Algebro-geometric solutions of the cosh-Gordon equation

Diploma Thesis

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Contents

1 Introduction

This thesis deals with the elliptic cosh-Gordon[1] equation

$$\Delta u = \cosh u. \tag{CG}$$

This equation is one of the real versions of the Sine-Gordon equation which plays an important role in various fields of physics (i.e. string theory, fluid dynamics, soliton theory) and also arises in the context of particular CMC surfaces. The goal of this document is to investigate the real-valued simply periodic solutions of this non-linear PDE using the methods of integrable systems. We will use the algebro-geometric approach, i.e. a solution of (CG) is connected to a Riemann surface, called the spectral curve, and a divisor on it. We are investigating finite type solutions, which are the solutions associated to Riemann surfaces of finite genus[2]. The development of these methods began in the 70s by Novikov, Dubrovin, Matveev, Its, Krichever and others.

The Gauss-Codazzi equations for surfaces in $\mathbb{H}^3$ with $|H| < 1$ can be reduced to (CG) under certain circumstances. There are only few publications devoted to CMC surfaces of this kind. The major result is the absence of compact CMC surfaces (without boundary). Dorfmeister et al. give a good overview on existing results in this and related areas in [DIK11]. In the same paper CMC surfaces, as described above, are constructed via the DPW method. In theory of CMC surfaces the sinh-Gordon equation

$$\Delta u = \sinh u \tag{SG}$$

plays a far more major role. For that reason there is a rich theory of it. The most important result is that all CMC tori are of finite type. This was

[1] Another name is cosh-Laplace equation.

[2] The terminology is not consistent: "finite gap" or "finite zone" (almost exclusively used in translations from Russian articles) are common names too.

achieved by Hitchin in [Hit90] and independently by Pinkall and Sterling in [PS89].
On the contrary the publications covering the cosh-Gordon equation are rather scarce. Babich studied all real versions of the Sine-Gordon equation from a unified point of view in a series of articles ([Bab91a, Bab91b, Bab91c]). Solutions in terms of theta functions and statements of their smoothness were presented with the result that solutions of (CG) are always singular unlike those of (SG). Using the results and methods presented in the series mentioned above, Babich and Bobenko constructed Willmore tori with umbilic lines in [BB93]. Novokshenov considered radial-symmetric solutions and its connection to minimal surfaces and the third Painlevé equation in [Nov96]. Babich and Bordag investigated genus 3 solutions with additional symmetries on the spectral curve in [BB05]. The sinh-Gordon equation is one of the major objects of interest of the group at the University of Mannheim, the author is writing this thesis at.
For a better understanding of (CG)'s solutions we first study the direct problem, i.e. we start with a solution u, define the spectral curve and the divisor and then investigate these two objects. Therefore this document is structured as follows: the second chapter introduces the cosh-Gordon equation in the context of CMC surfaces in hyperbolic 3-space. It is based on the author's bachelor thesis [Bin10]. The third chapter deals with the spectral curves associated to the solution u. The fourth chapter discusses the isoperiodic deformations and the moduli space of these Riemann surfaces. The fifth chapter covers the inverse problem, i.e. how to construct solutions u in terms of Baker-Akhiezer functions given a spectral curve and a divisor. The relevant results from Babichs publications mentioned above are discussed. Differences compared to (SG) are presented throughout this document.

2 Surface theory in $\mathbb{H}^3$

The aim of the current chapter is to introduce the cosh-Gordon equation as the Gauss equation of CMC surfaces in the hyperbolic three-space $\mathbb{H}^3$ with mean curvature H attaining values in $(-1, 1)$. All necessary concepts and terms are presented.The contents are based on the author's bachelor thesis and are revised and slightly extended, e.g. the Sym-Bobenko formula 2.3.2. Some (longer) proofs with straightforward calculations are shortened, for the complete proofs see [Bin10].

2.1 Representations of $\mathbb{H}^3$

Hyperbolic 3-space $\mathbb{H}^3$ is the unique complete simply connected three-dimensional Riemannian manifold with sectional curvature being constantly -1. There exist several models, each of them with different purposes and (dis)advantages. First, we want to present two models of $\mathbb{H}^3$ which will be useful in later studying CMC of surfaces in $\mathbb{H}^3$.

2.1.1 Minkowski model for $\mathbb{H}^3$

In the first model $\mathbb{H}^3$ is a subset of the Euclidean 4-space with a special bilinear form.

Definition 2.1.1. *The linear space $\mathbb{R}^4$ equipped with the metric $\langle\cdot,\cdot\rangle_{\mathbb{R}^{3,1}}$ induced by the matrix $\tilde{I} := \operatorname{diag}(-1, 1, 1, 1)$ [3] is the **Minkowski 4-space** $\mathbb{R}^{3,1}$. That is for $x^T = (x_0, x_1, x_2, x_3)$ and $y^T = (y_0, y_1, y_2, y_3)$ in $\mathbb{R}^4$ there holds*

$$\langle x, y\rangle_{\mathbb{R}^{3,1}} = x^T \tilde{I} y = -x_0 y_0 + \sum_{j=1}^{3} x_j y_j.$$

[3] Precisely $\langle\cdot,\cdot\rangle_{\mathbb{R}^{3,1}}$ is a constant metric tensor.

Remark 2.1.2. *The space $\mathbb{R}^{3,1}$ is a model for spacetime where x_0 is the time coordinate and $x_{1,2,3}$ the spatial coordinates. Therefore a vector x in $\mathbb{R}^{3,1}$ is called*

$$\begin{aligned} \textit{spacelike if } & \langle x, x\rangle_{\mathbb{R}^{3,1}} > 0, \\ \textit{timelike if } & \langle x, x\rangle_{\mathbb{R}^{3,1}} < 0, \\ \textit{lightlike if } & \langle x, x\rangle_{\mathbb{R}^{3,1}} = 0. \end{aligned}$$

Later on the subscript $\mathbb{R}^{3,1}$ will be omitted, since it is the only metric used. Since the metric's signature is $(3,1)$ it is a Lorentzian space. For this reason there are no orthonormal bases in the classical sense but "almost orthonormal" bases (b_0, b_1, b_2, b_3) such that all b_i are pairwise orthogonal. The first entry b_0 is normalized to have $\langle b_0, b_0\rangle_{\mathbb{R}^{3,1}} = -1$ and the others to have unit length. From now on such a basis is just called orthonormal. Later on in section 2.2.2 we will see that it is possible to use the immersed surface we consider to obtain an orthonormal basis of the Minkowski space $\mathbb{R}^{3,1}$.

Definition 2.1.3. *The Minkowski model for the hyperbolic 3-space is given by*

$$\mathbb{H}^3 := \{x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^{3,1} : \langle x, x\rangle_{\mathbb{R}^{3,1}} = -1,\ x_0 \geq 1\}.$$

Defined that way $\mathbb{H}^3$ is the set of certain timelike vectors which can be interpreted as a „sphere“ in $\mathbb{R}^{3,1}$ with radius i. Indeed it is the upper sheet of a hyperboloid. For the proof that the constant sectional curvature's value is -1 see [Lee97], chapter 8. We can also show that $\mathbb{H}^3$ is a Riemannian manifold, that is $\langle\cdot,\cdot\rangle_{\mathbb{R}^{3,1}}$ is positive definite on $T_p\mathbb{H}^3$ and thus induces a scalar product (see [Bin10], Proposition 2.1.3 for a proof).

Definition 2.1.4. *The metric preserving group of $\mathbb{R}^{3,1}$, i.e. all matrices $A \in \mathbb{R}^{4\times 4}$ with $\langle x, y\rangle_{\mathbb{R}^{3,1}} = \langle Ax, Ay\rangle_{\mathbb{R}^{3,1}}$ for all $x, y \in \mathbb{R}^{3,1}$, is the **Lorentz group** and is denoted by $O(3,1)$.*

Proposition 2.1.5. *For an element*

$$A := \begin{pmatrix} a_{00} & w^T \\ v & B \end{pmatrix} \ \textit{with}\ \ v, w \in \mathbb{R}^3,\ B \in \mathbb{R}^{3\times 3}$$

in $O(3,1)$ there holds $|a_{00}| \geq 1$ and $\det A = \pm 1$.

Proof.

$$\begin{aligned}\langle x,y\rangle_{\mathbb{R}^{3,1}} = \langle Ax,Ay\rangle_{\mathbb{R}^{3,1}} &\iff x^T\tilde{I}y = x^TA^T\tilde{I}Ay \\ &\iff \tilde{I} = A^T\tilde{I}A \qquad (*)\\ &\Rightarrow -1 = \det\tilde{I} = \det(A^T\tilde{I}A) = -(\det A)^2 \\ &\Rightarrow \det A = \pm 1\end{aligned}$$

Due to the condition $(*)$ for the the top left entry of $A^T\tilde{I}A$ there follows

$$\begin{aligned}-1 = v^Tv - a_{00}^2 &\iff a_{00}^2 = 1 + v^Tv \\ &\Rightarrow a_{00}^2 \geq 1 \Rightarrow |a_{00}| \geq 1\end{aligned}$$

■

Corollary 2.1.6. *The Lorentz group has four connected components.*

Sketch of proof. Since det is a continuous map and attains values in $\{-1,+1\}$ on $O(3,1)$, there are at least two components distinguished by the sign. Each of them splits up in two components since there is no continuous path between those matrices with $a_{00} \geq 1$ and $a_{00} \leq -1$. So there are at least four connected components. A complete proof can be found in [KM89], Theorem 12.11.

Proposition 2.1.7. $O^+(3,1) := \{A \in O(3,1) : a_{00} \geq 1\}$ *is the metric preserving group of* $\mathbb{H}^3$.

The crucial condition is to preserve the time coordinate x_0 from changing sign. $O^+(3,1)$ preserves the time coordinate of every vector $v \in \mathbb{R}^{3,1}$ with $\langle v,v\rangle_{\mathbb{R}^{3,1}} \leq 0$, therefore it acts also correctly on $\mathbb{H}^3$. A general proof for this result can be found in [Nab03], section 1.3.

The identity component $SO^+(3,1) := \{A \in O^+(3,1) \colon \det A = 1\}$, which is also called the **restricted Lorentz group**, additionally preserves the orientation of a basis and therefore is the most interesting subgroup of $O(3,1)$ for this document's purposes. To describe these isometries it is more elegant to use a different model for the hyperbolic 3-space, which is the topic of the next section.

2.1.2 Hermitian matrix model for $\mathbb{H}^3$

Definition 2.1.8.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

*are called the **Pauli matrices**.*

By adding $\sigma_0 := \mathbb{1}$ we get four linearly independent elements which define a linear subspace in $\mathbb{C}^{2\times 2}$. This subspace $\operatorname{span}_{\mathbb{R}}(\sigma_1, \sigma_2, \sigma_3, \sigma_0)$ is denoted by Herm(2). Using these matrices, $\mathbb{R}^{3,1}$ can be identified with 2×2 matrices using the linear map

$$\begin{aligned} \psi : \mathbb{R}^{3,1} &\rightarrow \operatorname{Herm}(2) \\ x &\mapsto \sum_{j=0}^{3} x_k \sigma_k = \begin{pmatrix} x_0 + x_3 & x_1 - i x_2 \\ x_1 + i x_2 & x_0 - x_3 \end{pmatrix}. \end{aligned} \tag{2.1}$$

From now on the image $\psi(x)$ will often be abbreviated as X. X is a Hermitian matrix, that is $X = \overline{X}^T =: X^*$ and hence has the form

$$\begin{pmatrix} a_{11} & a_{12} \\ \overline{a_{12}} & a_{22} \end{pmatrix}$$

with $a_{11}, a_{22} \in \mathbb{R}$ and $a_{12} \in \mathbb{C}$. The next step is to figure out how the metric $\langle \cdot, \cdot \rangle_{\mathbb{R}^{3,1}}$ looks like in Herm(2). First recall that for a 2×2 matrix

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

the adjugate matrix is defined as

$$\operatorname{adj}(M) := \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix}.$$

and for any matrix M there holds

$$M \cdot \operatorname{adj}(M) = \operatorname{adj}(M) \cdot M = \det(M) \cdot \mathbb{1}.$$

Now we can show an important property of the map ψ.

Proposition 2.1.9. *For the metric* $\langle \cdot, \cdot \rangle_H$ *on* Herm(2) *defined by*

$$\langle X, Y \rangle_H := -\frac{1}{2} \mathrm{tr}(X \sigma_2 Y^T \sigma_2)$$

the map ψ *is an isometry, i.e. there holds* $\langle \psi(x), \psi(y) \rangle_H = \langle x, y \rangle_{\mathbb{R}^{3,1}}$ *for all* $x, y \in \mathbb{R}^{3,1}$.

Proof. A simple calculation yields $\sigma_2 Y^T \sigma_2 = \mathrm{adj}(Y)$ for an arbitrary 2×2 matrix Y. Due to this formula we get

$$\begin{aligned} \langle \psi(x), \psi(y) \rangle_H = \langle X, Y \rangle_H &= -\frac{1}{2} \mathrm{tr}(X \sigma_2 Y^T \sigma_2) = -\frac{1}{2} \mathrm{tr}(X \mathrm{adj}(Y)) \\ &= -\frac{1}{2} \mathrm{tr} \left(\begin{pmatrix} x_0 + x_3 & x_1 - i x_2 \\ x_1 + i x_2 & x_0 - x_3 \end{pmatrix} \begin{pmatrix} y_0 - y_3 & -(y_1 - i y_2) \\ -(y_1 + i y_2) & y_0 + y_3 \end{pmatrix} \right) \\ &= -\frac{1}{2} (2(x_0 y_0 - x_3 y_3 - (x_1 y_1 + x_2 y_2)) \\ &= \langle x, y \rangle_{\mathbb{R}^{3,1}}. \end{aligned}$$

∎

From now on $\langle \cdot, \cdot \rangle$ stands either for $\langle \cdot, \cdot \rangle_H$ or for $\langle \cdot, \cdot \rangle_{\mathbb{R}^{3,1}}$ depending on the context.

Proposition 2.1.10. *The matrices* $(\sigma_1, \sigma_2, \sigma_3, \sigma_0)$ *form an orthonormal basis of* Herm(2) *with*

$$\langle \sigma_1, \sigma_1 \rangle = \langle \sigma_2, \sigma_2 \rangle = \langle \sigma_3, \sigma_3 \rangle = 1, \;\; \langle \sigma_0, \sigma_0 \rangle = -1$$

Proof. By definition of ψ it is a surjective linear map. Because ψ is also isometric it is injective and therefore an isomorphism. Considering the standard basis (e_0, e_1, e_2, e_3) of $\mathbb{R}^4$, we notice $\psi(e_i) = \sigma_i$. Now the claim follows directly.

∎

Theorem 2.1.11. *The hyperbolic 3-space* $\mathbb{H}^3$ *is diffeomorphic to*

$$\mathcal{H}^3 := \{X \in \mathrm{Herm}(2) : \;\det(X) = 1, \;\mathrm{tr}(X) \geq 2\}.$$

Proof. First we want to show $\psi(\mathbb{H}^3) \subset \mathcal{H}^3$. Let x be an element of $\mathbb{H}^3$.

$$\begin{aligned} -1 &= \langle x, x \rangle = \langle \psi(x), \psi(x) \rangle = -\frac{1}{2}\mathrm{tr}(X \mathrm{adj}(X)) \\ &= -\det(X) \end{aligned}$$

The condition for the trace of $\psi(x)$ follows from

$$\mathrm{tr}(X) = (x_0 + x_3) + (x_0 - x_3) = 2x_0.$$

Conversely we can express X with respect to the basis σ_i obtaining the scalars x_0, x_1, x_2, x_3. In total we have $\psi(\mathbb{H}^3) = \mathcal{H}^3$.
The map ψ is an isomorphism from $\mathbb{R}^{3,1}$ to Herm(2) hence it is also a diffeomorphism as well as its restriction on $\mathbb{H}^3$.

■

Lemma 2.1.12. *There holds*

$$\mathcal{H}^3 = \{FF^* : F \in SL_2\mathbb{C}\}.$$

Sketch of proof. An element X of the set on the right hand side is clearly Hermitian and its determinant equals one. Since Hermitian matrices have real eigenvalues there holds $\mathrm{tr}(X) \geq 2$ and hence X is an element of $\mathcal{H}^3$. On the other hand we can diagonalize an element of $\mathcal{H}^3$ and get the corresponding matrix F from $SL_2\mathbb{C}$ completing the proof.

This product representation is not unique because any right-multiplication with a matrix $M \in SU_2$ does not change anything, since

$$FF^* = FMM^*F^* = FM(FM)^*$$

proving the following result.

Corollary 2.1.13. $\mathbb{H}^3$ *is diffeomorphic to* $SL_2\mathbb{C}/SU_2$.

Theorem 2.1.14. *The orientation preserving isometry group* $SO^+(3,1)$ *of* $\mathbb{H}^3$ *is isomorphic to the projective special linear group* $PSL_2\mathbb{C} := SL_2\mathbb{C}/\{\pm\mathbb{1}\}$ *and acts as*

$$X \mapsto M \cdot X \cdot M^*$$

on an arbitrary $X \in \mathcal{H}^3$.

Proof. There exists a two to one homomorphism $\phi : SL_2\mathbb{C} \to SO^+(3,1)$ (cf. [Car00], chapter 3). Now let M be an element in $SL_2\mathbb{C}$ and $X \in \mathcal{H}^3$. The action $M \cdot X \cdot M^*$ then corresponds to $\phi(M) \cdot \psi^{-1}(X)$ in the Minkowski model. Since $M \cdot X \cdot M^* = -M \cdot X(-M)^*$ there holds $\ker \phi = \{\pm \mathbb{1}\}$. Dividing by $\{\pm \mathbb{1}\}$ turns ϕ into a group isomorphism.

■

In other words: all rotations of the hyperbolic 3-space can be implemented using conjugations with an element of $SL_2\mathbb{C}$. Since we will use the isometry group only to transform (oriented) bases, the reflections are of no interest.

2.2 Surface theory in $\mathbb{H}^3$

This section is structured as follows: in the first part basic terms and facts of differential geometry and surface theory in three-dimensional space forms are presented in a general context. The other two parts are specific to $\mathbb{H}^3$ and deal with the frame and the Lax pairs.

2.2.1 General surface theory

Definition 2.2.1. *A differentiable mapping*

$$f : \Sigma \to M$$

between two manifolds is an ***immersion*** *if its differential is injective at every point p in Σ.*

In the setting of surface theory Σ is an orientable two-dimensional manifold and M is one of the three-dimensional space forms like $\mathbb{R}^3, \mathbb{S}^3$ or $\mathbb{H}^3$. Then f is a representation of a surface in M in the sense of differential geometry.

The vector space $\mathbb{R}^2$ can be identified with the complex plane. By fixing an orientation of Σ and equipping it with a complex structure, it turns to a Riemann surface. Every coordinate chart then defines a complex coordinate $z = x + iy$. Since f is an immersion, the partial derivatives

$$f_x := \left(\frac{\partial f}{\partial x}\right)_p \text{ and } f_y := \left(\frac{\partial f}{\partial y}\right)_p$$

with respect to this chart are linearly independent and thus provide a basis for a linear subspace of the tangent space $T_{f(p)}M$.

Using the metric $\langle \cdot, \cdot \rangle_M$ of the manifold M we can define a metric $ds^2 = g$ on $T_p\Sigma$ by a pullback.

Definition 2.2.2. *The bilinear map $g_p : T_p\Sigma \times T_p\Sigma \to \mathbb{R}$ defined by*

$$g_p := \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \langle f_x, f_x \rangle_M & \langle f_x, f_y \rangle_M \\ \langle f_y, f_x \rangle_M & \langle f_y, f_y \rangle_M \end{pmatrix}$$

is the ***first fundamental form*** *of the immersion f.*

The determinant of g is the Gram's determinant for f hence the map f is an immersion if and only if $\det g > 0$.

Definition 2.2.3. *In the case $g_{11} = g_{22}$ and $g_{21} = 0 = g_{12}$, i.e. $g = \lambda(p) \cdot \mathbb{1}$, $\lambda(p) > 0$, the immersion f is called* ***conformally parametrized****.*

Then the metric can be (locally) written as

$$ds^2 = 4e^{2u}(dx^2 + dy^2)$$

with the so-called conformal factor u. It is a real-valued map defined on some neighborhood of p in Σ. We assume that u is sufficiently differentiable. From now on only conformal immersions are considered.

Since T_pM is three-dimensional it is possible to define the unit normal vector N with respect to the partial derivatives f_x and f_y. Using N, the **second fundamental form** b of the immersion f can be defined by

$$b := \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} -\langle N_x, f_x \rangle_M & -\langle N_y, f_x \rangle_M \\ -\langle N_x, f_y \rangle_M & -\langle N_y, f_y \rangle_M \end{pmatrix} = \begin{pmatrix} \langle N, f_{xx} \rangle_M & \langle N, f_{xy} \rangle_M \\ \langle N, f_{yx} \rangle_M & \langle N, f_{yy} \rangle_M \end{pmatrix}$$

The linear form b can also be written with the help of the differential forms $dz := dx + idy$ and $d\bar{z} := dx - idy$ as

$$b = Qdz^2 + \tilde{H}dzd\bar{z} + \overline{Q}d\bar{z}^2,$$

where the functions Q and $\tilde{H}$ are defined as follows

$$Q := \frac{1}{4}(b_{11} - b_{22} - ib_{12} - ib_{21}), \quad \tilde{H} := \frac{1}{2}(b_{11} + b_{22}).$$

The 2-form Qdz^2 is called the **Hopf differential** of f.

Later we will see that f is determined uniquely up to a rigid motion in M by the two fundamental forms if they satisfy a certain pair of equations (cf. Section 2.2.3).

Definition 2.2.4. *The linear map $S := g^{-1}b : T_p\Sigma \to T_p\Sigma$ is the **shape operator**.*

The shape operator can be expressed with the help of the functions u, Q and $\tilde{H}$:

$$S = \frac{1}{4e^{2u}} \begin{pmatrix} \tilde{H} + Q + \overline{Q} & i(Q - \overline{Q}) \\ i(Q - \overline{Q}) & \tilde{H} - Q - \overline{Q} \end{pmatrix}.$$

Definition 2.2.5. *The shape operator's eigenvalues κ_1, κ_2 are the **principal curvatures**, half of its trace is the **mean curvature** $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ and its determinant is the **Gaussian curvature** $K = \kappa_1\kappa_2$ of the immersion f.*

In the conformal case this leads to $H = \frac{1}{8}e^{-2u}\langle N, f_{xx} + f_{yy}\rangle$. We notice that the mean curvature is defined similarly as $\tilde{H}$, indeed the connection is $H = \frac{1}{4}e^{-2u}\tilde{H}$.

Definition 2.2.6. *If the map H is constant, then f is called a **constant mean curvature (CMC) surface**. In the special case of $H \equiv 0$, f is named **minimal surface**.*

Using Wirtinger differential operators

$$\partial_z := \frac{1}{2}(\partial_x - i\partial_y) \text{ and } \partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y)$$

and letting $\langle \cdot, \cdot \rangle$ be the complex bilinear extension of the metric $\langle \cdot, \cdot \rangle_M$ we get to the following elegant result:

Proposition 2.2.7. *Using the complex coordinate z, the maps H, Q and the conformality condition can be expressed by*

$$H = \frac{1}{2}e^{-2u}\langle f_{z\bar{z}}, N\rangle, \tag{2.2}$$

$$Q = \langle f_{zz}, N\rangle, \tag{2.3}$$

$$\langle f_z, f_z\rangle = \langle f_{\bar{z}}, f_{\bar{z}}\rangle = 0, \quad \langle f_z, f_{\bar{z}}\rangle = 2e^{2u}. \tag{2.4}$$

The proof is a straightforward calculation using the correspondences $\partial_x = \partial_z + \partial_{\bar{z}}$ respectively $\partial_y = i(\partial_z - \partial_{\bar{z}})$. The next step in the classical theory is the construction of a so-called moving frame which is a basis of $\mathbb{R}^3$ or of the ambient space in the case of other space forms. The frame is derived directly from the immersion f and with its help we obtain the so-called Gauss-Codazzi equations. It can be shown that those are the only conditions u, H and Q have to satisfy (cf. Theorem 2.2.14).
Since these equations are different from one space form to another, the next section will deal with the specific situation of f being an immersion in $\mathbb{H}^3$.

2.2.2 The extended frame in $\mathbb{H}^3$

Now we use the Minkowski model for $\mathbb{H}^3$, i.e. the conformal immersion $f : \Sigma \to \mathbb{H}^3$ is considered as a $\mathbb{R}^{3,1}$-valued map. Beginning with f the aim is to construct the frame i.e. a basis of $\mathbb{R}^{3,1}$ which is well adjusted to the surface.

Differentiating $\langle f, f \rangle = -1$ yields

$$\langle f, f_x \rangle = 0, \quad \langle f, f_y \rangle = 0$$

Additionally the conformality of f gives the orthogonality $\langle f_x, f_y \rangle = 0$ of the partial derivatives. Now you can also define the normal $\tilde{N}$ using the formal determinant

$$\tilde{N} := f \times f_x \times f_y := \det(E, f, f_x, f_y)$$

where $e_1, \dots, e_4$ is an orthonormal basis of $\mathbb{R}^{3,1}$, f, f_x, f_y are expressed with its help and $E = (e_1, e_2, e_3, e_4)^T$ is a vector containing the basis elements as single entries. Since f_x, f_y and $\tilde{N}$ are elements of $T_{f(p)}\mathbb{H}^3$ and $\mathbb{H}^3$ is a Riemannian manifold, their metric is positive. By setting $N := \frac{\tilde{N}}{\|\tilde{N}\|}$ we obtain the unit normal.

Definition 2.2.8. *The map*

$$\tilde{\mathcal{F}} : \Sigma \to \mathbb{R}^{4\times 4}, \; p \mapsto (f(p), f_x(p), f_y(p), N(p))$$

*is the **(extended) moving frame** of the immersion f.*

Remark 2.2.9. *The normalized frame*

$$\tilde{\mathcal{F}}_{on} = \left(f, \frac{f_x}{\|f_x\|}, \frac{f_y}{\|f_y\|}, N \right)$$

is an positive oriented orthonormal basis of $\mathbb{R}^{3,1}$ *and an element of the isometry group.*

In the case of using the complex coordinate z instead of x and y the extended frame is defined as

$$\mathcal{F} := (f, f_z, f_{\bar{z}}, N).$$

A straightforward calculation proves the following:

Proposition 2.2.10. *Every* $v \in \mathbb{R}^{3,1}$ *can be expressed with respect to* $\mathcal{F}$ *as follows*

$$v = -\langle v, f\rangle f + \frac{\langle v, f_{\bar{z}}\rangle}{2e^{2u}} f_z + \frac{\langle v, f_z\rangle}{2e^{2u}} f_{\bar{z}} + \langle v, N\rangle N.$$

2.2.3 The Lax pairs

Now we will describe the frame's behavior towards differentiation. This is possible in terms of matrix differential equations, which leads to the definition:

Definition 2.2.11. *The two matrix partial differential equations*

$$\mathcal{F}_z = \mathcal{F} \cdot \mathcal{U}, \quad \mathcal{F}_{\bar{z}} = \mathcal{F} \cdot \mathcal{V} \tag{2.5}$$

are called the ***Lax pair*** *of the immersion* f*.*

Since $\mathbb{H}^3$ can be represented either as subset of the Minkowski space $\mathbb{R}^{3,1}$ or using Hermitian matrices there are two "flavors": $\mathcal{U}$ and $\mathcal{V}$ being 4×4 respectively 2×2 matrices.

2.2.3.1 The Lax pair in terms of 4×4 matrices

Proposition 2.2.12. *Using the Minkowski model for* $\mathbb{H}^3$*, the Lax pair is described by the matrices*

$$\mathcal{U} = \begin{pmatrix} 0 & 0 & 2e^{2u} & 0 \\ 1 & 2u_z & 0 & -H \\ 0 & 0 & 0 & -\frac{1}{2}Qe^{-2u} \\ 0 & Q & 2He^{2u} & 0 \end{pmatrix} \tag{2.6}$$

$$\mathcal{V} = \begin{pmatrix} 0 & 2e^{2u} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\overline{Q}e^{-2u} \\ 1 & 0 & 2u_{\bar{z}} & -H \\ 0 & 2He^{2u} & \overline{Q} & 0 \end{pmatrix}. \tag{2.7}$$

Sketch of proof. Since the entries of

$$\mathcal{F}_z = (f_z, f_{zz}, f_{z\bar{z}}, N_z) \text{ and } \mathcal{F}_{\bar{z}} = (f_{\bar{z}}, f_{z\bar{z}}, f_{\bar{z}\bar{z}}, N_{\bar{z}}).$$

are $\mathbb{R}^{3,1}$-valued, they can be expressed with the frame $\mathcal{F}$ using the Proposition 2.2.10. Plugging the identities (2.3) and (2.2) (if applicable) into the obtained coefficients yields the columns of $\mathcal{U}$ and $\mathcal{V}$ proving the claim.

Differentiating the Lax pairs leads to the following result:

Proposition 2.2.13. *The compatibility condition $\mathcal{F}_{z\bar{z}} = \mathcal{F}_{\bar{z}z}$ is equivalent to*

$$\mathcal{U}_{\bar{z}} - \mathcal{V}_z - [\mathcal{U}, \mathcal{V}] = 0. \tag{2.8}$$

By computing this equation we get the **Gauss-Codazzi equations** for surfaces in $\mathbb{H}^3$:

$$2u_{z\bar{z}} + 2(H^2 - 1)e^{2u} - \frac{1}{2}Q\bar{Q}e^{-2u} = 0 \tag{2.9}$$

$$Q_{\bar{z}} = 2H_z e^{2u} \tag{2.10}$$

Now the Codazzi equation leads directly to the well known fact: f is a CMC surface if and only if Q is holomorphic.
Now we are prepared to formulate the fundamental theorem of surface theory:

Theorem 2.2.14. *If the mappings*

$$\begin{aligned} u &: \Sigma \to \mathbb{R}, \\ H &: \Sigma \to \mathbb{R}, \\ Q &: \Sigma \to \mathbb{C} \end{aligned}$$

defined on a simply connected two-dimensional manifold Σ satisfy the Gauss-Codazzi equations, there exists a conformal immersion $f : \Sigma \to \mathbb{H}^3$ with these maps as the conformal factor, mean curvature and Hopf differential. The surface f is unique up to a rigid motion of $\mathbb{H}^3$.

Remark 2.2.15. *The theorem is applicable to all three space forms. The only difference is the specific appearance of the Gauss-Codazzi equations.*

There is a connection between the Gauss equation and the so-called **cosh-Gordon** equation

$$(2u)_{w\overline{w}} - \cosh(2u) = 0. \tag{CG}$$

The trivial one is that in case the mean curvature H attains $\pm\frac{\sqrt{3}}{2}$ the equation (2.9) turns into the so-called reduced Gauss equation

$$2u_{z\bar{z}} - \frac{1}{2}e^{2u} - \frac{1}{2}Q\bar{Q}e^{-2u} = 0. \tag{2.11}$$

In some neighborhood of a non-umbilical point we can introduce a new coordinate w turning the latter equation into the cosh-Gordon equation as in (CG)[4]. A more deep connection will be presented in the next section in form of the Sym-Bobenko formula.
The cosh-Gordon equation is an elliptic non-linear PDE. In case of CMC surfaces in $\mathbb{H}^3$ with $|H| > 1$ (and in other space forms) we have the elliptic sinh-Gordon equation

$$(2u)_{w\overline{w}} + \sinh(2u) = 0 \tag{SG}$$

instead. Both are real variants of the Sine-Gordon equation but behave rather differently.

2.2.3.2 The Lax pair in terms of 2×2 matrices

Now we use the Hermitian matrix approach to represent the immersion and the frame. In this section f, f_x, f_y and N are considered as their images $\psi(f)$, $\psi(f_x)$, $\psi(f_y)$ and $\psi(N)$ under the diffeomorphism ψ.

Proposition 2.2.16. *There exists a unique matrix $F \in SL_2\mathbb{C}$ obeying $F(z_0) = \mathbb{1}$ such that there holds*

$$f = FF^*, \ \frac{f_x}{\|f_x\|} = F\sigma_1 F^*, \ \frac{f_y}{\|f_y\|} = F\sigma_2 F^*, \ N = F\sigma_3 F^*.$$

Proof. As already shown the normalized frame $\mathcal{F}_{on}$ is an orthonormal basis of $\mathbb{R}^{3,1}$. Since ψ is a linear isometry $\psi(\mathcal{F}_{on})$ constitutes an orthonormal basis too. By virtue of Theorem 2.1.14 there exists a unique matrix $F \in PSL_2\mathbb{C}$ which transforms (σ_i) into $\psi(\mathcal{F}_{on})$ which then have the form stated in the

[4]Since the map Q is holomorphic and locally nonzero, $w(z,\bar{z}) = \int \sqrt{Q(z,\bar{z})}dz$ accomplishes this task. Additionally there holds $Q(w,\bar{w}) \equiv 1$.

claim. By specifying F to be the identity matrix at a particular point z_0 it becomes unique in $SL_2\mathbb{C}$.

■

Later we will also need f_z and $f_{\bar{z}}$ for various computations. Using formulae for f_x and f_y from above a short calculation gives

$$f_z = \frac{1}{2}(f_x - i f_y) = e^u F(\sigma_1 - i\sigma_2)F^* = 2e^u F \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} F^* =: 2e^u F \varepsilon_1 F^*$$
$$f_{\bar{z}} = \frac{1}{2}(f_x + i f_y) = e^u F(\sigma_1 + i\sigma_2)F^* = 2e^u F \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} F^* =: 2e^u F \varepsilon_2 F^*.$$

Due to the preceding proposition it is sufficient to know the matrix F to describe the immersion f and its derivatives as well as the unit normal. Therefore calling F the (moving) frame in this model of the hyperbolic 3-space is justified.

Define $U := F^{-1}F_z$ and $V := F^{-1}F_{\bar{z}}$. These matrices always exist since $F \in SL_2\mathbb{C}$ and f is C^∞.

Proposition 2.2.17. *The Lax pair*

$$F_z = F \cdot U, \quad F_{\bar{z}} = F \cdot V$$

is described by the matrices

$$U = \frac{1}{2}\begin{pmatrix} -u_z & Qe^{-u} \\ 2(1-H)e^u & u_z \end{pmatrix} \quad \textit{and} \quad V = \frac{1}{2}\begin{pmatrix} u_{\bar{z}} & 2(1+H)e^u \\ -\bar{Q}e^{-u} & -u_{\bar{z}} \end{pmatrix}. \tag{2.12}$$

Sketch of proof. First we express the derivatives f_z and $f_{\bar{z}}$ with the help of the previous proposition. Now the general strategy is to differentiate f_z, $f_{\bar{z}}$ as well as N once more and compare the result to second derivatives obtained in the 4×4 case. Using $f_{z\bar{z}} = f_{\bar{z}z}$ leads to this particular form of U and V entry by entry.

As for the Lax pair in terms of 4×4 matrices, the compatibility condition as well as the Gauss-Codazzi equations (2.9) and (2.10) remain unchanged. They are the only conditions to be satisfied to obtain a solution of the Lax pair, more precisely there holds:

Proposition 2.2.18. *Let $O \subset \Sigma$ be an open and simply connected set and $U, V : O \to \mathfrak{sl}_2(\mathbb{C})$. There exists a unique solution $F : O \to SL_2\mathbb{C}$ of the Lax pair*

$$F_z = F \cdot U, \; F_{\bar{z}} = F \cdot V$$

for any initial condition $F(z_0) \in SL_2\mathbb{C}$ if and only if U and V satisfy the compatibility condition

$$U_{\bar{z}} - V_z - [U, V] = 0.$$

A proof using standard methods can be found in [Lei09], Proposition 2.3.1.

2.3 The Sym-Bobenko formula

The aim of this section is to describe a construction method for CMC immersions f with prescribed constant mean curvature $H \in (-1, 1)$. From the last section we know that the 2×2 frame F is an element of $SL_2\mathbb{C}$ and describes the surface f entirely. For that reason the strategy will be to use the Lax pair (2.12) in order to construct such a matrix F.

Instead of using the two Lax equations we will use a slightly different approach using differential forms with values in some Lie algebra. This approach allows us to perform some computations in a more elegant way. For an overview see [SKKR07]. Let $U, V : O \to \mathfrak{sl}_2(\mathbb{C})$ be the matrices describing the Lax pair and O an open and simply connected subset of Σ. The Lax differential equations transform into

$$dF = F\alpha$$

with $\alpha := Udz + Vd\bar{z} \in \Omega^1(O, \mathfrak{sl}_2(\mathbb{C}))$. Using the commutator $[\cdot, \cdot]$ of $\mathfrak{sl}_2(\mathbb{C})$ for two forms α and β in $\Omega^1(O, \mathfrak{sl}_2(\mathbb{C}))$ we define

$$[\alpha \wedge \beta](X, Y) = [\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)], \; X, Y \in T\Sigma.$$

The resulting object $[\alpha \wedge \beta]$ is then an element of $\Omega^2(O, \mathfrak{sl}_2(\mathbb{C}))$. In this setting the compatibility condition (2.8) is equivalent to the **Maurer-Cartan equation**

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0. \tag{2.13}$$

Compared to related works (e.g. [KS10]) α has a slightly different appearance. In order to change it we conjugate α with the Pauli matrix

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For the frame F it means the gauging with σ_1. The new frame $\tilde{F} = F\sigma_1$ obeys $d\tilde{F} = \tilde{F}\tilde{\alpha}$ with $\tilde{\alpha} = \sigma_1 \alpha \sigma_1$. The surface f itself is unaffected since $\tilde{F}\tilde{F}^* = FF^*$ but the formulae from Section 2.2.3.2 are a bit different[5] .We will omit the tilde hereafter.

Now we want to construct a family of CMC surfaces parametrized by a so-called spectral parameter $\lambda \in \mathbb{C}^*$ with cosh-Gordon equation as the Gauss equation. We start with a connection α associated to a certain CMC surface. We set $\lambda = \sqrt{\frac{1-H}{1+H}}$ and define α_λ as follows:

$$\begin{aligned} \alpha_\lambda &= U_\lambda dz + V_\lambda d\bar{z} \\ &= \frac{1}{2}\begin{pmatrix} u_z & \lambda e^u \\ Qe^{-u} & -u_z \end{pmatrix} dz + \frac{1}{2}\begin{pmatrix} -u_{\bar{z}} & -\bar{Q}e^{-u} \\ \lambda^{-1}e^u & u_{\bar{z}} \end{pmatrix} d\bar{z} \\ &= \frac{1}{2}\begin{pmatrix} u_z dz - u_{\bar{z}} d\bar{z} & \lambda e^u dz - \bar{Q}e^{-u} d\bar{z} \\ Qe^{-u} dz + \lambda^{-1} e^u d\bar{z} & -u_z dz + u_{\bar{z}} d\bar{z} \end{pmatrix}. \end{aligned} \tag{2.14}$$

The mean curvature of the corresponding surface remains unchanged. Applying (2.13) to α_λ yields the following reduced Gauss-Codazzi equations for CMC surfaces:

$$u_{z\bar{z}} - \frac{1}{4}e^{2u} - \frac{1}{4}Q\overline{Q}e^{-2u} = 0, \ \frac{1}{2}Q_{\bar{z}}e^{-u} = 0$$

As discussed previously in Section 2.2.3.1 we can turn the reduced Gauss equation into the cosh-Gordon equation

$$(2u)_{z\bar{z}} = \cosh(2u)$$

by change of coordinates.
We note that the zero curvature condition for α_λ is independent of λ. We can extend the definition allowing λ to be chosen from $\mathbb{C}^*$.

[5]Now there holds $f_z = 2e^u \tilde{F}\varepsilon_2\tilde{F}^*$, $f_{\bar{z}} = 2e^u\tilde{F}\varepsilon_1\tilde{F}^*$ and $N = -\tilde{F}\sigma_3\tilde{F}^*$.

Remark 2.3.1. *Because of Theorem 5.3.1 all solutions of the* cosh-*Gordon equation have singularities. We restrict the domain of u to regular points hereafter. It is an open subset of $\mathbb{C}$.*

Now we are able to formulate the main result of this section.

Theorem 2.3.2 (Sym-Bobenko formula)**.** *Consider u a real solution of the* cosh-*Gordon equation defined on an open and simply connected subset O of Σ. Let α_λ be as in* (2.14) *and F_λ the corresponding frame. Then for every $\lambda = e^{q+2i\psi} \in \mathbb{C}^*$ with $q, \psi \in \mathbb{R}$ the surface $f\colon O \to \mathbb{H}^3$ is defined by*

$$f = F_\lambda F_\lambda^{*\,6}$$

a conformally parametrized immersion with

$$\begin{aligned} H_f &= -\tanh(q) \\ e^{2u_f} &= \frac{1}{4}\cosh^2(q)e^{2u} = \frac{e^{2u}}{4-4H_f^2} \\ Q_f &= \frac{1}{2}\cosh(q)e^{2i\psi}Q = \frac{e^{2i\psi}Q}{2\sqrt{1-H_f^2}}. \end{aligned}$$

Same results were already obtained in [BB93].

The chosen point λ is called the **Sym point** of the immersion f. Unlike the case $|H| > 1$ in our situation λ can be chosen arbitrarily in $\mathbb{C}^*$. Especially there are no problems with $\mathbb{H}^3$ degenerating to $\mathbb{R}^3$ if $\lambda \in \mathbb{S}^1$.

To prove the theorem we need some auxiliary definitions and statements first. For an arbitrary differential form $\omega \in \Omega^1(O, \mathfrak{sl}_2(\mathbb{C}))$ the splitting into the (1,0) part and (0,1) part is denoted by

$$\omega = \omega' + \omega''.$$

Definition 2.3.3. *The Hodge star operator $*$ for $\omega \in \Omega^1(O, \mathfrak{sl}_2(\mathbb{C}))$ is defined as*

$$*\omega = -i\omega' + i\omega''.$$

[6] We will later see (Proposition 3.2.4) that there holds $F_\lambda F_\lambda^* = F_\lambda F^{-1}_{-\bar\lambda^{-1}}$.

Now the mean curvature H of a CMC immersion $f = FF^*\colon \Sigma \to \mathbb{H}^3$ can be computed using Lemma 3 from [SKKR07]:

Proposition 2.3.4. *Setting $\omega := f^{-1}df$ for the mean curvature H, there holds*

$$2d * \omega = -iH[\omega \wedge \omega].^{7}$$

Proof of Theorem 2.3.2. Since in this setting the Maurer-Cartan equation is equivalent to the cosh-Gordon equation, we can solve $dF_\lambda = F_\lambda \alpha_\lambda$ uniquely by virtue of Proposition 2.2.18. That is the immersion $f = F_\lambda F_\lambda^*$ is well-defined.
To be able to apply the previous proposition we first compute $\omega = f^{-1}df$:

$$\begin{aligned}\omega &= (F_\lambda F_\lambda^*)^{-1} d(F_\lambda F_\lambda^*)\\ &= F_\lambda^{*-1}(\alpha_\lambda + \alpha_\lambda^*)F_\lambda^*\\ &= \frac{1}{2}F_\lambda^{*-1}\begin{pmatrix} 0 & (\lambda + \overline{\lambda}^{-1})e^u dz\\ (\lambda^{-1} + \overline{\lambda})e^u d\bar{z} & 0\end{pmatrix} F_\lambda^*.\end{aligned}$$

Decomposing ω into dz-part ω' and $d\bar{z}$-part ω'' yields

$$\begin{aligned}\omega' &= f^{-1}f_z dz = \frac{1}{2}F_\lambda^{*-1}\begin{pmatrix} 0 & (\lambda + \overline{\lambda}^{-1})e^u dz\\ 0 & 0\end{pmatrix} F_\lambda^*\\ \omega'' &= f^{-1}f_{\bar{z}} d\bar{z} = \frac{1}{2}F_\lambda^{*-1}\begin{pmatrix} 0 & 0\\ (\lambda^{-1} + \overline{\lambda})e^u d\bar{z} & 0\end{pmatrix} F_\lambda^*.\end{aligned}$$

To prove the conformality we have to check

$$\langle f_z, f_z\rangle dz d\bar{z} = \langle f_z dz, f_z dz\rangle = \langle f\omega', f\omega'\rangle = -\det(F_\lambda F_\lambda^* \omega').$$

Since $\det(\omega') = 0$, the inner product $\langle f_z, f_z\rangle$ vanishes. Same approach yields $\langle f_{\bar{z}}, f_{\bar{z}}\rangle = 0$. We want to use equality $\langle f_z, f_{\bar{z}}\rangle = 2e^{2u_f}$ next.

$$\begin{aligned}\langle f_z, f_{\bar{z}}\rangle dz d\bar{z} &= -\frac{1}{2}\mathrm{tr}(f\omega' \mathrm{adj}(f\omega'')) = -\frac{1}{2}\mathrm{tr}(\omega' \mathrm{adj}(\omega''))\\ &= \frac{1}{8}(\lambda + \overline{\lambda}^{-1})(\lambda^{-1} + \overline{\lambda})e^{2u}\mathrm{tr}\left(\begin{pmatrix} 0 & dz\\ 0 & 0\end{pmatrix}\begin{pmatrix} 0 & d\bar{z}\\ 0 & 0\end{pmatrix}\right)\\ &= \frac{e^{2u}}{8}\left(|\lambda| + |\lambda|^{-1}\right)^2 dz d\bar{z}\end{aligned}$$

[7] Since we use a slightly different α with exchanged positions of the terms with H, we have to add a minus sign on the right hand side.

This leads to

$$e^{2u_f} = \frac{1}{16}\left(|\lambda| + |\lambda|^{-1}\right)^2 e^{2u}. \tag{2.15}$$

In order to apply the previous proposition we need the quantities $[\omega\wedge\omega]$ and $d * \omega$.

$$\begin{aligned}
[\omega \wedge \omega](X,Y) &= 2\,[\omega(X), \omega(Y)] \\
&= \frac{(|\lambda| + |\lambda|^{-1})^2}{2} e^{2u} F_\lambda^{*-1} \begin{pmatrix} (dz\wedge d\bar{z})(X,Y) & 0 \\ 0 & (d\bar{z}\wedge dz)(X,Y) \end{pmatrix} F_\lambda^* \\
&= \frac{1}{2}(|\lambda| + |\lambda|^{-1})^2 e^{2u} F_\lambda^{*-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} F_\lambda^* dz\wedge d\bar{z} \qquad (2.16)\\
d(*\omega) &= d(-i\omega' + i\omega'') \\
&= i\cdot d\left(F_\lambda^{*-1}\frac{1}{2}\begin{pmatrix} 0 & -(\lambda+\bar{\lambda}^{-1})e^u dz \\ (\lambda^{-1}+\bar{\lambda})e^u d\bar{z} & 0 \end{pmatrix} F_\lambda^*\right) \\
&=: i\cdot d\left(F_\lambda^{*-1} M F_\lambda^*\right) \\
&= iF_\lambda^{*-1}(-\alpha_\lambda^*\wedge M + dM - M\wedge\alpha_\lambda^*)F_\lambda^* \\
&= -\frac{i}{4} e^{2u}(|\lambda|^{-2} - |\lambda|^2) F_\lambda^{*-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} F_\lambda^* dz\wedge d\bar{z} \qquad (2.17)
\end{aligned}$$

Applying the formula from the previous proposition yields

$$H_f = \frac{|\lambda|^{-2} - |\lambda|^2}{(|\lambda| + |\lambda|^{-1})^2} = \frac{|\lambda|^{-1} - |\lambda|}{|\lambda|^{-1} + |\lambda|}. \tag{2.18}$$

Recall that there holds $Q_f = \langle f_{zz}, N\rangle$. But this identity is not very suitable for direct calculations. Thus we differentiate $\langle f_z, N\rangle = 0$ to obtain $\langle f_{zz}, N\rangle = -\langle f_z, N_z\rangle$. A short calculation yields

$$\begin{aligned}
N_z &= (-F_\lambda \sigma_3 F_\lambda^*)_z \\
&= \frac{1}{2} F_\lambda \begin{pmatrix} 0 & -(-\lambda + \bar{\lambda}^{-1})e^u \\ -2Qe^{-u} & 0 \end{pmatrix} F_\lambda^*.
\end{aligned}$$

Now we continue with

$$\begin{aligned} Q_f &= -\langle f_z, N_z \rangle \\ &= \frac{1}{8}\mathrm{tr}\left(\begin{pmatrix} 0 & (\lambda + \overline{\lambda}^{-1})e^u dz \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & (-\lambda + \bar{\lambda}^{-1})e^u \\ 2Qe^{-u} & 0 \end{pmatrix} \right) \\ &= \frac{1}{4}(\lambda + \overline{\lambda}^{-1})Q. \end{aligned} \tag{2.19}$$

Plugging $\lambda = e^{q+2i\psi}$ into the three formulae above leads to the statement of the theorem. Using $\cosh^2 - \sinh^2 = 1$, we also obtain $1 - H_f^2 = \cosh^{-2}(q)$. ■

2.4 Non-existence of CMC tori

It is known (see e.g. [Bob91]) that there are no compact surfaces in $\mathbb{H}^3$ with $|H| \leq 1$ due to the maximum principle. We will prove a special case of this fact, the non-existence of tori. Our strategy will be to apply certain version of the maximum principle for elliptic differential equations to the conformal factor u.

Let f be an immersion of a CMC torus, hence maps H and Q are constant. Then the domain Σ of f can be represented as $\mathbb{C}/\Lambda$ with Λ being the lattice generated by the two periods. Along with the immersion itself we can then consider the map u to be defined on $\mathbb{C}$. Identifying the complex plane with $\mathbb{R}^2$, we obtain a doubly periodic map from $\mathbb{R}^2$ to $\mathbb{R}$.

Proposition 2.4.1. *The conformal factor u of an immersion of a torus is a strictly subharmonic map on $\mathbb{R}^2$, i.e. $-\Delta u < 0$.*

Proof. First recall the Gauss equation (2.9) in the general form

$$\begin{aligned} &2u_{z\bar{z}} + 2(H^2 - 1)e^{2u} - \frac{1}{2}Q\overline{Q}e^{-2u} = 0 \iff \\ &4u_{z\bar{z}} = -4(H^2 - 1)e^{2u} + |Q|^2 e^{-2u}. \end{aligned}$$

Since $|H| < 1$, we have $H^2 - 1 < 0$ which leads to

$$4u_{z\bar{z}} = 4(1 - H^2)e^{2u} + |Q|^2 e^{-2u} > 0.$$

In the current setting z is a global chart and we can express it using the coordinates x,y and the Laplacian. Then there holds $u_{z\bar{z}} = \frac{1}{4}\Delta u$ yielding

$$\Delta u > 0$$

everywhere on $\mathbb{R}^2$.

■

Lemma 2.4.2. *Let O be an open subset of $\mathbb{R}^n$. A twice-differentiable map*

$$w : O \to \mathbb{R} \textit{ with } \Delta w > 0 \textit{ everywhere in } O$$

cannot attain its maximum in O.

Proof. We assume w has a local maximum at a point $p_0 \in O$.
The Hessian matrix $H(w, p_0)$ at the point p_0 is negative semidefinite then. According to Schwartz's theorem $H(w, p_0)$ is symmetric and therefore it has only non-positive eigenvalues. Since a matrix's trace is the sum of its eigenvalues λ_i we have

$$0 \geq \sum_{i=1}^{n} \lambda_i = \mathrm{tr}(H(w, p_0)) = \Delta w$$

which contradicts the premise $\Delta w > 0$.

■

Theorem 2.4.3 (Non-existence of tori)**.** *There are no CMC tori in $\mathbb{H}^3$ with $|H| < 1$.*

Proof. Assume f is an immersion of such a torus. By virtue of Proposition 2.4.1 f's conformal factor u is subharmonic. Since u is doubly periodic, it attains its maximum at some point $p_0 \in \mathbb{R}^2$ which is impossible due to the previous lemma.

■

3 The Spectral Curve

Starting with the situation as in the setting of the Sym-Bobenko formula we will construct a smooth algebraic curve associated with the frame F_λ. We will derive the properties of such spectral curves. This curve is the main component of the integrable systems approach. We will use curves of that kind to perform some deformations in the next chapter and to construct solutions of the cosh-Gordon equation in Chapter 5.

First we start with some standard facts from the theory of algebraic curves and Riemann surfaces (cf. e.g. [FK92]). We will use the example of hyperelliptic surfaces because it is the only kind occurring in this document.

3.1 Brief introduction to algebraic curves and Riemann surfaces

Definition 3.1.1 (Hyperelliptic curve)**.** *Consider a polynomial $P \in \mathbb{C}[z]$ with degree $2g+1$ or $2g+2$, $g \in \mathbb{N}_0$. The set*

$$C^* = \{(w,z) \in \mathbb{C}^2 : w^2 = P(z)\}$$

*is called a **hyperelliptic curve**.*

We can introduce a complex structure on C^* using the implicit function theorem. We redefine the curve C^* as the zero set of the map $R(w,z) := w^2 - P(z)$. Then on some neighborhood U of a point (w_0, z_0) in $\mathbb{C}^2$ with

$$\frac{\partial R}{\partial w}(w_0, z_0) \neq 0$$

the curve can be represented by using z only

$$C^*|_U = \{z \in U : R(w(z), z) = 0\}$$

and vice versa. Using z and w as charts turns C^* in a complex one-dimensional manifold, i.e. a Riemann surface. Therefore we can speak of C^* as a hyperelliptic surface. Of course points where this theorem can't be applied to may exist which motivates the following definition.

Definition 3.1.2 (Singularity)**.** *A point (w_0, z_0) of C^* is called singular if*

$$\text{grad } R(w_0, z_0) = \left(\frac{\partial R}{\partial w}(w_0, z_0), \frac{\partial R}{\partial z}(w_0, z_0)\right) = 0.$$

In our case C^* is singular if and only if the polynomial $P(z)$ has double or higher order roots. Therefore, we will assume P to have simple roots only. In general, every singular algebraic curve has a corresponding regular curve which is called normalization.

The map $z\colon C^* \to \mathbb{C}, (w, z) \mapsto z$ defines a two-sheeted covering of $\mathbb{C}$. Its branch points are the points (w_0, z_0) with

$$\frac{\partial R}{\partial w}(w_0, z_0) = 0.$$

In case of hyperelliptic curves this is equivalent to $w_0 = 0$ which means that the branch points are

$$(0, z_j) \text{ with } P(z_j) = 0,\ j \in \{1, \dots, \deg P\}.$$

Compactifying C^* turns z into a covering of $\hat{\mathbb{C}}$. In order to compactify using the Alexandroff compactification we have to add point(s) over ∞ to the curve C^*. Since the total number of the branch points needs to be even, we add

$$\begin{array}{lll} (\infty, \infty) & \text{if } \deg P & \text{is odd or} \\ (\infty, \infty^{\pm}) & \text{if } \deg P & \text{is even.} \end{array}$$

This means that $z^{-1}(\infty)$ is a branch point if $\deg P$ is odd. We denote the compactification of the curve C^* by C. This makes the number of branch points to be $2g + 2$.

In a neighborhood of a branch point $(0, z_j)$ the local parameter is w and it is equivalent to $\sqrt{z - z_j}$. The chart near the point(s) $z^{-1}(\infty)$ is either $\frac{1}{\sqrt{z}}$ or $\frac{1}{z}$.

The branch points are the fixed points of the **hyperelliptic involution**

$$\sigma\colon C \to C,\ (w,z) \mapsto (-w,z)$$

which exchanges the two sheets of the covering.

Use of the Riemann-Hurwitz formula yields the following result.

Proposition 3.1.3. *The number $g \in \mathbb{N}_0$ specifying the degree of the polynomial P is the* ***genus*** *of the hyperelliptic[8] surface C.*

The number g is also the dimension of the complex vector space of holomorphic differentials $\Omega(C)$. A standard choice for a basis of $\Omega(C)$ is given by

$$\omega_j = \frac{z^{j-1}}{w}dz,\ j \in \{1,\dots,g\}.$$

Topologically the surface C is a sphere with g handles. The group of homotopic cycles is called the fundamental group and is denoted by $\pi(C)$. Its abelianization is the homology group $H_1(C,\mathbb{Z})$. There are many possibilities to specify a basis of $H_1(C,\mathbb{Z})$. We want to specify the most important class of choices.

Definition 3.1.4. *A basis of the homology group $H_1(C,\mathbb{Z})$ $a_1, b_1, \dots, a_g, b_g$ is called canonical if the intersection numbers comply with*

$$\begin{aligned} a_j \circ a_k &= 0 \\ b_j \circ b_k &= 0 \\ a_j \circ b_k &= \delta_{jk} \end{aligned}$$

with $j,k \in \{1,\dots,g\}$. By cutting along the cycles of a canonical basis a Riemann surface is transformed into a 4g-gon with the cycles as edges.

Definition 3.1.5. *A basis of $\Omega(C)$ $\omega_1,\dots,\omega_g$ normalized with respect to a canonical basis of $H_1(C,\mathbb{Z})$ in the following sense*

$$\int_{a_j} \omega_k = 2\pi i \delta_{jk}$$

is called canonical.

[8]Usually a surface is just called elliptic if its genus is one.

Definition 3.1.6. *A **divisor** D on a compact Riemann surface is a map $D\colon C \to \mathbb{Z}$ with discrete support.*

Divisors are useful tools in theory of compact Riemann surfaces. We will write divisors as sums of points on the surface. The most important examples are principal divisors. For a meromorphic function f with roots $P_1, \ldots, P_n$ and poles $Q_1, \ldots, Q_n$ the principal divisor is given by

$$(f) = \sum_{j=1}^{n} P_j - \sum_{k=1}^{n} Q_k.$$

3.2 The monodromy and its properties

Since we are interested in the spectral curve of a CMC cylinder we start with u being periodic with period $\tau \in \mathbb{C}^*$ and u a solution of the cosh-Gordon equation. Via the Sym-Bobenko formula 2.3.2 we obtain a frame $F_\lambda\colon O \to SL_2\mathbb{C}$ for every $\lambda \in \mathbb{C}^*$ which is a solution of the initial value problem (IVP)

$$dF_\lambda = F_\lambda \alpha_\lambda, \; F_\lambda(z_0) = \mathbb{1} \tag{3.1}$$

with a periodic α_λ. But there is no reason for the frame F_λ to be periodic itself. We want to quantify the frame's deviation from being periodic with the so-called **monodromy** M defined by

$$F_\lambda(z+\tau) = M(\lambda)F_\lambda(z) \iff M(\lambda) = F_\lambda(z+\tau)F_\lambda(z)^{-1}. \tag{3.2}$$

Remark 3.2.1. *As we will see in Theorem 5.3.1 all solutions u are singular. Anticipating some results from Chapter 5, we sketch that the monodromy is a well-defined object. The monodromy can be reconstructed from the spectral curve and the divisor away from the singularities of u.*

The map M depends holomorphically on λ and has essential singularities at 0 and ∞ since same holds for the frame F_λ as a solution of the IVP (3.1). The monodromy M depends on the initial point z_0 from the IVP so that the notation $M(\lambda, z_0)$ is more appropriate. Since M does not depend on z we can plug $z = z_0$ into the right hand side of (3.2) leading to $M(\lambda, z_0) = F_\lambda(z_0+\tau)$.

The question how the monodromy changes with respect to the initial point is answered by the following statement.

Proposition 3.2.2. *There holds*

$$M(\lambda, z) = F_\lambda(z)^{-1} M(\lambda, z_0) F_\lambda(z).$$

Proof. Let $z_0 \neq z_1$ be some points in the open and simply connected domain O and F_λ, $\hat{F}_\lambda$ be the solutions of the IVPs

$$dF_\lambda = F_\lambda \alpha_\lambda,\ F_\lambda(z_0) = \mathbb{1} \text{ and } d\hat{F}_\lambda = \hat{F}_\lambda \alpha_\lambda,\ \hat{F}_\lambda(z_1) = \mathbb{1}.$$

We denote the associated monodromies by $M(\lambda, z_0)$ and $M(\lambda, z_1)$. A simple calculation shows that $F(z_1)^{-1}F(z)$ solves the second IVP, hence, $\hat{F}_\lambda(z) = F(z_1)^{-1}F(z)$. Using the definition (3.2), we see

$$\begin{aligned} M(\lambda, z_1) &= \hat{F}_\lambda(z+\tau)\hat{F}_\lambda(z)^{-1} \\ &= F_\lambda(z_1)^{-1} F_\lambda(z+\tau) F_\lambda(z)^{-1} F_\lambda(z_1) \\ &= F_\lambda(z_1)^{-1} M(\lambda, z_0) F_\lambda(z_1). \end{aligned}$$

■

We omit the initial point z_0 of $M(\lambda, z_0)$ hereafter. It is easy to show that the monodromy satisfies $dM(\lambda, z) = [M(\lambda, z), \alpha_\lambda(z)]$ and $M(\lambda, z) = F_\lambda(z)^{-1} F_\lambda(z+\tau)$.

We will now investigate the behavior of the connection α_λ, the frame F_λ and the monodromy $M(\lambda)$ towards the map

$$\lambda \mapsto -\bar{\lambda}^{-1}.$$

The corresponding properties of these three objects are called **reality conditions**, because they will induce an antiholomorphic involution on the spectral curve.

Proposition 3.2.3 (Reality condition for α_λ)**.** *There holds*

$$\alpha_\lambda = -\bar{\alpha}^T_{-\bar{\lambda}^{-1}}.$$

Proof. Since u is real, simple transformations yield

$$\begin{aligned} \alpha_{\bar{\lambda}^{-1}} &= \begin{pmatrix} u_z dz - u_{\bar{z}} d\bar{z} & \bar{\lambda}^{-1} e^u dz - e^{-u} d\bar{z} \\ e^{-u} dz + \bar{\lambda} e^u d\bar{z} & -u_z dz + u_{\bar{z}} d\bar{z} \end{pmatrix} \\ &= -\begin{pmatrix} \overline{u_z dz - u_{\bar{z}} d\bar{z}} & -\overline{\lambda^{-1} e^u d\bar{z} + e^{-u} dz} \\ \overline{-e^{-u} d\bar{z} - \lambda e^u dz} & \overline{-u_z dz + u_{\bar{z}} d\bar{z}} \end{pmatrix} \\ &= -\bar{\alpha}^T_{-\lambda}, \end{aligned}$$

which is equivalent to the claim.

■

Since α_λ passes this property to F_λ we have the following result.

Proposition 3.2.4 (Reality condition for F_λ)**.** *There holds*

$$F_\lambda = \left(\bar{F}^T_{-\bar{\lambda}^{-1}}\right)^{-1}.$$

Proof. Again we will use the uniqueness of solutions of an IVP.

$$\begin{aligned} d\left(\left(\bar{F}^T_{-\bar{\lambda}^{-1}}\right)^{-1}\right) &= -\left(\bar{F}^T_{-\bar{\lambda}^{-1}}\right)^{-1} d\left(\bar{F}^T_{-\bar{\lambda}^{-1}}\right)\left(\bar{F}^T_{-\bar{\lambda}^{-1}}\right)^{-1} \\ &= -\left(\bar{F}^T_{-\bar{\lambda}^{-1}}\right)^{-1} \bar{\alpha}^T_{-\bar{\lambda}^{-1}} \bar{F}^T_{-\bar{\lambda}^{-1}} \left(\bar{F}^T_{-\bar{\lambda}^{-1}}\right)^{-1} \\ &\overset{3.2.3}{=} \left(\bar{F}^T_{-\bar{\lambda}^{-1}}\right)^{-1} \alpha_\lambda \end{aligned}$$

Since $F_\lambda(z_0) = \mathbb{1} = \left(\bar{F}^T_{-\bar{\lambda}^{-1}}(z_0)\right)^{-1}$, the proof is complete.

■

Applying the previous proposition to $F_\lambda(z_0 + \tau) = M(\lambda)$ we obtain:

Proposition 3.2.5 (Reality condition for $M(\lambda)$)**.** *There holds*

$$M(\lambda) = \left(\bar{M}^T(-\bar{\lambda}^{-1})\right)^{-1}.$$

The question when does f define a CMC cylinder is answered by the following statement.

Proposition 3.2.6 (Closing condition)**.** *Let $f\colon O \subset \Sigma \to \mathbb{H}^3$ be a conformal CMC immersion given by the Sym-Bobenko formula 2.3.2, $f(z) = F_{\lambda_0}(z)F_{\lambda_0}(z)^*$ with the Sym point $\lambda_0 \in \mathbb{C}^*$ and $u : O \to \mathbb{R}$ a simply periodic solution of the* cosh-*Gordon equation with period τ. Then f ist τ-periodic if and only if*

$$M(\lambda_0) = M(-\bar{\lambda}_0^{-1}) = \pm\mathbb{1}.$$

Proof. If f is periodic then there holds $f(z) = f(z+\tau)$. Conversely applying the Sym-Bobenko formula to both sides yields

$$\begin{aligned} f(z) &= F_{\lambda_0}(z)\overline{F_{\lambda_0}(z)}^T \overset{3.2.4}{=} F_{\lambda_0}(z)(F_{-\bar{\lambda}_0^{-1}}(z))^{-1} \text{ and} \\ f(z+\tau) &= F_{\lambda_0}(z+\tau)\overline{F_{\lambda_0}(z+\tau)}^T \\ &= M(\lambda_0)F_{\lambda_0}(z)\overline{M(\lambda_0,z)F_{\lambda_0}(z)}^T \\ &\overset{3.2.5}{=} M(\lambda_0)F_{\lambda_0}(z)(F_{-\bar{\lambda}_0^{-1}}(z))^{-1}M(-\bar{\lambda}_0^{-1})^{-1}. \end{aligned}$$

Now it is easy to see that the periodicity is equivalent to the claim.

■

3.3 Monodromy's eigenvalue curve

The monodromy encodes the information about the potential u. Now we take a closer look at the eigenvalues of the monodromy $M(\lambda)$. The eigenvectors will be investigated in Chapter 5.

Definition 3.3.1. *The eigenvalue curve*[9] *of* $M(\lambda)$ *is defined as*

$$\Gamma^* = \{(\lambda, \mu) \in \mathbb{C}^* \times \mathbb{C}^* : \det(\mu\mathbb{1} - M(\lambda)) = 0\}.$$

By virtue of Proposition 3.2.2 the eigenvalues of $M(\lambda)$ do not depend on z justifying to speak of Γ^* as associated to a simply periodic solution of the cosh-Gordon equation CG.

We can interpret Γ^* as a two-sheeted covering using the map

$$\lambda\colon \Gamma^* \to \mathbb{C}^*,\ (\lambda, \mu) \mapsto \lambda. \tag{3.3}$$

Since $\det M(\lambda) = 1$, the eigenvalues are μ and μ^{-1}. We denote

$$R(\lambda, \mu) := \det(\mu\mathbb{1} - M(\lambda)) = \mu^2 - \Delta(\lambda)\mu + 1 \tag{3.4}$$

and

$$\Delta(\lambda) := \mathrm{tr}(M(\lambda)) = \mu + \mu^{-1}$$

[9]Some authors use the term multiplier curve instead.

for abbreviation purposes. The curve Γ^* is the zero set of the function R but since it is a transcendental function rather than an algebraic one, there are some differences compared to the theory presented previously. They will be pointed out on occurrence while investigating the structure of Γ^*.

We start with the branch points of the covering λ. At these points the two eigenvalues of $M(\lambda)$ coincide, i.e. $\mu(\lambda) = \mu(\lambda)^{-1} = \pm 1$. These points also have to obey the condition

$$\frac{\partial R}{\partial \mu}(\lambda, \mu) = 2\mu - \Delta(\lambda) = 0 \iff \mu = \frac{\Delta(\lambda)}{2}.$$

But these points could also be singularities. Since we do not know much about $\Delta(\lambda)$, we can't study $\frac{\partial R}{\partial \lambda} = -\frac{\partial \Delta(\lambda)}{\partial \lambda}$ to distinguish the singular from the branch points. For that reason we continue to analyze $\frac{\partial R}{\partial \mu}$.
Using

$$\mu = \frac{\Delta(\lambda)}{2} \pm \frac{\sqrt{\Delta(\lambda)^2 - 4}}{2}, \tag{3.5}$$

we conclude

$$\mu = \frac{\Delta(\lambda)}{2} \iff \Delta(\lambda)^2 - 4 = 0.$$

Consider γ_0 to be a root of $\Delta^2 - 4$ of order n. Let k be the local chart near γ_0, without restrictions we can assume $k(\gamma_0) = 0$. Then we can write the Taylor expansion of $\Delta^2 - 4$ as

$$\Delta^2 - 4 = ck^n + O(k^{n+1}) \, , c \in \mathbb{C}.$$

Taking the square root we get

$$\sqrt{\Delta^2 - 4} = \sqrt{c}\sqrt{k^n}\sqrt{1 + O(k)}.$$

A branch point is characterized by the fact that if we “walk” a full loop around it we will change the sheet once. Here $\sqrt{k^n}$ changes sign once only if n is odd. Greater odd numbers $n \geq 3$ describe points which have characteristics of branch points as well as singular points. We don't want to deal with them, hence we assume from now on, that $\Delta(\lambda)^2 - 4$ has roots with order at most two. In conclusion, we describe branch points by the following properties.

Definition 3.3.2. *A point $\gamma = (\lambda, \mu)$ of the curve Γ^* is a* ***branch point*** *of* (3.3) *if and only if $\mu = \pm 1$ and the root of $\Delta(\lambda)^2 - 4$ is simple.*

Now we want to investigate the involutions of the eigenvalue curve Γ^*. Because of the reality condition 3.2.5 we can expect Γ^* to have more structure as in the general case.

First we look at the analog of the hyperelliptic involution.

Proposition 3.3.3. *The map*

$$\sigma\colon \Gamma^* \to \Gamma^*, (\lambda, \mu) \mapsto \left(\lambda, \frac{1}{\mu}\right)$$

is a holomorphic involution of Γ^* and exchanges the two sheets of the covering λ.

Proof. Obviously this map is holomorphic and swaps the two eigenvalues μ and μ^{-1} of $M(\lambda)$ which are lying on different sheets.

■

The fixed points of σ are points with $\mu(\lambda) = \pm 1$, i.e. branch and singular points. The additional structure of Γ^* is reflected by the following proposition.

Theorem 3.3.4. *There exist two antiholomorphic involutions*

$$\eta\colon \Gamma^* \to \Gamma^*,\ (\lambda, \mu) \mapsto \left(-\frac{1}{\bar{\lambda}}, \bar{\mu}\right)$$
$$\rho\colon \Gamma^* \to \Gamma^*,\ (\lambda, \mu) \mapsto \left(-\frac{1}{\bar{\lambda}}, \frac{1}{\bar{\mu}}\right)$$

and there holds

$$\eta = \sigma \circ \rho = \rho \circ \sigma \text{ and } \rho = \sigma \circ \eta = \eta \circ \sigma. \tag{3.6}$$

Proof. Both maps are clearly antiholomorphic. The condition (3.6) is obvious too. It is also easy to see that there holds $\eta^2 = \rho^2 = id$.
For the remaining part we first want to show

$$M(\lambda) = \sigma_2 \bar{M}(-\bar{\lambda}^{-1})\sigma_2 \tag{3.7}$$

with the Pauli matrix $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$. A straightforward calculation yields $\alpha_\lambda = \sigma_2 \bar{\alpha}_{-\bar{\lambda}^{-1}} \sigma_2$. Following the steps of the proof of Propositions 3.2.4 and 3.2.5 then leads to (3.7).
Since there holds $\sigma_2 = \sigma_2^{-1}$, we see that the eigenvalues of $M(\lambda)$ and $\bar{M}(-\bar{\lambda}^{-1})$ coincide, meaning that both mappings η and ρ are involutions of Γ^*.

∎

Due to the existence of these additional involutions, we will call Γ^* a real curve. Since $\lambda = -\bar{\lambda}^{-1}$ is equivalent to $|\lambda|^2 = -1$, we have the following result.

Corollary 3.3.5. *The anti-holomorphic involutions η and ρ have no fixed points.*

Remark 3.3.6. *This gives a distinction compared to the* sinh*-Gordon case because only one antiholomorphic involution is free from fixed points there. In conformity with similar works we will use η although most properties hold for ρ too.*

Definition 3.3.7. *The involutions' action on a map f on Γ^* is denoted by*

$$\sigma^* f(\lambda, \mu) = f(\sigma(\lambda, \mu)) \ \textit{etc.}$$

Proposition 3.3.8. *The set of branch points B is invariant under the involution η.*

Proof. Let γ be a branch point. By definition it is a fixed point of σ. By virtue of condition (3.6) we see

$$\rho(\gamma) = \eta(\sigma(\gamma)) = \eta(\gamma) \text{ and } \rho(\gamma) = \sigma(\eta(\gamma)) \Rightarrow \sigma(\eta(\gamma)) = \eta(\gamma).$$

As a fixed point of σ the point $\eta(\gamma)$ is either a branch point or a singularity (cf. 3.3.2). Since there holds $\eta^*\Delta = \overline{\Delta}$, the order of the root of $\Delta^2 - 4$ remains simple. In total, we have $\eta(\gamma) \in B$ leading to $\eta(B) = B$.

∎

Corollary 3.3.9. *All branch points can be clustered into pairs $(\gamma_j,\ \eta(\gamma_j))$ and if B is finite the number of branch points is even.*

Proof. Since η does not have any fixed points we have $\eta(\gamma_j) \neq \gamma_j$. Because η is an involution we have $\eta(\eta(\gamma_j)) = \gamma_j$.

■

We will call $\eta(\gamma)$ the η-cousin of the point γ.

Our aim is to have a compact Riemann surface to work with in the end. Therefore, we will not investigate Riemann surfaces of infinite genus (as done in [Sch96]). This premise leads to the following definition.

Definition 3.3.10 (Solution of finite type)**.** *A simply periodic solution u of the* cosh-*Gordon equation, which generates an eigenvalue curve Γ^* with finite number of branch points, is called a solution of finite type. The curve Γ^* will be called to be of finite type too.*

Unfortunately Γ^* turns out to be hard to handle due to the next result.

Proposition 3.3.11. *Every finite type curve Γ^* has infinitely many singularities over every neighborhood of 0 and ∞.*

Proof. The map M depends holomorphically on λ and has essential singularities at 0 and ∞ because same holds for the frame F_λ as it solves (3.1). For that reason, we consider the eigenvalue function μ to be a holomorphic map from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$ in λ with essential singularities at 0 and ∞ as well.
Let U_0 and U_∞ be arbitrary punctured neighborhoods of 0 and ∞ respectively. By virtue of the Big Picard Theorem, μ restricted to U_0 or U_∞ attains every value in $\mathbb{C}$ infinitely often with at most one exception. Without loss of generality let this (possible) exception be -1. Since we have only a finite number of branch points, almost all of the points $(\lambda, \mu(\lambda))$ with $\mu(\lambda) = 1$ are singularities of Γ^*.

■

The most important consequence is that Γ^* can't be compactified because the singularities have accumulation points at 0 and ∞, which are the points ought to be included. We will construct the normalization of Γ^* in the next section in order to deal with this issue.

3.4 Normalization of the eigenvalue curve

In this section we will construct the normalization, i.e. a smooth hyperelliptic curve of a singular finite type curve Γ^*. This procedure does not change the structure of the Riemann surface itself.

We sketch the construction strategy first. The curve Γ^* is parametrized by μ and λ. Since λ already is a meromorphic function, there are no problems with it. But because μ has essential singularities at 0 and ∞, we need a replacement. First, we introduce a meromorphic parameter κ, which depends on μ and λ and express Γ^* as the zero set of some function in κ and λ. Then, we use this function to introduce a new parameter ν such that

$$\nu^2 = a(\lambda),$$

with some polynomial a having the branch points of Γ^* as roots.

Consider the map $p := \ln \mu$[10]. It is a meromorphic function but is multivalued, hence, it is not a good replacement for μ.
In [GS95] the same problem in case of the KdV equation is solved by introducing a new parameter $\tilde{\kappa} := \frac{dp}{d\lambda}$. Since the space of meromorphic differentials is one dimensional, $\tilde{\kappa}$ is a meromorphic function. But in case of cosh-Gordon (as well as sinh-Gordon), it will turn out that it is appropriate to use the 1-form $dq := d\ln\lambda = d\lambda/\lambda$ instead of $d\lambda$ because of the additional outstanding point 0. Hence, we define the new parameter as

$$\kappa = \frac{dp}{dq} = \frac{d\ln\mu}{d\ln\lambda} = \frac{\lambda}{\mu}\frac{d\mu}{d\lambda}. \tag{3.8}$$

Using the total derivative of $R(\lambda, \mu)$

$$\frac{\partial R}{\partial \lambda}d\lambda + \frac{\partial R}{\partial \mu}d\mu\bigg|_{\Gamma^*} = \frac{\partial \Delta}{\partial \lambda}\mu d\lambda + (2\mu - \Delta)d\mu\bigg|_{\Gamma^*} = 0,$$

we get

$$d\mu = \frac{\mu\Delta'}{2\mu - \Delta}d\lambda \text{ and } d\lambda = \frac{2\mu - \Delta}{\mu\Delta'}d\mu \tag{3.9}$$

[10]It is often called the quasimomentum, especially in physics-related publications.

denoting differentiation with respect to λ by prime. Now it follows that

$$\kappa = \frac{\lambda \Delta'}{2\mu - \Delta}.$$

Using $(2\mu - \Delta)^2 = \Delta^2 - 4$, we obtain

$$\kappa^2 = \frac{\lambda^2 (\Delta')^2}{\Delta^2 - 4}.$$

Since Δ' and $\Delta^2 - 4$ encode the roots of dp and dq, we need more information about them. We are not interested in common roots of dp and dq, since they are not branch points but singularities (cf. 3.3.2). Because we have a finite type curve there are only finitely many individual roots. We denote the values of the function λ at these zeros of $d\lambda$ by α_j and those of dp by β_k. Since λ does not vanish on Γ^*, the zeros of dq are those of $d\lambda$, which are the branch points of Γ^*. Since they are distinct, all roots of $d\lambda$ are simple. Because of the Corollary 3.3.9 there is some $g \in \mathbb{N}_0$ such that there are $2g$ branch points. We set the number of individual roots of dp to be m.

We define $\tilde{a}(\lambda) = \prod_{j=1}^{2g}(\lambda - \alpha_j)$ and $\tilde{b}(\lambda) = \prod_{k=1}^{m}(\lambda - \beta_k)$. Since we have a finite type curve dp and dq are essentially described by these polynomials. Now we have to know how the polynomial $\tilde{a}$ behaves towards the action of η.

Lemma 3.4.1. *Consider a polynomial $p \in \mathbb{C}[\lambda]$ of even degree $\deg p = 2n$. If all of its roots λ_j are nonzero and obey the condition*

$$p(\lambda_j) = 0 \Leftrightarrow p(-\overline{\lambda}_j^{-1}) = 0 \qquad (*)$$

then the polynomial satisfies the following ***reality condition***

$$p(\lambda) = C_p \lambda^{2n} \overline{p(-1/\bar{\lambda})} \qquad (3.10)$$

with $C_p = \frac{p_{2n}}{\bar{p}_{2n}} \prod_{j=1}^{2n} \lambda_j$.

Proof. The condition $(*)$ implies that the roots are clustered in pairs. Because a polynomial is defined by its roots up to a constant, there holds

$$\begin{aligned} p(\lambda) &= \sum_{k=1}^{2n} p_k \lambda^k = p_{2n} \prod_{j=1}^{2n} (\lambda - \lambda_j) \\ &= p_{2n} \left(\prod_{l=1}^{n} \bar{\lambda}_l^{-1} \right) \prod_{l=1}^{n} (\lambda - \lambda_l)(1 + \lambda \bar{\lambda}_l) \\ &=: \tilde{C}_p \; \tilde{p}(\lambda). \end{aligned}$$

Simple transformations now yield

$$\begin{aligned} \tilde{p}(\lambda) &= \prod_{l=1}^{n} (\lambda - \lambda_l)(1 + \lambda \bar{\lambda}_l) \\ &= \lambda^{2n} \prod_{l=1}^{n} (1 - \lambda^{-1} \lambda_l)(\lambda^{-1} + \bar{\lambda}_l) \\ &= (-1)^n \lambda^{2n} \prod_{l=1}^{n} (-\lambda^{-1} - \bar{\lambda}_l)(1 - \lambda^{-1} \lambda_l) \\ &= (-1)^n \lambda^{2n} \overline{\tilde{p}(-1/\bar{\lambda})}. \end{aligned}$$

For the original polynomial we have

$$\begin{aligned} p(\lambda) &= \tilde{C}_p (-1)^n \lambda^{2n} \overline{\tilde{p}(-1/\bar{\lambda})} \\ &= \frac{\tilde{C}_p}{\bar{\tilde{C}}_p} (-1)^n \lambda^{2n} \overline{p(-1/\bar{\lambda})}. \end{aligned}$$

Due to $(*)$ there holds $\frac{\tilde{C}_p}{\bar{\tilde{C}}_p} = (-1)^n \frac{p_{2n}}{\bar{p}_{2n}} \prod_{l=1}^{n} -\lambda_l \bar{\lambda}_l^{-1} = (-1)^n \frac{p_{2n}}{\bar{p}_{2n}} \prod_{j=1}^{2n} \lambda_j$ completing the proof.

■

We will normalize the polynomial $\tilde{a}$ by setting the highest coefficient to $i\sqrt{\prod_{j=1}^{2g} \overline{\alpha_j}}$ and call the resulting polynomial a. Now the highest coefficient a_{2g} is located on the unit circle[11] and a is uniquely determined by its roots up to a multiplication with a real constant. The reality condition for a then reads

[11] Every product $\alpha_j \eta(\alpha_j)$ has already absolute value 1.

$$a = -\lambda^{2g}\overline{\eta^* a}. \tag{3.11}$$

We now set

$$\kappa^2 = \frac{b^2}{a}.$$

Note that this also normalizes b. Because we can recover μ from $\int \kappa dq = \ln \mu$ away from the branch points, the set

$$\left\{(\lambda, \kappa) \in \mathbb{C} \times \hat{\mathbb{C}} : \kappa^2 = \frac{b(\lambda)^2}{a(\lambda)}\right\}$$

completely describes Γ^* excluding the singularities.
Introducing the new parameter $\nu := \frac{b}{\kappa}$ leads to the following result.

Proposition 3.4.2. *The algebraic curve defined by*

$$\nu^2 = \lambda a(\lambda)$$

is a normalization[12] *of the eigenvalue curve.*

The compactification of this curve is denoted by Y. The covering $\lambda\colon Y \to \hat{\mathbb{C}}$ has the $2g$ zeros of the polynomail a, as well as the points $y_+ := \lambda^{-1}(0)$ and $y_- := \lambda^{-1}(\infty)$ as branch points. We call the curve Y the **spectral curve** of the potential u. Its genus g is called the **spectral genus**. The set $Y \setminus \{y^+, y^-\}$ is denoted by Y^*.

Next, it is natural to ask for the manifestation of the involutions of the eigenvalue curve on Y.

Proposition 3.4.3. *The involutions on Y are*

$$\begin{aligned} \sigma &: (\lambda, \nu) \mapsto (\lambda, -\nu) \\ \eta &: (\lambda, \nu) \mapsto (-\bar{\lambda}^{-1}, \bar{\lambda}^{-g-1}\bar{\nu}) \\ \rho &: (\lambda, \nu) \mapsto (-\bar{\lambda}^{-1}, -\bar{\lambda}^{-g-1}\bar{\nu}). \end{aligned}$$

[12] Recall that a normalization is only unique up to an isomorphism.

Proof. Since σ is the holomorphic involution swapping the sheets, it now has the form $\sigma(\lambda,\nu)=(\lambda,-\nu)$.
As already seen the polynomial a satisfies the following reality condition

$$\overline{\eta^* a} = -\lambda^{-2g} a.$$

Since the involution η has to uphold $\nu^2=\lambda a(\lambda)$, we have

$$\overline{\eta^*\nu}^2 = \lambda a(\lambda)\lambda^{-2g-2}.$$

Considering that η does not swap the sheets of the covering, leads to

$$\eta^*\nu = \bar{\lambda}^{-g-1}\bar{\nu}. \tag{3.12}$$

The property $\rho=\eta\circ\sigma$ gives the involution ρ.

■

Corollary 3.4.4. *The spectral genus is odd.*

Proof. We check the involution condition $\eta^2 = id$:

$$\begin{aligned}\eta(\eta(\lambda,\nu)) &= \eta(-\bar{\lambda}^{-1}, \bar{\nu}\bar{\lambda}^{-g-1})\\ &= (\lambda, \lambda^{-g-1}(-1)^{g+1}\lambda^{g+1}\nu)\\ &= (\lambda, (-1)^{g+1}\nu).\end{aligned}$$

For that reason η, is an involution if and only if the spectral genus g is odd.

■

This result is well known (cf. e.g. [Bab91a]).

From now on, we will work with the normalization of the original eigenvalue curve.

3.5 Properties of the eigenvalue function

Because of the importance of the map μ, we want to summarize its properties in this section for later use.

First we want to examine the properties of $\ln \mu$ near the points 0 and ∞. We start with a connection α_λ such that the Hopf differential Qdz^2 is again turned into dz^2 by a coordinate transform near a non-umbilic point leading to

$$\alpha_\lambda = \frac{1}{2}\begin{pmatrix} u_z & \lambda e^u \\ e^{-u} & -u_z \end{pmatrix} dz + \frac{1}{2}\begin{pmatrix} -u_{\bar z} & -e^{-u} \\ \lambda^{-1} e^u & u_{\bar z} \end{pmatrix} d\bar z. \tag{3.13}$$

As seen in Section 2.3 we can change α_λ by gauging F_λ with some matrix:

$$F_\lambda \mapsto F_\lambda A =: G_\lambda.$$

Now our aim is to represent α_λ as an expansion near 0 and ∞ in terms of local charts $\sqrt{\lambda}$ or $\frac{1}{\sqrt{\lambda}}$ with a constant leading term. Therefore we need A to depend on z. Then there holds

$$\begin{aligned} dG_\lambda &= d(F_\lambda A) \\ &= G_\lambda(A^{-1}\alpha_\lambda A + A^{-1}dA) \\ &=: G_\lambda \beta_\lambda. \end{aligned}$$

That is we now have

$$dG_\lambda = G_\lambda \beta_\lambda \ , G_\lambda(z_0) = A(z_0)$$

instead of the original IVP. For the associated monodromy $\tilde M(\lambda)$ there holds

$$\tilde M(\lambda) = F_\lambda(z+\tau)A(z+\tau)A(z)^{-1}F_\lambda(z)^{-1},$$

hence, in case A is τ-periodic the monodromy does not change at all. We use

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{\frac{u}{2}} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} e^{-\frac{u}{2}} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{3.14}$$

near ∞ and

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{\lambda} e^{-\frac{u}{2}} & 0 \\ 0 & e^{\frac{u}{2}} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{3.15}$$

near 0. The results (after some tedious calculations) are

$$2\beta_\lambda = \sqrt{\lambda}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} dz - \begin{pmatrix}0 & 2u_z\\ 2u_z & 0\end{pmatrix} dz + \frac{1}{\sqrt{\lambda}}\begin{pmatrix}\sinh(2u) & -\cosh(2u)\\ \cosh(2u) & -\sinh(2u)\end{pmatrix} d\bar{z}$$
$$2\beta_\lambda = \frac{1}{\sqrt{\lambda}}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} d\bar{z} - \begin{pmatrix}0 & 2u_{\bar{z}}\\ 2u_{\bar{z}} & 0\end{pmatrix} d\bar{z} + \sqrt{\lambda}\begin{pmatrix}\cosh(2u) & \sinh(2u)\\ -\sinh(2u) & -\cosh(2u)\end{pmatrix} dz$$

at ∞ and at 0 respectively. Using

$$\tilde{A} = A\begin{pmatrix}i & -i\\ 1 & 1\end{pmatrix},$$

the latter leading term turns into

$$\frac{1}{\sqrt{\lambda}}\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}.$$

Now if we assume that u and its derivatives are bounded, we can derive some asymptotic analysis for $\ln\mu$. With $z_0 = 0$ we have $M(\lambda) = F_\lambda(\tau)$. Then for very small (or large) λ we obtain the following result:

$$\ln\mu = \frac{\tau}{2}\sqrt{\lambda} + O(\frac{1}{\sqrt{\lambda}}) \text{ near } \infty \tag{3.16}$$
$$\ln\mu = \frac{i\tau}{2\sqrt{\lambda}} + O(\sqrt{\lambda}) \text{ near } 0. \tag{3.17}$$

In conclusion we have the following results.

Proposition 3.5.1. *The map μ has the following attributes:*

(i) μ is a holomorphic function on Y^*

(ii) μ is nonzero on Y^* and attains ± 1 at the branch points of Y

(iii) $d\ln\mu$ is a differential of the second kind i.e. it has no residues.

(iv) $d\ln\mu$ obeys the conditions $\sigma^*(d\ln\mu) = -d\ln\mu$, $\eta^*(d\ln\mu) = \overline{d\ln\mu}$ and $\rho^*(d\ln\mu) = -\overline{d\ln\mu}$

(v) $\ln\mu$ has simple poles at y_+ and y_-

(vi) $d\ln\mu$ has double poles at y_+ and y_- and is regular elsewhere

(vii) all periods of $d \ln \mu$ are integer multiples of $2\pi i$ and integrals of the form $\int_{\alpha_j}^{\alpha_{j+1}} d \ln \mu$ vanishes for all j.

Proof. (i) The monodromy is holomorphic on $\mathbb{C}^*$ thus μ is holomorphic as its eigenvalue function.

(ii) Because the monodromy's determinant is equal to one, there holds $\mu \cdot \mu^{-1} = 1$. At a branch point, there holds $\mu = \mu^{-1}$, which leads to $\mu = \pm 1$.

(iii) Differentials of meromorphic functions do not have any residues.

(iv) Apply the involutions to $d \ln \mu = \frac{d\mu}{\mu}$.

(v) Cf. (3.16) and (3.17).

(vi) Because of (iii) only poles of order two and higher are possible. The form $d \ln \mu$ has second order poles over 0 and ∞ due to (v). Assuming a pole of order at least two somewhere else leads to $\ln \mu$ having a pole of order at least one at this point. But this contradicts (i) since $\mu = e^{\ln \mu}$.

(vii) The first part is trivial. Using (iv) we obtain

$$\int_{\alpha_j}^{\alpha_{j+1}} d \ln \mu = \int_{\sigma(\alpha_j)}^{\sigma(\alpha_{j+1})} \sigma^*(d \ln \mu) = -\int_{\alpha_j}^{\alpha_{j+1}} d \ln \mu.$$

■

Proposition 3.5.2. *The differential dp can be written as*

$$dp = \frac{b}{\nu} dq$$

with a polynomial $b \in \mathbb{C}^{g+1}[\lambda]$ which obeys the following reality condition

$$\overline{\eta^* b} = -\lambda^{-(g+1)} b. \tag{3.18}$$

Proof. Using the definition of ν, we get the formula as stated above.
To determine the degree of b we take a look at the canonical divisor of Y. It has degree $2g - 2$ and the same holds for the divisor of dp. Because dp has double poles over 0 and ∞ only (see Proposition 3.5.1, (vi)), there are

$2g+2$ zeros over $\mathbb{C}^*$. They are distinct from branch points because otherwise the latter won't be branch points but singularities. For that reason they are not cancelled with the zeros of ν. Since the roots are located outside of the branch points and there holds $\sigma^* dp = -dp$, the form dp vanishes simultaneously on both sheets. In total, we have $g+1$ roots β_k in the λ-plane and therefore $\deg b$ is $g+1$.

To prove the last part we apply η to dp:

$$\begin{aligned}\overline{\eta^* dp} &= \frac{\overline{\eta^* b}}{\lambda^{-g-1}\nu}(-dq)\\ &\stackrel{!}{=} dp = \frac{b}{\nu}dq.\end{aligned}$$

■

Corollary 3.5.3. *Roots of the polynomial b are clustered in pairs $(\beta_k, \eta(\beta_k))$ with $\eta(\beta_k) \neq \beta_k$.*

This property also holds for $\beta_k \in S^1$ which is specific for the cosh-Gordon equation compared to the sinh-Gordon case.

Now we specify how the period τ is encoded in the polynomial b. The form dp is a global object and the constant of the leading term at 0 is $\frac{b(0)}{\sqrt{a(0)}}$. At the same time, because of (3.17), we know that

$$dp = -\frac{i\tau}{2}\frac{d\sqrt{\lambda}}{\lambda} + \text{ higher order terms}$$

in a neighborhood of 0. A comparison now yields

$$\frac{b(0)}{\sqrt{a(0)}} = -\frac{i\tau}{2}. \tag{3.19}$$

4 Isoperiodic deformations of spectral curves

In this chapter we continue to investigate the spectral curve or to be more precise the moduli space of these curves. We want to analyze it using non-isospectral deformations. First we will choose a representation for the spectral curve Y suitable for deformations. It will turn out that it is convenient to describe Y using the polynomials a and b derived in the previous chapter. Then a deformation is implemented by moving the roots of a and b such that all the nature of a spectral curve is preserved.

This chapter follows [GS95]. We want to embed our setting into the classification of integrable systems presented in this publication. In our situation we have two outstanding functions on the spectral curve Y, namely λ and μ. The meromorphic map λ induces a trivial flow on the Jacobian. The function μ has essential singularities at y_+ and y_- and is nonzero on Y^*. It induces a periodic flow on the Jacobian. This behavior corresponds to the second case discussed. Since there is only one periodic flow on the Jacobian it is called the simply periodic case.

4.1 Definition of spectral data and the deformation

We want to describe a spectral curve in terms of the polynomials a and b.

Definition 4.1.1. *The polynomials* $(a, b) \in \mathbb{C}^{2g}[\lambda] \times \mathbb{C}^{g+1}[\lambda]$ *having simple roots only and no common roots that satisfy the following conditions*

(i) $\overline{\eta^* a} = -\lambda^{-2g} a$, $|a(0)| = 1$

(ii) $\overline{\eta^* b} = -\lambda^{-(g+1)} b$

(iii) *the periods of the form* $\phi := \frac{b}{\nu\lambda} d\lambda$ *defined on the curve* $\nu^2 = \lambda a(\lambda)$ *are integer multiples of* $2\pi i$ *(closing condition)*

(iv) $\frac{b(0)}{\sqrt{a(0)}} = \frac{-i\tau}{2}$

are called the ***spectral data***[13] *of a real simple periodic solution of the* cosh-*Gordon equation with a period* $\tau \in \mathbb{C}$.

We denote the set of all spectral data by $\mathcal{M}_g(\tau)$. If the property (*iv*) is violated then the set of corresponding spectral data is denoted by $\mathcal{M}_g$. Both sets have the structure of a manifold. These are rather complicated subsets of the set of all polynomials a and b. Therefore we will derive vector fields, induced by deformations, such that $\mathcal{M}_g$ and $\mathcal{M}_g(\tau)$ are invariant under their action.

Remark 4.1.2. *The spectral data describe the function* μ *completely. The form* ϕ *obeys the same reality conditions as* $d \ln \mu$ *and is therefore the candidate for it. The differential* $d \ln \mu$ *is uniquely defined by its properties, because the difference of two such forms is holomorphic and has vanishing a-periods it has to be zero by Riemann's bilinear identity. Therefore we have* $\phi = d \ln \mu$. *On* $\hat{Y} := Y \setminus \bigcup_{j=1}^{g} [\alpha_j, \eta(\alpha_j)]$ *it has a meromorphic primitive* h *since* $\int_{\alpha_j}^{\eta(\alpha_j)} \phi = 0$. *The integration constant is uniquely determined by the reality conditions. The map* h *can be extended to* Y *and is multi-valued then. The eigenvalue function* μ *can be recovered by* $\mu = e^h$. *It is single-valued and attains the values* ± 1 *at every* α_j *because of the closing condition (iii).*

Now we can define a deformation of a particular point of the moduli space. Consider an expansion of a fixed spectral curve Y with genus g:

$$Y(t) = Y + \frac{\partial Y}{\partial t} t + O(t^2).$$

Such a deformation of a spectral curve Y defines a continuous one-parameter family of Riemann surfaces $Y(t)$ with $Y(0) = Y$.

[13]More exactly: we consider aquivalence classes with respect to Möbius transformations of the type $\lambda \mapsto e^{i\phi}\lambda$ because the solution u associated to a spectral curve is invariant under their actions.

We are interested in infinitesimal variations and restrict ourselves to first order variations

$$\left.\frac{\partial Y(t)}{\partial t}\right|_{t=0} = \frac{\partial Y}{\partial t}$$

only. The left hand side will be denoted by $\dot{Y}$ hereafter. The interesting question is how many linear independent flows exist and how can they be described.

Since a spectral curve is entirely determined by the functions μ and λ we will use the following **deformation equation**

$$\left.\frac{\partial p}{\partial t}\right|_{t=0} dq - \left.\frac{\partial q}{\partial t}\right|_{t=0} dp, \tag{D}$$

where p is defined as $\ln \mu$ and q as $\ln \lambda$.

This equation defines a 1-form ω on Y which satisfies

$$\overline{\eta^* \omega} = -\omega.$$

Conversely specifying such a form ω gives us a certain deformation. Such forms describe the tangent space of the moduli space $\mathcal{M}_g(\tau)$ or $\mathcal{M}_g$ at a fixed point.
The equation describes normal variation only. Since we are interested in non-isospectral deformations, tangent variations have no use for us.

Remark 4.1.3. *The deformation equation* (D) *is different compared to [GS95]. Here* $\ln \lambda$ *is used instead of* λ*.*

Next we want to investigate the properties of the deformation to be able to describe flows corresponding to different deformations.

4.2 Properties of the deformation

Considering (D) to define a deformation of the eigenvalue curve Γ^* helps to understand the deformations using the following result:

Proposition 4.2.1. *There holds*

$$\left.\frac{\partial p}{\partial t}\right|_{t=0} dq - \left.\frac{\partial q}{\partial t}\right|_{t=0} dp = -\frac{1}{\mu}\left.\frac{\partial R}{\partial t}\right|_{t=0} \frac{\partial R}{\partial \mu}^{-1} dq = \frac{1}{\lambda}\left.\frac{\partial R}{\partial t}\right|_{t=0} \frac{\partial R}{\partial \lambda}^{-1} dp.$$

Proof. Let $R(\lambda, \mu)$ and $R(\lambda(t), \mu(t), t)$ be the functions defining the surfaces Γ^* and $\Gamma^*(t)$ respectively. Using the total derivative of $R(\lambda, \mu)$

$$\left.\frac{\partial R}{\partial \lambda} d\lambda + \frac{\partial R}{\partial \mu} d\mu\right|_{\Gamma^*} = 0$$

we get

$$d\mu = -\frac{\partial R}{\partial \mu}\frac{\partial R}{\partial \lambda}^{-1} d\lambda \text{ and } d\lambda = -\frac{\partial R}{\partial \lambda}\frac{\partial R}{\partial \mu}^{-1} d\mu. \tag{$*$}$$

Using this result and the total derivative of $R(\lambda(t), \mu(t), t)$

$$\left.\frac{\partial R}{\partial \lambda}\dot{\lambda} dt + \frac{\partial R}{\partial \mu}\dot{\mu} dt + \dot{R} dt\right|_{\Gamma^*(t)} = 0 \tag{$\sharp$}$$

yields

$$\begin{aligned}
\dot{p} dq - \dot{q} dp &= \frac{1}{\mu\lambda}\left(\dot{\mu} d\lambda - \dot{\lambda} d\mu\right) \\
&\overset{(*)}{=} \frac{1}{\mu\lambda}\left(\dot{\mu} d\lambda + \dot{\lambda}\frac{\partial R}{\partial \lambda}\frac{\partial R}{\partial \mu}^{-1} d\lambda\right) \\
&= \frac{1}{\mu\lambda}\frac{\partial R}{\partial \mu}^{-1}\left(\frac{\partial R}{\partial \mu}\dot{\mu} + \frac{\partial R}{\partial \lambda}\dot{\lambda}\right) d\lambda \\
&\overset{(\sharp)}{=} -\frac{\dot{R}}{\mu}\frac{\partial R}{\partial \mu}^{-1} dq.
\end{aligned}$$

Substituting $d\lambda$ instead of $d\mu$ in the first step leads to the second part of the statement.

■

This result is the reason why we can choose $\dot{q} = \dot{\lambda}/\lambda = 0$ or $\dot{p} = \dot{\mu}/\mu = 0$ in the deformation equation. If we choose the map μ to be independent of t then λ becomes a multi-valued function of μ and t (and without restriction vice versa). We can consider the t-family of surfaces as a covering of some subset of $\mathbb{C}\times\mathbb{R}$ containing tuples (λ, t). Using this identification it is possible to compare surfaces corresponding to different values of t.

A proper deformation has to respect the conditions (i)-(iv) from the definition of the spectral data (4.1.1). The first two are not problematic because deformations are compatible with the involutions' action. The closing condition is crucial for the existence of the differential $d\ln\mu$ and motivates the following definition. The last condition will be treated later. Luckily, the deformation equation implies the following result.

Proposition 4.2.2. *Every meromorphic form ω defines a deformation that preserves the closing condition from 4.1.1.*

Proof. It is necessary to show that the first order variation of $d(p(t))$'s periods vanish. We consider a deformation described by a certain form ω. We choose $\dot{\lambda} = 0$ leading to $\dot{p} = \frac{\omega}{dq}$. The map $\dot{p}$ is a single-valued meromorphic function as a quotient of 1-forms on a compact Riemann surface. For an arbitrary cycle c a simple calculation then shows

$$\frac{\partial}{\partial t}\left(\int_c d(p(t))\right)\Big|_{t=0} = \int_c d\dot{p} = 0,$$

since $\dot{p}$ is a primitive of $d\dot{p}$.

■

But not every meromorphic form is admissible, because other properties of dp should be preserved too. For this reason we are interested in the precise properties of the map $\dot{p}$.

Proposition 4.2.3. *The map $\dot{p}$ is a meromorphic function and has the form*

$$\dot{p} = \frac{c}{\nu},$$

where c is a complex polynomial with degree less or equal to $g+1$.

Proof. This proof is based on arguments presented in [HKS12], Section 9. The map $\dot{p}$ is a single-valued meromorphic function as shown in the previous proof. It attains values in $\{i\pi k, k \in \mathbb{Z}\}$ at the branch points α_j, hence using the local chart $\sqrt{\lambda - \alpha_j}$ we can write

$$p(\lambda) = p_{\alpha_j}(\lambda)\sqrt{\lambda - \alpha_j} + i\pi k, k \in \mathbb{Z}$$

with $p_{\alpha_j}(\alpha_j) \neq 0$ in some neighborhood of α_j. Differentiating yields

$$\dot{p}(\lambda) = \dot{p}_{\alpha_j}(\lambda)\sqrt{\lambda - \alpha_j} - \frac{\dot{\alpha}_j p_{\alpha_j}(\lambda)}{2\sqrt{\lambda - \alpha_j}}.$$

That is the map $\dot{p}$ has poles at every α_j. The map p has poles at 0 and ∞ only. Since we can choose $\dot{\lambda} = 0$ the map $\dot{p}$ can't have higher order poles at these points (but they can be absent). In total we conclude that there holds $\dot{p} = \frac{c}{\nu}$, since the branch points α_j are the roots of the map ν as well as 0. Since ν has a pole of order $2g+1$ at ∞, the degree of the polynomial c is not greater than $g+1$.

■

The polynomial c is real in the following sense:

Proposition 4.2.4. *For the polynomials c there holds the following* ***reality condition***

$$c(-\bar{\lambda}^{-1}) = \bar{\lambda}^{-(g+1)}\overline{c(\lambda)}. \tag{4.1}$$

Proof. Applying η to $\dot{p}$ yields the claim:

$$\overline{\eta^*\dot{p}} = \frac{\overline{\eta^* c}}{\lambda^{-g-1}\nu} \stackrel{!}{=} \dot{p} = \frac{c}{\nu}.$$

■

Using the reality condition and equating the coefficients lead to the following result.

Proposition 4.2.5. *For the coefficients of the polynomial* $c(\lambda) = \sum_{i=0}^{g+1} c_i \lambda^i$ *there holds*

$$c_n = \begin{cases} \bar{c}_{g+1-n} & , n \text{ even} \\ -\bar{c}_{g+1-n} & , n \text{ odd.} \end{cases}$$

This result enables us to investigate the regularity of the form $\dot{p}dq$ in a more precise way:

Proposition 4.2.6. *The form* $\dot{p}dq$ *has double poles over* 0 *and* ∞ *in case* $\deg c = g+1$ *and is holomorphic if and only if* $\deg c \leq g$*.*

Proof. Let n be the degree of the polynomial c and $\gamma_1, \dots, \gamma_n$ its roots. Then the divisor of $\dot{p}dq$ (as a map in λ) is

$$(\dot{p}dq) = 2\sum_{j=1}^{n} \gamma_j - 2 \cdot 0 + (2g - 2n)\infty$$

The maximum degree of c is $g+1$, then there is a double pole over ∞. Because of the reality condition, there holds $c_0 = \overline{c}_{g+1}$, hence c has roots at 0 if and only if its degree is less than $g+1$. In case $\deg c \leq g$ all the poles at ∞ are cancelled with the result that $\dot{p}dq$ is free from singularities.

■

Now we see that deformations are completely described by 1-forms with prescribed singularities or equivalently by polynomials c with certain properties. Such polynomials c determine a deformation completely. Using the polynomial c can often be more convenient than working with the form ω. One important tool for this approach is introduced in the following proposition.

Proposition 4.2.7 (Compatibility condition)**.** *For the function* p *there holds* $\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial \lambda}\right)\Big|_{t=0} = \frac{\partial \dot{p}}{\partial \lambda}$ *if and only if the polynomials* a*,* b *and* c *obey the so-called* ***Whitham equation***

$$2a\dot{b} - \dot{a}b = 2\lambda a c' - ac - \lambda a' c. \tag{W}$$

Proof. We compute both sides of the equation:

$$\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial \lambda}\right)\Big|_{t=0} = \frac{\partial}{\partial t}\left(\frac{b}{\nu\lambda}\right)\Big|_{t=0} \overset{\dot{\lambda}=0}{=} \frac{\dot{b}\nu\lambda - \dot{\nu}\lambda b}{\nu^2\lambda^2} \overset{\nu=\sqrt{\lambda a}}{=} \frac{2\dot{b}a\lambda^2 - \lambda^2\dot{a}b}{2\nu^3\lambda^2} = \frac{2\dot{b}a - \dot{a}b}{2\nu^3}$$
$$\frac{\partial \dot{p}}{\partial \lambda} = \frac{\partial}{\partial \lambda}\left(\frac{c}{\nu}\right) = \frac{c'\nu - \nu' c}{\nu^2} = \frac{2c'a\lambda - ac - \lambda a' c}{2\nu^3}.$$

Equating both expressions completes the proof.

■

If there are no common roots of a and b, then the Whitham equation (W) determines the values of $\dot{a}$ and $\dot{b}$ at the roots α_j and β_k via the formulae

$$\dot{a}(\alpha_j) = \frac{\alpha_j a'(\alpha_j)c(\alpha_j)}{b(\alpha_j)} \text{ and} \tag{4.2}$$

$$\dot{b}(\beta_k) = \frac{2\beta_k a(\beta_k)c'(\beta_k) - a(\beta_k)c(\beta_k) - \beta_k a'(\beta_k)c(\beta_k)}{2a(\beta_k)}, \tag{4.3}$$

with the result that (W) can be uniquely solved. In case the polynomial b has higher order roots we have to consider its higher order derivatives at β_j too.
Using the Whitham equation, we obtain the following result regarding $\dot{\tau}$:

Proposition 4.2.8. *Holomorphic forms ω describe deformations such that the period τ of the associated solution u is preserved.*

Proof. We want to express $\dot{\tau}$ in terms of the polynomials a, b and c. Because of (3.19) we obtain

$$\frac{-i\dot{\tau}}{2} = \frac{\partial}{\partial t}\left(\frac{b_0}{\sqrt{a_0}}\right)\Bigg|_{t=0} \overset{\text{(W)}}{=} -\frac{c_0}{2\sqrt{a_0}}.$$

By virtue of Proposition 4.2.4, there holds $c_0 = \bar{c}_{g+1}$, hence deformations corresponding to polynomials c with $\deg c < g+1$ do not change the period τ infinitesimally. These polynomials define holomorphic forms due to Proposition 4.2.6.

∎

4.3 Increasing the genus

Deformations can also be used to increase the genus of an existing spectral curve. This procedure is sometimes called “opening the double points”. We will sketch the necessary steps. In order to increase the genus from g to $g+2$ for a fixed $(a, b) \in \mathcal{M}_g$, the following operations need to be performed:

- choose $\alpha^* \in \mathbb{C}^*$ such that $a(\alpha^*) \neq 0$ and $\mu(\alpha^*) = \pm 1$
- add the factor $(\lambda - \alpha^*)^2(\lambda - \eta(\alpha^*))^2$ to the polynomial a

- add the factor $(\lambda - \alpha^*)(\lambda - \eta(\alpha^*))$ to the polynomial b
- choose a polynomial c with $\deg c = g + 3$ and $c(\alpha^*) \neq 0$.

Note that the map μ is not changed by these modifications. Recall that the Whitham equation (W) is not uniquly solvable anymore. For details cf. [HKS12], Section 9.2.

4.4 Parametrization of the moduli space

4.4.1 General set of parameters

In this section we present a general set of parameters of the moduli space. We choose a canonical basis (a, b) of $H_1(Y, \mathbb{Z})$. Let $\omega_1, \ldots, \omega_g$ be the corresponding canonical basis of $\Omega(Y)$. These forms define g deformations via the equation (D). We choose $\dot{\lambda} = 0$ yielding

$$\left.\frac{\partial p}{\partial t_j}\right|_{t_j=0} dq = \omega_j. \tag{4.4}$$

Since ω_j are holomorphic we have deformations which preserve the period τ. By definition, these parameters t_j are g independent variables. For that reason, the space of such deformations is of real dimension g. Since these deformations describe the tangent space of $\mathcal{M}_g(\tau)$ at a fixed point, the moduli space itself can be considered as a g-dimensional subset of the space of all real polynomials a and b. If we take meromorphic forms ω (with singularities as described above), then we obtain $g + 1$ parameters.

The parameters t_j are given only locally since they are defined only in some neighborhood of Y in $\mathcal{M}_g(\tau)$. Due to the fact that ω_j is a canonical basis, the parameters t_j can be given explicitly:

Proposition 4.4.1. *The parameters t_j are (locally) given by the formula*

$$t_j = \frac{1}{2\pi i} \int_{a_j} p dq, \; j \in \{1, \ldots, g\}.$$

Proof. A simple calculation shows

$$\left.\frac{\partial t_j}{\partial t_k}\right|_{t_k=0} = \frac{\partial}{\partial t_k}\left(\frac{1}{2\pi i}\int_{a_j} pdq\right)\Bigg|_{t_k=0} = \int_{a_j}\left.\frac{\partial p}{\partial t_k}\right|_{t_k=0} dq = \int_{a_j}\omega_k = \delta_{jk}.$$

■

As mentioned in [GS95] the coordinates t_j are action variables and the associated flows commute. Unfortunately these parameters are rather complicated since they are expressed by some integrals.

4.4.2 More elegant set of parameters

We introduce a different set of (local) parameters. We will use the values of the trace function Δ of the monodromy at the roots of its derivative Δ' i. e. the roots β_k of the polynomial b for some fixed point in the moduli space. Such coordinates $t_k := \Delta(\beta_k)$ were first discussed in [MO75]. These will be parameters of $\mathcal{M}_g$, since the corresponding flows change the period of the solution τ.

We assume that all β_k are distinct so that we have a complete set of $g + 1$ parameters and reorder the roots of the polynomial b such that $\eta(\beta_k) = \beta_{k+1}$ for $k \in \{1, \ldots, g\}$. Since all the roots β_k are clustered in pairs with their η-cousins, we can only achieve that a parameter t_k is independent of the parameters induced by the remaining β_l with the exception of its own η-cousin. This fact leads to a polynomial c vanishing at all roots of b except for β_k and β_{k+1}.

The positive side effect is that these deformations are also very suitable for numerical computations. Choosing $\dot{p} = 0$ and rearranging the terms of the deformation equation (D) leads to

$$\dot{\lambda} = -\frac{\lambda\omega}{dp}.$$

This equation holds away from the poles of the right hand side. Since we already know that $\omega = \frac{c}{\nu}dq$ from Proposition 4.2.3, the last equation is equivalent to

$$\dot{\lambda} = -\frac{\lambda c(\lambda)}{b(\lambda)}. \tag{4.5}$$

The problematic points are the roots β_k of b such that $c(\beta_k) \neq 0$. Using (4.5) we can now precisely state how the roots α_j of a and β_k of b are changed by a certain deformation. The polynomial c has to be a minor modification of the polynomial b because of the behavior at the roots β_k as requested above. This leads to a quite simple right hand side of (4.5).

Because the polynomial c has to obey the reality condition

$$c(-\bar{\lambda}^{-1}) = \bar{\lambda}^{-(g+1)}\overline{c(\lambda)}$$

and there holds

$$\overline{\eta^*\left(\frac{1}{\lambda - \beta_k}\right)} = \frac{1}{-\lambda^{-1} - \bar{\beta}_k} = -\frac{\lambda}{1 + \lambda\bar{\beta}_k},$$

we define c as

$$c(\lambda) = \frac{i\lambda}{(\lambda - \beta_k)(1 + \lambda\bar{\beta}_k)} b(\lambda)$$

for a fixed root β_k of b for the moment. This way we obtain $\frac{g+1}{2}$ holomorphic forms

$$\omega_k = \frac{i\lambda}{(\lambda - \beta_k)(1 + \lambda\bar{\beta}_k)} dp,$$

that is the induced deformations do not alter the period τ of the solution u.

Due to the reality condition, the forms ω_k can be multiplied by real constants only. For that reason we have only a partial set of g parameters of $\mathcal{M}_g(\tau)$ and the normalization $\frac{dt_k}{dt_k} = 1$ cannot be achieved.

Both of these problems can be solved using the ansatz that for any function f in λ the expression $f - \overline{\eta^* f}$ changes the sign by the action of the involution η which fixes the reality condition of b. Following this idea yields the polynomial

$$c_k(\lambda) := \left(\frac{\gamma_k}{\lambda - \beta_k} + \frac{\bar{\gamma}_k\lambda}{1 + \bar{\beta}_k\lambda}\right) b(\lambda), \tag{4.6}$$

where γ_k is some complex constant. It will be used to normalize the parameter t_k.

The degree of the polynomials c_k is $g+1$ which means $\dot{\tau} \neq 0$. Because the space of differentials is a real space, we have $g+1$ linear independent forms and therefore a complete set of parameters of $\mathcal{M}_g$.

Now it is easy to show that the parameters t_k and t_{k+1} do not depend on any other parameter as intended.

Proposition 4.4.2. *There holds* $\frac{dt_k}{dt_l}\Big|_{t_l=0} = 0$ *for all* $l \neq k, k+1$.

Proof.

$$\frac{dt_k}{dt_l}\bigg|_{t_l=0} = \frac{d(2\cosh(p(\beta_k)))}{dt_l}\bigg|_{t_l=0} = 2\sinh(p(\beta_k))\,\frac{\partial p(\beta_k)}{\partial t_l}\bigg|_{t_l=0}$$
$$= 2\sinh(p(\beta_k))\frac{c_l(\beta_k)}{\nu(\beta_k)}$$

Since c_l vanishes for all β except for β_l and β_{l+1}, the claim is proven. ∎

Using the last expression, it is also possible to determine the value of the normalization constant γ_k.

Proposition 4.4.3. *For the coordinate* $t_k := \Delta(\beta_k)$ *there holds* $\frac{dt_k}{dt_k} = 1$ *if the corresponding polynomial* c_k *is normalized by setting*

$$\gamma_k = \frac{\nu(\beta_k)}{2\sinh(p(\beta_k))}\frac{1}{b'(\beta_k)}.$$

Proof.

$$\frac{dt_k}{dt_k}\bigg|_{t_k=0} = 2\sinh(p(\beta_k))\frac{c_k(\beta_k)}{\nu(\beta_k)} \stackrel{!}{=} 1 \iff$$
$$\gamma_k = \frac{\nu(\beta_k)}{2\sinh(p(\beta_k))}\; b_{g+1}^{-1}\prod_{\substack{l=1\\ l\neq k}}^{g+1}(\beta_k - \beta_l)^{-1}$$

Additionally we can write the last expression using the derivative of b due to

$$b'(\lambda) = b_{g+1}\sum_{l=1}^{g+1}\prod_{\substack{m=1\\ m\neq l}}^{g+1}(\lambda - \beta_m) = \sum_{l=1}^{g+1}\frac{b(\lambda)}{\lambda - \beta_l}$$

as $b'(\beta_k)^{-1}$.

■

Because of the reality condition $\eta^*\Delta = \bar{\Delta}$ there also holds $\bar{t}_k = t_{k+1}$ leading to $\gamma_k = \bar{\gamma}_{k+1}$ and $\frac{d\bar{t}_{k+1}}{dt_k} = 1$. In this sense the coordinates are complex.

Remark 4.4.4. *If we choose a point (a, b) in $\mathcal{M}_g$ such that b has higher order roots then this situation corresponds to a point such that some of the parameters t_k coincide at this particular point. Since we can move the t_k separately, we can find a polynomial b with only simple roots within a small neighborhood of (a, b).*
These parameters are local coordinates of $\mathcal{M}_g$ as mentioned above. They can be turned into global coordinates if we add some information about the covering of $\mathcal{M}_g$ described by the map Δ as discussed in the third part of [GS95]. This information, the so-called glueing rules, are represented as a graph and encode which sheets of the covering meet at which branch point. The latter are exactly the parameters t_k.

Using these polynomials, we can state the formulae which describe the deformations of the roots of a and b suitable for simulations.

Proposition 4.4.5. *A deformation defined by a polynomial c_k as in* (4.6) *leads to the following variations of the roots of the polynomials a and b:*

$$\dot{\alpha}_j = -\alpha_j \left(\frac{\gamma_k}{\alpha_j - \beta_k} + \frac{\bar{\gamma}_k \alpha_j}{1 + \bar{\beta}_k \alpha_j} \right), \ \forall j \in \{1, \dots, 2g\}$$

$$\dot{\beta}_l = -\beta_l \left(\frac{\gamma_k}{\beta_l - \beta_k} + \frac{\bar{\gamma}_k \beta_l}{1 + \bar{\beta}_k \beta_l} \right), \ \forall l \in \{1, \dots, g+1\}, l \neq k, k+1$$

$$\dot{\beta}_k = \frac{\gamma_k}{2} \left(-2\beta_k \sum_{\substack{l=1 \\ l \neq k}}^{g+1} \frac{1}{\beta_k - \beta_l} + 1 + \beta_k \sum_{j=1}^{2g} \frac{1}{\beta_k - \alpha_j} \right) - \bar{\gamma}_k \frac{\beta_k^2}{1 + |\beta_k|^2}$$

$$\dot{\beta}_{k+1} = -\gamma_k \frac{\beta_{k+1}}{\beta_{k+1} - \beta_k} - \frac{\bar{\gamma}_k \beta_{k+1}^2}{2} \left(-2\beta_{k+1} \sum_{\substack{l=1 \\ l \neq k+1}}^{g+1} \frac{1}{\beta_{k+1} - \beta_l} - 1 + \beta_{k+1} \sum_{j=1}^{2g} \frac{1}{\beta_{k+1} - \alpha_j} \right).$$

Before proving these results we want to reformulate the last two statements. We split the polynomial c_k into two parts:

$$c_k(\lambda) = c_k^{(1)}(\lambda) + c_k^{(2)}(\lambda),$$

and denote by t_1 and t_2 the parameters induced by the polynomials $c_k^{(1)}$ and $c_k^{(2)}$ respectively. Then there holds

$$\dot{\beta}_k = \frac{\gamma_k}{2} - \beta_k \sum_{\substack{l=1 \\ l\neq k}}^{g+1} \frac{\partial_{t_1}\beta_l|_{t_1=0}}{\beta_l} + \frac{\beta_k}{2}\sum_{j=1}^{2g}\frac{\partial_{t_1}\alpha_j|_{t_1=0}}{\alpha_j} + \left.\frac{\partial\beta_k}{\partial t_2}\right|_{t_2=0}$$

$$\dot{\beta}_{k+1} = \left.\frac{\partial\beta_{k+1}}{\partial t_1}\right|_{t_1=0} + \frac{\bar{\gamma}_k\beta_{k+1}^2}{2} + \beta_{k+1}^2 \sum_{\substack{l=1 \\ l\neq k+1}}^{g+1} \frac{\partial_{t_2}\beta_l|_{t_2=0}}{\beta_l^2} + \frac{\beta_{k+1}}{2}\sum_{j=1}^{2g}\frac{\partial_{t_2}\alpha_j|_{t_2=0}}{\alpha_j^2}.$$

These formulae are similar to equation (13) in [GS95] although there, in case of the KdV equation, it originates from fact that the quantity $\sum\alpha_j - 2\sum\beta_k$ is independent of t.

Proof. First two statements are direct applications of (4.5). The last two can't be determined in the same way since the right hand side of the formula has a pole. Hence, we need an expression which contains $\dot{\beta}_k$ and not $\dot{\beta}_{k+1}$. For the derivative with respect to some parameter t there holds:

$$\begin{aligned}
\dot{b}(\lambda) &= \left.\frac{\partial}{\partial t}\left(b_{g+1}\prod_{l=1}^{g+1}(\lambda-\beta_l)\right)\right|_{t=0} \\
&= \dot{b}_{g+1}\prod_{l=1}^{g+1}(\lambda-\beta_l) + b_{g+1}\sum_{l=1}^{g+1}(-\dot{\beta}_l)\prod_{\substack{m=1 \\ m\neq l}}^{g+1}(\lambda-\beta_m) \\
&= \dot{b}_{g+1}\prod_{l=1}^{g+1}(\lambda-\beta_l) - \sum_{l=1}^{g+1}\dot{\beta}_l\frac{b(\lambda)}{\lambda-\beta_l}.
\end{aligned}$$

This leads to $\dot{b}(\beta_k) = -\dot{\beta}_k b'(\beta_k)$ for some root β_k. Plugging it into the Whitham equation (W) yields

$$\begin{aligned}
\dot{\beta}_k &= \frac{1}{2b'(\beta_k)}\left(-2\beta_k c_k'(\beta_k) + c_k(\beta_k) + \beta_k c_k(\beta_k)\frac{a'(\beta_k)}{a(\beta_k)}\right) \qquad (*) \\
&= \frac{1}{2b'(\beta_k)}\left(-2\beta_k(c_k^{(1)\prime}(\beta_k) + c_k^{(2)\prime}(\beta_k)) + c_k^{(1)}(\beta_k) + \beta_k c_k^{(1)}(\beta_k)\frac{a'(\beta_k)}{a(\beta_k)}\right).
\end{aligned}$$

We now split the polynomial c_k into two parts:

$$c_k(\lambda) =: c_k^{(1)}(\lambda) + c_k^{(2)}(\lambda)$$

and perform some auxiliary calculations (most of them are straightforward, sometimes L'Hôpital's rule is needed):

$$\begin{aligned}
c_k(\beta_k) &= c_k^{(1)}(\beta_k) = \gamma_k b'(\beta_k) \\
c_k'(\beta_k) &= c_k^{(1)\prime}(\beta_k) + c_k^{(2)\prime}(\beta_k) = \gamma_k \frac{b''(\beta_k)}{2} + \bar{\gamma}_k \frac{\beta_k b'(\beta_k)}{1+|\beta_k|^2} \\
a'(\lambda) &= \frac{\partial}{\partial\lambda}\left(a_{2g}\prod_{j=1}^{2g}(\lambda-\alpha_j)\right) = a_{2g}\sum_{j=1}^{2g}\prod_{\substack{k=1\\k\neq j}}^{2g}(\lambda-\alpha_k) \\
\frac{a'(\lambda)}{a(\lambda)} &= \sum_{j=1}^{2g}\frac{1}{\lambda-\alpha_j} \\
\frac{a'(\beta_k)}{a(\beta_k)} &= \sum_{j=1}^{2g}\frac{1}{\beta_k-\alpha_j} = -\frac{1}{\gamma_k}\sum_{j=1}^{2g}\frac{c_k^{(1)}(\alpha_j)}{b(\alpha_j)} = \frac{1}{\gamma_k}\sum_{j=1}^{2g}\frac{\partial_{t_1}\alpha_j|_{t_1=0}}{\alpha_j} \\
b'(\lambda) &= b_{g+1}\sum_{l=1}^{g+1}\prod_{\substack{m=1\\m\neq l}}^{g+1}(\lambda-\beta_m) = \sum_{l=1}^{g+1}\frac{b(\lambda)}{\lambda-\beta_l} \\
b''(\lambda) &= \sum_{l=1}^{g+1}\frac{b'(\lambda)(\lambda-\beta_l)-b(\lambda)}{(\lambda-\beta_l)^2} \\
b''(\beta_k) &= 2\sum_{\substack{l=1\\l\neq k}}^{g+1}\frac{b'(\beta_k)}{\beta_k-\beta_l} \\
\frac{b''(\beta_k)}{b'(\beta_k)} &= 2\sum_{\substack{l=1\\l\neq k}}^{g+1}\frac{1}{\beta_k-\beta_l} = -\frac{2}{\gamma_k}\sum_{\substack{l=1\\l\neq k}}^{g+1}\frac{c_k^{(1)}(\beta_l)}{b(\beta_l)} = \frac{2}{\gamma_k}\sum_{\substack{l=1\\l\neq k}}^{g+1}\frac{\partial_{t_1}\beta_l|_{t_1=0}}{\beta_l}
\end{aligned}$$

We can now return to $(*)$ and get:

$$\dot{\beta}_k = \frac{1}{2b'(\beta_k)}\left(-2\beta_k {c_k}'(\beta_k) + c_k(\beta_k) + \beta_k c_k(\beta_k)\frac{a'(\beta_k)}{a(\beta_k)}\right)$$

$$= \frac{1}{2b'(\beta_k)} \left(-2\beta_k (c_k^{(1)\prime}(\beta_k) + c_k^{(2)\prime}(\beta_k)) + c_k^{(1)}(\beta_k) + \beta_k c_k^{(1)}(\beta_k) \sum_{j=1}^{2g} \frac{1}{\beta_k - \alpha_j} \right)$$
$$= \frac{\gamma_k}{2} \left(-2\beta_k \sum_{\substack{l=1 \\ l\neq k}}^{g+1} \frac{1}{\beta_k - \beta_l} + 1 + \beta_k \sum_{j=1}^{2g} \frac{1}{\beta_k - \alpha_j} \right) - \bar{\gamma}_k \frac{\beta_k^2}{1 + |\beta_k|^2}$$
$$= \frac{\gamma_k}{2} - \beta_k \sum_{\substack{l=1 \\ l\neq k}}^{g+1} \frac{\partial_{t_1}\beta_l|_{t_1=0}}{\beta_l} + \frac{\beta_k}{2} \sum_{j=1}^{2g} \frac{\partial_{t_1}\alpha_j|_{t_1=0}}{\alpha_j} + \left.\frac{\partial \beta_k}{\partial t_2}\right|_{t_2=0}$$

The computation of $\dot{\beta}_{k+1}$ is quite similar. Using

$$\frac{\lambda}{1+\bar{\beta}_k\lambda} = \frac{1}{\lambda^{-1}+\bar{\beta}_k} = \frac{1}{\lambda^{-1} - \beta_{k+1}^{-1}} = -\frac{\beta_{k+1}\lambda}{\lambda - \beta_{k+1}}$$
$$c_k(\beta_{k+1}) = c_k^{(2)}(\beta_{k+1}) = -\bar{\gamma}_k \beta_{k+1}^2 b'(\beta_{k+1})$$
$$c_k{}'(\beta_{k+1}) = \gamma_k \frac{b'(\beta_{k+1})}{\beta_{k+1} - \beta_k} - \frac{\bar{\gamma}_k\beta_{k+1}}{2}(2b'(\beta_{k+1}) + \beta_{k+1}b''(\beta_{k+1}))$$
$$\frac{a'(\beta_{k+1})}{a(\beta_{k+1})} = \sum_{j=1}^{2g} \frac{1}{\beta_{k+1} - \alpha_j} = \frac{1}{\bar{\gamma}_k\beta_{k+1}} \sum_{j=1}^{2g} \frac{c_k^{(2)}(\alpha_j)}{\alpha_j b(\alpha_j)} = -\frac{1}{\bar{\gamma}_k\beta_{k+1}} \sum_{j=1}^{2g} \frac{\partial_{t_2}\alpha_j|_{t_2=0}}{\alpha_j^2}$$
$$\frac{b''(\beta_{k+1})}{b'(\beta_{k+1})} = \sum_{\substack{l=1 \\ l\neq k+1}}^{g+1} \frac{2}{\beta_{k+1} - \beta_l} = \frac{2}{\bar{\gamma}_k\beta_{k+1}} \sum_{\substack{l=1 \\ l\neq k+1}}^{g+1} \frac{c_k^{(2)}(\beta_l)}{\beta_l b(\beta_l)} = \frac{-2}{\bar{\gamma}_k\beta_{k+1}} \sum_{\substack{l=1 \\ l\neq k+1}}^{g+1} \frac{\partial_{t_2}\beta_l|_{t_2=0}}{\beta_l^2},$$

we get

$$\dot{\beta}_{k+1} = \frac{1}{2b'(\beta_{k+1})} \left(-2\beta_{k+1} c_k{}'(\beta_{k+1}) + c_k(\beta_{k+1}) + \beta_{k+1} c_k(\beta_{k+1}) \frac{a'(\beta_{k+1})}{a(\beta_{k+1})} \right)$$
$$= -\frac{1}{2b'(\beta_{k+1})} \Bigg(-\beta_{k+1}^2 \bar{\gamma}_k (2b'(\beta_{k+1}) + \beta_{k+1}b''(\beta_{k+1})) + \bar{\gamma}_k \beta_{k+1}^2 b'(\beta_{k+1})$$
$$+ \bar{\gamma}_k \beta_{k+1}^3 b'(\beta_{k+1}) \sum_{j=1}^{2g} \frac{1}{\beta_{k+1} - \alpha_j} \Bigg) + \frac{1}{b'(\beta_{k+1})} \left(\gamma_k \frac{\beta_{k+1} b'(\beta_{k+1})}{\beta_{k+1} - \beta_k} \right)$$
$$= -\gamma_k \frac{\beta_{k+1}}{\beta_{k+1} - \beta_k}$$

$$- \frac{\bar{\gamma}_k \beta_{k+1}^2}{2} \left(-2\beta_{k+1} \sum_{\substack{l=1 \\ l \neq k+1}}^{g+1} \frac{1}{\beta_{k+1} - \beta_l} - 1 + \beta_{k+1} \sum_{j=1}^{2g} \frac{1}{\beta_{k+1} - \alpha_j} \right)$$

$$= \left. \frac{\partial \beta_{k+1}}{\partial t_1} \right|_{t_1=0} + \frac{\bar{\gamma}_k \beta_{k+1}^2}{2} + \beta_{k+1}^2 \sum_{\substack{l=1 \\ l \neq k+1}}^{g+1} \frac{\partial_{t_2} \beta_l |_{t_2=0}}{\beta_l^2} + \frac{\beta_{k+1}}{2} \sum_{j=1}^{2g} \frac{\partial_{t_2} \alpha_j |_{t_2=0}}{\alpha_j^2}.$$

■

5 Solutions in terms of Baker-Akhiezer functions

In this chapter we want to express solutions u of the cosh-Gordon equation (CG) in terms of Baker-Akhiezer functions. Afterwards, we will discuss the singularities of these solutions.

A solution u of the cosh-Gordon equation is described by a spectral curve Y and a divisor on this curve. The spectral curve was the object of interest in the chapters 3 and 4, now we address ourselves to the divisor because with its help we can define a Baker-Akhiezer function. Before defining the Baker-Akhiezer function in the classical way, i.e. by prescribing its analytical properties (cf. Definition 5.2.3), we will derive these properties by introducing the Baker-Akhiezer function as the eigenvector function of the monodromy $M(\lambda, z)$ and a solution of a certain differential equation (cf. Definition 5.2.1). To distinguish these two functions rigorously we will call the latter one the Pseudo-Baker-Akhiezer function. We will refrain from using theta functions to express the Baker-Akhiezer function. This approach was covered in several articles by Babich ([Bab91a], [Bab91b]) in quite a thorough way. But we will use results from these papers to discuss the singularities of u in the last section.

5.1 Monodromy's eigenvectors and the associated divisor

We start with the monodromy $M(\lambda, z)$ as given in Proposition 3.2.2. The eigenvectors of $M(\lambda, z)$ depend on the spatial variable z in contrast to its eigenvalues (cf. Section 3.2). We will use the eigenvectors to define the divisor mentioned above.

We start the investigation with the eigenvectors of $M(\lambda)$. The eigenspace of $M(\lambda)$ belonging to the eigenvalue μ is denoted by $\mathrm{Eig}(M(\lambda), \mu)$ and its elements by $v(\lambda, \mu)$ hereafter. Besides the standard eigenvector v we are also considering "the transposed" eigenvector w^T, which satisfies $w^T M(\lambda) = \mu w^T$. We will call w^T the dual eigenvector. The latter equation is equivalent to $w \in \mathrm{Eig}(M(\lambda)^T, \mu)$.

Remark 5.1.1. *For all points $y = (\lambda, \nu)$ of the spectral curve Y, the spaces* $\mathrm{Eig}(M(\lambda), \mu(\lambda))$ *are one-dimensional. Away from the branch points there are two distinct eigenvalues μ and μ^{-1} and therefore all corresponding eigenspaces have dimension one. At a branch point there is only a single eigenvalue but the monodromy fails to semisimple, i.e. the eigenspace is one-dimensional as well. Since Y is smooth there are no points left.*

We can specify the eigenvectors of the monodromy $M(\lambda)$ in terms of its entries. We assume the monodromy has the following general form:

$$M(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix}.$$

Consider μ to be an eigenvalue of $M(\lambda)$ and $\tilde{v} = (\alpha, \beta)^T$ is the corresponding eigenvector with α and β being functions of λ. Then $\tilde{v}$ has to suffice the equality

$$\begin{pmatrix} \alpha a + \beta b \\ \alpha c + \beta d \end{pmatrix} = \begin{pmatrix} \mu\alpha \\ \mu\beta \end{pmatrix}.$$

Setting $\alpha = b$ and transforming the equations yield $\tilde{v} = (b, \mu - a)^T$. Normalizing $\tilde{v}$ gives us

$$v = \begin{pmatrix} 1 \\ \frac{\mu - a}{b} \end{pmatrix}. \tag{5.1}$$

It is possible to express $M(\lambda)$ in a more precise way:

Proposition 5.1.2. *The monodromy $M(\lambda)$ has the form*

$$M(\lambda) = \begin{pmatrix} a & b \\ -\eta^* b & \eta^* a \end{pmatrix}.$$

Proof. Due to the monodromy's reality condition 3.2.5 we have

$$\bar{M}(-\bar{\lambda}^{-1}) = (M(\lambda)^T)^{-1} \Leftrightarrow \begin{pmatrix} \overline{\eta^* a} & \overline{\eta^* b} \\ \overline{\eta^* c} & \overline{\eta^* d} \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

Since $\overline{\eta^* \overline{\eta^* f}} = f$, the last equality directly leads to the claim.

■

With this representation of $M(\lambda)$ and since the eigenspaces are one-dimensional, we obtain

$$v = \begin{pmatrix} 1 \\ \frac{\overline{\eta^* b}}{\overline{\eta^* a} - \mu} \end{pmatrix}.$$

Applying the same procedure allows us to state for the dual eigenvector:

$$w = \begin{pmatrix} 1 \\ \frac{a-\mu}{\overline{\eta^* b}} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{b}{\mu - \overline{\eta^* a}} \end{pmatrix}. \tag{5.2}$$

The eigenvectors define two divisors:

$$D := -(v) \text{ and } D^T := -(w^T). \tag{5.3}$$

Due to the normalization of the eigenvectors v and w^T, their principal divisors are equal to the negative of their polar divisors, i.e. $D = (v)_\infty$ and $D^T = (w^T)_\infty$.
Considering the formulae for v and w^T from above there holds:

$$\begin{aligned} D &= \sum_j (\lambda_j, \nu_j) \text{ with } b(\lambda) = 0 \text{ and } \overline{\eta^* a}(\lambda) = \mu(\lambda, \nu) \\ D^T &= \sum_j (\lambda_j, \nu_j) \text{ with } \overline{\eta^* b}(\lambda) = 0 \text{ and } a(\lambda) = \mu(\lambda, \nu). \end{aligned}$$

Proposition 5.1.3. *The three involutions transform the divisors D and D^T as follows*

$$\begin{aligned} \rho \circ D &= D^T \\ \eta \circ D &= D - (f), \ \textit{with } f\overline{\eta^* f} = -1 \\ \sigma \circ D &= D^T - \rho \circ (f), \end{aligned} \tag{5.4}$$

where f is a meromorphic function on Y.

The equation (5.4) is the **reality condition** of the divisor D. We call such divisors **quaternionic**[14]. To prove this proposition we need to know how involutions act on the eigenvectors first.

Lemma 5.1.4. *For an eigenvector* $v \in \operatorname{Eig}(M(\lambda), \mu)$ *there holds*

$$\begin{aligned}\sigma^* v &\in \operatorname{span}_{\mathbb{C}}(\sigma_2 w)\\ \rho^* v &\in \operatorname{span}_{\mathbb{C}}(\bar{w})\\ \eta^* v &\in \operatorname{span}_{\mathbb{C}}(\sigma_2 \bar{v}),\end{aligned}$$

where $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ *is the second Pauli matrix.*

Proof. Since all eigenspaces are one-dimensional, it is sufficient that the vectors on both sides are eigenvectors associated with the same eigenvalue.

The eigenvector $\sigma^* v = v(\lambda, \mu^{-1})$ is an element of $\operatorname{Eig}(M(\lambda)^{-1}, \mu)$. Using the monodromy's reality condition 3.2.5 and (3.7), we get $M(\lambda)^{-1} = \sigma_2 M^T(\lambda)\sigma_2$, which leads to

$$\sigma_2 M(\lambda)^T \sigma_2 v(\lambda, \mu^{-1}) = \mu v(\lambda, \mu^{-1}) \Leftrightarrow \ \sigma_2 v(\lambda, \mu^{-1}) \in \operatorname{Eig}(M(\lambda)^T, \mu).$$

For the vector $\rho^* v = v(-\bar{\lambda}^{-1}, \bar{\mu}^{-1})$ there holds

$$\begin{aligned}\bar{M}(-\bar{\lambda}^{-1})^{-1}\bar{v}(-\bar{\lambda}^{-1}, \bar{\mu}^{-1}) &= \overline{M(-\bar{\lambda}^{-1})^{-1} v(-\bar{\lambda}^{-1}, \bar{\mu}^{-1})}\\ &= \overline{\bar{\mu} v(-\bar{\lambda}^{-1}, \bar{\mu}^{-1})}\\ &= \mu \bar{v}(-\bar{\lambda}^{-1}, \bar{\mu}^{-1}).\end{aligned}$$

Plugging the reality condition in yields $\bar{v}(-\bar{\lambda}^{-1}, \bar{\mu}^{-1}) \in \operatorname{Eig}(M(\lambda)^T, \mu)$.

Now we look at $\eta^* v = v(-\bar{\lambda}^{-1}, \bar{\mu})$. There holds

$$\begin{aligned}\bar{M}(-\bar{\lambda}^{-1})\bar{v}(-\bar{\lambda}^{-1}, \bar{\mu}) &= \overline{M(-\bar{\lambda}^{-1}) v(-\bar{\lambda}^{-1}, \bar{\mu})} = \overline{\bar{\mu} v(-\bar{\lambda}^{-1}, \bar{\mu})}\\ &= \mu \bar{v}(-\bar{\lambda}^{-1}, \bar{\mu}).\end{aligned}$$

[14]The map f induces a quaternionic structure j on $H^0(Y, \mathcal{O}_D)$, i.e. an antilinear map with $j^2 = -1$ defined by $g \mapsto \frac{\overline{\eta^* g}}{f}$ (cf. [Hit90], p. 636).

Consequently we get

$$\begin{aligned}
&\mu\bar{v}(-\bar{\lambda}^{-1},\bar{\mu}) = \bar{M}(-\bar{\lambda}^{-1})\bar{v}(-\bar{\lambda}^{-1},\bar{\mu}) \overset{(3.7)}{=} \sigma_2 M(\lambda)\sigma_2\bar{v}(-\bar{\lambda}^{-1},\bar{\mu})\\
\Leftrightarrow\ &\mu\sigma_2\bar{v}(-\bar{\lambda}^{-1},\bar{\mu}) = M(\lambda)\sigma_2\bar{v}(-\bar{\lambda}^{-1},\bar{\mu})
\end{aligned}$$

which leads to $\sigma_2\bar{v}(-\bar{\lambda}^{-1},\bar{\mu}) \in \mathrm{Eig}(M(\lambda),\mu)$.

■

Proof of Proposition 5.1.3. First we know from the Lemma 5.1.4 and due to the normalization that $\rho^* v = \bar{w}$. Therefore $\rho \circ D = D^T$ holds.
The same lemma states that in a point $y \in Y$ there holds $\eta^* v = \sigma_2 \bar{v}$ up to a constant. This constant depends on y and thus defines a meromorphic function f so that $v = \sigma_2 f \overline{\eta^* v}$. This leads to $D = (f) + \eta \circ D$. From the same equation for v we additionally get

$$\begin{aligned}
-v = -\mathbb{1}v = \bar{\sigma}_2\sigma_2 v = \bar{\sigma}_2\mathbb{1}f\overline{\eta^* v} = f\overline{\eta^*\sigma_2 v}\\
= f\overline{\eta^*(f\overline{\eta^* v})} = f\eta^*(\bar{f}\eta^* v) = f\overline{\eta^*(f)}\eta^*(\eta^* v)\\
= f\overline{\eta^*(f)}v
\end{aligned}$$

and therefore $f\overline{\eta^* f} = -1$.
The last identity follows from

$$\sigma \circ D = (\rho \circ \eta) \circ D = \rho(D - (f)) = D^T - \rho \circ (f).$$

■

Remark 5.1.5. *The map f from Proposition 5.1.3 is $\frac{\mu - a}{b}$, i.e. the non-constant entry of the normalized eigenvector v.*

Proposition 5.1.6. *There holds*

$$\deg D = g + 1.$$

Proof. We define the projector $P = \frac{vw^T}{w^T v}$ first. The divisor of P is the negative of the branch divisor $B = 0 + \infty + \sum_{j=1}^{2g} \alpha_j$ of the covering map λ. Now we get

$$B = -(P) = -(v) - (w^T) + (w^T v).$$

Applying deg to this equation and using the definition of D and D^T yields

$$\deg B = 2g + 2 = \deg D + \deg D^T$$

The divisors D and D^T are connected by $D^T = \rho \circ D$ by Lemma 5.1.4 and have the same degree for this reason, which completes the proof.

■

5.2 The Baker-Akhiezer function

The introduction of the Pseudo-B.-A. function is guided by [KS10], p. 237ff.

Definition 5.2.1. *A function $\tilde{\Psi}$ which solves the equations*

$$\begin{aligned} d\tilde{\Psi} &= -\alpha_\lambda \tilde{\Psi} \\ M\tilde{\Psi} &= \mu\tilde{\Psi} \end{aligned}$$

is called the ***Pseudo-Baker-Akhiezer function****. The function $\tilde{\Phi}$ which solves*

$$\begin{aligned} d\tilde{\Phi}^T &= \tilde{\Phi}^T \alpha_\lambda \\ \tilde{\Phi}^T M &= \mu\tilde{\Phi}^T \end{aligned}$$

is its dual counterpart.

Since the frame F_λ solves $dF_\lambda = F_\lambda \alpha_\lambda$ and by using the formula $M(\lambda, z) = F_\lambda(z)^{-1} M(\lambda) F_\lambda(z)$ (cf. Proposition 3.2.2), we have the following result:

Proposition 5.2.2. *There holds*

$$\begin{aligned} \tilde{\Psi} &= F_\lambda^{-1} v \\ \tilde{\Phi}^T &= w^T F_\lambda. \end{aligned}$$

Because the eigenspaces are of dimension one, the functions $\tilde{\Psi}$ and $\tilde{\Phi}$ are unique up to multiplication with a constant. We will focus on $\tilde{\Psi}$ hereafter. The divisors of $\tilde{\Psi}$ and v are linearly equivalent and have therefore the same degree.

We want to use the Pseudo-B.-A. function to derive a formula for the potential u from the equation $d\tilde{\Psi} = -\alpha_\lambda \tilde{\Psi}$. It is more convenient to proceed with a gauged version of α_λ (cf. Section 3.5 for details). As a consequence of gauging with a matrix A we have the transformation

$$F_\lambda \mapsto F_\lambda A,$$

which leads to

$$\tilde{\Psi} \mapsto A^{-1}\tilde{\Psi} =: \hat{\Psi}.$$

The resulting function solves the equations from the Definition 5.2.1 but with gauged versions of α_λ and monodromy.

We use the matrices

$$A_0 = \begin{pmatrix} i\sqrt{\lambda}e^{\frac{-u}{2}} & 0 \\ 0 & e^{\frac{u}{2}} \end{pmatrix}, \ A_\infty = \begin{pmatrix} e^{\frac{u}{2}} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}}e^{\frac{-u}{2}} \end{pmatrix}$$

separately with the following results:

$$\alpha_\lambda^0 = \frac{1}{2}\begin{pmatrix} 0 & -i\sqrt{\lambda}e^{2u} \\ \sqrt{\lambda}e^{-2u} & 0 \end{pmatrix} dz + \frac{1}{2}\begin{pmatrix} -2u_{\bar{z}} & \frac{i}{\sqrt{\lambda}} \\ \frac{i}{\sqrt{\lambda}} & 2u_{\bar{z}} \end{pmatrix} d\bar{z} \tag{5.5}$$

$$\alpha_\lambda^\infty = \frac{1}{2}\begin{pmatrix} 2u_z & \sqrt{\lambda} \\ \sqrt{\lambda} & -2u_z \end{pmatrix} dz + \frac{1}{2}\begin{pmatrix} 0 & -\frac{1}{\sqrt{\lambda}}e^{-2u} \\ \frac{1}{\sqrt{\lambda}}e^{2u} & 0 \end{pmatrix} d\bar{z}. \tag{5.6}$$

Using (5.5) we can derive a Its-Matveev-type formula for u in terms of entries of $\hat{\Psi} =: (\hat{\psi}_1, \hat{\psi}_2)^T$:

$$\begin{aligned} d\hat{\Psi} &= -\alpha_\lambda^0 \hat{\Psi} \\ \Rightarrow \partial_z \hat{\psi}_1 &= \frac{i\sqrt{\lambda}}{2}e^{2u}\hat{\psi}_2, \ \partial_z\hat{\psi}_2 = -\frac{\sqrt{\lambda}}{2}e^{-2u}\hat{\psi}_1 \\ \Leftrightarrow u &= \frac{1}{2}\ln\left(-\frac{2i}{\sqrt{\lambda}}\frac{\partial_z\hat{\psi}_1}{\hat{\psi}_2}\right) = -\frac{1}{2}\ln\left(\frac{2}{\sqrt{\lambda}}\frac{\partial_z\hat{\psi}_2}{\hat{\psi}_1}\right). \end{aligned} \tag{5.7}$$

The formulae (5.5) and (5.6) lead to the asymptotic expansions:

$$\hat{\Psi} = \exp\left(-\frac{i\bar{z}}{2\sqrt{\lambda}}\right)\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + O(\sqrt{\lambda})\right] \text{ near } \lambda = 0 \tag{5.8}$$

$$\hat{\Psi} = \exp\left(-\frac{\sqrt{\lambda}z}{2}\right)\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} + O(\frac{1}{\sqrt{\lambda}})\right] \text{ near } \lambda = \infty. \tag{5.9}$$

We now present the classical definition of the Baker-Akhiezer function prescribing its analytical properties (cf. e.g. [Dub81], Chapter III).

Definition 5.2.3. *Let D be a positive divisor on the real spectral curve Y (as in Chapter 3) with degree $g+1$, $y^+, y^- \notin D$ and that satisfies the reality condition* (5.4)*. A **Baker-Akhiezer function** $\Psi(y,z,\bar{z})$ is a vector-valued map with the following properties:*

(i) for fixed $(z^*, \bar{z}^*)$ the entries $\Psi_{1,2}(y,z^*,\bar{z}^*)$ are meromorphic maps in y on Y^* with the divisor $(\Psi_{1,2}|_{Y^*}) \geq -D$, i.e. $\Psi_{1,2}(\cdot, z^*, \bar{z}^*)$ is a holomorphic section in $\mathcal{O}_D$ on Y^*

(ii) the products

$$\Psi(y,z,\bar{z})\exp\left(\frac{i\bar{z}}{2\sqrt{\lambda}}\right) \text{ and } \Psi(y,z,\bar{z})\exp\left(\frac{\sqrt{\lambda}z}{2}\right) \tag{5.10}$$

are holomorphic in a neighborhood of y^+ and y^- respectively, i.e. $\Psi_{1,2}$ are holomorphic sections in $\mathcal{O}_D \otimes L(z)$, where $L(z)$ is the line bundle defined by the cocycles $\exp\left(\frac{i\bar{z}}{2\sqrt{\lambda}}\right)$ and $\exp\left(\frac{\sqrt{\lambda}z}{2}\right)$.

*We say the Baker-Akhiezer function is **normalized** if there holds:*

(iii) the products as in (*ii*) *satisfy the following asymptotics*

$$\begin{pmatrix}1\\1\end{pmatrix} + O(\sqrt{\lambda}) \text{ at } y^+$$
$$\begin{pmatrix}1\\-1\end{pmatrix} + O(\frac{1}{\sqrt{\lambda}}) \text{ at } y^-.$$

Consider the set

$$S := \{z \in \mathbb{C} : \dim H^0(Y, \mathcal{O}_{D-0-\infty} \otimes L(z)) \neq 0\}. \tag{S}$$

We will see that it is crucial for the existence and uniqueness of the B.-A. function.
The divisor D is quaternionic with respect to the function f, hence, same holds for $D - 0 - \infty$. We call the corresponding line bundle quaternionic as well. The line bundle $L(z)$ is real due to the reality condition $\eta^*\mu = \bar{\mu}$.

The tensor product of a real and a quaternionic bundle is quaternionic (cf. [Hit90], p. 667). For that reason the vector space $\dim H^0(Y, \mathcal{O}_{D-0-\infty} \otimes L(z))$ is of even dimension for all $z \in \mathbb{C}$. Then S is discrete by Martens' Theorem (see [ACGH85], Theorem 5.1).
Compared to sinh-Gordon, our situation is more complicated. In that case the bundle $\mathcal{O}_{D-0-\infty} \otimes L(z)$ has degree $g-1$ because $L(z)$ has degree zero. Due to its quaternionic structure with respect to the different involution $\eta(\lambda) = \bar{\lambda}^{-1}$, the set S is empty ([Hit90], Proposition 7.15).

Theorem 5.2.4. *The conditions* $(i) - (iii)$ *from Definition 5.2.3 define the Baker-Akhiezer function uniquely on* $Y \times \mathbb{C} \setminus S$.

To prove this theorem we need the Riemann-Roch Theorem:

Theorem 5.2.5. *Consider a divisor* D *on a compact Riemann surface* X *of genus* g. *Then the dimensions of the vector spaces* $H^0(X, \mathcal{O}_D)$ *and* $H^1(X, \mathcal{O}_D)$ *satisfy*

$$\dim H^0(X, \mathcal{O}_D) = 1 - g + \deg D + \dim H^1(X, \mathcal{O}_D). \tag{RR}$$

For a proof see [For81], p. 129ff.

The quantity $\dim H^1(X, \mathcal{O}_D)$ is called **index of speciality** and is denoted by $i(D)$. A divisor D is called **special** if $i(D) > 0$.

Proof of Theorem 5.2.4. The proof is based on the ideas presented in [Kew08], p. 11f. We prove the existence first. The set $\mathbb{C} \setminus S$ is open. Consider z_0 to be a point in this set. For a fixed $z_0 \in \mathbb{C} \setminus S$ we define $D(z_0)$ as the divisor corresponding to the line bundle $\mathcal{O}_D \otimes L(z_0)$. Then, the divisor $\tilde{D}(z_0) := D(z_0) - 0 - \infty$ corresponds to $\mathcal{O}_{D-0-\infty} \otimes L(z_0)$. Because the bundle $L(z)$ is of degree zero there holds $\deg \tilde{D}(z_0) = \deg(D - 0 - \infty) = g - 1$. Using Riemann-Roch Theorem we obtain

$$0 = 1 - g + \deg \tilde{D}(z_0) + i(\tilde{D}(z_0)) = i(\tilde{D}(z_0)),$$

i.e. the divisor $\tilde{D}(z_0)$ is non-special. Since there holds $D(z_0) \geq \tilde{D}(z_0)$ the divisor D is non-special as well. Applying (RR) again yields

$$\dim H^0(Y, \mathcal{O}_D \otimes L(z_0)) = 1 - g + \deg D(z_0) + i(D(z_0)) = 2.$$

Because we have two linearly independent Ψ_j, we can normalize them as requested.
We now assume to have two normalized functions Ψ and $\hat{\Psi}$. The difference $\Psi_{1,2} - \hat{\Psi}_{1,2}$ of their entries vanishes at y^+ and y^- because the singular parts are identical. Hence it is a section in $\mathcal{O}_{D-0-\infty} \otimes L(z)$ and for all z in $\mathbb{C} \setminus S$ the difference has to be zero.

■

Remark 5.2.6. *We still have to prove the differentiability in the spatial variables. But the B.-A. functions are known to be differentiable because there are explicit formulas using Riemann's theta functions.*

The monodromy and the other objects can be given in terms of the entries of the Baker-Akhiezer function Ψ.

5.3 The solution u in terms of B.-A. functions and its singularities

We can express solutions of the cosh-Gordon equation using the formula (5.7):

$$u = \frac{1}{2} \ln \left(-\frac{2i}{\sqrt{\lambda}} \frac{\partial_z \hat{\psi}_1}{\hat{\psi}_2} \right).$$

The Baker-Akhiezer function is periodic because the vector bundle $L(z)$ is. For that reason the solution u is periodic as well.

What can we say about the regularity of u? From general theory it is known that the Baker-Akhiezer function Ψ has poles in the spatial variable at points of the set (S), which leads to singularities of u at these points.

Theorem 5.3.1 ([Bab91b],[Bab91c])**.** *All real solutions of the* cosh*-Gordon equation are singular on some line in the complex plane.*

Sketch of Babich's proof. Starting with a curve of genus g, which has a real structure with respect to η, a real solution of the cosh-Gordon equation is given in terms of theta functions:

$$u = -2 \ln \frac{\theta_1}{\theta_2} + \text{ regular parts.}$$

The reality condition for the arguments of the theta functions are translated into a condition for divisors using Riemann's Theorem. If the divisor contains y^+, y^- or it is special, then at least one of the theta functions vanish which leads to a singularity of u. Now using the map λ, the set of quaternionic divisors is constructed.[15] Then let G be the set of quaternionic divisors that lead to singularities. It is a subset of the real part of the Jacobian M. One of the components of G has codimension 1 in the corresponding component of M, i.e. the set of singularities is one-dimensional in the complex plane.

■

The B.-A. function in the paper is a little bit different, a divisor of degree g is used. In our situation the lines of singularities therefore correspond to the condition

$$\dim H^0(Y, \mathcal{O}_{D-2\cdot 0} \otimes L(z)) \neq 0. \tag{S'}$$

Once again in sinh-Gordon case condition (S′) is never satisfied because the divisor $\tilde{D} := D - 2 \cdot 0$ is quaternionic:

$$\eta \circ \tilde{D} = \tilde{D} - (\lambda f).$$

Then Hitchins argument is applicable (as mentioned in the previous section). Therefore all solutions are smooth.

[15] Because there holds $\lambda \overline{\eta^* \lambda} = -1$, the divisors $2 \cdot y^+ + \sum_j (y_j + \eta(y_j))$ and $2 \cdot y^- + \sum_j (y_j + \eta(y_j))$ are quaternionic.

6 Conclusions and open questions

The main part of this document was devoted to the direct problem and it was covered in detail. The inverse problem was treated less extensively and there are several ways to continue the investigations.
One interesting question is to find out more about the properties of the singularities and the corresponding conditions on the divisors. For that reason the structure of $\mathrm{Pic}_{g+1}(Y)$, i.e. the component of the Picard variety containing divisors of degree $g+1$, is of interest. In addition it is worthwhile to learn more about the behavior of the monodromy M and the connection α at the singular points of the solution u and the consequences for the corresponding CMC surfaces.
A different point of continuation would be the moduli space and isoperiodic deformations. The presented ansatz can be extended to the moduli space of CMC cylinders. The moduli space of curves with different genus can be studied or the deformations can be implemented numerically using the given formulae. The investigation of the moduli space using the covering induced by the trace map Δ (cf. [GS95], Section 3) would improve the understanding as well.

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