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Deformation of constant mean curvature tori in a three sphere

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Abstract

This thesis deals with deformation of spectral curves of constant mean curvature tori in $\mathbb{S}^3$. At first the key points of the theory of surfaces of constant mean curvature are brought together. The connection between conformal maps into different space forms an the sinh-Gordon equation is established. Then the periodic solutions of the sinh-Gordon equation are analyzed and the notion of a spectral curve is explained. After that a particular deformation preserving certain parameters of the spectral curve is described. This deformation produce a one-parameter family of spectral curves of tori in $\mathbb{S}^3$ from a given spectral curve of a torus in $\mathbb{S}^3$. It will be shown where this deformation can branch from spectral genus 0 to a higher genus in particular to genus 2. The spectral curves at this branch points are computed explicitly. A special class of homogeneous tori is defined and it will be shown that if a spectral curve of some the tori in this class is taken as the start point of the deformation the mean curvature is going to infinity during the deformation. The exact conditions for this behavior are established and it will also be proved that this conditions are in fact not only necessary but also sufficient. This result will give a deformation path of spectral curves from a homogeneous torus in $\mathbb{S}^3$ through some tori of spectral genus 2 in $\mathbb{S}^3$ to a torus in $\mathbb{R}^3$.

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CHAPTER 1

Introduction

The surfaces with constant mean curvature have been of big interest to mathematicians since the middle of the nineteenth century. In the eighties of the twentieth century this topic regained new interest with the discovery of the first immersed but not embedded tori in $\mathbb{R}^3$ by Wente [11]. The theory was developed further by Hitchin [5], by Pinkall and Sterling [7] and by Bobenko [1, 2, 3]. In particular the latter worked out the relation between constant mean curvature tori in different space forms, periodic solutions of the sinh-Gordon equation and hyperelliptic Riemann surfaces. These surfaces are called spectral curves. A deformation on spectral curves of constant mean curvature tori in $\mathbb{S}^3$ was introduced by Kilian and Schmidt [6]. The underlying idea to this deformation was established by Grinevich and Schmidt [4]. This deformation generates for a given spectral curve of a constant mean curvature torus a one-parameter family of spectral curves of constant mean curvature tori. The closing conditions are fulfilled by all the members of this family. Certain spectral curves in such families may have double points and at these curves it is possible to branch the deformation to a higher spectral genus. The study of endpoints of such deformations reveal some interesting behavior. In particular some deformation of spectral curves of constant mean curvature tori in $\mathbb{S}^3$ end in spectral curves of constant mean curvature tori in $\mathbb{R}^3$. This gives an explicit example of how some CMC tori in $\mathbb{R}^3$ can be seen as limits of CMC tori in $\mathbb{S}^3$. This is also true in general, see Umehara and Yamada [10].

The main part of this thesis is divided into three chapters. In the chapter 2 we recall some key points of the theory of constant mean curvature surfaces. We look at conformal immersions into the euclidean space and into the three sphere and show the definition of extended frames. We point out the relation between constant mean curvature surfaces and solutions of the sinh-Gordon equation and at the end of the chapter we also recall the formulas which construct an immersion from an extend frame.

The chapter 3 will deal with periodic solutions of the sinh-Gordon equation. We will investigate the monodromy of such solutions and explain the notion of a spectral curve. Here we will obtain the important conditions that have to be fulfilled in order for a hyperelliptic Riemann surface to be a spectral curve of a constant mean curvature torus in either $\mathbb{S}^3$ or $\mathbb{R}^3$. We also show an efficient way to encode the spectral data in few real values.

The chapter 4 is the main chapter of the thesis. Here we will introduce the already mentioned deformation on the spectral curves of constant mean

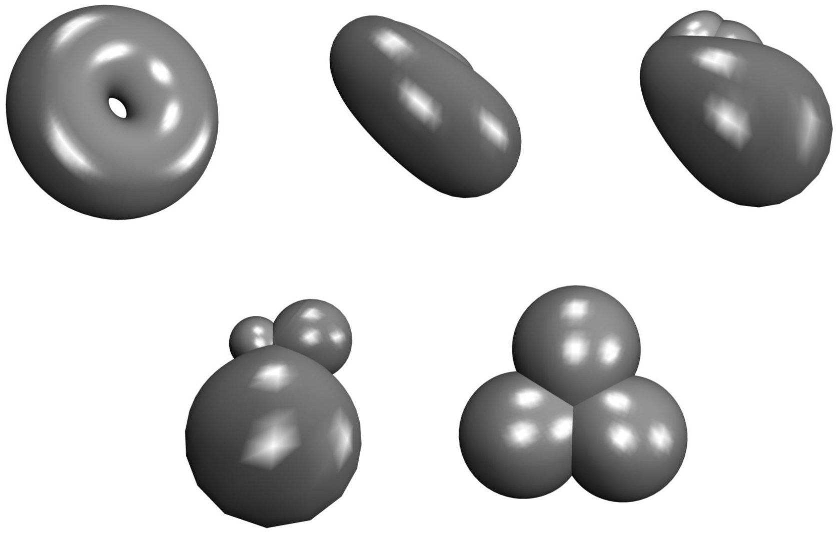

FIGURE 1-1. Deformation path starting from a homogeneous torus in $\mathbb{S}^3$ and ending in a Wente torus in $\mathbb{R}^3$ with some typical tori on the path.

curvature tori and analyze the behaviour of this deformation. We will show at which point it is possible for the deformation to branch from spectral genus $g = 0$ to $g = 2$ and we will compute the spectral data of these branch points. In the second part of this chapter we will focus our interest on a special class of spectral curves and prove some properties of the deformation on these curves. The last part of this chapter will deal with the problem of finding the necessary and sufficient conditions on the spectral curves of the mentioned special class so that during the deformation the mean curvature reaches infinity at the end of the deformation. We will see that the spectral curve at the end of such deformation is actually a spectral curve of a CMC torus in $\mathbb{R}^3$. A numeric computation will also show that one of those tori is the already mentioned Wente torus. This means that we will show a deformation path from a homogeneous CMC torus in $\mathbb{S}^3$ to a Wente torus in $\mathbb{R}^3$. In the course of this presentation we will show some images of CMC tori. All those images were created witch Nick Schmitt's CMClab [8] based on numerical spectral data obtained by computing the deformation.

CHAPTER 2

Conformal immersions, sinh-Gordon equation, and Sym-Bobenko formulas

In the following we will gather some useful standard facts about conformal immersions into the three dimensional space forms $\mathbb{S}^3$ and $\mathbb{R}^3$. In particular we will look at an equation for the mean curvature of these immersions. Then we will define an extended frame and illustrate its relation to the sinh-Gordon equation. At the end of the chapter we will also recall the Sym-Bobenko formulas which construct an immersion from a given extended frame. The exposition follows the one made in [9].

2.1. Conformal immersions into $\mathbb{S}^3$ and $\mathbb{R}^3$

We will look at the matrix Lie group SU_2. The Lie algebra $\mathfrak{su}_2$ of this group is equipped with a commutator $[\cdot,\cdot]$. Let $\alpha,\beta \in \Omega(T\mathbb{R}^2, \mathfrak{su}_2)$ be smooth 1-forms on $\mathbb{R}^2 \cong T\mathbb{R}^2$ with values in $\mathfrak{su}_2$. We define now a $\mathfrak{su}_2$-valued 2-form

$$[\alpha \wedge \beta](X, Y) = [\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)] \tag{2.1}$$

for $X, Y \in T\mathbb{R}^2$. Let $L_g : h \mapsto gh$ be the left multiplication in SU_2. By left translation we obtain an isomorphism of the tangential bundle $T\,\mathrm{SU}_2 \cong \mathrm{SU}_2 \times \mathfrak{su}_2$. We also have a Maurer-Cartan form

$$\theta : T\,\mathrm{SU}_2 \to \mathfrak{su}_2, v_g \mapsto (dL_{g^{-1}})_g v_g$$

which satisfies the Maurer-Cartan equation

$$2d\theta + [\theta \wedge \theta] = 0. \tag{2.2}$$

For a map $F : \mathbb{R}^2 \to \mathrm{SU}_2$, the pullback $\alpha = F^*\theta$ satisfies (2.2) as well. The converse is also true, every solution $\alpha \in \Omega^1(\mathbb{R}^2, \mathfrak{su}_2)$ of (2.2) integrates to a smooth map $F : \mathbb{R}^2 \to \mathrm{SU}_2$ with $\alpha = F^*\theta$.

We now complexify the tangent bundle $T\mathbb{R}^2$ and decompose $(T\mathbb{R}^2)^{\mathbb{C}} = T'\mathbb{R}^2 \oplus T''\mathbb{R}^2$ into $(1,0)$ and $(0,1)$ tangent spaces and write $d = \partial + \bar{\partial}$. We also decompose

$$\Omega^1(\mathbb{R}^2, \mathfrak{sl}_2(\mathbb{C})) = \Omega'(\mathbb{R}^2, \mathfrak{sl}_2(\mathbb{C})) \oplus \Omega''(\mathbb{R}^2, \mathfrak{sl}_2(\mathbb{C}))$$

using ${\mathfrak{su}_2}^{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$. We split $\omega \in \Omega^1(\mathbb{R}^2, \mathfrak{sl}_2(\mathbb{C}))$ accordingly into the $(1,0)$ part ω' and the $(0,1)$ part ω'' writing $\omega = \omega' + \omega''$. Finally, we set the $*$-operator on $\Omega^1(\mathbb{R}^2, \mathfrak{sl}_2(\mathbb{C}))$ to $*\omega = -i\omega' + i\omega''$.

For further computations we fix the following basis of $\mathfrak{sl}_2(\mathbb{C})$:

$$\epsilon_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad \epsilon_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \tag{2.3}$$

Moreover let $\langle \cdot, \cdot \rangle$ be the billinear extension of the Ad-invariant inner product on $\mathfrak{su}_2$ to $\mathfrak{su}_2{}^{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$, such that $\langle \epsilon, \epsilon \rangle = 1$. For $X, Y \in \mathfrak{su}_2$ we further have

$$\langle X, Y \rangle = -\frac{1}{2} \operatorname{tr} XY, \quad \|X\| = \sqrt{\det X}, \quad X \times Y = \frac{1}{2}[X, Y]. \tag{2.4}$$

So following relations arise

$$\begin{gathered} \langle \epsilon_-, \epsilon_- \rangle = \langle \epsilon_-, \epsilon \rangle = 0, \quad \epsilon_-^* = \epsilon_+, \\ [\epsilon, \epsilon_-] = 2i\epsilon_-, \quad [\epsilon_+, \epsilon] = 2i\epsilon_+, \quad [\epsilon_-, \epsilon_+] = i\epsilon. \end{gathered} \tag{2.5}$$

2.1.1. Euclidean three space. Now we shall prove a formula for the mean curvature of a conformal immersion into $\mathbb{R}^3$.

LEMMA 2.1. *The mean curvature H of a conformal immersion $f : \mathbb{R}^2 \to \mathfrak{su}_2$ is given by $2d * df = H[df \wedge df]$.*

PROOF. Let $U \subset \mathbb{R}^2$ be an open simply connected set with a coordinate $z : U \to \mathbb{C}$. Write $df' = f_z dz$ and $df'' = f_{\bar{z}} d\bar{z}$. The conformality of the map f is equivalent to $\langle f_z, f_z \rangle = \langle f_{\bar{z}}, f_{\bar{z}} \rangle = 0$ and the existence of a function $v \in C^\infty(U, \mathbb{R}^+)$, such that $\langle f_z, f_{\bar{z}} \rangle = \frac{1}{2} v^2$. Let $N : U \to \mathbb{S}^2$ be the Gauss map and $F : U \to \mathrm{SU}_2$ a frame such that $N = F \epsilon F^{-1}$, $f_z = vF\epsilon_- F^{-1}$ and $f_{\bar{z}} = vF\epsilon_+ F^{-1}$. The mean curvature is $H = 2v^{-2} \langle f_{z\bar{z}}, N \rangle$ and the Hopf differential Qdz^2 is given by $Q = \langle f_{zz}, N \rangle$. We compute $[df \wedge df] = 2iv^2 N dz \wedge \bar{z}$. Another computation shows

$$\begin{aligned} F^{-1}dF = \frac{1}{2v}\Big(&- (v^2 H dz + 2\bar{Q} d\bar{z}) i\epsilon_- + (2Q dz + v^2 H d\bar{z}) i\epsilon_+ \\ &- (v_z dz - v_{\bar{z}} d\bar{z}) i\epsilon \Big). \end{aligned}$$

Now we can compute $d * df = iv^2 HN dz \wedge d\bar{z}$ and by combining this with the result for $[df \wedge df]$ we have proved the claim. □

2.1.2. The three sphere. There exists of course also a similar equation for the mean curvature of a conformal immersion into $\mathbb{S}^3$ which we will prove in the following.

LEMMA 2.2. *Let $f : \mathbb{R}^2 \to \mathbb{S}^3$ be a conformal immersion and $\omega = f^{-1}df$ then the mean curvature H is given by $2d * \omega = H[\omega \wedge \omega]$.*

PROOF. Let $U \subset \mathbb{R}^2$ be an open simply connected set with a coordinate $z : U \to \mathbb{C}$. As in the case of euclidean three space we write $df' = f_z dz$ and $df'' = f_{\bar{z}} d\bar{z}$. The conformality of the map f is equivalent to $\langle f_z, f_z \rangle = \langle f_{\bar{z}}, f_{\bar{z}} \rangle = 0$ and the existence of a function $v \in C^\infty(U, \mathbb{R}^+)$, such that $\langle f_z, f_{\bar{z}} \rangle = \frac{1}{2} v^2$. The left invariance gives us $\langle \omega', \omega' \rangle = \langle df', df' \rangle$ so the conformality becomes $\langle \omega', \omega' \rangle = 0$.

Let be $F, G : U \to \mathrm{SU}_2$ two smooth maps which transform the basis $\{Id, \epsilon_-, \epsilon_+, \epsilon\}$ to a frame $\{f, f_z, f_{\bar{z}}, N\}$. In this case we have $f = FG^{-1}$, $df = vF(\epsilon_- dz + \epsilon_+ d\bar{z})G^{-1}$ and $N = F\epsilon G^{-1}$. Setting $\alpha = F^{-1}dF$ and $\beta = G^{-1}dG$ we obtain

$$\alpha = \left(-\frac{1}{2}v(H+i)dz - v^{-1}\bar{Q}d\bar{z}\right)i\epsilon_- + \left(v^{-1}Qdz + \frac{1}{2}v(H-i)d\bar{z}\right)i\epsilon_+ - \left(\frac{1}{2}v^{-1}v_z dz - \frac{1}{2}v^{-1}v_{\bar{z}}d\bar{z}\right)i\epsilon$$

and

$$\beta = \left(-\frac{1}{2}v(H-i)dz - v^{-1}\bar{Q}d\bar{z}\right)i\epsilon_- + \left(v^{-1}Qdz + \frac{1}{2}v(H+i)d\bar{z}\right)i\epsilon_+ - \left(\frac{1}{2}v^{-1}v_z dz - \frac{1}{2}v^{-1}v_{\bar{z}}d\bar{z}\right)i\epsilon$$

We can now compute

$$\begin{aligned}\omega = f^{-1}df &= (FG^{-1})^{-1}d(FG^{-1}) = GF^{-1}d(FG^{-1})\\ &= GF^{-1}dFG^{-1} + GF^{-1}FdG^{-1} = G(F^{-1}dF + dG^{-1}G)G^{-1}\\ &= G(F^{-1}dF + G^{-1}dGG^{-1}G)G^{-1} = G(\alpha-\beta)G^{-1}.\end{aligned}$$

Using $\omega = G(\alpha-\beta)G^{-1}$ together with the former computations we obtain

$$d * \omega = iv^2HG\epsilon G^{-1}dz \wedge d\bar{z}. \tag{2.6}$$

On the other hand a computation also reveals

$$[\omega \wedge \omega] = 2iv^2G\epsilon G^{-1}dz \wedge d\bar{z} \tag{2.7}$$

So by combining (2.6) and (2.7) we have proved the claim. □

2.1.3. Extended frames. In the following we will introduce the concept of an extended frame and explain its significance for conformal immersions.

Lemma 2.3. *Let $f : \mathbb{R}^2 \to \mathrm{SU}_2$ be a conformal immersion with non-zero mean curvature $H \neq 0$ then there exist a $\mathbb{S}^1$-family of conformal immersions.*

Proof. Let $\omega = f^{-1}df$. Let the mean curvature of f be H. We have seen that then both equations

$$2d * \omega = H[\omega \wedge \omega], \qquad 2d\omega + [\omega \wedge \omega] = 0 \tag{2.8}$$

are fulfilled. Combining these equations we obtain $d\omega + H^{-1}d * \omega = 0$ and after splitting up

$$(1 - iH^{-1})d\omega' + (1 + iH^{-1})d\omega'' = 0. \tag{2.9}$$

From (2.8) we obtain $2d\omega'' = -2d\omega' - [\omega \wedge \omega]$ and $2d\omega' = -2d\omega'' - [\omega \wedge \omega]$. Inserting these into (2.9) gives us $4d\omega' = -(1 - iH)[\omega \wedge \omega]$ and $4d\omega'' = -(1 + iH)[\omega \wedge \omega]$. Then we see that

$$\alpha_\lambda = \frac{1}{2}(1 - \lambda^{-1})(1 + iH)\omega' + \frac{1}{2}(1 - \lambda)(1 - iH)\omega''$$

satisfies $2d\alpha_\lambda + [\alpha_\lambda \wedge \alpha_\lambda] = 0$ for all $\lambda \in \mathbb{C}^\times$. So there exists a solution $F_\lambda : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathfrak{su}_2$ for $dF_\lambda = F_\lambda \alpha_\lambda$ with the initial condition $F_\lambda(0) = \mathbb{1}$. This solution is called an *extended frame*.

Let now $\lambda_0, \lambda_1 \in \mathbb{S}^1, \lambda_0 \neq \lambda_1$. We claim that $\hat{f} : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathrm{SU}_2$, $\hat{f} = F_{\lambda_1} F_{\lambda_0}^{-1}$ is a conformal immersion with a constant mean curvature

$$\hat{H} = i \frac{\lambda_0 + \lambda_1}{\lambda_0 - \lambda_1}.$$

Let $\hat{\omega} = \hat{f}^{-1} d\hat{f} = F_{\lambda_0}(\alpha_{\lambda_1} - \alpha_{\lambda_0})F_{\lambda_0}^{-1}$. It follows then

$$\hat{\omega} = \frac{1}{2} F_{\lambda_0}\Big((\lambda_0^{-1} - \lambda_1^{-1})(1 + iH)\omega' + (\lambda_0 - \lambda_1)(1 - iH)\omega''\Big)F_{\lambda_0}^{-1}.$$

The left invariance leads to $\langle \omega', \omega' \rangle = \langle df', df' \rangle$ and so we obtain $\langle \hat{\omega}', \hat{\omega}' \rangle = 0$. This confirms that $\hat{f}$ is a conformal map since f was assumed to be conformal.

We now compute

$$d * \hat{\omega} = \frac{i}{4}(\lambda_1\lambda_0^{-1} - \lambda_0\lambda_1^{-1})(1 + H^2)F_{\lambda_0}[\omega' \wedge \omega'']F_{\lambda_0}^{-1}$$

and

$$[\hat{\omega} \wedge \hat{\omega}] = \frac{1}{2}(1 - \lambda_1\lambda_0^{-1})(1 - \lambda_0\lambda_1^{-1})(1 + H^2)F_{\lambda_0}[\omega' \wedge \omega'']F_{\lambda_0}^{-1}.$$

Both equations together show that $\hat{H} = i\frac{\lambda_0+\lambda_1}{\lambda_0-\lambda_1}$ is the curvature of $\hat{f}$. □

2.2. The sinh-Gordon equation

This section will illustrate the relation between conformal immersions with constant mean curvature and solutions of the sinh-Gordon equation.

Proposition 2.4. *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. Let*

$$\alpha_\lambda = \frac{1}{2}\begin{pmatrix} u_z dz - u_{\bar{z}} d\bar{z} & i\lambda^{-1}e^u dz + ie^{-u}d\bar{z} \\ ie^{-u}dz + i\lambda e^u d\bar{z} & -u_z dz + u_{\bar{z}} d\bar{z} \end{pmatrix}.$$

Then the equation $2d\alpha_\lambda + [\alpha_\lambda \wedge \alpha_\lambda]$ holds exactly if u solves the sinh-Gordon-equation

$$u_{z\bar{z}} + \frac{1}{2}\sinh(2u) = 0.$$

Proof. We decompose $\alpha_\lambda = \alpha'_\lambda dz + \alpha''_\lambda d\bar{z}$ into the corresponding $(1,0)$ and $(0,1)$ parts. Then we compute

$$\bar{\partial}\alpha'_\lambda = \frac{1}{2}\begin{pmatrix} u_{z\bar{z}} & i\lambda^{-1}u_{\bar{z}}e^u \\ -iu_{\bar{z}}e^{-u}dz & -u_{z\bar{z}} \end{pmatrix}, \quad \partial\alpha''_\lambda = \frac{1}{2}\begin{pmatrix} -u_{z\bar{z}} & -iu_z e^{-u} \\ i\lambda u_z e^u dz & u_{z\bar{z}} \end{pmatrix}$$

and

$$[\alpha'_\lambda, \alpha''_\lambda] = \frac{1}{4}\begin{pmatrix} -e^{2u} + e{-}2u & 2i\lambda^{-1}u_{\bar{z}}e^u + iu_z e^{-u} \\ -2i\lambda u_z e^u - 2iu_{\bar{z}}e^{-u} & e^{2u} - e{-}2u \end{pmatrix}.$$

The equation $2d\alpha_\lambda + [\alpha_\lambda \wedge \alpha_\lambda]$ is equivalent to $\bar{\partial}\alpha'_\lambda - \partial\alpha''_\lambda = [\alpha'_\lambda, \alpha''_\lambda]$. The last equation holds if and only if u solves the sinh-Gordon equation. □

2.3. The Sym-Bobenko formulas

In the following we will explain the Sym-Bobenko formulas which allow us to compute an conformal immersion from a given extended frame into different space forms.

Proposition 2.5. *Let M be a simply connected Riemann surface and F_λ an extended frame.*

(i) *For every $\lambda_0, \lambda_1 \in \mathbb{S}^1, \lambda_0 \neq \lambda_1$ the map $f_\lambda : M \to \mathbb{S}^3$ defined by*

$$f_\lambda = F_{\lambda_0} F_{\lambda_1}^{-1}$$

is a conformal immersion $M \to \mathbb{S}^3$ with constant mean curvature $H = i(\lambda_0 + \lambda_1)/(\lambda_0 - \lambda_1)$.

(ii) *Let $H \in \mathbb{R}^*$. For every $\lambda, \in \mathbb{S}^1$ the map $f_\lambda : M \to \mathbb{R}^3$ defined by*

$$f_\lambda = -2i\lambda H^{-1}(\partial_\lambda F_\lambda)F_\lambda^{-1} - \frac{1}{H} F_\lambda \epsilon F_\lambda^{-1}$$

is a conformal immersion $M \to \mathbb{R}^2$ with constant mean curvature H

Proof. We already have proved the case of the three sphere in section 2.1.3. We will now proceed with the case of euclidean space. Let $F_\lambda : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathfrak{su}_2$ be an extenden frame. And let $\alpha_\lambda = F_\lambda^{-1} dF_\lambda$. We can then decompose α_λ in the following way

$$\alpha_\lambda = (\alpha_1' + \lambda\alpha_1'')\epsilon_- + (\lambda^{-1}\alpha_2' + \alpha_2'')\epsilon_+ + (\alpha_3' + \alpha_3'')\epsilon, \tag{2.10}$$

with α_j' and α_j'' independent of λ. Since $\alpha_\lambda \in \mathfrak{su}_2$ for $\lambda \in \mathbb{S}$ we also obtain $\bar{\alpha_1}'' = \alpha_2'$, $\bar{\alpha_1}' = \alpha_2''$ and $\bar{\alpha_3}' = \alpha_3''$. We can now decouple the integrability $2d\alpha_\lambda = [\alpha_\lambda \wedge \alpha_\lambda]$ into the ϵ_-, ϵ_+ and ϵ components

$$\lambda d\alpha_1'' + 2i\lambda\alpha_3' \wedge \alpha_1'' = 2i\alpha_1' \wedge \alpha_3'' - d\alpha_1', \tag{2.11a}$$

$$\lambda^1 d\alpha_2' + 2i\lambda^{-1}\alpha_2' \wedge \alpha_3'' = 2i\alpha_3' \wedge \alpha_2'' - d\alpha_2'', \tag{2.11b}$$

$$d\alpha_3' + i\alpha_1' \wedge \alpha_2'' = i\alpha_2' \wedge \alpha_1'' - d\alpha_3''. \tag{2.11c}$$

The left sides of the equations (2.11a) and (2.11b) depend on λ, the right sides do not so both sides of these equations have to be identically zero. Using this relations we compute

$$df_\lambda = 2iH^{-1}F_\lambda(-\alpha_1' e_- + \alpha_2''\epsilon_+)F_\lambda^{-1}. \tag{2.12}$$

We can see now $\left\langle df_\lambda', df_\lambda' \right\rangle = 0$ by using (2.5) so the conformality is clear. On the other hand we use (2.12) to obtain

$$d * df_\lambda = -4iH^{-1}\alpha_2'' \wedge \alpha_1' F_\lambda \epsilon F_\lambda^{-1}$$

and

$$[df_\lambda \wedge df_\lambda] = -8iH^{-2}\alpha_2'' \wedge \alpha_1' F_\lambda \epsilon F_\lambda^{-1}.$$

These two equations together with lemma 2.1 prove the claim. □

CHAPTER 3

Spectral curves of constant mean curvature tori

In the last chapter we have seen the connection between conformal immersions of constant mean curvature surfaces and solutions of the sinh-Gordon equation. In this thesis we are interested in immersions of tori into $\mathbb{S}^3$ so we are looking for periodic solutions of the sinh-Gordon equation. More precisely in order to obtain an immersion of a torus $f : \mathbb{R}^2/\Gamma \to \mathbb{S}^3$ we need a doubly periodic solution of the sinh-Gordon equation. In this chapter we will analyze periodic solutions. We will introduce the notion of monodromy and construct a hyperelliptic Riemann surface from a solution of the sinh-Gordon equation. This surface is the so called spectral curve. We will also see that a given spectral curve define a solution of the sinh-Gordon equation in turn. Then we will show the closing conditions needed to obtain a double periodic solution from a periodic one. The results of this section are mainly due to Bobenko [2], see also [1, 3].

The second part of the chapter will deal with the question how to encode the spectral curve with additional functions needed to obtain a solution of the sinh-Gordon equation to a set of complex numbers. We will call this set the spectral data. This representation of the spectral curve will allow us to define a deformation on the spectral curve and represent this deformation by a system of ordinary differential equations later. This representation is taken from [6].

3.1. The spectral curve

We have seen that for an open subset $U \subset \mathbb{C}$ and a function $u : U \to \mathbb{R}$ the sinh-Gordon equation

$$u_{z\bar{z}} + \frac{1}{2}\sinh(2u) = 0$$

is the integrability condition of $dF = F\alpha$ with

$$\alpha = \frac{1}{2}\begin{pmatrix} u_z dz - u_{\bar{z}} d\bar{z} & i\lambda^{-1}e^{u}dz + ie^{-u}d\bar{z} \\ ie^{-u}dz + i\lambda e^{u}d\bar{z} & -u_z dz + u_{\bar{z}} d\bar{z} \end{pmatrix}.$$

If U is simply connected and u solves the sinh-Gordon equation there exists a unique solution F to $dF = F\alpha$ with the initial condition $F(z_0) = \mathbb{1}$ for some fixed $z_0 \in \mathbb{C}$.

$$\begin{aligned} F : U \times \mathbb{C}^* &\to \mathrm{SU}_2 \\ (z, \lambda) &\to F(z, \lambda) \end{aligned}$$

F is holomorphic in λ. Assume u is a periodic solution of the sinh-Gordon equation. So there exists a $p \in \mathbb{C}$ such that

$$u(z+p) = u(z) \quad \text{for all } z \in \mathbb{C}.$$

Then we also have

$$\alpha(z+p) = \alpha(z) \quad \text{for all } z \in \mathbb{C}.$$

It follows

$$F(z+p) = F(z_0+p)F(z) \quad \text{for all } z \in \mathbb{C}.$$

Definition 3.1. Let $M(p,\lambda) = F(z_0+p,\lambda)$. The map

$$M : U \times \mathbb{C}^* \to \mathrm{SU}_2$$

is called the *monodromy* of F.

The monodromy is a holomorphic map in λ. It turns out that even though the monodromy depends on the choice of the base point z_0 the conjugacy class of it and hence eigenvalues do not. Let $\Delta(\lambda) = \mathrm{tr}(M(\lambda))$, then the eigenvalues μ of M are solutions of $\det(\mu\mathbb{1} - M(\lambda)) = \mu^2 - \Delta(\lambda)\mu + 1 = 0$. Let us define now the *spectral curve* Y

$$Y = \{(\mu,\lambda) \in \mathbb{C}^* \times \mathbb{C}^* \;:\; \mu^2 - \Delta(\lambda)\mu + 1 = 0\}. \tag{3.1}$$

Now we will diagonalise $M(\lambda)$. We can regard $M(\lambda)$ as a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and we obtain

$$\det(\mu\mathbb{1} - M(\lambda)) = \det(\mu\mathbb{1} - \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = \mu^2 - (a+d)\mu + (ad-bc) = 0.$$

We will look for $V = (v_1, v_2)^t$ and $W = (w_1, w_2)$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} V = \mu V \quad \text{and} \quad W \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mu W.$$

A computation shows that $V = (b, \mu - a)^t$ and $V = (\mu - d, c)^t$ are solutions for V and $W = (c, \mu - a)$ and $W = (m - d, b)$ are the solutions for W respectively. We set

$$P = \frac{V \cdot W}{W \cdot V} = \frac{1}{2\mu - a - d}\begin{pmatrix} \mu - d & b \\ c & \mu - a \end{pmatrix}.$$

The last term is independent of the choice for V and W. We also see that $P^2 = P$ and $P\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \mu P = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)P$. So P is a projector.

Since $M(\lambda)$ is holomorphic in λ the functions a, b, c and d are holomorphic in $\lambda \in \mathbb{C}^*$ as well. We also have $\mu^2 - \Delta(\lambda)\mu + 1 = 0$, by differentiation we obtain

$$(2\mu - \Delta(\lambda))d\mu - \Delta'(\lambda)\mu d\lambda = 0$$

We assume that the spectral curve does not have any singularities. So there are no points (μ, λ) such that $(2\mu - \Delta(\lambda)) = 0$ and $\Delta'(\lambda)\mu = 0$ simultaneously. This gives

$$\frac{d\lambda}{2\mu - \Delta(\lambda)} = \frac{d\mu}{\Delta'(\lambda)}$$

and $Pd\lambda$ is a holomorphic one-form on Y. It turns out that Y is a double cover over $\lambda \in \mathbb{C}^*$ with branch points at $\Delta(\lambda)^2 = 4$. So Y is a hyperelliptic Riemann surface genus g. It can then be represented by the equation $\nu^2 = a(\lambda)$, where a is polynomial and the branch points correspond to the roots of a. We denote the branch points

$$\alpha_0 = 0, \alpha_1, \ldots, \alpha_{2g+1}, \alpha_{2g+2} = \infty$$

and obtain

$$\nu^2 = \lambda(\lambda - \alpha_1)\cdots(\lambda - \alpha_{2g+1}).$$

There are several important involutions on Y. First of all there is the hyperelliptic involution σ. Besides that there are two more anti-holomorphic involutions η and ρ. These involutions are described by

$$\sigma : (\mu, \lambda) \to (\mu^{-1}, \lambda), \tag{3.2a}$$

$$\eta : (\mu, \lambda) \to (\bar{\mu}, \bar{\lambda}^{-1}), \tag{3.2b}$$

$$\rho : (\mu, \lambda) \to (\bar{\mu}^{-1}, \bar{\lambda}^{-1}). \tag{3.2c}$$

We see that $\rho = \eta \circ \sigma$ and we also can show that η does not have any fixed points.

We already introduced the map $W : Y \to \mathbb{C}^2$. We normalise this map so that $W_1 = 1$ and thus $W = (1, w)$ then w is a meromorphic function on Y. The poles of W and the poles of w are the same. They are described by $\{(\mu, \lambda) : W_1 = 0\}$. These poles define a divisor D. There is a connection between the spectral curve Y with a divisor D and solutions of the sinh-Gordon equation.

Proposition 3.2. *Let Y be a hyperelliptic Riemann surface so that $\lambda : Y \to \mathbb{CP}^1$ with branch points y^+ over $\lambda = 0$ and y^- over $\lambda = \infty$. Furthermore let Y possess besides the hyperelliptic involution σ an anti-holomorphic involution η without fixed points such that $\eta^*\bar{\lambda} = \lambda^{-1}$. Let D be a divisor of degree $g + 1$ and $\eta(D) - D = (f)$ and $f\eta^* f = -1$. Then there exists a real solution of the sinh-Gordon equation for this data.*

Proof. In the following we will give a sketch of the proof of this proposition. Over every point $(\mu, \lambda) \in Y$ lies $W(\mu, \lambda)$ in the eigenspace of $M(\lambda)$ with the eigenvalue μ. This eigenbundle is a holomorphic line bundle on Y or the compactification of Y with the additional points y^+ over $\lambda = 0$ and y^- over $\lambda = \infty$ respectively. The map $(w_1, w_2) \to w_1$ is a linear map which induces a global section of the dual of the eigenbundle. The divisor D gives the zeros of this global section.

Let define the map $\Psi : \mathbb{C} \times Y \to \mathbb{C}^2$ by

$$\Psi(z, \mu(\lambda)) = W(\mu, \lambda)F(z, \lambda)$$

with the normalized function $W(\mu, \lambda) = (1, w(\mu, \lambda))$. The map Ψ is the so called *Baker-Akhiezer* function. Let $T = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)$. One can show then that Ψ has the following properties.

(i) The function Ψ is meromorphic on $Y \setminus \{y^+, y^-\}$ and has its only poles at D.

(ii) Let $k = \sqrt{\lambda}$ be a local parameter at $\lambda = 0$ then the function

$$\Psi \begin{pmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{pmatrix} T^{-1} \exp\left(\frac{i}{4k} \begin{pmatrix} \bar{z} & 0 \\ 0 & -\bar{z} \end{pmatrix} \right)$$

is holomorphic in a neighbourhood of y^+ and takes the value $(1, 1)$ at y^+.

(iii) Let $k = \sqrt{\lambda}$ be a local parameter at $\lambda = \infty$ then the function

$$\Psi \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} T^{-1} \exp\left(\frac{ik}{4} \begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix} \right)$$

is holomorphic in a neighborhood of y^- and takes the value $(1, -1)$ at y^-.

We obtain for every z and the Mittag-Leffler distribution $\frac{i}{4k}z$ at $\lambda = 0$ and $\frac{ik}{4}z$ at $\lambda = \infty$ an element of $H^1(Y, O)$ and by $\exp : z \to L_z$ a 2-dimensional subgroup of $H^1(Y, O^*)$.

The mentioned properties ensure that Ψ_1 is a holomorphic section of $O_D \otimes L_z$ with the values $1, 1$ at $\lambda = 0$ and $\lambda = \infty$. and that Ψ_2 is a holomorphic section of $O_D \otimes L_z$ with the values $1, -1$ at $\lambda = 0$ and $\lambda = \infty$. So these properties determine Ψ uniquely if every holomorphic section $O_D \otimes L_z$ is sufficiently described by its values at y^+ and y^-. This is equivalent to

$$H^0(Y, O_{D-y^+-y^-} \otimes L_z) = 0$$

The Baker-Akhiezer function leads to complex solutions of the sinh-Gordon equation. In order to obtain real solutions of the sinh-Gordon equation the monodromy has to obey the following

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bar{M}\left(\frac{1}{\bar{\lambda}}\right) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = M(\lambda).$$

In this case the spectral curve is invariant under the involution $\eta : (\mu, \lambda) \to (\bar{\mu}, \bar{\lambda}^{-1})$. The branch points of this spectral curve have to be invariant as a subset of $\mathbb{CP}^1$ under the involution $\lambda \to \frac{1}{\bar{\lambda}}$. We have then

$$\begin{aligned} \bar{W}\left(\bar{\mu}, \frac{1}{\bar{\lambda}}\right) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} M(\lambda) &= \bar{W}\left(\bar{\mu}, \frac{1}{\bar{\lambda}}\right) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bar{M}\left(\frac{1}{\bar{\lambda}}\right) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \bar{W}\left(\bar{\mu}, \frac{1}{\bar{\lambda}}\right) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \bar{\mu}. \end{aligned}$$

And it follows

$$\bar{W}\left(\bar{\mu}, \frac{1}{\bar{\lambda}}\right) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = f W(\mu, \lambda)$$

or equivalently

$$\eta * \bar{W} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = f W.$$

The poles of $\eta^*\bar{W}$ are given by $\eta(D)$. One can also see that $f\eta^*\bar{f} = -1$ and thus that η has no fixed points. We also have $\eta(D) - D = (f)$.

We use now the following lemma.

Lemma 3.3. *Let Y be a hyperelliptic Riemann surface with $\lambda : Y \to \mathbb{CP}^1$ of degree 2. Let Y have an anti-holomorphic involution η with $\eta^*(\bar{\lambda}) = \lambda^{-1}$. The points y^+ and y^- over $\lambda = 0$ and $\lambda = \infty$ shall be branch points and η has no fixed points. Let D be a divisor of degree $\deg(D) \leq g - 1$ such that $D - \eta(D) = (f)$ and $f\eta^* f = -1$. Then it follows that $H^0(Y, O_D) = 0$.*

The line bundle L_z fulfills $\eta^*\bar{L}_z = L_z$ for every $z \in \mathbb{C}$ since the transition functions are invariant under η. $O_{D-y^+-y^-} \otimes L_z$ is a line bundle for a divisor D' of degree $g - 1$. And we also have $\eta(D') - D' = (f)$ with $f\eta^* f = -1$. So if the divisor D obeys the conditions $\deg(D) = g + 1$ and $\eta(D) - D = (f)$ with $f\eta^* f = -1$ we can use the lemma to see $H^0(Y, O_{D-y^+-y^-} \otimes L_z) = 0$ for all $z \in \mathbb{C}$. So the Baker-Akhiezer function is uniquely defined for all $z \in \mathbb{C}$ for such divisors. □

Proposition 3.2 establishes a connection between spectral curves and periodic solutions of the sinh-Gordon equation. We are interested in spectral curves of tori so we are looking for doubly periodic solutions of sinh-Gordon equation. The spectral curves of doubly periodic solutions fulfill additional closing conditions.

Proposition 3.4. *Let F_λ be an extended frame. The formulas in proposition 2.5 define doubly periodic solutions of the sinh-Gordon equation only if the following holds. Let $\mu_i(\lambda)$, $i = 1, 2$ be eigenvalues of the monodromies corresponding to the two periods.*

(i) *In case of $f_\lambda : M \to \mathbb{S}^3$ there are two distinct $\lambda_0, \lambda_1 \in \mathbb{S}^1$ such that $\mu_i(\lambda_0) = \mu_i(\lambda_1) = \pm 1$.*
(ii) *In case of $f_\lambda : M \to \mathbb{R}^3$ there exist $\lambda_0 \in \mathbb{S}^1$ such that $\mu_i(\lambda_0) = \pm 1$ and $\partial_{\lambda_0}\mu_i(\lambda_0) = 0$.*

The restrictions in this proposition can be regarded as additional conditions to the conditions set up in the proposition 3.2, so that the resulting solutions of the sinh-Gordon equation become doubly periodic.

3.2. Representation of the spectral data

First of all we will reformulate the definition and properties of a spectral curve of a torus from the last section.

Proposition 3.5. *Let Y be a hyperelliptic Riemann surface $\kappa : Y \to \mathbb{CP}^1$ with branch points over $\kappa = i(y^+)$ and $\kappa = -i(y^-)$. Then Y must obey the following conditions to be a spectral curve of an immersed torus in $\mathbb{S}^3$.*

(i) *Besides the hyperelliptic involution σ Y possesses two more anti-holomorphic involutions η and $\rho = \eta \circ \sigma = \sigma \circ \eta$. η has no fixed points and it interchanges y^+ and y^-, $\eta(y^+) = y^-$.*

(ii) *There are two non vanishing holomorphic functions* μ_1, μ_2 *on* $Y \setminus \{y^+, y^-\}$ *which fulfill the following conditions*

$$\sigma^*\mu_i = \mu_i^{-1},\ \eta^*\bar{\mu}_i = \mu_i,\ \rho^*\bar{\mu}_i = \mu_i^{-1}.$$

(iii) *The 1-forms* $d\ln\mu_i$ *are meromorphic differentials of the second kind with double poles on* $y^\pm$*. The principal parts at* y^+ *respectively* y^- *of these differentials are linearly independent.*

(iv) *There are four fixed points* y_1, $y_2 = \sigma(y_1)$, y_3, $y_4 = \sigma(y_3)$ *of* ρ*, so that* μ_1 *and* μ_2 *attain the value* 1 *or* -1 *there.*

We choose the parameter $\kappa : Y \to \mathbb{CP}^1$ so that $y^\pm$ correspond to $\kappa = i$ and $\kappa = -i$ respectively. We also choose κ so that it fulfills

$$\sigma^*\kappa = \kappa,\ \eta^*\bar{\kappa} = \kappa,\ \rho^*\bar{\kappa} = \kappa.$$

We describe the spectral curve which is a hyperelliptic surface by the equation

$$\nu^2 = (\kappa^2 + 1)a(\kappa).$$

Here $a(\kappa)$ is a polynomial defined by

$$a(\kappa) = \prod_{i=1}^{g}(\kappa - \alpha_i)(\kappa - \bar{\alpha}_i)$$

with pairwise different branch points $\alpha_1, \dots, \alpha_g \in \{\kappa \in \mathbb{C} : \Im\{\kappa\} > 0\}$. Therefore we have $\eta^*\bar{a} = a$ and $\rho^*\bar{a} = a$. We also have for κ with $\Im\{\kappa\} = 0$ that $a(\kappa) > 0$ and the following transformations hold

$$\eta^*\bar{\nu} = -\nu,\ \rho^*\bar{\nu} = \nu,\ \rho^*\nu = -\nu.$$

One can see that a is a real polynomial with $\deg(a) = 2g$ and a leading coefficient $a_{2g} = 1$.

We now define two functions b_1, b_2 by

$$b_i(\kappa) = \frac{1}{\pi i}\nu(\kappa^2 + 1)\partial_\kappa \ln\mu_i. \tag{3.3}$$

We see that b_i are polynomial in κ of degree $g + 1$ and satisfy

$$\eta^*\bar{b}_i = -b_i, \quad \mathrm{ord}_{\kappa=i,-i}\, b_i = g/2$$

So the differentials $d\ln\mu_i$ can be written as

$$d\ln\mu_i = \pi i \frac{b_i(\kappa)}{\nu(\kappa^2 + 1)} d\kappa.$$

with polynomials b_i of degree $\deg(b_i) = g + 1$. We also see that all of the coefficients of b_i are real.

There is still a freedom in the choice of the parameter κ so we pick $\kappa_0, \kappa_1 \in \mathbb{R}$, $\kappa_0 \neq \kappa_1$ and take the unique parameter κ such that y_1 and $y_2 = \sigma(y_2)$ correspond to the two points over $\kappa = \kappa_0$ and y_3 and $y_4 = \sigma(y_3)$ to the two points over $\kappa = \kappa_1$ respectively.

Summarizing these observations we can say

Proposition 3.6. *The spectral curve of a torus in $\mathbb{S}^3$ is sufficiently described by the choice of the parameter κ, the coefficients of the polynomial $a(\kappa)$ and the two polynomials $b_i(\kappa)$, and by the two points k_0 and k_1.*

In the next chapter we will mainly use the parameter κ as it suits most of the further computations. But there is also a parameter λ on the spectral curve Y which is used in many works we reference. In some situations this parameter will seem to be more natural than κ, therefore we will also introduce it. The parameter λ is chosen so that the points $y^{\pm}$ correspond to the values 0 and ∞. The parameter λ transforms under the anti-holomorphic involution η as $\eta^*\bar{\lambda} = \lambda^{-1}$, and attains $\lambda = 1$ at $\kappa = 0$. So in all it obeys the following transformation rules

$$\sigma^*\lambda = \lambda, \ \eta^*\bar{\lambda} = \lambda^{-1}, \ \rho^*\bar{\lambda} = \lambda^{-1}.$$

The points λ_0 and λ_1 are those that correspond to κ_0 and κ_1. One can then obtain the following relations between the both parameters

$$\lambda = \frac{i-\kappa}{i+\kappa} \quad \text{and} \quad \kappa = i\frac{1-\lambda}{1+\lambda}.$$

We see that the points on the real line in κ correspond to the unit circle in λ.

CHAPTER 4

Deformation of constant mean curvature tori

In the last chapter we explained how a spectral curve of constant mean curvature torus in $\mathbb{S}^3$ can be sufficiently described by a set of coefficients of certain polynomials and two more complex numbers. In this chapter we will introduce a deformation on this spectral data which will give us one-parameter family of spectral curves for every given spectral curve. This deformation can be represented by a system of ordinary differential equations. The deformation can branch at certain spectral curves to families of spectral curves with a higher spectral genus. In particular we will be interested in spectral curves of genus $g = 0$ and their branch points to $g = 2$. In order to find those we will analyze tori with spectral curves of genus $g = 0$ and compute their spectral data. Some of these spectral curves have double points. These are exactly those curves where the deformation can branch to higher genus. We will describe an algorithm how to find such curves in a family of genus $g = 0$ spectral curves. We will look at the spectral curves of tori with a rectangular conformal class in particular and compute these spectral curves with double points explicitly. Then we will show how to obtain the initial conditions for the deformation of tori with spectral genus $g = 2$ from those curves. After that we will explicitly show the deformation ODE for this class of spectral curves and investigate some properties of this ODE. In the end of the chapter we will use these properties to formulate and prove the conditions which have to be fulfilled so that during the deformation the mean curvature goes to infinity. We see then that an endpoint of such a family of spectral curves from tori in $\mathbb{S}^3$ obtained by the deformation ODE is a spectral curve of a torus in $\mathbb{R}^3$.

4.1. Deformation of spectral curves

We will now establish the period preserving deformation of the spectral curve introduced first by Kilian and Schmidt [6]. This deformation is defined by a system of ordinary differential equations on the spectral data we introduced in the previous chapter.

Let us look closely at the differentials $d\ln\mu_i$. We have seen that we can write $d\ln\mu_i = \pi i\nu^{-1}(\kappa^2+1)^{-1}b_i(\kappa)d\kappa$. We observe first that

$$\oint d\ln\mu_i \in 2\pi i\mathbb{Z}$$

for all the cycles in $H_1(Y, \mathbb{Z})$. So these path integrals can not continuously depend on the deformation parameter t. The μ_i are locally algebraic functions of the parameter κ and continuously differentiable with respect to t, so $\partial_t \ln \mu_i$ are global meromorphic functions on Y with only possible poles at the branch points of Y.

We can now make the following ansatz

$$\partial_t \ln \mu_i = \pi i \frac{c_i(\kappa)}{\nu}. \tag{4.1}$$

Here the c_i are polynomial in κ of the degree $\deg(c_i) = g + 1$ with real coefficients.

We define the differential ω as

$$\omega = (\partial_t \ln \mu_1) d \ln \mu_2 - (\partial_t \ln \mu_2) d \ln \mu_1. \tag{4.2}$$

This is a meromorphic 1-form on Y with poles at most of order three at $\kappa = i$ and $\kappa = -i$ and roots at $\kappa = \kappa_0$ and $\kappa = \kappa_1$. We further see that

$$\sigma^* \omega = \omega, \ \eta^* \bar{\omega} = \omega, \ \rho^* \bar{\omega} = \omega.$$

So we can conclude that ω as defined in (4.2) has to obey the following relation

$$\omega \sim \frac{(\kappa - \kappa_0)(\kappa - \kappa_1)}{(\kappa^2 + 1)^2} d\kappa. \tag{4.3}$$

Now we can absorb the factor of proportionality by a reparametrization of the deformation parameter t and we obtain

$$(\partial_t \ln \mu_1) d \ln \mu_2 - (\partial_t \ln \mu_2) d \ln \mu_1 = \frac{(\kappa - \kappa_0)(\kappa - \kappa_1)}{(\kappa^2 + 1)^2} d\kappa,$$

or with our ansatz

$$-\pi^2 \frac{c_1(\kappa) b_2(\kappa) - c_2(\kappa) b_1(\kappa)}{\nu^2 (\kappa^2 + 1)^2} = \frac{(\kappa - \kappa_0)(\kappa - \kappa_1)}{(\kappa^2 + 1)^2}. \tag{4.4}$$

We also need to ensure the integrability $\partial^2_{tk} \ln \mu_i = \partial^2_{kt} \ln \mu_i$ of the deformation ODE. So a second equation arises

$$\frac{\partial}{\partial_t} \pi i \frac{b_i(\kappa)}{\nu(\kappa^2 + 1)} = \frac{\partial}{\partial_\kappa} \pi i \frac{c_i(\kappa)}{\nu}. \tag{4.5}$$

Now we use the defining equation $\nu^2 = (\kappa^2 + 1) a(\kappa)$ and denote the differentiation of a with respect to κ by a' and the differentiation with respect to t by $\dot{a}$. Of course the same notation is used also for b_i and c_i. As all polynomials and also the parameter κ_0, κ_1 and λ_0, λ_1 respectively depend on t we usually omit this in the notation. But we still show the dependence on κ and λ respectively to avoid confusion as we some times switch between these parameters. After an algebraic manipulation of (4.4) and (4.5) we obtain

$$\begin{aligned} 2\dot{b}_i(\kappa) a(\kappa) - b_i(\kappa) \dot{a}(\kappa) = & -2\kappa a(\kappa) c_i(\kappa) \\ & + (\kappa^2 + 1)(2a(\kappa) c_i'(\kappa) - c_i(\kappa) a'(\kappa), \end{aligned} \tag{4.6a}$$

$$c_1(\kappa) b_2(\kappa) - c_2(\kappa) b_1(\kappa) = -\frac{1}{\pi^2} (\kappa - \kappa_0)(\kappa - \kappa_1) a(\kappa). \tag{4.6b}$$

Proposition 4.1. *Let Y be a spectral curve with genus g of a CMC torus in $\mathbb{S}^3$. If the differentials $d \ln \mu_i$ for $i = 1, 2$ do not have any common roots the deformation described above is well defined and Y is contained in an open family of spectral curves of CMC tori in $\mathbb{S}^3$ with genus g.*

Proof. If the differentials do not have any common roots, b_1 and b_2 also do not have common roots. In this case the equation (4.6b) evaluated at these $2g + 2$ roots uniquely determine c_1 and c_2. So there are functions $\gamma_{i,j}(t)$ for all coefficients of c_i so that

$$c_i(\kappa, t) = \sum_{j=0}^{g+1} \gamma_{i,j}(t)\kappa^j.$$

Now we look at the roots of α_j of a. At these roots (4.6a) reads as

$$b_i(\alpha_j, t)\dot{a}(\alpha_j, t) = (\alpha_j^2 + 1)c_i(\alpha_j, t)a'(\alpha_j, t). \tag{4.7}$$

On the other hand at these roots (4.6b) turns to

$$c_1(\alpha_j, t)b_2(\alpha_j, t) = c_2(\alpha_j, t)b_1(\alpha_j, t). \tag{4.8}$$

The polynomials b_1 and b_2 have no common roots, so one of the ratios $c_i(\alpha_j, t)/b_i(\alpha_j, t)$ has to be well defined. This in turn ensures that one of the differential equations (4.7) is also well defined and thus uniquely determines $\dot{a}$. But then (4.6a) also uniquely determine $\dot{b}_1$ and $\dot{b}_2$. So we have seen that given the initial data a, b_1, b_2, κ_0, κ_1 the deformation equations uniquely determine $\dot{a}$, $\dot{b}_1$, $\dot{b}_2$, c_1, c_2. □

Now we need to determine $\dot{\kappa}_i(t)$ so that the closing conditions are preserved during the deformation. More precisely we need

$$\partial_t(\ln \mu_i(\kappa_j(t), t)) \mid_{\kappa_j} = (\partial_t \ln \mu_i + \dot{k}_j \partial_\kappa \ln \mu_i) \mid_{\kappa_j} = 0,$$

and by (3.3) and (4.1) we obtain

$$\dot{\kappa}_j = -\frac{\partial_t \ln \mu_i}{\partial_\kappa \ln \mu_i}\Big|_{\kappa=\kappa_j} = -(\kappa_j^2 + 1)\frac{c_i(\kappa_j)}{b_i(\kappa_j)}.$$

We have seen that as long as $d \ln \mu_1$ and $d \ln \mu_2$ have no common roots, at least one of the ratios $c_i(\kappa)/b_i(\kappa)$ exists and then the other has also to exist and both must coincide by (4.6b). We see that in the open family of spectral curves discussed in the previous proposition it is possible to let κ_j change in such a way that also the closing conditions are preserved in the whole family.

4.2. Constant mean curvature tori with spectral genus $g = 0$

The deformation we described before has some bifurcation points. If the one-parameter family of spectral curves reaches a spectral curve which has double points it is possible to branch the deformation to a higher spectral genus by opening one ore more double points. This is possible if we treat a spectral curve with double points as a limit curve of a family of spectral curves in which one or more pairs of branch points fall together. In

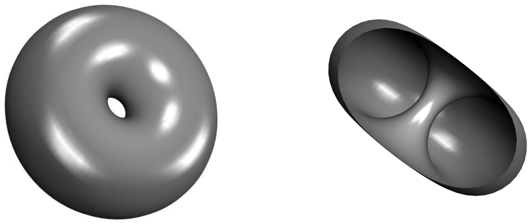

FIGURE 4-1. Homogeneous torus in $\mathbb{S}^3$, normal view and a cross section

particular we are interested in the case where we start with a spectral curve of genus $g = 0$ and open two double points, branching to a spectral genus $g = 2$. In the following we will describe how to compute all the needed data for the spectral curve of genus $g = 0$ with a pair of double points. At the end of the section we will see how the actual branching is performed and we obtain the spectral data which we can use as initial condition for deformation of spectral curves of genus $g = 2$ later.

4.2.1. Spectral data of flat tori. We start the computation of the spectral data by showing some general facts about tori with spectral genus $g = 0$.

PROPOSITION 4.2. *For every surface $f : \mathbb{R}^2 \to \mathbb{S}^3$ with constant mean curvature H and spectral genus $g = 0$ there exists a $t_0 \in \mathbb{R}$ with $H = \cot(2t_0)$ and $\lambda_0 = e^{it_0}$ such that*

$$f = F_{\lambda_0^{-1}} F_{\lambda_0}^{-1} \quad \textit{with} \quad F_\lambda(z) = \exp\left(\frac{i}{2}\begin{pmatrix} 0 & z\lambda^{-1} + \bar{z} \\ z + \bar{z}\lambda & 0 \end{pmatrix}\right). \tag{4.9}$$

PROOF. Let us recall that every constant mean curvature surface could be described as $f = F_{\lambda_1} F_{\lambda_0}^{-1}$ and F_λ was a solution of $dF_\lambda = F_\lambda \alpha_\lambda$. Let us also recall from 2.2 that α_λ has the following form

$$\alpha_\lambda = \frac{1}{2}\begin{pmatrix} u_z dz - u_{\bar{z}} d\bar{z} & i\lambda^{-1} e^u dz + ie^{-u} d\bar{z} \\ ie^u dz + i\lambda e^{-u} d\bar{z} & -u_z dz + u_{\bar{z}} d\bar{z} \end{pmatrix}. \tag{4.10}$$

In case of a flat surface we have $u \equiv 0$ and solving $dF_\lambda = F_\lambda \alpha_\lambda$ with such an α_λ we obtain an F_λ such as in (4.9). So there exist $\lambda_0, \lambda_1 \in \mathbb{S}^1$ and a frame in the given form such that $f = F_{\lambda_1} F_{\lambda_0}^{-1}$. The parameter λ_0 and λ_1 are determined only up to a rotation and it is possible to find a rotation such that $\lambda_0 = \lambda_1^{-1}$ hold. □

REMARK. The freedom in rotating the λ-plane which we use to obtain $\lambda_0 = \lambda_1^{-1}$ can be used to ensure that λ_0 lies in the upper half of the λ-plane or $\Im\{\lambda_0\} \geq 0$.

Let us now suppose that f defined as in (4.9) is an immersion of a torus into $\mathbb{S}^3$. This means that it factors through a lattice $\omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ and certain conditions are ensured on the eigenvalues of the monodromy with respect to ω_1 and ω_2. Precisely the following conditions must hold

$$\mu(\omega_j, \lambda_0) = \mu(\omega_j, \lambda_1) \in \{-1, 1\}.$$

This is equivalent to the following. There exist integers $K_1, K_2, L_1, L_2 \in \mathbb{Z}$ such that $K_1 \equiv L_1,\ K_2 \equiv L_2 \mod 2$ and

$$\ln \mu(\omega_j, \lambda_0) = \pi i K_j, \quad \ln \mu(\omega_j, \lambda_1) = \pi i L_j. \tag{4.11}$$

The integers K_1, K_2, L_1, L_2 are the *wrapping numbers*, they encode how many times F_{λ_0} and F_{λ_1} return to their initial value $\mathbb{1}$ as they traverse a period in z.

We have seen that the choice of the wrapping numbers together with a mean curvature uniquely determine a flat torus in $\mathbb{S}^3$ up to an isometry.

Now we will establish formulas for the eigenvalues and periods of $F_\lambda(z)$. The eigenvalues of $F_\lambda(z)$ compute to

$$\mu(z, \lambda) = \exp(\pm\frac{i}{2}\sqrt{(z\lambda^{-1} + \bar{z})(z + \bar{z}\lambda)})$$

or

$$\ln \mu(z, \lambda) = \pm\frac{i}{2}(z\lambda^{-\frac{1}{2}} + \bar{z}\lambda^{\frac{1}{2}}). \tag{4.12}$$

Now we take a look at the periods. We have $F_\lambda(0) = \mathbb{1}$ as the initial condition and we know that $F_\lambda(\omega_j) = \pm\mathbb{1}$, so we are looking for z such that $F_\lambda(z) = \pm\mathbb{1}$. This is the case if and only if $\ln \mu(z, \lambda) \in \pi i\mathbb{Z}$, or equivalently

$$z\lambda^{-\frac{1}{2}} + \bar{z}\lambda^{\frac{1}{2}} \in 2\pi i\mathbb{Z}$$

We now combine (4.11) and (4.12) and see that these are equivalent to the periods satisfying

$$\omega_j = 2\pi\frac{K_j\lambda_0^{-\frac{1}{2}} - L_j\lambda_1^{-\frac{1}{2}}}{\lambda_0^{-1} - \lambda_1^{-1}}. \tag{4.13}$$

We have seen in section 3.2 that the spectral curve is sufficiently described by the polynomials a and b and the parameters κ_0, κ_1. We are interested in the spectral data of a spectral genus $g = 0$ torus, so a is a constant polynomial $a \equiv 1$. In the following we will compute the polynomials $b_1(\kappa)$ and $b_2(\kappa)$ for given spectral parameters κ_0 and κ_1.

We recall that for the case of genus $g = 0$ we have an explicit formula for the immersion given in the proposition 4.2. With the help of this proposition we have also established a formula (4.12) for the eigenvalues of the function F_z. This formula is given with respect to the parameter λ but we are interested in a formula with respect to parameter κ as this will be more convenient later. Rewriting the formula we obtain

$$\ln \mu(z, \kappa) = \frac{i}{2}\left(z\left(\sqrt{\frac{i-\kappa}{i+\kappa}}\right)^{-1} + \bar{z}\sqrt{\frac{i-\kappa}{i+\kappa}}\right). \tag{4.14}$$

We have seen that we also have explicit formulas for the periods for given winding numbers

$$\omega_i = 2\pi \frac{K_i \lambda_0^{-\frac{1}{2}} - L_i \lambda_1^{-\frac{1}{2}}}{\lambda_0^{-1} - \lambda_1^{-1}}. \tag{4.15}$$

Now we will obtain the polynomials b_i. As $b_i(\kappa)$ are defined by the equation

$$b_i(\kappa) = \frac{1}{\pi i} \nu(\kappa^2 + 1)\partial_\kappa \ln \mu_i,$$

we just use (4.14) and (4.15) to evaluate this formula. This formula also contains the expression ν. We remember that using the parameter κ we have $\nu^2 = (\kappa^2 + 1)a(\kappa)$. Here $\nu^2 = \kappa^2 + 1$, so $\nu = \sqrt{(\kappa^2 + 1)}$. Our preliminary result for b_i will be

$$b_i(\kappa) = \frac{1}{\pi i}(\kappa^2 + 1)^{3/2}\partial_\kappa \ln \mu(\omega_i, \kappa). \tag{4.16}$$

Let us look at $\ln \mu_i(\omega_j, \kappa)$ more closely. We have

$$\begin{aligned}
\ln \mu(\omega_i, \kappa) &= \frac{i}{2}\left(\omega_i \left(\sqrt{\frac{i-\kappa}{i+\kappa}}\right)^{-1} + \bar{\omega}_i \sqrt{\frac{i-\kappa}{i+\kappa}}\right) \\
&= \pi i \left(\frac{K_j \lambda_0^{-\frac{1}{2}} - L_j \lambda_1^{-\frac{1}{2}}}{\lambda_0^{-1} - \lambda_1^{-1}} \left(\sqrt{\frac{i-\kappa}{i+\kappa}}\right)^{-1} + \frac{K_j \overline{\lambda_0^{-\frac{1}{2}}} - L_j \overline{\lambda_1^{-\frac{1}{2}}}}{\overline{\lambda_0^{-1}} - \overline{\lambda_1^{-1}}} \sqrt{\frac{i-\kappa}{i+\kappa}}\right)
\end{aligned}$$

since K_i and L_i are both integers. As λ_0 and λ_1 lie on the unit circle further algebraic transformation lead to

$$\ln \mu(\omega_i,\kappa) = \frac{\pi\left(L_i \sqrt{\lambda_1}\,(i(\lambda_0 + 1)\kappa - (\lambda_0 - 1)) + K_i \sqrt{\lambda_0}\,(-i(\lambda_1 + 1)\kappa + (\lambda_1 - 1))\right)}{(\lambda_0 - \lambda_1)\sqrt{\kappa^2 + 1}}.$$

Now we use the relation $\lambda_0 = \lambda_1^{-1}$ we have established earlier to obtain

$$\ln \mu(\omega_i, \kappa) = \frac{\pi\sqrt{\lambda_0}\,(L_i\,((\lambda_0 + 1)\kappa + i(\lambda_0 - 1)) - K_i\,((\lambda_0 + 1)\kappa - i(\lambda_0 - 1)))}{(\lambda_0^2 - 1)\sqrt{\kappa^2 + 1}}.$$

After computing the derivative $\partial_\kappa \ln \mu(\omega_i, \kappa)$ we use (4.16) to obtain

$$b_i(\kappa) = -\frac{(K_i + L_i)\sqrt{\lambda_0}}{\lambda_0 + 1}\kappa + \frac{i(K_i - L_i)\sqrt{\lambda_0}}{\lambda_0 - 1}. \tag{4.17}$$

We have $|\lambda_0| = 1$ and $\lambda_0 = \frac{i-\kappa_0}{i+\kappa_0}$ which as we have seen ensures $\kappa_0 \in \mathbb{R}$. We obtain

$$\frac{\sqrt{\lambda_0}}{\lambda_0+1} = \frac{1}{2}\sqrt{\kappa_0^2+1},$$
$$\frac{i\sqrt{\lambda_0}}{\lambda_0-1} = \frac{1}{2\kappa_0}\sqrt{\kappa_0^2+1}.$$

So (4.17) becomes

$$b_i(\kappa) = -\left(\frac{K_i+L_i}{2}\sqrt{\kappa_0^2+1}\right)\kappa + \frac{K_i-L_i}{2\kappa_0}\sqrt{\kappa_0^2+1}. \tag{4.18}$$

Let us summarize the last calculations in a proposition.

Proposition 4.3. *The spectral data (a, b_1, b_2) of a flat torus with given winding numbers K_i, L_i and the spectral parameter k_0 and $k_1 = -k_0$ is*

$$a(\kappa) = 1, \qquad b_i(\kappa) = -\left(\frac{K_i+L_i}{2}\sqrt{\kappa_0^2+1}\right)\kappa + \frac{K_i-L_i}{2\kappa_0}\sqrt{\kappa_0^2+1}.$$

In the further treatment we will concentrate on tori with a rectangular conformal class. We assume that one period of such a torus is real and the other is purely imaginary. This assumption will impose certain conditions on the winding numbers. To see these conditions we make the following observations. We use the formula (4.15) for the periods but rewrite it for the parameter κ, we have

$$\omega_i = \frac{\pi i}{2\kappa_0}\left(\kappa_0^2+1\right)\left(K_i\left(\sqrt{\frac{i-\kappa_0}{i+\kappa_0}}\right)^{-1} - L_i\left(\sqrt{\frac{i+\kappa_0}{i-\kappa_0}}\right)^{-1}\right).$$

Now we use fact that κ_0 is real and obtain

$$\omega_i = \frac{\pi}{2\kappa_0}\sqrt{\kappa_0^2+1}((K_i+L_i)\kappa_0 + i(K_i-L_i)).$$

As $\frac{\pi}{2\kappa_0}\sqrt{\kappa_0^2+1}$ is real as long κ_0 is real, we see that ω_i is real whenever $K_i = L_i$ and purely imaginary whenever $K_i = -L_i$. So we can assume $K_1 = L_1$ and $K_2 = -L_2$. We will use this assumption in all further computations.

Definition 4.4. From here on we look at homogeneous tori with one real and one purely imaginary period. These tori are described by the winding numbers $K_1 = L_1$ and $K_2 = -L_2$.

If we use these relations for the data in proposition (4.3) we see

$$\begin{aligned} b_1(\kappa) &= -\left(\frac{K_1 + L_1}{2}\sqrt{\kappa_0^2+1}\right)\kappa + \frac{K_1 - L_1}{2\kappa_0}\sqrt{\kappa_0^2+1} \\ &= -\left(K_1\sqrt{\kappa_0^2+1}\right)\kappa, \end{aligned} \tag{4.19a}$$

$$\begin{aligned} b_2(\kappa) &= -\left(-\frac{K_2 + L_2}{2}\sqrt{\kappa_0^2+1}\right)\kappa + \frac{K_2 - L_2}{2\kappa_0}\sqrt{\kappa_0^2+1} \\ &= \frac{K_2}{\kappa_0}\sqrt{\kappa_0^2+1}. \end{aligned} \tag{4.19b}$$

So we have proved the following proposition.

Proposition 4.5. *If one period of an flat torus is real and the other purely imaginary it is possible to write the polynomials $b_i(\kappa)$ so that $b_1(\kappa)$ is an odd polynomial and $b_2(\kappa)$ is an even polynomial. Both have real coefficients. The spectral data (a, b_1, b_2,κ_0) of such torus is given by*

$$a(\kappa) = 1, \qquad b_1(\kappa) = -\left(K_1\sqrt{\kappa_0^2+1}\right)\kappa,$$
$$b_2(\kappa) = \frac{K_2}{\kappa_0}\sqrt{\kappa_0^2+1}.$$

Remark. So we can write $b_1(\kappa) = f_1\kappa$ and $b_2(\kappa) = f_2$ with real numbers f_1 and f_2. Obviously the spectral data of such a torus is sufficiently determined by the real triple (f_1,f_2,κ_0).

4.2.2. Algorithm for finding double points on a spectral curve of a flat torus. In the following we will describe the algorithm for finding a spectral curve from a torus of a genus $g = 0$ with double points on it. This algorithm is presented in [6].

We will work with the parameter λ in the following. The double points of the spectral curve are those λ_d for which there exist two additional integers $M_j \in \mathbb{Z}$ such that we have

$$\ln\mu(\omega_j, \lambda_d) = \pi i M_j \tag{4.20}$$

for the both periods. We want to prove an important observation concerning double points first.

Proposition 4.6. *The possible double points on the spectral curve of a flat torus in $\mathbb{S}^3$ have to lie on the unit circle $\mathbb{S}^1$ in the λ-plane.*

Proof. Let $\lambda_d \in \mathbb{C}^\times$ be a double point on the spectral curve of a flat torus in $\mathbb{S}^3$. Then there exist two integers $M_j \in \mathbb{Z}$ such that condition (4.20) holds for both periods ω_j. By (4.12) this reads

$$\omega_1\lambda_d^{-\frac{1}{2}} + \bar{\omega}_1\lambda_d^{\frac{1}{2}} = 2\pi i M_1,$$
$$\omega_2\lambda_d^{-\frac{1}{2}} + \bar{\omega}_2\lambda_d^{\frac{1}{2}} = 2\pi i M_2$$

or

$$\omega_1 + \bar{\omega}_1 \lambda_d = 2\pi i M_1 \lambda_d^{\frac{1}{2}},$$
$$\omega_2 + \bar{\omega}_2 \lambda_d = 2\pi i M_2 \lambda_d^{\frac{1}{2}}.$$

Now we combine both equations by eliminating λ_d and we obtain

$$\lambda_d^{\frac{1}{2}} = \frac{\omega_1 \bar{\omega}_2 - \omega_2 \bar{\omega}_1}{2\pi(\bar{\omega}_2 M_1 - \bar{\omega}_1 M_2)}.$$

On the other hand, we can eliminate λ_d^0 and obtain

$$\lambda_d^{\frac{1}{2}} = \frac{2\pi(\omega_2 M_1 - \omega_1 M_2)}{\bar{\omega}_1 \omega_2 - \bar{\omega}_2 \omega_1}.$$

Both solutions together lead to $\sqrt{\lambda_d} = \overline{1/\sqrt{\lambda_d}}$ and so $|\sqrt{\lambda_d}| = 1$ or equivalently $\lambda_d \in \mathbb{S}^1$. □

Now it is possibly to write down an algorithm for finding double points. This algorithm will allow us to compute the value of the parameter λ_0 and therefore for $\lambda_1 = \lambda_0^{-1}$ and the value of the double point λ_d for the given winding numbers.

We have seen that the defining equations for a double point are given by (4.20) for two integers $M_j \in \mathbb{Z}$. We also have established formulas for the eigenvalues and the periods of a flat torus, namely (4.12) and (4.13). Using the equation for eigenvalues to rewrite the equation (4.20) we obtain

$$\frac{i}{2}(\omega_j \lambda_d^{-\frac{1}{2}} + \bar{\omega}_j \lambda_d^{\frac{1}{2}}) = \pi i M_j.$$

Now we insert the formula (4.13) for the periods and obtain after some algebraic simplifications the following equations

$$K_j \lambda_0^{\frac{1}{2}}(\lambda_d - \lambda_1) + L_j \lambda_1^{\frac{1}{2}}(\lambda_0 - \lambda_d) + M_j \lambda_d^{\frac{1}{2}}(\lambda_1 - \lambda_0) = 0.$$

Substituting $x = \sqrt{\lambda_1/\lambda_0}$ and $y = \sqrt{\lambda_d/\lambda_0}$ these equations read as

$$K_j(x^2 - y^2) + L_j(xy^2 - x) + M_j(y - x^2 y) = 0. \tag{4.21}$$

We observe first that if (x, y) is a solution of (4.21) there exist another solution (x^{-1}, y^{-1}) as well. Now we set

$$a = K_2 L_1 - K_1 L_2, \quad b = K_2 M_1 - K_1 M_2, \quad c = L_2 M_1 - L_1 M_2,$$
$$p = \frac{a^2 - b^2 - c^2}{bc}, \quad q = \frac{a^2 - b^2 + c^2}{ac}.$$

and

$$x^2 + px + 1 = 0, \quad y^2 + qy + 1 = 0. \tag{4.22}$$

A computation shows that the solutions $x_{1,2}$ and $y_{1,2}$ of (4.22) correspond to solutions of (4.21) in a way that (x_1, y_2) and (x_2, y_1) solve (4.21).

Proposition 4.7. *It is possible to assume $\kappa_0 > 0$.*

Proof. As we have seen it is possible to assume that $\lambda_0 = \lambda_1^{-1}$ and thus $x = \sqrt{1/\lambda_0^2}$. So $\lambda_0 = \pm x^{-1}$. With the two solutions $x_{1,2}$ there are actually four possible values but only two of them lie in the upper half of the λ-plane and thus lead to really different spectral curves as the other two can be derived from the first by a rotation. λ_0 in the upper half of the λ-plane is equivalent to $\kappa_0 > 0$ after the transformation to the parameter κ. □

In the following we transform the solutions $\lambda_0, \lambda_1, \lambda_d$ to $\kappa_0, \kappa_1, \kappa_d$.

Proposition 4.8. *Let κ_d be a double point of on a spectral curve of a torus with winding numbers $K_1 = L_1$ and $K_2 = -L_2$. If $\kappa_d \neq 0$ then $-\kappa_d$ is also a double point on this spectral curve.*

Proof. When the winding numbers obey $K_1 = L_1$ and $K_2 = -L_2$ the period ω_1 is real and ω_2 is purely imaginary. If κ_d is a double point so there are numbers M_1 and M_2 so that

$$\ln \mu(\omega_j, \kappa_d) = \pi i M_j.$$

On the other hand since κ_d is real we have

$$\ln \mu(\omega_j, \kappa_d) = \frac{i}{2}\left(\omega_j \sqrt{\frac{i+\kappa_d}{i-\kappa_d}} + \bar{\omega}_j \sqrt{\frac{i-\kappa_d}{i+\kappa_d}}\right).$$

and a computation reveals

$$\ln \mu(\omega_1, -\kappa_d) = \pi i M_1,$$
$$\ln \mu(\omega_2, -\kappa_d) = -\pi i M_2.$$

So $-\kappa_d$ is also a double point. □

4.2.3. Branching from spectral genus $g = 0$ to $g = 2$. In the previous sections we have seen how one can obtain spectral data of a torus from spectral genus $g = 0$ with prescribed winding numbers and spectral parameter κ_0. We also have seen how to find the spectral curves which have two double points κ_d and $-\kappa_d$. Now we will use this data to branch the deformation to a family of spectral genus $g = 2$ curves.

We start with the polynomial $a(\kappa)$. In the case of spectral genus $g = 0$ the polynomial $a(\kappa)$ is a constant polynomial. For $g = 2$ the polynomial $a(\kappa)$ must have double zeros at the double points. As it still has to have a leading coefficient 1 we obtain

$$a(\kappa) = (\kappa - \kappa_d)^2(\kappa + \kappa_d)^2 = \kappa^4 + (-2\kappa_d^2)\kappa^2 + \kappa_d^4. \tag{4.23}$$

In order for $d \ln \mu_i$ to have the right roots and poles the polynomials $b_i(\kappa)$ need to have roots at the double points. Let $\tilde{b}_i$ denote the b_i from the case $g = 0$ we have computed earlier. We obtain

$$b_i(\kappa) = \tilde{b}_i(\kappa)(\kappa^2 - \kappa_d^2) \tag{4.24}$$

as the new b_i. Using proposition 4.5 we obtain

$$\begin{aligned} b_1(\kappa) &= -\left(K_1\sqrt{\kappa_0^2+1}\right)(\kappa^2-\kappa_d^2)\kappa,\\ b_2(\kappa) &= \left(\frac{K_2}{\kappa_0}\sqrt{\kappa_0^2+1}\right)(\kappa^2-\kappa_d^2). \end{aligned} \tag{4.25}$$

One important observation for this data is that a is an even polynomial, b_1 is an odd polynomial and b_2 is again an even polynomial. This observation will be very helpful in the further treatment. We can summarize the symmetries of this spectral curve as follows.

Proposition 4.9. *The spectral data of a flat torus at a branch point from genus $g = 0$ to $g = 2$ has the following properties.*

(i) $a(\kappa) = \kappa^4 + a_2\kappa^2 + a_0$ *with* $a_0, a_2 \in \mathbb{R}$.
(ii) $b_1(\kappa) = f_1(\kappa^2 - \beta_1)\kappa$ *with* $f_1, \beta_1 \in \mathbb{R}$.
(iii) $b_2(\kappa) = f_2(\kappa^2 - \beta_2)$ *with* $f_2, \beta_2 \in \mathbb{R}$.
(iv) $\kappa_0 = -\kappa_1$.

Besides the branch points at i and $-i$ there are four other roots of $a(\kappa)$ α_1, α_2, α_3 and α_4 which also fulfill the relation $\alpha_1 = -\alpha_3$ in addition to the relations $\alpha_1 = \bar{\alpha}_2$ and $\alpha_3 = \bar{\alpha}_4$ which are always fulfilled as we have seen in section 3.2. We also see that b_1 has roots at 0, $\sqrt{\beta_1}$ and $-\sqrt{\beta_1}$ and b_2 has roots at $\sqrt{\beta_1}$ and $-\sqrt{\beta_1}$ respectively.

Now we use (4.23) and (4.25) to write down the coefficients of a, b_1 and b_2 and we obtain the following proposition.

Proposition 4.10. *The spectral data of a flat torus at a branch point from genus $g = 0$ to $g = 2$ is*

$$\begin{aligned} a_0 &= \kappa_d^4, \quad a_2 = -2\kappa_d^2,\\ \beta_1 &= \kappa_d^2, \quad f_1 = -\left(K_1\sqrt{\kappa_0^2+1}\right),\\ \beta_2 &= \kappa_d^2, \quad f_2 = \left(\tfrac{K_2}{\kappa_0}\sqrt{\kappa_0^2+1}\right). \end{aligned}$$

4.3. Deformation of spectral curves from tori of rectangular type

In this section we will compute the deformation ODE explicitly for the tori of rectangular type. Then we will investigate the properties of this ODE in particular we are interested in possible endpoints of the deformation and monotony of it. We will see that certain symmetries which were present in the initial conditions will be preserved during the deformation. In the next section we will also see that these symmetries will occur in the endpoints of the deformation path we are interested in. This will retroactively give the reason why it was possible to look only on a very restricted class of spectral curves while searching a path to a spectral curve in $\mathbb{R}^3$. The last part of this section we deal with the problem that the initial condition presented in the last section constitute a singularity for the deformation ODE. We will compute the derivatives of the functions in the ODE at this point and show that there is only one direction in which the deformation can take place so that the family of spectral curves we obtain in this way will be a family of spectral curves of tori in $\mathbb{S}^3$.

4.3.1. The deformation ODE for tori of rectangular type. We have seen in the previous section that the spectral curve of a rectangular torus has several additional symmetries right at the branch point compared to symmetries already present in the general case. These symmetries are summarized in proposition 4.9. We will now show that we can solve the equation defining the deformation preserving these symmetries. Thus we are coming to the conclusion that the deformation defined in the section 4.1 preserves these symmetries in the following way. If such symmetry occurs in one curve of the family given by the deformation it has also to occur in all other curves of the family.

Let us gather the important information first. We have seen in proposition 4.9 that the spectral curve at the initial point is defined by

$$\begin{aligned} a(\kappa) &= \kappa^4 + a_2\kappa^2 + a_0, \\ b_1(\kappa) &= f_1(\kappa^2 - \beta_1)\kappa, \\ b_2(\kappa) &= f_2(\kappa^2 - \beta_2), \end{aligned} \tag{4.26}$$

and the parameter κ_0, $\kappa_1 = -\kappa_0$ whereas $a_0, a_2, \beta_1, f_1, \beta_2, f_2, \kappa_0 \in \mathbb{R}$. On the other hand we have defined in section 4.1 a deformation ODE which essentially break down to two equations

$$\begin{aligned} 2\dot{b}_i(\kappa)a(\kappa) - b_i(\kappa)\dot{a}(\kappa) = &- 2\kappa a(\kappa)c_i(\kappa) \\ &+ (\kappa^2+1)(2a(\kappa)c_i'(\kappa) - c_i(\kappa)a'(\kappa), \end{aligned} \tag{4.27a}$$

$$c_1(\kappa)b_2(\kappa) - c_2(\kappa)b_1(\kappa) = -\frac{1}{\pi^2}(\kappa-\kappa_0)(\kappa-\kappa_1)a(\kappa). \tag{4.27b}$$

These equations are accompanied by a third one for κ_j,

$$\dot{\kappa}_j = -(\kappa_j^2+1)\frac{c_i(\kappa_j)}{b_i(\kappa_j)}. \tag{4.28}$$

Let us now rewrite the equation (4.27b) by using $\kappa_0 = -\kappa_1$ and we obtain

$$c_1(\kappa)b_2(\kappa) - c_2(\kappa)b_1(\kappa) = -\frac{1}{\pi^2}(\kappa^2-\kappa_0^2)a(\kappa). \tag{4.29}$$

When applying the initial conditions (4.26) to (4.29) we see that we have only even powers of κ at the right side. In order to solve this equation c_1 has to be an odd function as b_1 is assumed to be odd and c_2 has to be an even function as b_2 is assumed to be even. It is possible to write down c_1 and c_2 as solutions of this equation as long as b_1 and b_2 do not have common roots. This condition is equivalent to $\beta_1 \neq \beta_2$ and $\beta_2 \neq 0$. Let us now assume that we can write c_i as

$$\begin{aligned} c_1(\kappa) &= \gamma_{1,2}\kappa^2 + \gamma_{1,0}, \\ c_2(\kappa) &= \gamma_{2,3}\kappa^3 + \gamma_{2,1}\kappa. \end{aligned} \tag{4.30}$$

A computation shows that under the restrictions mentioned above $\gamma_{i,j}$ turn out to be

$$\begin{aligned}
\gamma_{1,2} =& \frac{-\beta_1^2\beta_2 + \beta_1\beta_2\kappa_0^2 - a_2\beta_1\beta_2 + a_2\beta_2\kappa_0^2 - a_0\beta_2 + a_0\kappa_0^2}{\pi^2 f_2\beta_2(\beta_1-\beta_2)},\\
\gamma_{1,0} =& \frac{-a_0\kappa_0^2}{\pi^2\beta_2 f_2},\\
\gamma_{2,3} =& \frac{1}{\pi^2 f_1},\\
\gamma_{2,1} =& \frac{-\beta_1\beta_2^2 + \beta_2^2\kappa_0^2 - a_2\beta_2^2 + a_2\beta_2\kappa_0^2 - a_0\beta_2 + a_0\kappa_0^2}{\pi^2 f_1\beta_2(\beta_1-\beta_2)}.
\end{aligned}$$

Now we use this result to compute the actual deformation ODE,

$$\begin{aligned}
\dot{\kappa}_0 &= -\frac{(a_0 + (a_2+\beta_1)\beta_2)\kappa_0(\kappa_0^2+1)}{\pi^2(\beta_1-\beta_2)\beta_2 f_1 f_2},\\
\dot{a}_0 &= \frac{2a_0(2a_0(\beta_2-\kappa_0^2) + 2\beta_2(\kappa_0^2-\beta_1) + a_2(\beta_2(\beta_1-\kappa_0^2-1)+\kappa_0^2))}{\pi^2(\beta_1-\beta_2)\beta_2 f_1 f_2},\\
\dot{a}_2 &= \frac{2((a_2-1)a_2\beta_2(\beta_1-\kappa_0^2) + a_0(\beta_2(a_2-2\beta_1+2\kappa_0^2-2) - (a_2-2)\kappa_0^2))}{\pi^2(\beta_1-\beta_2)\beta_2 f_1 f_2},\\
\dot{\beta}_1 &= \frac{\left(a_2-\beta_1^2\right)\beta_2(\beta_1-\kappa_0^2) + a_0(\beta_2(\beta_1+\kappa_0^2+2) - 2(\beta_1+1)\kappa_0^2)}{\pi^2(\beta_1-\beta_2)\beta_2 f_1 f_2},\\
\dot{f}_1 &= \frac{(a_2+\beta_1)\beta_2(\beta_1-\kappa_0^2) + a_0(\beta_2-\kappa_0^2)}{\pi^2(\beta_1-\beta_2)\beta_2 f_2},\\
\dot{\beta}_2 &= \frac{(-2\beta_1(\beta_2+1) + (a_2+1)\kappa_0^2 + \beta_2(-a_2+2\kappa_0^2+1))\beta_2^2 + a_0(\beta_2-\kappa_0^2)}{\pi^2(\beta_1-\beta_2)\beta_2 f_1 f_2},\\
\dot{f}_2 &= \frac{2a_0(\beta_2-\kappa_0^2) + \beta_2(\beta_1(a_2+2\beta_2+1) + \beta_2(a_2-2\kappa_0^2-1) - 2a_2\kappa_0^2)}{\pi^2(\beta_1-\beta_2)\beta_2 f_1}.
\end{aligned} \tag{4.31}$$

We have obtained the equation for $\dot{\kappa}_0$ by rewriting (4.28) and using the solutions for c_2.

4.3.2. Properties of the deformation ODE. In this section we will look more closely at the deformation ODE (4.31) from the previous section and investigate which symmetries are preserved by it.

We first observe that if κ_0, a_0, a_2 β_1, f_1, β_2, f_2 are real numbers so are $\dot{\kappa}_0$, $\dot{a}_0$, $\dot{a}_2$, $\dot{\beta}_1$, $\dot{f}_1$, $\dot{\beta}_2$, $\dot{f}_2$. This shows that κ_0, a_0, a_2 β_1, f_1, β_2, f_2 stay real throughout the deformation if they are real initially.

Proposition 4.11. *If it is possible to write the spectral data of a torus in the form*

$$a(\kappa) = \kappa^4 + a_2\kappa^2 + a_0,$$
$$b_1(\kappa) = f_1(\kappa^2 - \beta_1)\kappa,$$
$$b_2(\kappa) = f_2(\kappa^2 - \beta_2),$$

plus the parameter κ_0 whereas $a_0, a_2, \beta_1, f_1, \beta_2, f_2, \kappa_0 \in \mathbb{R}$, then this special form will be preserved throughout the whole deformation.

Proof. By being able to solve (4.27) and for a, b_1 and b_2 of the form as defined in the proposition and being able to write down an explicit ODE on the coefficients we have actually proved that a, b_1 and b_2 keep their special form throughout the whole deformation. □

We see that the deformation is described by a system of ordinary differential equations with real coefficients. We start the deformation with some known initial conditions and the derivatives are continuous functions of the coefficients so it is clear that this system has a solution for every t in some interval I. Now let I be the maximal interval for which the solution to the ODE exists. This interval either extends to infinity or is bounded by some $t_{max} < \infty$. If it is bounded the only reason for this can be that some derivative in the equation (4.31) has a pole at t_{max}. By inspecting the denominators of the terms on the right side of the deformation ODE we see that a pole is possible if one of the following is true.

$$\beta_1 - \beta_2 = 0, \quad \beta_2 = 0, \quad f_1 = 0, \quad f_2 = 0.$$

Assuming that during the deformation and also at a possible endpoint t_{max} f_i never vanishes we see that the following is true.

Proposition 4.12. *The maximal interval where a solution of the deformation ODE is defined either extends to infinity or ends at t_{max} and then either $\beta_1(t_{max}) - \beta_2(t_{max}) = 0$ or $\beta_2(t_{max}) = 0$ must hold.*

As neither $\beta_1(t) - \beta_2(t) = 0$ or $\beta_2(t) = 0$ can be true for $t < t_{max}$, we also have the following.

Corollary 4.13. *During the deformation either $\beta_1 - \beta_2 > 0$ or $\beta_1 - \beta_2 < 0$ is preserved. The same is also true for $\beta_2 > 0$ or $\beta_2 < 0$.*

Now we have to show that the assumption that f_i never vanishes is indeed correct.

Proposition 4.14. *During the deformation and also at a possible endpoint $f_i \neq 0$ is preserved.*

Proof. On a spectral curve of a torus the differentials $d \ln \mu_i$ are not identically zero. So there exist cycles in $H_1(Y, \mathbb{Z})$ such that $\oint d \ln \mu_i \neq 0$. As the values of $\oint d \ln \mu_i$ are preserved during the deformation f_i cannot become zero during the deformation and at a possible endpoint. □

Proposition 4.15. *The functions f_1, f_2 impose only a reparametrization of the time axis for the functions κ_0, a_0, a_2 β_1, β_2 compared to the case where f_1 and f_2 assumed to be constant, as long as f_1 and f_2 do not change their sign.*

Proof. The functions f_1 and f_2 are only factors in all terms of the deformation ODE. More precisely the terms defining $\dot{\kappa}_0$, $\dot{a}_0$, $\dot{a}_2$, $\dot{\beta}_1$, $\dot{\beta}_2$ contain f_1 and f_2 only as the factor $1/(f_1 f_2)$. Thus as long as f_1 and f_2 do not change their sign their actual value only define a reparametrization of the time axis for the functions κ_0, a_0, a_2 β_1, β_2. □

Corollary 4.16. *Changing f_1 to $-f_1$ at the initial conditions corresponds to the change from t to $-t$, so it reverses the direction of the deformation. The same applies to the change from f_2 to $-f_2$.*

Since we will be interested in the endpoints of the deformation later on this observation tells us that the exact values of f_1 and f_2 do not change the other values at the endpoint of the deformation.

Proposition 4.17. *The values $\ln \mu_1(\infty)$ and $\ln \mu_2(0)$ remain constant during the deformation.*

Proof. The derivatives of $\partial_t \ln \mu_i$ are defined as

$$\begin{aligned}\partial_t \ln \mu_i &= \pi i \frac{c_i(\kappa)}{\nu} \\ &= \pi i \frac{c_i(\kappa)}{\sqrt{(\kappa^2+1)(\kappa^4+a_2\kappa^2+a_0)}}.\end{aligned}$$

We have seen in (4.30) that $c_2(0) = 0$ so $\partial_t \ln \mu_2(0) = 0$. On the other hand we also have seen that $\deg(c_1(\kappa)) = 2$ and so we have

$$\lim_{\kappa\to\pm\infty} \partial_t \ln \mu_1 = \lim_{\kappa\to\pm\infty} \pi i \frac{c_1(\kappa)}{\sqrt{(\kappa^2+1)(\kappa^4+a_2\kappa^2+a_0)}} = 0.$$

□

4.3.3. Solving the problem with singular initial conditions. In section 4.2.3 we have computed the spectral data of a homogeneous torus right at the branch point from spectral genus $g = 0$ to $g = 2$. We have seen that this spectral data depends on the parameter κ_0, the position of one double point κ_d and the two winding numbers $K_1 = L_1$ and $K_2 = -L_2$. From this we obtained

$$\begin{aligned} a_0 &= \kappa_d^4, & a_2 &= -2\kappa_d^2, \\ \beta_1 &= \kappa_d^2, & f_1 &= -\left(K_1\sqrt{\kappa_0^2+1}\right), \\ \beta_2 &= \kappa_d^2, & f_2 &= \left(\frac{K_2}{\kappa_0}\sqrt{\kappa_0^2+1}\right). \end{aligned} \tag{4.32}$$

We see that $\beta_1 = \beta_2$. So the initial condition is a singularity of the deformation ODE, the derivatives are not defined here. We will nevertheless see that it is possible to find a solution for the deformation equation at this point by the means of power series. We will now compute the first derivatives of the spectral data for the initial conditions. In order to compute the derivatives we use the ansatz

$$\begin{aligned} &\kappa_0(t) = \kappa_0(0) + \dot\kappa_0(0)t, \\ &a_0(t) = a_0(0) + \dot a_0(0)t, \quad a_2(t) = a_2(0) + \dot a_2(0)t, \\ &\beta_1(t) = \beta_1(0) + \dot\beta_1(0)t, \quad f_1(t) = f_1(0) + \dot f_1(0)t, \\ &\beta_2(t) = \beta_2(0) + \dot\beta_2(0)t, \quad f_2(t) = f_2(0) + \dot f_2(0)t. \end{aligned}$$

Now we insert these functions in the deformation ODE and solve the resulting equation for $t = 0$. We obtain the derivatives of the spectral data depending on the data of the initial homogeneous torus. This also proves the existence of a solution to the ODE which has the initial data as its value at $t = 0$. The resulting system of equations has two solutions. The first is

$$\begin{aligned} &\dot\kappa_0(0) = \tfrac{\delta_1}{2\kappa_0(0)}, \\ &\dot a_0(0) = \delta_2, \quad \dot a_2(0) = -\tfrac{\delta_2}{\kappa_d^2}, \\ &\dot\beta_1(0) = \tfrac{\delta_2}{2\kappa_d^2}, \quad \dot f_1(0) = \delta_3, \\ &\dot\beta_2(0) = \tfrac{\delta_2}{2\kappa_d^2}, \quad \dot f_2(0) = \delta_4, \\ &\delta_1, \delta_2, \delta_3, \delta_4 \in \mathbb{R}. \end{aligned} \tag{4.33}$$

We can dismiss this first solution as it obeys $\dot\beta_1(0) = \dot\beta_2(0)$ and $2\dot a_0(0) = -\kappa_d^2 \dot a_2(0) = a_2(0)\dot a_2(0)$. This has the consequence that $4a_0 = a_2^2$ will be preserved, so (4.33) does not open the double points on the spectral curve. So let us look at the second solution

$$\begin{aligned} &\dot\kappa_0(0) = \frac{3\big(\kappa_0(0)^3+\kappa_0(0)\big)}{\pi^2 f_1(0) f_2(0)}, \\ &\dot a_0(0) = -\frac{4\kappa_d^2\big(\kappa_d^4+\kappa_0(0)^2\big)}{\pi^2 f_1(0) f_2(0)}, \quad \dot a_2(0) = \frac{4\big(5\kappa_d^4-4\big(\kappa_0(0)^2-1\big)\kappa_d^2-3\kappa_0(0)^2\big)}{\pi^2 f_1(0) f_2(0)}, \\ &\dot\beta_1(0) = \frac{2\big(\kappa_d^4-2\big(\kappa_0(0)^2+2\big)\kappa_d^2+3\kappa_0(0)^2\big)}{\pi^2 f_1(0) f_2(0)}, \quad \dot f_1(0) = \frac{3\kappa_0(0)^2-4\kappa_d^2}{\pi^2 f_2(0)}, \\ &\dot\beta_2(0) = \frac{2\big(3\kappa_d^4-2\big(2\kappa_0(0)^2+1\big)\kappa_d^2+\kappa_0(0)^2\big)}{\pi^2 f_1(0) f_2(0)}, \quad \dot f_2(0) = \frac{-8\kappa_d^2+8\kappa_0(0)^2+1}{\pi^2 f_1(0)}. \end{aligned} \tag{4.34}$$

This solution obviously does not obey the relations we mentioned during the discussion of the first solution, so we will investigate this solution further.

Now we will see that given the values κ_d for the double point there is only one direction for t in which the deformation can take place. The deformation in the other direction will lead to spectral curves with branch points on the real line in the κ-plane or on the unit circle in the λ-plane and thus these spectral curves will not be spectral curves of tori in $\mathbb{S}^3$ as they do not meet the reality conditions, see [2].

As we have seen in the corollary 4.16 we can change the direction of the deformation by changing the sign of f_1 or f_2 respectively at the initial conditions.

Definition 4.18. In the following let us assume that $f_1(0) < 0$ and $f_2(0) > 0$.

If the assumption does not hold we have seen that we can just change the signs accordingly.

The branch points of the spectral curve must be either off the real line in the κ-plane or there can be two pairs of branch points on the real line as in the case of the homogeneous torus. The branch points are at the roots of $a(\kappa) = \kappa^4 + a_2\kappa^2 + a_0$. So there are two cases allowed

(i) $4a_0 > a_2^2$: four distinct branch points off the real line,

(ii) $4a_0 = a_2^2$ and $a_2 < 0$: two pairs of branch points on the real line.

As we see in (4.32) $4a_0 = a_2^2$ and $a_2 < 0$ hold on the initial conditions so only the deformation direction is allowed which will preserve $4a_0 \geq a_2^2$. We apply

$$4a_0 \geq a_2^2 \Rightarrow 4\dot{a}_0 \geq 2a_2\dot{a}_2$$

to the derivatives we have obtained in (4.34). So

$$4\dot{a}_0(0) = -\frac{16\kappa_d^2\left(\kappa_d^4 + \kappa_0(0)^2\right)}{\pi^2 f_1(0)f_2(0)},$$

$$2a_2(0)\dot{a}_2(0) = -\frac{16\kappa_d^2\left(5\kappa_d^4 - 4\left(\kappa_0(0)^2 - 1\right)\kappa_d^2 - 3\kappa_0(0)^2\right)}{\pi^2 f_1(0)f_2(0)}.$$

Our assumption $f_1 f_2 < 0$ gives then that $4\dot{a}_0 \geq 2a_2\dot{a}_2$ is fulfilled when

$$\kappa_d^2(\kappa_d^4 + \kappa_0(0)^2) \geq \kappa_d^2(5\kappa_d^4 - 4\left(\kappa_0(0)^2 - 1\right)\kappa_d^2 - 3\kappa_0(0)^2).$$

There are three solutions to this equation

$$\kappa_d = 0, \quad \kappa_d = \pm i, \quad \kappa_d^2 \leq \kappa_0(0)^2.$$

The first two solutions can be ruled out since $\kappa_d \in \mathbb{R}$ and in the special class of tori we investigate $\kappa_d \neq 0$ is assumed. The last solution $\kappa_d^2 \leq \kappa_0(0)^2$ is the one we are interested in. We see that if $\kappa_d^2 < \kappa_0(0)^2$ holds t has to be increasing in order for the reality condition to be preserved. If $\kappa_d^2 > \kappa_0(0)^2$ holds t has to be decreasing. Thus we have proved the following.

Proposition 4.19. *Given the spectral data as in (4.32) the reality conditions are preserved by the deformation only in one direction of the deformation.*

In the following we will only analyze the case $\kappa_0(0) > \kappa_d > 0$. We have already seen that the cases with $\kappa_0(0) < 0$ can be reduced to the cases with $\kappa_0(0) > 0$. We recall how we introduced the κ parameter in the section 3.2. We demanded that κ attains $\kappa = 0$ at $\lambda = 1$. Now it is possible to change this definition so that $\kappa = 0$ is attained at $\lambda = -1$. The resulting transformation rule is

$$\kappa = i\frac{1+\lambda}{1-\lambda}$$

Compared to the previous definition of κ the transition to this parameter corresponds to the change $\kappa \to -1/\kappa$. With this transformation we can make sure that $\kappa_0(0) > \kappa_d$. The formula for H has to be changed accordingly of course. It will follow that $\kappa_0 \to 0$ correspond to $H \to -\infty$ now. So we see that we can deduce the behaviour in all other cases from the behaviour in the case $\kappa_0(0) > \kappa_d > 0$.

PROPOSITION 4.20. *It is possible to assume $\kappa_0(0) > \kappa_d > 0$. All other cases can be reduced to this case.*

4.4. Endpoints of the deformation

In the last sections we computed the initial spectral data for homogeneous tori of rectangular type, we established the explicit form of the deformation ODE and gathered some properties of it. In this section we will use these properties to investigate the possible endpoints of the deformation studied in the sections before. We are especially interested in the case that the mean curvature goes to infinity during the deformation. In the first part of this section we will develop a necessary condition on the initial data in order for this to happen. The second part of this section will investigate the deformation paths the deformation take if this necessary condition is held. We will show that the only possible endpoints of such paths are those where the mean curvature is infinity. We will exclude all other possibilities. This will show that the necessary condition found in the first part of this section is in fact a sufficient condition for the mean curvature to go to infinity. We will see that the spectral curve at the end of such deformation is a spectral curve of a torus in $\mathbb{R}^3$. So in this section we will achieve the main goal of the thesis by showing a class of tori in $\mathbb{S}^3$ which deform to tori in $\mathbb{R}^3$.

We first look at the differentials $d \ln \mu_i$. We recall that these are obtained by

$$\partial_\kappa \ln \mu_i = \pi i \frac{b_i(\kappa)}{(\kappa^2+1)\sqrt{(\kappa^2+1)a(\kappa)}}.$$

So in our case we have

$$\partial_\kappa \ln \mu_1 = \pi i \frac{f_1(\kappa^2-\beta_1)\kappa}{(\kappa^2+1)\sqrt{(\kappa^2+1)(\kappa^4+a_2\kappa^2+a_0)}}, \tag{4.36a}$$

$$\partial_\kappa \ln \mu_2 = \pi i \frac{f_2(\kappa^2-\beta_2)}{(\kappa^2+1)\sqrt{(\kappa^2+1)(\kappa^4+a_2\kappa^2+a_0)}}. \tag{4.36b}$$

We have already seen that $4a_0 \geq a_2^2$ is preserved during the deformation so the following holds

$$a(\kappa) \geq 0 \quad \forall \kappa \in \mathbb{R}.$$

This leads to $\sqrt{(\kappa^2+1)a(\kappa)} \in \mathbb{R}$. We even have

$$(\kappa^2+1)\sqrt{(\kappa^2+1)a(\kappa)} > 0 \quad \text{for all } \kappa \in \mathbb{R}.$$

Therefore we obtain

$$\frac{f_1(\kappa^2-\beta_1)\kappa}{(\kappa^2+1)\sqrt{(\kappa^2+1)(\kappa^4+a_2\kappa^2+a_0)}} \in \mathbb{R} \quad \forall\kappa \in \mathbb{R},$$
$$\frac{f_2(\kappa^2-\beta_2)}{(\kappa^2+1)\sqrt{(\kappa^2+1)(\kappa^4+a_2\kappa^2+a_0)}} \in \mathbb{R} \quad \forall\kappa \in \mathbb{R},$$

since f_i and β_i are real as well. This gives us on the other hand

$$\frac{1}{\pi i}\ln\mu_1, \frac{1}{\pi i}\ln\mu_2 \in \mathbb{R} \quad \forall\kappa \in \mathbb{R}.$$

and we have proved the following proposition.

Proposition 4.21. *The map* Ψ *restricted to* $\kappa \in \mathbb{R}$ *and defined by*

$$\Psi : \kappa \mapsto \left(\frac{1}{\pi i}\ln\mu_1, \frac{1}{\pi i}\ln\mu_2\right)$$

is a map $\Psi : \mathbb{R} \to \mathbb{R}^2$.

As every condition in the previous argument is preserved by deformation the proposition also holds at every point of the deformation.

Proposition 4.22. *The value* $(\pi i)^{-1}\ln\mu_2(\infty)$ *is either monotonously decreasing or monotonously increasing throughout the whole deformation.*

Proof. We have seen that $c_2(\kappa) = \gamma_{2,3}\kappa^3 + \gamma_{2,1}\kappa$ with $\gamma_{2,3} = (\pi^2 f_1)^{-1}$ so we obtain

$$\lim_{\kappa\to\pm\infty} \partial_t(\pi i)^{-1}\ln\mu_2 = \lim_{\kappa\to\pm\infty} \frac{c_2(\kappa)}{\sqrt{(\kappa^2+1)(\kappa^4+a_2\kappa^2+a_0)}} = \frac{1}{\pi^2 f_1} \neq 0.$$

In proposition 4.14 we have seen that $f_1 \neq 0$ during the deformation, so it cannot change its sign as well. Therefore $(\pi i)^{-1}\ln\mu_2(\infty)$ is either strictly monotonic decreasing or monotonic increasing throughout the whole deformation. □

Corollary 4.23. *During the deformation of spectral curves every curve can occur only once.*

Proof. The value of $\Psi(\infty)$ changes strictly monotonic and thus is different for every t. Different values of $\Psi(\infty)$ in a family of spectral curves can only result form different spectral curves. □

Let us now analyze this map Ψ. We first start with this map in case of spectral genus $g = 0$ before the branching to $g = 2$. In this case we actually have

$$\partial_\kappa \ln\tilde{\mu}_1 = \pi i\frac{f_1\kappa}{(\kappa^2+1)\sqrt{(\kappa^2+1)}}, \tag{4.37a}$$
$$\partial_\kappa \ln\tilde{\mu}_2 = \pi i\frac{f_2}{(\kappa^2+1)\sqrt{(\kappa^2+1)}}. \tag{4.37b}$$

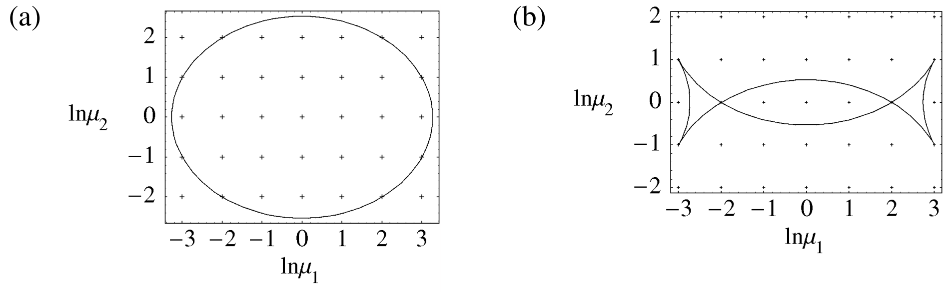

FIGURE 4-2. Imaginary parts of $\ln \mu_i$ of the homogeneous torus. The plot (a) shows the imaginary parts of $\ln \mu_i$ on the spectral curve of genus $g = 0$. The plot (b) shows the imaginary parts of $\ln \mu_i$ on the spectral curve of genus $g = 2$ right after opening the double points.

When we set $\ln \tilde{\mu_1}(\infty) = 0$ and $\ln \tilde{\mu_2}(0) = 0$ we can easily compute

$$\frac{1}{\pi i} \ln \tilde{\mu_1} = -\frac{f_1}{\sqrt{(\kappa^2 + 1)}}, \tag{4.38a}$$

$$\frac{1}{\pi i} \ln \tilde{\mu_2} = \frac{f_2 \kappa}{\sqrt{(\kappa^2 + 1)}}. \tag{4.38b}$$

One can show that these functions produce the right values for $\ln \mu_i(\kappa_0)$ and $\ln \mu_i(\kappa_d)$ if the values f_i are obtained by the formulas from the section 4.2.1. So in the case $g = 0$ we have

$$\Psi_0(\kappa) = (-\frac{f_1}{\sqrt{(\kappa^2 + 1)}}, \frac{f_2 \kappa}{\sqrt{(\kappa^2 + 1)}}).$$

The components of Ψ_0 obey to the following equation

$$(\frac{1}{f_1})^2(-\frac{f_1}{\sqrt{(\kappa^2 + 1)}})^2 + (\frac{1}{f_2})^2(\frac{f_2 \kappa}{\sqrt{(\kappa^2 + 1)}})^2 = \frac{1}{\kappa^2 + 1} + \frac{\kappa^2}{\kappa^2 + 1} = 1.$$

We can see that the graph of Ψ_0 is the one half of an ellipse with its center at 0 and the axes f_1 and f_2. If we let κ go twice from $-\infty$ to ∞, the second time using the function $\Psi_0(\kappa) = (f_1(\sqrt{(\kappa^2 + 1)})^{-1}, -f_2\kappa(\sqrt{(\kappa^2 + 1)})^{-1})$, we actually obtain a full ellipse. The second half of the ellipse corresponds to the Ψ_0 after the sheet interchange σ which causes $\sigma^*\kappa = \kappa$ and $\sigma^* \ln \mu_i = -\ln \mu_i$. The figure 4-2a shows such a graph.

LEMMA 4.24. *The ellipse in previous graph together with the value κ_0 and κ_d completely determine the spectral data of a spectral genus $g = 0$ torus.*

PROOF. In section 4.2.1 we have seen that we can determine the spectral data by the triple (f_1, f_2, κ_0) and the values f_1 and f_2 are determined by the sizes of the axes of the ellipse. □

Now let us investigate what happens when we branch to $g = 2$. At the starting point we have the spectral data as shown in (4.32), so equation (4.36) becomes

$$\begin{aligned}\partial_\kappa \ln \mu_1 &= \pi i \frac{f_1(\kappa^2 - \kappa_d^2)\kappa}{(\kappa^2+1)\sqrt{(\kappa^2+1)(\kappa-\kappa_d)^2(\kappa+\kappa_d)^2}} \\ &= \pi i \frac{f_1 \kappa}{(\kappa^2+1)\sqrt{(\kappa^2+1)}} \frac{\kappa^2-\kappa_d^2}{|\kappa^2-\kappa_d^2|},\end{aligned} \tag{4.39a}$$

$$\begin{aligned}\partial_\kappa \ln \mu_2 &= \pi i \frac{f_2(\kappa^2 - \kappa_d^2)}{(\kappa^2+1)\sqrt{(\kappa^2+1)(\kappa-\kappa_d)^2(\kappa+\kappa_d)^2}} \\ &= \pi i \frac{f_2}{(\kappa^2+1)\sqrt{(\kappa^2+1)}} \frac{\kappa^2-\kappa_d^2}{|\kappa^2-\kappa_d^2|}.\end{aligned} \tag{4.39b}$$

We see that the only difference between (4.37) and (4.39) is the additional factor $(\kappa^2 - \kappa_d^2)\,|\kappa^2 - \kappa_d^2|^{-1}$. This factor obviously has the following behavior on the real line

$$\frac{\kappa^2-\kappa_d^2}{|\kappa^2-\kappa_d^2|} = \begin{cases} 1 & \kappa^2 > \kappa_d^2 \\ -1 & \kappa^2 < \kappa_d^2 \end{cases} \tag{4.40}$$

We use this behavior together with (4.38) which we have computed for (4.37) to compute the $\ln \mu_i$ for real κ. As in case of $g = 0$ we still want $\ln \mu_1(\infty) = 0$ and $\ln \mu_2(0) = 0$. We obtain then

$$\ln \mu_1(\kappa) = \begin{cases} \ln \tilde{\mu}_1(\kappa) & \kappa^2 \geq \kappa_d^2 \\ -\ln \tilde{\mu}_1(\kappa) + 2\ln \tilde{\mu}_2(|\kappa_d|) & \kappa^2 < \kappa_d^2 \end{cases} \tag{4.41a}$$

$$\ln \mu_2(\kappa) = \begin{cases} \ln \tilde{\mu}_2(\kappa) - 2\ln \tilde{\mu}_2(|\kappa_d|) & \kappa^2 > \kappa_d^2,\ \kappa > 0 \\ -\ln \tilde{\mu}_2(\kappa) & \kappa^2 \leq \kappa_d^2 \\ \ln \tilde{\mu}_2(\kappa) - 2\ln \tilde{\mu}_2(-|\kappa_d|) & \kappa^2 > \kappa_d^2,\ \kappa < 0 \end{cases} \tag{4.41b}$$

Here we extended the map to the points $\kappa = \kappa_d$ and $\kappa = -\kappa_d$ continuously. Using this functions we can draw a graph as in figure 4-2b.

Now we will analyze the branching procedure. We look at the following four points closely.

$$P_1 = \Psi_0(\kappa_d),\ P_2 = \Psi_0(-\kappa_d),\ P_3 = \sigma(P_1),\ P_4 = \sigma(P_2).$$

We have

$$P_1 = (P_1', P_1'') = ((\pi i)^{-1} \ln \tilde{\mu}_1(\kappa_d), (\pi i)^{-1} \ln \tilde{\mu}_2(\kappa_d)). \tag{4.42}$$

A computation shows then that

$$\begin{aligned} P_1 &= (P_1', P_1''), & P_2 &= (P_1', -P_1''), \\ P_3 &= (-P_1', -P_1''), & P_4 &= (-P_1', P_1''). \end{aligned} \tag{4.43}$$

These four points are the vertexes of a rectangle. The branching procedure interchanges P_1 with P_2 and P_3 with P_4. The relations in stated in (4.43) are preserved by this procedure. After this procedure the points are still vertexes of the same rectangle. Now we can see what happens with the arcs of the ellipse from Ψ_0 joining these points. The arc between P_1 with P_2 is

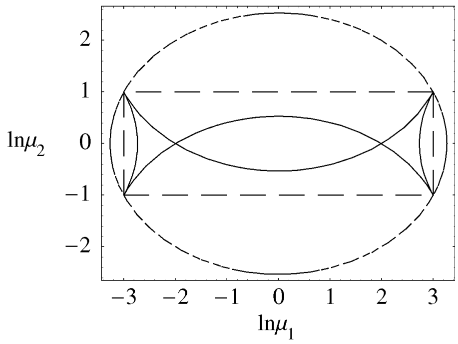

FIGURE 4-3. Imaginary parts of $\ln \mu_i$ of the homogeneous torus before and after the branching.

reflected in the center of the segment form P_1 to P_2. Same happens to the arc between P_3 and P_4. The arc between P_1 and P_4 and the arc between P_2 and P_3 are translated according to translation of endpoints (P_1,P_4) to (P_2,P_3) and vice versa. The figure 4-3 shows the image of Ψ_0 (the ellipse) and of Ψ after the branching and also the rectangle defined by P_1, P_2, P_3 and P_4.

Besides the points P_i on the graph of Ψ and Ψ_0 there is one more distinguished point $P_0 = \Psi_0(\kappa_0)$ and after the branching the point $P = \Psi(\kappa_0)$. Given the points P_1, P_2, P_3 and P_4 there is a one parameter family of ellipses joining all these points. If we also take into account that the point P_0 which is not equal to one of the former points also lies on the ellipse this ellipse is completely determined and thus using the lemma 4.24 also the spectral data of the spectral genus $g = 0$ torus. After the branching the five points P, P_1, P_2, P_3 and P_4 do not completely determine the spectral data. The ambiguity is that P can lie on the arc between P_1 and P_2 or on the arc between P_2 and P_3. So there are exactly two different spectral genus $g = 0$ curves which lead to the same data for P, P_1, P_2, P_3 and P_4. Now we can show the following

PROPOSITION 4.25. *There cannot be more than two different genus $g = 0$ spectral curves in a family of spectral curves from spectral genus $g = 2$ tori of rectangular type. If there are two such curves one of these genus $g = 0$ spectral curves completely determines the other.*

PROOF. The value of P_1 was defined in (4.42) by the value of the functions $\ln \mu_i$ at the double point. This double point of the spectral curve is opened to two branch points of the spectral curve. The values of $\ln \mu_i$ at the branch points of the spectral curve remain constant during the deformation. So P_1 now defined by the values of $\ln \mu_i$ at a branch point of the spectral curve remains constant. The point P also remains constant during the deformation as the deformation was constructed so that the values of $\ln \mu_i$ at κ_0 remain constant. If a family of genus $g = 2$ spectral curves also contains one genus $g = 0$ spectral curve which lead to points P, P_1, P_2, P_3 and P_4 every other genus $g = 0$ spectral curve which may be contained in this family will produce the same P, P_1, P_2, P_3 and P_4. The argument in the previous paragraph showed that for given P, P_1, P_2, P_3 and P_4 there can be only two different genus $g = 0$ spectral curves which lead to this points. □

We now will turn to our main task, to develop criteria for the initial conditions which ensure that during the deformation the mean curvature increases to infinity $H \to \infty$. The graph we introduced earlier will now be very helpful in developing this criteria. We remember first that

$$H = \frac{1}{2}(1 - \kappa_0^2)/\kappa_0$$

so $H \to \infty$ is equivalent to $\kappa_0 \to 0$ and $H \to -\infty$ to $\kappa_0 \to \infty$. The deformation is constructed in such a way that $\partial_t \ln \mu_2(\kappa_0) = 0$. We also have seen that $\partial_t \ln \mu_2(0) = 0$ during the deformation (proposition 4.17). Obviously it is necessary that $\ln \mu_2(\kappa_0) = \ln \mu_2(0)$ holds at the initial conditions so that $\kappa_0 \to 0$ is possible and all coefficients of the spectral data remain finite. This is of course equivalent to $\int_0^{\kappa_0} d \ln \mu_2 = 0$. Thus we have found and proved the necessary condition for $H \to \infty$.

Proposition 4.26. *The deformation of the spectral curves from tori of rectangular type can only end in a spectral curve of a torus with $H = \infty$ if the initial conditions of the deformation are such that the following holds.*

$$\int_0^{\kappa_0} d \ln \mu_2 = 0.$$

We have seen that we can compute $\ln \mu_2$ at the initial conditions using the equations (4.39b), (4.40) and (4.37b). With our assumption $\kappa_0 > \kappa_d$ we then obtain

$$\begin{aligned}\int_0^{\kappa_0} d \ln \mu_2 &= \int_0^{\kappa_d} d \ln \mu_2 + \int_{\kappa_d}^{\kappa_0} d \ln \mu_2 \\ &= \frac{f_2 \kappa}{\sqrt{(\kappa^2+1)}}\Big|_{\kappa=0} - \frac{f_2 \kappa}{\sqrt{(\kappa^2+1)}}\Big|_{\kappa=\kappa_d} - \frac{f_2 \kappa}{\sqrt{(\kappa^2+1)}}\Big|_{\kappa=\kappa_d} + \frac{f_2 \kappa}{\sqrt{(\kappa^2+1)}}\Big|_{\kappa=\kappa_0} \\ &= -2\frac{f_2 \kappa_d}{\sqrt{(\kappa_d^2+1)}} + \frac{f_2 \kappa_0}{\sqrt{(\kappa_0^2+1)}} = 0\end{aligned}$$

So the condition in proposition 4.26 break down to κ_0 and κ_d solving the equation

$$-2\frac{\kappa_d}{\sqrt{(\kappa_d^2+1)}} + \frac{\kappa_0}{\sqrt{(\kappa_0^2+1)}} = 0$$

There are two solutions for this equation

$$\kappa_0 = \frac{2\kappa_d}{\sqrt{1-3\kappa_d^2}} \quad \kappa_0 = -\frac{2\kappa_d}{\sqrt{1-3\kappa_d^2}}.$$

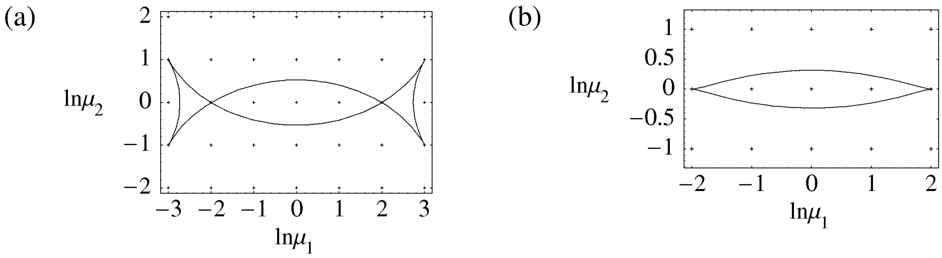

FIGURE 4-4. Imaginary parts of $\ln\mu_i$ during the deformation of the spectral curve. The (a) shows the imaginary parts of $\ln\mu_i$ on the spectral curve right after opening the double points. The (b) shows the imaginary parts of $\ln\mu_i$ on the spectral curve at the end of the deformation when the spectral curve of a torus in $\mathbb{R}^3$ is reached.

We have shown that it is possible to assume $\kappa_0 > 0$, so only the first solution is of interest for us.

COROLLARY 4.27. *The necessary condition for $\kappa \to 0$ is equivalent to*

$$\kappa_0 = \frac{2\kappa_d}{\sqrt{1 - 3\kappa_d^2}}$$

being fulfilled at the initial conditions.

REMARK. The parameter κ_0 needs to be real, so $\kappa_d^2 < 1/3$ has to hold. The value of the denominator is bounded when this inequality holds. Namely $\sqrt{1 - 3\kappa_d^2} \leq 1$. So the condition of the corollary also imply $\kappa_0 \geq \kappa_d$. Thus given such initial conditions the mean curvature will be decreasing at the start of the deformation.

Figure 4-4 shows the graphs of Ψ for one such deformation. The spectral data of the initial homogeneous torus was obtained by the algorithm from section 4.2.2. We started with the parameters $K_1 = 1$, $K_2 = 2$ and $M_1 = 3$, $M_2 = 1$ and obtained $\kappa_0 = \sqrt{5/3}$ and $\kappa_d = 5/27$. The rest of the spectral data was obtained by the formulas in section 4.2.3. The graph at the end of the deformation was obtained by numerical integration of the deformation ODE.

PROPOSITION 4.28. *If the initial spectral data for the deformation fulfill*

$$\int_0^{\kappa_0} d\ln\mu_2 = 0$$

then the inequality

$$\kappa_0 \geq \sqrt{\beta_2}$$

is fulfilled as well and both conditions stay fulfilled throughout the deformation. If the deformation ODE ends with $\beta_2 = 0$ *then* $\kappa_0 = 0$ *must hold at this endpoint as well.*

Proof. We have seen that $\partial_t \ln \mu_2(\kappa_0) = 0$ and $\partial_t \ln \mu_2(0) = 0$ so the condition $\int_0^{\kappa_0} d \ln \mu_2 = 0$ stays fulfilled during the deformation. On the other hand

$$\partial_\kappa \ln \mu_2 = \pi i \frac{f_2(\kappa^2 - \beta_2)}{(\kappa^2 + 1)\sqrt{(\kappa^2 + 1)(\kappa^4 + a_2\kappa^2 + a_0)}}.$$

As we have seen the denominator stays positive for all real κ and $\pm\sqrt{\beta_2}$ are obviously the only roots of the numerator. So the mean value theorem shows us that the condition $\int_0^{\kappa_0} d \ln \mu_2 = 0$ implies the condition $\kappa_0 \geq \sqrt{\beta_2}$. In case $\beta_2 = 0$ the derivative $\partial_\kappa \ln \mu_2$ has only a double root at $\kappa = 0$ and is either positive or negative depending on f_2 for all other κ. So $\int_0^{\kappa_0} d \ln \mu_2 = 0$ can only be fulfilled with $\kappa_0 = 0$. This means $\int_0^{\kappa_0} d \ln \mu_2 = 0$ and $\beta_2 = 0$ imply $\kappa_0 = 0$. □

We have seen in the proposition 4.25 that in a family of genus $g = 2$ spectral curves, that we are looking at, no more than two different spectral curves of genus $g = 0$ with double points can occur. Now we will show that in the family of spectral curves which hold the condition from proposition 4.26 only one spectral curves of genus $g = 0$ with double points can occur. We have seen that these genus $g = 0$ curves are defined by the points P and P_1 to P_4. The two different curves arise as there are two different possibilities for the arc on which P can lie. Either it is the arc between P_1 and P_2 or the arc between P_2 and P_3. We have $P = (P', P'') = ((\pi i)^{-1} \ln \mu_1(\kappa_0), (\pi i)^{-1} \ln \mu_2(\kappa_0))$. The condition from proposition 4.26 leads to $P'' = 0$. This means that in case that P lies on the arc between P_1 and P_2 also $P_0 = (P'_0, 0)$. This can only be fulfilled when $\kappa_0 = 0$ and then $\kappa_0 < \sqrt{\beta_2}$. This obviously contradicts to the previous proposition. As a spectral curve can occur only once during the deformation and the spectral curve at the start of the deformation is already a spectral curve of genus $g = 0$ with double points it has to be the only such curve in the family holding the condition from proposition 4.26.

Proposition 4.29. *In the family of spectral curves which hold the condition from proposition 4.26 only one spectral curve of genus* $g = 0$ *with double points can occur. This is the spectral curve at start of the deformation.*

We have proved so far that $\int_0^{\kappa_0} d \ln \mu_2 = 0$ has to hold at the start and during the deformation in order for k_0 to be able to go to 0. We will now look at the possible endpoints of a family of spectral curves which hold $\int_0^{\kappa_0} d \ln \mu_2 = 0$. In the last section in the proposition 4.12 we have seen that there are three possibilities how the deformation can end. It can extend to infinity or it can end with t_{max} such that $\beta_1 - \beta_2 = 0$ or $\beta_2 = 0$ at t_{max}. The next computations will show that $\beta_1 - \beta_2 = 0$ is not possible when $\int_0^{\kappa_0} d\ln\mu_2 = 0$ holds at the starting point.

For further treatment we will look at a slightly different variant of the map Ψ we investigate earlier. We extend the domain of this map from $\kappa \in \mathbb{R}$ to all $\kappa \in \mathbb{C}$ in such a way that we obtain a map $\tilde{\Psi} : Y \mapsto \mathbb{R}^2$. We redefine $\tilde{\Psi}$ to

$$\tilde{\Psi} = (\Re\{\ln\mu_1\}, \Re\{\ln\mu_2\}). \tag{4.44}$$

In the following we will prove some properties of this map.

Lemma 4.30. *The map $\tilde{\Psi}$ is a well defined map $\tilde{\Psi} : Y \mapsto \mathbb{R}^2$. This map is an immersion if*

$$\Im\left\{\frac{d\ln\mu_1}{d\ln\mu_2}\right\} \neq 0$$

Proof. The involution representing the sheet interchange is subject to $\sigma^*\kappa = \kappa$ and $\sigma^*\mu_i = \mu_i^{-1}$ so $\sigma^* \ln\mu_i(\kappa) = -\ln\mu_i(\kappa)$.

Now we compute the derivative of $\tilde{\Psi}$ and obtain

$$d\tilde{\Psi} = \begin{pmatrix} \Re\{\partial\ln\mu_1\} & \Re\{\partial\ln\mu_2\} \\ -\Im\{\partial\ln\mu_1\} & -\Im\{\partial\ln\mu_2\} \end{pmatrix}$$

So $\tilde{\Psi}$ is an immersion only if

$$\Re\{\partial\ln\mu_1\}\Im\{\partial\ln\mu_2\} + \Re\{\partial\ln\mu_2\}\Im\{\partial\ln\mu_1\} = 0.$$

This is equivalent to

$$\Im\left\{\frac{d\ln\mu_1}{d\ln\mu_2}\right\} \neq 0.$$

□

Let us now look at $d\ln\mu_1/d\ln\mu_2$. In case both differentials have a non zero common root we have $\beta_1 = \beta_2$. Then we can use this to obtain

$$\frac{d\ln\mu_1}{d\ln\mu_2} = \frac{f_1(\kappa^2-\beta)\kappa}{f_2(\kappa^2-\beta)} = \frac{f_1}{f_2}\kappa.$$

We see that in this case $d\ln\mu_1/d\ln\mu_2 \in \mathbb{R}$ exactly for $\kappa \in \mathbb{R}$. Using the previous lemma we see that the $\kappa \in \mathbb{R}$ are exactly the points where $\tilde{\Psi}$ is not an immersion.

Proposition 4.31. *In case that $\beta_1 = \beta_2$ the map $\tilde{\Psi} : Y \to \mathbb{R}^2$ is a double cover of $\mathbb{CP}^1$ with the only branch points at $\kappa = \pm i$.*

Proof. In this argument we will use the parameter λ. We look at $\ln\mu_i$ at the points $\lambda = 0$. Both maps have a pole at this point. We recall that there exists a local coordinate function z such that $z = 0$ correspond to $\lambda = 0$ and so that

$$\ln\mu_1 = m_{-1}z^{-1} + m_0 + zh(z),$$
$$\ln\mu_2 = \tilde{m}_{-1}\tau z^{-1} + \tilde{m}_0 + z\tilde{h}(z)$$

for $|\lambda| \leq 1$ with $\tau \notin \mathbb{R}$ and holomorphic functions h and $\tilde{h}$. We see therefore that for $|\lambda| \leq 1$ the map $\tilde{\Psi}$ is a continuous map to $\mathbb{CP}^1$. The points with $|\lambda| \leq 1$ constitute a manifold with boundary, call it Y'. The collar of this manifold is diffeomorphic to $\mathbb{S}^1 \times [0, 1)$. We identify the $\mathbb{S}^1 \times (0, 1)$ part of this collar with a punctured disk in polar coordinates which is also diffeomorphic to $\mathbb{S}^1 \times (0, 1)$. The map $\tilde{\Psi}$ is as the real part of a holomorphic map harmonic on this punctured disc and can be extended to the center o. We identified the boundary of Y' with a point. Thus we obtain a map $Y' \to \mathbb{CP}^1$.

We have seen that $d \ln \mu_1/d \ln \mu_2 \sim \kappa$ for $|\lambda| = 1$. This means that this quotient takes every value in $\mathbb{R}$. If we take into account that over every point λ with $|\lambda| = 1$ there are two points in Y, the quotient $d \ln \mu_1/d \ln \mu_2$ takes every value in $\mathbb{R}$ two times. This shows that the tangent space at the point o is isomorphic to the whole $\mathbb{R}^2$ and the map $\tilde{\Psi}$ is also an immersion at this point.

Now let us get back to the disc which we identifyed with the collar described in the previous paragraph. The lemma 4.30 showed us that $\tilde{\Psi}$ is an immersion on the punctured disc. The previous argument shows us the $\tilde{\Psi}$ is also an immersion at the center o of the disc. Thus we see that the map $Y' \to \mathbb{CP}^1$ is everywhere an immersion. It is open and closed map into $\mathbb{CP}^1$ and therefore has to be surjective. We also see that this map is a covering map. The preimage of $\lambda = 0$ consists of one point so we have an homeomorphism of degree 1. This shows that Y' is homeomorphic to $\mathbb{CP}^1$ after the identification of $|\lambda| = 1$ to one point. The same argument is also valid for $\{y \in Y : |\lambda(y)| \geq 1)\} = Y''$. Thus we see that the whole Y has genus $g = 0$. As Y is always a double cover of $\mathbb{CP}^1$ and we have shown that it has genus $g = 0$ it can only have branch points over $\lambda = 0$ and $\lambda = \infty$ this corresponds to $\kappa = \pm i$ and thus proves the claim. □

Proposition 4.32. *Let us assume the deformation hits a point where $\beta_1 = \beta_2 = \kappa_d^2$ is fulfilled for a $\kappa_d > 0$. Then the spectral data (a, b_1, b_2, κ_0) at this point is of the form*

$$a(\kappa) = \kappa^4 - (2\kappa_d^2)\kappa^2 + \kappa_d^4,$$
$$b_1(\kappa) = f_1(\kappa^2 - \kappa_d^2)\kappa,$$
$$b_2(\kappa) = f_2(\kappa^2 - \kappa_d^2).$$

Proof. As $\beta_1 = \beta_2 = \beta$ we use proposition 4.31 to see that the spectral curve which is represented by the hyperelliptic curve Y has only branch points at $\kappa = \pm i$. This curve is defined by the equation $\nu^2 = (\kappa^2 + 1)a(\kappa)$ with a polynomial $a(\kappa) = \kappa^4 + a_2\kappa^2 + a_0$. In order for Y to have only branch points at $\kappa = \pm i$ the roots of a have to be of the form $\alpha_1 = \alpha_2$ and $\alpha_3 = \alpha_4$. Therefore the relation $4a_0 = a_2^2$ has to be fulfilled. The differentials $d \ln \mu_i$ are not allowed to have poles outside the branch points of Y so they do not have other poles besides those at $\kappa = \pm i$. This leads to the relation $a_0 = 1/4a_2^2 = \beta_1^2 = \beta_2^2$. This spectral data corresponds to a flat torus with double points. We have seen that such tori can only have double points on the real axis in κ, this means $a_2 < 0$. On the other hand our deformation allowed only $\beta_2 \geq 0$. By setting $\beta_2 = \kappa_d^2$ and thus $\beta_1 = \kappa_d^2$ and $a_0 = -2\kappa_d^2$, $a_0 = \kappa_d^4$ we obtain the claim. □

Proposition 4.33. *In a family of genus $g = 2$ spectral curves from tori of rectangular type which fulfill $\int_0^{\kappa_0} d\ln\mu_2 = 0$ a spectral curve with $\beta_1 = \beta_2$ never occurs.*

Proof. We have now proved that if $\beta_1 = \beta_2$ occurs the spectral data at this point corresponds to spectral data of a flat torus with double points. On the other hand the proposition 4.29 showed that in a family holding the condition from proposition 4.26 the only such curve is the one at the start of the deformation. So the case $\beta_1 = \beta_2$ cannot occur in the endpoint. □

Collecting the previous observations we have the following situation. The solution of the deformation ODE can either extend to infinity or end with $\beta_1 = \beta_2$ or $\beta_2 = 0$. We have shown that given the initial condition $\int_0^{\kappa_0} d\ln\mu_2 = 0$ the endpoint $\beta_1 = \beta_2$ is not possible and we also have shown that in this case the endpoint $\beta_2 = 0$ implies κ_0 and thus $H = \infty$. So it remains to analyze what happens if the solution of the deformation ODE extends to infinity. To this end we return to the map Ψ from the beginning of the section. We want to look at the region enclosed by the arcs connecting $\Psi(\kappa)$ and $\sigma * \Psi(\kappa)$. We denote by γ the path which defines the boundary of this area. We can divide γ in four pieces and thus compute the whole path piecewise. Lets call this pieces γ_i, $i = 1, .., 4$. We define $\gamma_1 : [\kappa_0, \infty] \to \mathbb{R}^2$, $\gamma_1(\kappa) = \Psi(\kappa)$. We also denote by $\gamma_1 = (\gamma_1', \gamma_1“)$ the both components of γ_1. We use the relations $ln\mu_1(-\kappa) = ln\mu_1(\kappa)$, $ln\mu_2(-\kappa) = -ln\mu_1(\kappa)$ and the sheet interchange σ which has the effect $\sigma * \kappa = \kappa$, $\sigma * \ln\mu_1 = -\ln\mu_1$ and $\sigma * \ln\mu_2 = -\ln\mu_2$ to compute the remaining three pieces of γ. We can write then

$$\gamma_1 = (\gamma_1', \gamma_1''), \quad \gamma_2 = (\gamma_1', -\gamma_1''), \quad \gamma_3 = (-\gamma_1', -\gamma_1''), \quad \gamma_4 = (-\gamma_1', \gamma_1''). \tag{4.45}$$

The endpoints of γ_1 and γ_4 corresponding to $\kappa \to \infty$ coincide. The same is true for γ_2 and γ_3. The next proposition shows that also the endpoints of γ_1 and γ_2 and the endpoints of γ_3 and γ_4 corresponding to $\kappa = \kappa_0$ also coincide. This means that γ is a closed path.

Lemma 4.34. *If the initial spectral data fulfill $\int_0^{\kappa_0} d\ln\mu_2 = 0$ the path γ is closed for every t and encloses a simply connected region in $\mathbb{R}^2$.*

Proof. If $\int_0^{\kappa_0} d\ln\mu_2 = 0$ is fulfilled we have seen that $\Psi(-\kappa_0) = \Psi(\kappa_0)$ so γ is closed at the start of the deformation. The deformation is constructed so that $\partial_t\Psi(\kappa_0) = 0$ and $\partial_t\Psi(-\kappa_0) = 0$ hold. This ensures that γ stays closed during the deformation.

Now we have to show that γ does not have any self intersections since then the area it encloses has to be simply connected. This is true for the initial spectral curve of a homogeneous torus as one can see from the construction of this curve. We need to show that this also stays so during the deformation. We remember first that the zeros of both $\partial_k \ln \mu_i$ are inside the interval $(-\kappa_0, \kappa_0)$. This means that on γ_1 both components γ_1' and γ_1'' are monotonic and so are the components of γ_2, γ_3 and γ_4. This proves that no γ_i cannot have any self intersections. One endpoint of γ_1 is equal to $((\pi i)^{-1} \ln \mu_1(0), 0)$ and the other is equal to $(0, \lim_{\kappa\to\infty}(\pi i)^{-1} \ln \mu_1(\kappa))$ and thus γ_1 stays completely in the first quadrant. By (4.45) we see that every γ_i stays in a different quadrant and we follow that the whole γ has no self intersections and thus the region it encloses must be simply connected. □

We know now that γ encloses some region of finite area A. In the following we will compute the variation $\partial_t A$ of this area. Let $\dot\gamma_1$ be the variation of γ_1. The variation $\partial_t A_1$ of the area which is passed by γ_1 can be computed then by

$$\begin{aligned}
\partial_t A_1 &= \int_{\kappa_0}^{\infty} \dot\gamma(\kappa)^{\perp} \cdot \partial_\kappa \gamma(\kappa) d\kappa = \int_{\kappa_0}^{\infty} (-\dot{\gamma''}(\kappa), \dot{\gamma'}(\kappa)) \cdot (\partial_\kappa \gamma'(\kappa), \partial_\kappa \gamma''(\kappa) d\kappa \\
&= \int_{\kappa_0}^{\infty} \frac{1}{\pi i} \partial_t \ln \mu_1 \frac{1}{\pi i} \partial_\kappa \ln \mu_2 - \frac{1}{\pi i} \partial_t \ln \mu_2 \frac{1}{\pi i} \partial_\kappa \ln \mu_1 d\kappa \\
&= -\frac{1}{\pi^2} \int_{\kappa_0}^{\infty} \omega = -\frac{1}{\pi^2} \int_{\kappa_0}^{\infty} \frac{\kappa^2 - \kappa_0^2}{(\kappa^2+1)^2} d\kappa
\end{aligned}$$

The last integral can be easily computed to

$$\partial_t A_1 = -\frac{1}{\pi^2} \int_{\kappa_0}^{\infty} \frac{\kappa^2 - \kappa_0^2}{(\kappa^2+1)^2} d\kappa = -\frac{1}{4\pi^2} \left((\kappa_0^2 - 1)(2\arctan(\kappa_0) - \pi) + 2\kappa_0 \right)$$

As we constructed γ from four pieces we see that $\partial_t A = 4 \partial_t A_1$. This means that $\partial_t A$ is a function of κ_0 defined by

$$\partial_t A(\kappa_0) = -\frac{1}{\pi^2} \left((\kappa_0^2 - 1)(2\arctan(\kappa_0) - \pi) + 2\kappa_0 \right)$$

We also see that $\partial_t A(0) = -\pi^{-1}$ and $\partial_t A(\infty) = 0$. In between $\partial_t A(\kappa_0)$ is strictly monotonic increasing. This computation allows us now to analyze what behaviour κ_0 can have if the solution of the deformation ODE extends to infinity. We start with a finite area A and a finite κ_0. During the deformation the area A cannot become negative on the other hand

$$|\int_{o}^{\infty} \partial_t A| < \infty \tag{4.46}$$

is only possible if $\kappa_0 \to \infty$ as $t \to \infty$. This behavior has some consequences. In order to see them we use another path $\tilde\gamma$. We define this path by

$$\tilde\gamma : [-\kappa_0, \kappa_0] \to \mathbb{R}^2, \quad \tilde\gamma(\kappa) = \Psi(\kappa).$$

The same argument as in lemma 4.34 shows that also $\tilde\gamma$ is closed. At the start of the deformation this path has also no self intersections and thus encloses a simply connected area. If we assume that this stays so during the deformation we can compute the variation of the area this path encloses in the same way we did for γ. We obtain then

$$\partial_t \tilde{A}(\kappa_0) = \frac{1}{\pi^2}\int_{-\kappa_0}^{\kappa_0} \omega = \frac{1}{\pi^2}\int_{-\kappa_0}^{\kappa_0} \frac{\kappa^2-\kappa_0^2}{(\kappa^2+1)^2} d\kappa$$
$$= -\arctan(\kappa_0)\,\kappa_0^2 - \kappa_0 + \arctan(\kappa_0)$$

We have $\partial_t \tilde{A}(0) = 0$ and from that point on $\partial_t \tilde{A}$ is strictly monotonic decreasing and approach $-\infty$ as $\kappa_0 \to \infty$. As the area $\tilde{A}$ was also finite the following has to hold if the solution of the deformation ODE extends to infinity

$$|\int_o^\infty \partial_t \tilde{A}| < \infty.$$

This is only possible if $\kappa_0 \to 0$ as $t \to \infty$ contradicting that we have found earlier that $\kappa_0 \to \infty$ as $t \to \infty$ for (4.46) to be fulfilled.

So we have shown that under the assumption that $\tilde{\gamma}$ has no self intersections the assumption that the solution of the deformation ODE extends to infinity leads to a contradiction. Without assuming that $\tilde{\gamma}$ has no self intersections we still have shown that $\kappa_0 \to \infty$ as $t \to \infty$ and so $H \to -\infty$. With the mentioned assumption we have therefore the following.

Proposition 4.35. *The solution of the deformation ODE with initial spectral data fulfilling $\int_0^{\kappa_0} d\ln\mu_2 = 0$ can only be defined on a bounded interval. So there exists a value t_{max} beyond which the solution cannot be extended.*

The last proposition shows that the solution of the deformation ODE cannot extend to infinity. The proposition 4.33 shows that at the endpoint of the deformation $\beta_1 = \beta_2$ cannot be fulfilled. So by proposition 4.12 there exists a value t_{max} beyond which the solution cannot be extended and $\beta_2(t_{max}) = 0$ holds. By proposition 4.28 we also have $\kappa_0(t_{max}) = 0$ and therefore $H(t_{max}) = \infty$. We summarize this main result of this thesis in the following theorem

Theorem 4.36. *The deformation of the spectral curve of rectangular torus will always end in a spectral curve of a torus with $H = \infty$ if the initial conditions of the deformation are such that the following holds.*

$$\int_0^{\kappa_0} d\ln\mu_2 = 0.$$

The value t_{max} such that the spectral data $(a(t_{max}), b_1(t_{max}), b_2 t_{max}), \kappa_0(t_{max}))$ belongs to a spectral curve of a torus with $H = \infty$ will be finite.

Let us now recall the closing conditions presented in proposition 3.4. In case of $\mathbb{S}^3$ there has to exist two distinct $\lambda_0, \lambda_1 \in \mathbb{S}^1$ such that

$$\mu_i(\lambda_0) = \mu_i(\lambda_1) = \pm 1.$$

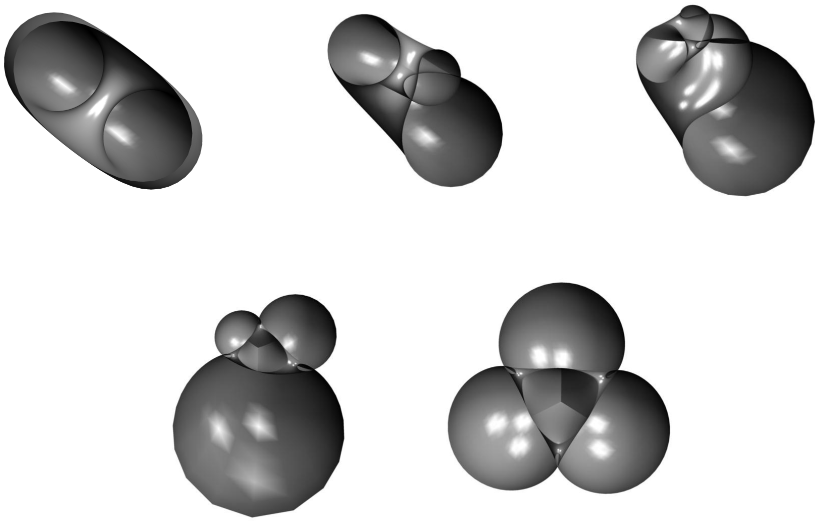

FIGURE 4-5. Deformation path starting from a homogeneous torus in $\mathbb{S}^3$ and ending in a Wente torus in $\mathbb{R}^3$ with some typical tori on the path. All images are cross sections. See also Figure 1-1 for normal view.

Let the initial spectral data be such that this condition is met. In case of $\mathbb{R}^3$ there has to exist $\lambda_0 \in \mathbb{S}^1$ such that

$$\mu_i(\lambda_0) = \pm 1 \text{ and } \partial_{\lambda_0}\mu_i(\lambda_0) = 0.$$

The deformation is constructed so that the values $\ln \mu_i(\lambda_0)$ remain constant. This ensures that $\mu_i(\lambda_0) = \pm 1$ is fulfilled also in the endpoint of the deformation. On the other hand $\mu_i(\lambda_0) = \mu_i(\lambda_1)$ is preserved during the deformation and at the end λ_0 and λ_1 come together to $\lambda_0 = \lambda_1 = 1$. This in turn ensures that $\partial_{\lambda_0}\mu_i(\lambda_0) = 0$ is fulfilled at the end of the deformation. We can also see this by another observation. $\partial_{\lambda_0}\mu_i(\lambda_0) = 0$ is equivalent to $\partial_{\kappa_0} \ln \mu_i(\kappa_0) = 0$. We have then for $\kappa_0 = 0$ that $\partial_{\kappa_0} \ln \mu_1(\kappa_0) = 0$ since $b_1(\kappa)$ is an odd polynomial and $\partial_{\kappa_0} \ln \mu_2(\kappa_0) = 0$ since $\beta_2 = 0$ at the end of the deformation and thus also $b_2(\kappa)$ has a zero at 0. Se we have seen that the spectral curve at the end of the deformation is in fact a spectral curve from a torus in $\mathbb{R}^3$. The following last proposition states this finding which was the main goal of this thesis.

PROPOSITION 4.37. *The spectral curve at the end of the deformation which starts with initial spectral data holding the condition in proposition 4.36 is a spectral curve of a torus in* $\mathbb{R}^3$.

Figure 4-5 and figure 1-1 from the introduction show a deformation from a homogeneous torus $\mathbb{S}^3$ to one in $\mathbb{R}^3$. The spectral data of the initial homogeneous torus was obtained by the algorithm from section 4.2.2. We started with the parameters $K_1 = 1$, $K_2 = 2$ and $M_1 = 3$, $M_2 = 1$ and obtained $\kappa_0 = \sqrt{5/3}$ and $\kappa_d = 5/27$. The rest of the spectral data was obtained by the formulas in section 4.2.3. So the initial spectral data was as follows.

$$\begin{pmatrix} a \\ b_1 \\ b_2 \\ \kappa_0 \end{pmatrix} = \begin{pmatrix} \kappa^4 - \frac{10\kappa^2}{27} + \frac{25}{729} \\ \left(\kappa^2 + \frac{5}{27}\right)\kappa \\ \kappa^2 + \frac{5}{27} \\ \sqrt{\frac{5}{3}} \end{pmatrix}.$$

Numerical computation shows then that the deformation ends at the value $t_{max} = 5.70048$ with the following spectral data

$$\begin{pmatrix} a \\ b_1 \\ b_2 \\ \kappa_0 \end{pmatrix} = \begin{pmatrix} \kappa^4 + 1.04303\kappa^2 + 0.324438 \\ \left(\kappa^2 - 0.442003\right)\kappa \\ \kappa^2 \\ 0 \end{pmatrix}.$$

The spectral curve at the end of the deformation has its branch points at

$$\lambda = 0.141272 - 0.101785i, \quad \lambda = 4.65969 + 3.35724i,$$
$$\lambda = 0.141272 + 0.101785i, \quad \lambda = 4.65969 - 3.35724i.$$

This spectral data corresponds to the spectral data of the famous Wente torus.

CHAPTER 5

Conclusion and Outlook

In this thesis we have shown a deformation path which starts at the spectral data of a homogeneous torus in $\mathbb{S}^3$ and ends in spectral data of a torus in $\mathbb{R}^3$. We proved for a specific family of homogeneous tori that this deformation path indeed ends in spectral data from a torus in $\mathbb{R}^3$. This deformation path gives an explicit example for how a torus in $\mathbb{R}^3$ can be a limit of a family of tori in $\mathbb{S}^3$.

There are still many open questions in this area. Numerical experiments during the development of the thesis suggested a relation between the data of the initial homogeneous torus such as the winding numbers and the data for the double points to open and the geometrical properties of the torus at the end of the deformation such as the number of lobes. Further the thesis showed only for a very limited family of initial conditions whether the deformation of them end in spectral data of a torus in $\mathbb{R}^3$ or not. It would be interesting to prove a similar statement for more general spectral data.

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Eidesstattliche Erklärung

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Mannhem, den 29.02.2008

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