \alpha_\lambda, \ \ F_\lambda(0) = \mathbb{1},$$

along the real axis. Considering a trivial solution of the Maurer-Cartan equation (the zero-curvature equation) related to α_λ it is possible to obtain the monodromy $M_0(\lambda)$ corresponding to the "vacuum"-solution $F_0(x, y, \lambda)$. The goal would be to find a bound for the monodromy $M(\lambda)$ in terms of $M_0(\lambda)$ as one approaches the critical points $\lambda = 0, \infty$.

It will also be convenient to reformulate the stated results in the language elucidated in [6] and [10], that is in the language of quaternionic holomorphic geometry. Considering a quaternionic line bundle V with complex structure S, it is natural to investigate the connection ∇ and to introduce a S^1-family of flat quaternionic connections

$$\nabla_\lambda = \nabla + (\lambda - 1)A$$

with $\lambda = e^{\theta S}$. One also obtains a smooth map

$$f : M \to S^3 \subset \mathbb{H}$$

satisfying

$$f^{-1}df = (\lambda - 1)A.$$

It would be interesting to reflect the results in this more general setting.

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