# CONSTANT MEAN CURVATURE SURFACES IN HYPERBOLIC 3-SPACE WITH $|H| < 1$

**Bachelor Thesis**

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# Contents

# 1 Introduction

CMC surfaces have been broadly studied in the field of differential geometry for a considerable time and are still subjects of current investigations. This document deals with CMC surfaces in hyperbolic three-space with the mean curvature's modulus restricted to be less than one. It enqueues in the list of theses previously supervised by Prof. Dr. Martin U. Schmidt: in 2008 Markus Knopf [8] examined CMC surfaces in the three-sphere, at the same time Vania Neugebauer [13] and Wjatscheslaw Kewlin [6] analyzed CMC tori in hyperbolic three-space respectively three-sphere. Finally in 2009 Matthias Leimeister [11] took a closer look at deformations of cylinders in $\mathbb{H}^3$ with $|H| > 1$.

In the next chapter two models for the hyperbolic 3-space are described which are used later throughout the document. In the following chapter an overview on surface theory and the concept of moving frames and Lax pairs is given. Readers who aren't very familiar with differential geometry may want to start there. The fourth chapter presents the Sym-Bobenko formula which allows to construct CMC surfaces with given mean curvature. In the subsequent chapter the argument for non-existence of compact CMC surfaces, as mentioned for example in [2], is developed. And finally a brief overview and some further ideas are presented in the last chapter.

# 2 Representations of $\mathbb{H}^3$

Hyperbolic 3-space $\mathbb{H}^3$ is the unique simply connected three-dimensional Riemannian manifold with sectional curvature being constantly -1. There exist several models, each of them with a different purpose and (dis)advantages. First we want to present two models of $\mathbb{H}^3$ which will be useful later for studying CMC surfaces in $\mathbb{H}^3$.

## 2.1 Minkowski model for $\mathbb{H}^3$

In this model $\mathbb{H}^3$ is a subset of the 4-space with a special metric.

**Definition 2.1.1.** $\mathbb{R}^4$ *equipped with the metric* $\langle \cdot, \cdot \rangle_{\mathbb{R}^{3,1}}$ *induced by the indefinite matrix* $\tilde{I} := \operatorname{diag}(-1, 1, 1, 1)$ *is the* ***Minkowski 4-space*** $\mathbb{R}^{3,1}$*. That means for* $x^T = (x_0, x_1, x_2, x_3)$ *and* $y^T = (y_0, y_1, y_2, y_3)$ *in* $\mathbb{R}^4$ *there holds*

$$\langle x, y \rangle_{\mathbb{R}^{3,1}} = x^T \tilde{I} y = -x_0 y_0 + \sum_{j=1}^{3} x_j y_j$$

$\mathbb{R}^{3,1}$ is a model for spacetime where $x_0$ is the time coordinate and $x_{1,2,3}$ the spatial coordinates. Later on the subscript $\mathbb{R}^{3,1}$ will be omitted, since it is the only metric used. Since the metric's signature is clearly $(3, 1)$ it is a Lorentzian space. For this reason there are no orthonormal bases in the classical sense but "almost orthonormal" bases $(b_0, b_1, b_2, b_3)$ with all $b_i$ pairwise orthogonal and only $b_0$ having $\langle b_0, b_0 \rangle_{\mathbb{R}^{3,1}} = -1$ and the others having unit length. From now on such a basis is just called orthonormal. Later on in section 3.2 we will see that it is possible to use the immersed surface one considers to obtain an orthonormal basis of the Minkowski space $\mathbb{R}^{3,1}$.

Now we can define the Minkowski model for the hyperbolic 3-space.

**Definition 2.1.2.**

$$\mathbb{H}^3 := \{x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^{3,1} : \langle x, x \rangle_{\mathbb{R}^{3,1}} = -1,\ x_0 \geq 1\}$$

Defined that way $\mathbb{H}^3$ can be interpreted as a sphere in $\mathbb{R}^{3,1}$ with radius $i$. Indeed it is the upper sheet of a hyperboloid. The fact that the Minkowski model has indeed constant sectional curvature -1 and therefore is the hyperbolic 3-space is proven in [4], Lemma 1.1.10.

**Proposition 2.1.3.** $\mathbb{H}^3$ *is a Riemannian manifold, i.e.*

$$\forall v \in T_p\mathbb{H}^3,\ v \neq 0 : \langle v, v\rangle_{\mathbb{R}^{3,1}} > 0.$$

*Proof.* Consider a differentiable curve $c : J \to \mathbb{H}^3,\ t \mapsto c(t)$, whereby $0 \in J \subset \mathbb{R}$ and $\alpha(0) = p$. By definition the tangent space $T_p\mathbb{H}^3$ consists of equivalence classes of all derivatives evaluated at 0 of these curves. Differentiating the equation $\langle c(t), c(t)\rangle_{\mathbb{R}^{3,1}} = -1$ yields $\langle c(t), c'(t)\rangle_{\mathbb{R}^{3,1}} = 0$. Therefore $p^\perp := \mathrm{span}_{\mathbb{R}}(p)^\perp = \{v \in \mathbb{R}^{3,1} : \langle p, v\rangle_{\mathbb{R}^{3,1}} = 0\} \subset T_p\mathbb{H}^3$. Since $p^\perp$ sits in $\mathbb{R}^{3,1}$ it is three-dimensional. Because $T_p\mathbb{H}^3$ has the same dimension, it coincides with $p^\perp$.
Now assume $v \in T_p\mathbb{H}^3$ a non-zero vector with $\langle v, v\rangle_{\mathbb{R}^{3,1}} \leq 0$. Because of this condition and $\langle p, p\rangle_{\mathbb{R}^{3,1}} = -1$ there follows

$$\sum_{i=1}^{3} v_i^2 \leq v_0^2 \text{ and } \sum_{j=1}^{3} p_i^2 < p_0^2 \tag{2.1}$$

Using the Cauchy-Schwarz inequality there holds

$$\left(\sum_{i=1}^{3} v_i p_i\right)^2 \leq \left(\sum_{i=1}^{3} v_i^2\right)\left(\sum_{j=1}^{3} p_i^2\right) \overset{2.1}{<} (v_0 p_0)^2 \iff \left|\sum_{i=1}^{3} v_i p_i\right| < |v_0 p_0|.$$

The latter inequality conflicts with $v \in p^\perp \Leftrightarrow \sum_{i=1}^{3} v_i p_i = v_0 p_0$ because it implies $|\sum_{i=1}^{3} v_i p_i| = |v_0 p_0|$. So $\langle v, v\rangle_{\mathbb{R}^{3,1}}$ has to be strictly positive. ■

In other words the proposition states that $\langle \cdot, \cdot\rangle_{\mathbb{R}^{3,1}}$ is positive definite on $T_p\mathbb{H}^3$.

**Definition 2.1.4.** *The metric preserving group of* $\mathbb{R}^{3,1}$*, i.e. all matrices* $A \in \mathbb{R}^{4\times 4}$ *with* $\langle x, y\rangle_{\mathbb{R}^{3,1}} = \langle Ax, Ay\rangle_{\mathbb{R}^{3,1}}$ *for all* $x, y \in \mathbb{R}^{3,1}$*, is the **Lorentz group** and is denoted as* $O(3,1)$*.*

**Proposition 2.1.5.** *For an element*

$$A := \begin{pmatrix} a_{00} & w^T \\ v & B \end{pmatrix} \ \textit{with}\ \ v, w \in \mathbb{R}^3,\ B \in \mathbb{R}^{3\times 3}$$

*in* $O(3,1)$ *there holds* $|a_{00}| \geq 1$ *and* $\det A = \pm 1$*.*

*Proof.*

$$\begin{aligned} \langle x, y\rangle_{\mathbb{R}^{3,1}} = \langle Ax, Ay\rangle_{\mathbb{R}^{3,1}} &\iff x^T \tilde{I} y = x^T A^T \tilde{I} A y \\ &\iff \tilde{I} = A^T \tilde{I} A \\ &\Rightarrow -1 = \det \tilde{I} = \det(A^T \tilde{I} A) = -(\det A)^2 \\ &\Rightarrow \det A = \pm 1 \end{aligned} \tag{2.2}$$

Due to the condition 2.2 for the the top left entry of $A^T \tilde{I} A$ there follows

$$\begin{aligned} -1 = v^T v - a_{00}^2 &\iff a_{00}^2 = 1 + v^T v \\ &\Rightarrow a_{00}^2 \geq 1 \Rightarrow |a_{00}| \geq 1 \end{aligned}$$

■

**Corollary 2.1.6.** *The Lorentz group has four connected components.*

*Sketch of proof.* Since det is a continuous map and takes only values in $\{-1, +1\}$ on $O(3,1)$ there are at least two components distinguished by the sign. Each of them splits up in two components since there is no continuous path between those matrices with $a_{00} \geq 1$ and $a_{00} \leq -1$. So there are at least four connnected components. A complete proof can be found in [9], theorem 12.11.

**Proposition 2.1.7.** $O^+(3,1) := \{A \in O(3,1) : a_{00} \geq 1\}$ *is the metric preserving group of* $\mathbb{H}^3$.

The crucial condition is to preserve the time coordinate $x_0$ from changing sign. $O^+(3,1)$ preserves the time coordinate of every vector $v \in \mathbb{R}^{3,1}$ with $\langle v, v \rangle_{\mathbb{R}^{3,1}} \leq 0$, therefore it acts also correctly on $\mathbb{H}^3$. A general proof for this result can be found in [12], section 1.3.

The identity component $SO^+(3,1) := \{A \in O^+(3,1) : \det A = 1\}$, which is also called the **restricted Lorentz group**, additionally preserves the orientation of a basis and therefore is the most interesting subgroup of $O(3,1)$ for this document's purposes. To describe these isometries it is more elegant to use a different model for the hyperbolic 3-space which is the subject of the next section.

## 2.2 Hermitian matrix model for $\mathbb{H}^3$

**Definition 2.2.1.**

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

*are called the **Pauli matrices**.*

By adding $\sigma_0 := id$ one gets four linearly independent elements which define a linear subspace in $\mathbb{C}^{2\times 2}$. This subspace $\operatorname{span}_{\mathbb{R}}(\sigma_1, \sigma_2, \sigma_3, \sigma_0)$ is denoted by $Herm(2)$. Using these matrices, $\mathbb{R}^{3,1}$ can be identified with $2 \times 2$ matrices using the linear map

$$\begin{aligned} \psi : \mathbb{R}^{3,1} &\to Herm(2) \\ x &\mapsto \sum_{j=0}^{3} x_k \sigma_k = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}. \end{aligned} \tag{2.3}$$

From now on the image $\psi(x)$ will often be abbreviated as $X$. $X$ is a Hermitian matrix, this means $X = \overline{X}^T =: X^*$ and hence has the form

$$\begin{pmatrix} a_{11} & a_{12} \\ \overline{a_{12}} & a_{22} \end{pmatrix}$$

with $a_{11}, a_{22} \in \mathbb{R}$ and $a_{12} \in \mathbb{C}$. The next step is to figure out how $\langle \cdot, \cdot \rangle_{\mathbb{R}^{3,1}}$ looks like in $Herm(2)$. To show that the following proposition will be helpful.

**Definition 2.2.2.** *For a $2 \times 2$ matrix $M$ the adjugate matrix ist defined as*

$$\operatorname{adj}(M) := \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix}.$$

Recall that in the case M is invertible there holds $\operatorname{adj}(M) = \det(M) M^{-1}$.

**Proposition 2.2.3.** *For an arbitrary $2 \times 2$ matrix $Y$ holds*

$$\sigma_2 Y^T \sigma_2 = \operatorname{adj}(Y).$$

*Proof.*

$$\begin{aligned} \sigma_2 Y^T \sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} y_{11} & y_{21} \\ y_{12} & y_{22} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} y_{21} i & -y_{11} i \\ y_{22} i & -y_{12} i \end{pmatrix} \\ &= \begin{pmatrix} y_{22} & -y_{12} \\ -y_{21} & y_{11} \end{pmatrix} = \operatorname{adj}(Y) \end{aligned}$$

■

**Proposition 2.2.4.** *For the metric $\langle \cdot, \cdot \rangle_H$ on $Herm(2)$ defined by*

$$\langle X, Y \rangle_H := -\frac{1}{2} \operatorname{tr}(X \sigma_2 Y^T \sigma_2)$$

*$\psi$ is an isometry, i.e. there holds $\langle \psi(x), \psi(y) \rangle_H = \langle x, y \rangle_{\mathbb{R}^{3,1}}$ for all $x, y \in \mathbb{R}^{3,1}$.*

*Proof.*

$$\begin{aligned} \langle \psi(x), \psi(y) \rangle_H &= \langle X, Y \rangle_H = -\frac{1}{2} \operatorname{tr}(X \sigma_2 Y^T \sigma_2) \overset{2.2.3}{=} -\frac{1}{2} \operatorname{tr}(X \operatorname{adj}(Y)) \\ &= -\frac{1}{2} \operatorname{tr} \left( \begin{pmatrix} x_0 + x_3 & x_1 - i x_2 \\ x_1 + i x_2 & x_0 - x_3 \end{pmatrix} \begin{pmatrix} y_0 - y_3 & -(y_1 - i y_2) \\ -(y_1 + i y_2) & y_0 + y_3 \end{pmatrix} \right) \\ &= -\frac{1}{2} (2(x_0 y_0 - x_3 y_3 - (x_1 y_1 + x_2 y_2)) \\ &= \langle x, y \rangle_{\mathbb{R}^{3,1}} \end{aligned}$$

■

From now on $\langle \cdot, \cdot \rangle$ stands either for $\langle \cdot, \cdot \rangle_H$ or for $\langle \cdot, \cdot \rangle_{\mathbb{R}^{3,1}}$ depending on the context.

**Proposition 2.2.5.** *The matrices* $(\sigma_1, \sigma_2, \sigma_3, \sigma_0)$ *form a orthonormal basis of* $Herm(2)$ *with*

$$\langle \sigma_1, \sigma_1 \rangle = \langle \sigma_2, \sigma_2 \rangle = \langle \sigma_3, \sigma_3 \rangle = 1, \ \langle \sigma_0, \sigma_0 \rangle = -1$$

*Proof.* By definition of $\psi$ it is a surjective linear map. Because $\psi$ is also isometric it is injective and therefore an isomorphism. Considering the standard basis $(e_0, e_1, e_2, e_3)$ of $\mathbb{R}^4$ one notices $\psi(e_i) = \sigma_i$. Now the claim follows directly. ∎

**Theorem 2.2.6.** *The hyperbolic 3-space* $\mathbb{H}^3$ *is diffeomorphic to*

$$\mathcal{H}^3 := \{X \in Herm(2) : \ \det(X) = 1, \ \operatorname{tr}(X) \geq 2\}$$

*Proof.* First we want to show $\psi(\mathbb{H}^3) = \mathcal{H}^3$. Let $x$ be an element of $\mathbb{H}^3$.

$$\begin{aligned} -1 = \langle x, x \rangle = \langle \psi(x), \psi(x) \rangle = \langle X, X \rangle &= -\frac{1}{2}\operatorname{tr}(X\sigma_2 X^T \sigma_2) \\ \overset{2.2.3}{=} -\frac{1}{2}\operatorname{tr}(X \cdot \operatorname{adj}(X)) &= -\frac{1}{2}\operatorname{tr}\begin{pmatrix} \det(X) & * \\ * & \det(X) \end{pmatrix} \\ = -\det(X) \end{aligned}$$

Now consider the trace of $\psi(x)$:

$$\begin{aligned} \operatorname{tr}(X) &= (x_0 + x_3) + (x_0 - x_3) = 2x_0 \geq 2 \\ &\Rightarrow \psi(\mathbb{H}^3) \subset \mathcal{H}^3 \end{aligned}$$

By expressing $X$ with respect to the basis $\sigma_i$ one gets the scalars $x_0, x_1, x_2, x_3$ yielding, because $\det(X) = 1$ and $\operatorname{tr}(X) \geq 2$, an element $x = \psi^{-1}(X) \in \mathbb{H}^3 \Rightarrow \mathcal{H}^3 \subset \psi(\mathbb{H}^3)$ and in total $\psi(\mathbb{H}^3) = \mathcal{H}^3$.

The last paragraph shows that $\tilde{\psi} := \psi|_{\mathbb{H}^3}$ is surjective and since $\psi$ is an isometry it is also bijective. Now only stating $\tilde{\psi}$ and $\tilde{\psi}^{-1}$ to be differentiable is left. As an isomorphism $\psi$ is differentiable and its differential is $\psi$ itself. Because $\tilde{\psi}$ is a restriction to a submanifold in the domain $\mathbb{R}^{3,1}$, it is differentiable too. For the differential there holds $d_p\tilde{\psi} = (d_p\psi)|_{T_p\mathcal{H}^3} = \psi|_{T_p\mathcal{H}^3}$. The determinant is a submersion and therefore according to the regular value theorem $\mathcal{H}^3$ is a submanifold of dimension three in $Herm(2)$. Both tangent spaces are three-dimensional and the differential of $\tilde{\psi}$ is injective, then by rank-nullity theorem $d_p\tilde{\psi}$ is bijective. This leads to $d_q(\tilde{\psi}^{-1}) = (d_p\tilde{\psi})^{-1}$ with $\psi(p) = q$. ∎

**Lemma 2.2.7.** $\mathcal{H}^3 = \{FF^* : F \in SL_2\mathbb{C}\}$

*Proof.* Define $P = \{FF^* : F \in SL_2\mathbb{C}\}$ and let $A$ be an element of $P$. One can easily see that the determinant of $A$ is 1 and it has the form

$$\begin{pmatrix} |f_{11}|^2 + |f_{12}|^2 & f_{11}\overline{f_{21}} + f_{12}\overline{f_{22}} \\ \overline{f_{11}}f_{21} + \overline{f_{12}}f_{22} & |f_{21}|^2 + |f_{22}|^2 \end{pmatrix}$$

and therefore is Hermitian. Because the characteristic polynomial looks like $\chi(\lambda) = \lambda^2 - \operatorname{tr}(A)\lambda + \det(A)$ the eigenvalues have the form

$$\lambda_{1,2} = \frac{\operatorname{tr}(A)}{2} \pm \sqrt{\frac{\operatorname{tr}(A)^2}{4} - 1}.$$

As a Hermitian matrix A has real eigenvalues which leads to

$$\frac{\operatorname{tr}(A)^2}{4} - 1 \geq 0 \iff \operatorname{tr}(A)^2 \geq 4 \iff |\operatorname{tr}(A)| \geq 2$$

Since the trace of A contains only absolute values it is non-negative with the consequence $\operatorname{tr}(A) \geq 2$. In total one has $A \in \mathcal{H}^3 \Rightarrow P \subset \mathcal{H}^3$.

Now it has to be proven that every $X \in \mathcal{H}^3$ can be decomposed in the product $FF^* \in P$. Let $\lambda_1$ and $\lambda_2$ be the real eigenvalues of $X$. Since $\det(X) = 1$ there holds $\lambda_2 = \frac{1}{\lambda_1}$ and hence both eigenvalues have the same sign. Due to the trace being positive, they are also positive. Like any Hermitian matrix $X$ can be diagonalised using unitary matrices $U$ and $U^{-1} = U^*$. Where $U = (v_1, v_2)$ with $v_1$ and $v_2$ being eigenvectors of $X$. Setting $D := \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and $\tilde{D} := \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix}$ one gets

$$X = UDU^* = U\tilde{D}\tilde{D}U^* = U\tilde{D}(U\overline{\tilde{D}})^* = \underbrace{U\tilde{D}}_{=:F}(U\tilde{D})^*$$

Now one has $\det(\tilde{D}) = \sqrt{\lambda_1}\sqrt{\lambda_2} = \sqrt{\det(X)} = 1$. As a unitary matrix, $U$ has $\det = \pm 1$. In the case $\det(U) = -1$, proceeding with $\tilde{U} = U\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = (v_1, -v_2)$ leads to the requested result, since $\det(\tilde{U}) = -\det(U) = 1$. In total we have $F \in SL_2\mathbb{C}$
$\Rightarrow X \in P \Rightarrow \mathcal{H}^3 \subset P$.

■

This product representation is not unique because any right-multiplication with a matrix $M \in SU_2$ doesn't change anything, since

$$FF^* = FMM^*F^* = FM(FM)^*.$$

This fact leads to

**Corollary 2.2.8.** *$\mathbb{H}^3$ can be identified with $SL_2\mathbb{C}/SU_2$*

**Theorem 2.2.9.** *The orientation preserving isometry group $SO^+(3,1)$ of $\mathbb{H}^3$ is isomorphic to the projective special linear group $PSL_2\mathbb{C} := SL_2\mathbb{C}/\{\pm id\}$ and acts as*

$$X \mapsto M \cdot X \cdot M^*$$

*on an arbitrary $X \in \mathcal{H}^3$.*

*Proof.* There exists a two to one homomorphism $\phi$ between $SL_2\mathbb{C}$ and $SO^+(3,1)$ (cf. [3], chapter 3). Let now $M$ be an element in $SL_2\mathbb{C}$ and $X \in \mathcal{H}^3$. The action $M \cdot X \cdot M^*$ then corresponds to $\phi(M) \cdot \psi^{-1}(X)$ in the Minkowski model. Since $M \cdot X \cdot M^* = -M \cdot X(-M)^*$ there holds $\ker \phi = \{\pm id\}$. Dividing by $\{\pm id\}$ turns $\phi$ into a group isomorphism. ∎

In other words: all rotations of the hyperbolic 3-space can be implemented using conjugations with an element of $SL_2\mathbb{C}$. Since we will use the isometry group only for transforming oriented bases the reflections are of no interest.

# 3 Surface theory in $\mathbb{H}^3$

## 3.1 General surface theory

This section is a brief introduction into surface theory in the three space forms $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$ and covers the basic terms of differential geometry used later in the text.

**Definition 3.1.1.** *A differentiable mapping*

$$f : \Sigma \to M$$

*between two manifolds is an* ***immersion*** *if its differential is injective at every point p in M.*

In the setting of surface theory $\Sigma$ is a orientable two-dimensional manifold and $M$ is one of the three-dimensional space forms like $\mathbb{R}^3, \mathbb{S}^3$ or $\mathbb{H}^3$. Then $f$ is a representation of a surface in $M$ in the sense of differential geometry.

Since $\mathbb{R}^2$ can be identified with the complex plane and by fixing an orientation of $\Sigma$ and equipping it with a complex structure it turns to a Riemann surface. Every coordinate chart then defines a complex coordinate $z = x + iy$. Since $f$ is an immersion the partial derivatives

$$f_x := \left(\frac{\partial f}{\partial x}\right)_p \text{ and } f_y := \left(\frac{\partial f}{\partial y}\right)_p$$

with respect to this chart are linearly independent and thus provide a basis for a linear subspace of the tangent space $T_{f(p)}M$.

Using the metric $\langle\cdot,\cdot\rangle_M$ of the manifold $M$ one can define a metric $ds^2 = g$ on $T_p\Sigma$:

**Definition 3.1.2.** *The bilinear map* $g_p : T_p\Sigma \times T_p\Sigma \to \mathbb{R}$ *defined by*

$$g_p := \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} \langle f_x, f_y\rangle_M & \langle f_x, f_y\rangle_M \\ \langle f_y, f_x\rangle_M & \langle f_y, f_y\rangle_M \end{pmatrix}$$

*is the* ***first fundamental form*** *of the immersion* $f$.

**Proposition 3.1.3.** $f$ *is an immersion* $\iff \det g > 0$

*Proof.* Since $g$ is Gram's determinant for $f_x$ and $f_y$, $\det g > 0$ is equivalent to the linear independence of the partial derivatives.

■

**Definition 3.1.4.** *In the case $g_{11} = g_{22}$ and $g_{21} = 0 = g_{12}$, i.e. $g = \lambda(p) \cdot id$, $\lambda(p) > 0$, the immersion $f$ is called **conformally parameterized**.*

The metric can be then written as

$$ds^2 = 4e^{2u}(dx^2 + dy^2)$$

with the so-called conformal factor $u \in C^\infty(\Sigma, \mathbb{R})$. Practically the coordinate chart $z$ can be always chosen in a way that $g$ is conformal because $\Sigma$ is orientable (cf. [14], Theorem 1.6.5). Therefore from now on only conformal immersions are considered.

Since $T_pM$ is three-dimensional it is possible to define the unit normal vector $N$ with respect to the partial derivatives $f_x$ and $f_y$. Using $N$ the **second fundamental form** $b$ of the immersion $f$ can be defined by

$$b := \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} -\langle N_x, f_x \rangle_M & -\langle N_y, f_x \rangle_M \\ -\langle N_x, f_y \rangle_M & -\langle N_y, f_y \rangle_M \end{pmatrix} = \begin{pmatrix} \langle N, f_{xx} \rangle_M & \langle N, f_{xy} \rangle_M \\ \langle N, f_{yx} \rangle_M & \langle N, f_{yy} \rangle_M \end{pmatrix}$$

The linear form $b$ can be also written with the help of the differential forms $dz := dx + idy$ and $d\overline{z} := dx - idy$

$$b = Qdz^2 + \tilde{H}dzd\overline{z} + \overline{Q}d\overline{z}^2$$

whereby the functions $Q$ and $\tilde{H}$ are defined as follows

$$Q := \frac{1}{4}(b_{11} - b_{22} - ib_{12} - ib_{21}), \quad \tilde{H} := \frac{1}{2}(b_{11} + b_{22})$$

$Qdz^2$ is the **Hopf differential** of $f$.

We will later see that $f$ is determined uniquely up to a rigid motion in $M$ by the two fundamental forms if they satisfy a certain pair of equations (cf. section 3.3).

**Definition 3.1.5.** *The linear map $S := g^{-1}b : T_p\Sigma \to T_p\Sigma$ is the **shape operator**.*

The shape operator can be expressed with the help of the functions $u, Q$ and $\tilde{H}$:

$$S = \frac{1}{4e^{2u}} \begin{pmatrix} \tilde{H} + Q + \overline{Q} & i(Q - \overline{Q}) \\ i(Q - \overline{Q}) & \tilde{H} - Q - \overline{Q} \end{pmatrix}$$

**Definition 3.1.6.** *The shape operator's eigenvalues $\kappa_1, \kappa_2$ are the **principal curvatures**, its half-trace is the **mean curvature** $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ and its determinant is the **Gaussian curvature** $K = \kappa_1\kappa_2$ of the immersion $f$.*

In the conformal case this leads to $H = \frac{1}{8}e^{-2u}\langle N, f_{xx} + f_{yy}\rangle$. One notices that the mean curvature is defined similarly as $\tilde{H}$, indeed there is a coherence $H = \frac{1}{4}e^{-2u}\tilde{H}$ among them.

**Definition 3.1.7.** *If $H$ is constant, then $f$ is called a **constant mean curvature (CMC) surface**. In the special case of $H \equiv 0$, $f$ is named **minimal surface**.*

Using Wirtinger differential operators

$$\partial_z := \frac{1}{2}(\partial_x - i\partial_y) \text{ and } \partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y)$$

and letting $\langle\cdot,\cdot\rangle$ be the complex bilinear extension of the metric $\langle\cdot,\cdot\rangle_M$ we get to the following elegant result:

**Proposition 3.1.8.** *Using the complex coordinate $z$, $H$, $Q$ and the conformality condition can be expressed by*

$$H = \frac{1}{2}e^{-2u}\langle f_{z\bar{z}}, N\rangle, \tag{3.1}$$

$$Q = \langle f_{zz}, N\rangle \tag{3.2}$$

$$\langle f_z, f_z\rangle = \langle f_{\bar{z}}, f_{\bar{z}}\rangle = 0, \quad \langle f_z, f_{\bar{z}}\rangle = 2e^{2u} \tag{3.3}$$

The proof is a straight forward calculation using the correspondences $\partial_x = \partial_z + \partial_{\bar{z}}$ respectively $\partial_y = i(\partial_z - \partial_{\bar{z}})$. The next step in the classical theory is the construction of a so-called moving frame which is a basis of $\mathbb{R}^3$ or of the ambient space in the case of $\mathbb{S}^3$ and $\mathbb{H}^3$. The frame is derived directly from the immersion $f$ and with its help one obtains the so-called Gauss-Codazzi equations. It can be shown that those are the only conditions $u$, $H$ and $Q$ have to satisfy (cf. theorem 3.3.5).
Since these equations are different from one space form to another, the next section will deal with the specific situation of $f$ being an immersion in $\mathbb{H}^3$.

## 3.2 The extended frame in $\mathbb{H}^3$

In this section $\mathbb{H}^3$ is seen with the help of the Minkowski model, i.e. the (conformal) immersion $f : M \to \mathbb{H}^3$ is considered as a $\mathbb{R}^{3,1}$-valued map. Starting with $f$ the aim is to construct the frame i.e. a basis of $\mathbb{R}^{3,1}$ which is well adjusted to the surface.

Differentiating $\langle f, f\rangle = -1$ yields

$$\langle f, f_x\rangle = 0, \quad \langle f, f_y\rangle = 0$$

Additionally the conformality of $f$ gives the orthogonality $\langle f_x, f_y\rangle = 0$ of the partial derivatives. Now you can also define the normal $\tilde{N}$ using the formal determinant

$$\tilde{N} := f \times f_x \times f_y := \det(E, f, f_x, f_y)$$

where $e_1, \ldots, e_4$ is an orthonormal basis of $\mathbb{R}^{3,1}$, $f, f_x, f_y$ are expressed with its help and $E = (e_1, e_2, e_3, e_4)^T$ is a vector containing the basis elements as single entries. Since $f_x, f_y$ and $\tilde{N}$ are elements of $T_p\mathbb{H}^3$ and $\mathbb{H}^3$ is a Riemannian manifold, their metric is positive. By setting $N := \frac{\tilde{N}}{\|\tilde{N}\|}$ one obtains the unit normal.

**Definition 3.2.1.** *The map*

$$\tilde{\mathcal{F}} : \Sigma \to \mathbb{R}^{4\times 4}, p \mapsto (f(p), f_x(p), f_y(p), N(p))$$

*is the* ***extended*** *or* ***moving frame*** *of the immersion $f$.*

**Proposition 3.2.2.** *The normalized frame*

$$\tilde{\mathcal{F}}_{on} = \left(f, \frac{f_x}{\|f_x\|}, \frac{f_y}{\|f_y\|}, N\right) =: (f, e_1, e_2, N)$$

*is an positive oriented orthonormal basis of $\mathbb{R}^{3,1}$ and an element of the isometry group.*

In the case of using the complex coordinate $z$ instead of $x$ and $y$ the extended frame is defined as

$$\mathcal{F} := (f, f_z, f_{\bar{z}}, N)$$

**Proposition 3.2.3.** *Every $v \in \mathbb{R}^{3,1}$ can be expressed with respect to $\mathcal{F}$:*

$$v = -\langle v, f\rangle f + \frac{\langle v, f_{\bar{z}}\rangle}{2e^{2u}} f_z + \frac{\langle v, f_z\rangle}{2e^{2u}} f_{\bar{z}} + \langle v, N\rangle N$$

*Proof.* Since the $\tilde{\mathcal{F}}_{on}$ is a orthonormal basis of $\mathbb{R}^{3,1}$ $v$ can be represented as

$$v = -\langle v, f\rangle f + \frac{\langle v, f_x\rangle}{4e^{2u}} f_x + \frac{\langle v, f_y\rangle}{4e^{2u}} f_y + \langle v, N\rangle N$$

which is quite similar to the asserted formula. Now plugging the correspondences

$$f_x = f_z + f_{\bar{z}} \text{ and } f_y = i(f_z - f_{\bar{z}})$$

into $\frac{\langle v, f_x\rangle}{4e^{2u}} f_x + \frac{\langle v, f_y\rangle}{4e^{2u}} f_y$ yields the claim.

■

## 3.3 The Lax pairs

The goal of this section is to describe the frame's behavior towards differentiation. This is possible in terms of matrix differential equations, which leads to the definition:

**Definition 3.3.1.** *The two matrix partial differential equations*

$$\mathcal{F}_z = \mathcal{F} \cdot \mathcal{U}, \quad \mathcal{F}_{\bar{z}} = \mathcal{F} \cdot \mathcal{V} \tag{3.4}$$

*are the* ***Lax pair*** *of the immersion $f$.*

Since $\mathbb{H}^3$ can be represented either as subset of the Minkowski space $\mathbb{R}^{3,1}$ or using Hermitian matrices there are two "flavors": $\mathcal{U}$ and $\mathcal{V}$ being $4 \times 4$ respectively $2 \times 2$ matrices.

### 3.3.1 The Lax pair in terms of $4 \times 4$ matrices

**Proposition 3.3.2.** *Using the Minkowski model for $\mathbb{H}^3$ the Lax pair is described by the matrices*

$$\mathcal{U} = \begin{pmatrix} 0 & 0 & 2e^{2u} & 0 \\ 1 & 2u_z & 0 & -H \\ 0 & 0 & 0 & -\frac{1}{2}Qe^{-2u} \\ 0 & Q & 2He^{2u} & 0 \end{pmatrix} \quad \textit{and} \quad \mathcal{V} = \begin{pmatrix} 0 & 2e^{2u} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\overline{Q}e^{-2u} \\ 1 & 0 & 2u_{\overline{z}} & -H \\ 0 & 2He^{2u} & \overline{Q} & 0 \end{pmatrix} \tag{3.5}$$

*Proof.* Since the columns of

$$\mathcal{F}_z = (f_z, f_{zz}, f_{z\overline{z}}, N_z) \text{ and } \mathcal{F}_{\overline{z}} = (f_{\overline{z}}, f_{z\overline{z}}, f_{\overline{z}\overline{z}}, N_{\overline{z}}).$$

are $\mathbb{R}^{3,1}$-valued they can be expressed with the frame $\mathcal{F}$ using the proposition 3.2.3. The obtained coefficients then will form the columns of $\mathcal{U}$ resp. $\mathcal{V}$.

First we look at $\mathcal{F}_z$:
In the case $f_z$ there is nothing to compute. For the remaining entries one has to take advantage of the pairwise orthogonality of the elements of $\mathcal{F}$ and differentiate the inner products with respect to $z$.

$$\begin{aligned}
\langle f_z, f\rangle &= 0 \overset{\partial_z}{\Rightarrow} 0 = \langle f_{zz}, f\rangle + \langle f_z, f_z\rangle = \langle f_{zz}, f\rangle \\
\langle f_z, f_{\overline{z}}\rangle &= 2e^{2u} \overset{\partial_z}{\Rightarrow} 2u_z 2e^{2u} = \langle f_{zz}, f_{\overline{z}}\rangle + \langle f_z, f_{\overline{z}z}\rangle = \langle f_{zz}, f_{\overline{z}}\rangle \\
&\quad \text{since } \langle f_z, f_z\rangle = 0 \overset{\partial_{\overline{z}}}{\Rightarrow} 0 = 2\langle f_{z\overline{z}}, f_z\rangle \\
\langle f_z, f_z\rangle &= 0 \overset{\partial_z}{\Rightarrow} 0 = 2\langle f_{zz}, f_z\rangle \\
\langle f_{zz}, N\rangle &= Q \text{ (cf. equation 3.2)} \\
&\Longrightarrow f_{zz} = 2u_z f_z + QN
\end{aligned}$$

$$\begin{aligned}
\langle f_z, f\rangle &= 0 \overset{\partial_{\overline{z}}}{\Rightarrow} 0 = \langle f_{z\overline{z}}, f\rangle + \langle f_z, f_{\overline{z}}\rangle \iff -\langle f_{z\overline{z}}, f\rangle = 2e^{2u} \\
\langle f_{z\overline{z}}, f_z\rangle &= 0 \text{ as shown above} \\
\langle f_{\overline{z}}, f_{\overline{z}}\rangle &= 0 \overset{\partial_z}{\Rightarrow} 0 = 2\langle f_{\overline{z}z}, f_{\overline{z}}\rangle \\
\langle f_{z\overline{z}}, N\rangle &= 2He^{2u} \text{ (cf. equation 3.1)} \\
&\Longrightarrow f_{z\overline{z}} = 2e^{2u} f + 2He^{2u} N
\end{aligned}$$

$$\begin{aligned}
\langle N, f\rangle &= 0 \overset{\partial_z}{\Rightarrow} 0 = \langle N_z, f\rangle + \langle N, f_z\rangle = \langle N_z, f\rangle \\
\langle N_z, f_{\overline{z}}\rangle &= 0 \overset{\partial_z}{\Rightarrow} 0 = \langle N_z, f_{\overline{z}}\rangle + \langle N, f_{\overline{z}z}\rangle = \langle N_z, f_{\overline{z}}\rangle + 2He^{2u} \\
\langle N, f_z\rangle &= 0 \overset{\partial_z}{\Rightarrow} 0 = \langle N_z, f_z\rangle + \langle N, f_{zz}\rangle = \langle N_z, f_z\rangle + Q \\
\langle N, N\rangle &= 1 \overset{\partial_z}{\Rightarrow} 0 = 2\langle N_z, N\rangle \\
&\Longrightarrow N_z = -Hf_z - \frac{1}{2}Qe^{-2u} f_{\overline{z}}
\end{aligned}$$

The same procedure for $\mathcal{F}_{\overline{z}}$ yields the matrix $\mathcal{V}$:

$$\begin{aligned}
\langle f_z, f\rangle &= 0 \overset{\partial_{\overline{z}}}{\Rightarrow} 0 = \langle f_{z\overline{z}}, f\rangle + \langle f_z, f_{\overline{z}}\rangle \iff -\langle f_{z\overline{z}}, f\rangle = 2e^{2u} \\
\langle f_{\overline{z}}, f_{\overline{z}}\rangle &= 0 \overset{\partial_z}{\Rightarrow} 0 = 2\langle f_{z\overline{z}}, f_{\overline{z}}\rangle \\
\langle f_{z\overline{z}}, f_z\rangle &= 0 \text{ and } \langle f_{z\overline{z}}, N\rangle = 2He^{2u} \text{ as shown above} \\
&\implies f_{z\overline{z}} = 2e^{2u}f + 2He^{2u}N
\end{aligned}$$

$$\begin{aligned}
\langle f_{\overline{z}}, f\rangle &= 0 \overset{\partial_{\overline{z}}}{\Rightarrow} 0 = \langle f_{\overline{z}\overline{z}}, f\rangle + \langle f_{\overline{z}}, f_{\overline{z}}\rangle = \langle f_{\overline{z}\overline{z}}, f\rangle \\
\langle f_{\overline{z}}, f_{\overline{z}}\rangle &= 0 \overset{\partial_{\overline{z}}}{\Rightarrow} 0 = 2\langle f_{\overline{z}\overline{z}}, f_{\overline{z}}\rangle \\
\langle f_{\overline{z}}, f_z\rangle &= 2e^{2u} \overset{\partial_{\overline{z}}}{\Rightarrow} 2u_{\overline{z}}2e^{2u} = \langle f_{\overline{z}\overline{z}}, f_z\rangle + \langle f_{\overline{z}}, f_{z\overline{z}}\rangle \\
\langle f_{\overline{z}\overline{z}}, N\rangle &= \overline{\langle f_{zz}, N\rangle} = \overline{Q} \\
&\implies f_{\overline{z}\overline{z}} = 2u_{\overline{z}}f_{\overline{z}} + \overline{Q}N
\end{aligned}$$

$$\begin{aligned}
\langle N, f\rangle &= 0 \overset{\partial_{\overline{z}}}{\Rightarrow} 0 = \langle N_{\overline{z}}, f\rangle + \langle N, f_{\overline{z}}\rangle = \langle N_{\overline{z}}, f\rangle \\
\langle N, f_{\overline{z}}\rangle &= 0 \overset{\partial_{\overline{z}}}{\Rightarrow} 0 = \langle N_{\overline{z}}, f_{\overline{z}}\rangle + \langle N_{\overline{z}}, f_{\overline{z}\overline{z}}\rangle \iff \langle N, f_{\overline{z}}\rangle = -\overline{Q} \\
\langle N, f_z\rangle &= 0 \overset{\partial_{\overline{z}}}{\Rightarrow} 0 = \langle N_{\overline{z}}, f_z\rangle + \langle N, f_{z\overline{z}}\rangle \\
\langle N, N\rangle &= 1 \overset{\partial_{\overline{z}}}{\Rightarrow} 0 = 2\langle N_{\overline{z}}, N\rangle \\
&\implies N_{\overline{z}} = -\frac{1}{2}\overline{Q}e^{-2u}f_z - Hf_{\overline{z}}
\end{aligned}$$

■

**Proposition 3.3.3.** *The compatibility condition $\mathcal{F}_{z\overline{z}} = \mathcal{F}_{\overline{z}z}$ is equivalent to the* ***Maurer-Cartan equation***

$$\mathcal{U}_{\overline{z}} - \mathcal{V}_z - [\mathcal{U}, \mathcal{V}] = 0 \tag{3.6}$$

*Proof.* Using the Lax pair yields

$$\begin{aligned}
\mathcal{F}_{z\overline{z}} &= (\mathcal{F}\mathcal{U})_{\overline{z}} = \mathcal{F}_{\overline{z}}\mathcal{U} + \mathcal{F}\mathcal{U}_{\overline{z}} = \mathcal{F}\mathcal{V}\mathcal{U} + \mathcal{F}\mathcal{U}_{\overline{z}} \\
\mathcal{F}_{\overline{z}z} &= (\mathcal{F}\mathcal{V})_z = \mathcal{F}_z\mathcal{V} + \mathcal{F}\mathcal{V}_z = \mathcal{F}\mathcal{U}\mathcal{V} + \mathcal{F}\mathcal{V}_z
\end{aligned}$$

so plugging them into the compatibility condition leads to the claim:

$$\begin{aligned}
\mathcal{F}_{z\overline{z}} = \mathcal{F}_{\overline{z}z} &\iff \mathcal{F}\mathcal{V}\mathcal{U} + \mathcal{F}\mathcal{U}_{\overline{z}} = \mathcal{F}\mathcal{U}\mathcal{V} + \mathcal{F}\mathcal{V}_z \\
&\iff \mathcal{V}\mathcal{U} + \mathcal{U}_{\overline{z}} - \mathcal{U}\mathcal{V} - \mathcal{V}_z = 0 \\
&\iff \mathcal{U}_{\overline{z}} - \mathcal{V}_z - [\mathcal{U}, \mathcal{V}] = 0
\end{aligned}$$

■

By computing the Maurer-Cartan equation one gets the **Gauss-Codazzi equations** for surfaces in $\mathbb{H}^3$:

$$2u_{z\bar{z}} + 2(H^2 - 1)e^{2u} - \frac{1}{2}Q\bar{Q}e^{-2u} = 0, \qquad Q_{\bar{z}} = 2H_z e^{2u} \tag{3.7}$$

The Codazzi equation now leads directly to the well known fact:

**Corollary 3.3.4.** *$f$ is a CMC surface $\iff$ $Q$ is holomorphic*

Now we are equipped with all tools to formulate the

**Theorem 3.3.5** (Fundamental theorem of surface theory)**.** *If the mappings*

$$\begin{aligned} u &: \Sigma \to \mathbb{R}, \\ H &: \Sigma \to \mathbb{R}, \\ Q &: \Sigma \to \mathbb{C} \end{aligned}$$

*defined on a simply connected two-dimensional manifold $\Sigma$ satisfy the Gauss-Codazzi equations, there exists a conformal immersion $f : \Sigma \to \mathbb{H}^3$ with these maps as the conformal factor, mean curvature and Hopf differential. $f$ is unique up to a rigid motion of $\mathbb{H}^3$.*

**Remark 3.3.6.** *The theorem is applicable to all three space forms. The only difference is the specific appearance of the Gauss-Codazzi equations.*

Using the following two transformations

$$\begin{aligned} z &\mapsto (2\sqrt{1 - H^2})^{-1} w \\ Q &\mapsto 2\sqrt{1 - H^2} e^{2i\psi} \tilde{Q} \end{aligned}$$

where $\psi$ is a real constant, the Gauss equation turns into

$$4(1 - H^2) \cdot 2u_{w\overline{w}} + 2(H^2 - 1)e^{2u} - 4(1 - H^2)e^{2i\phi}e^{-2i\phi} \cdot \frac{1}{2}\tilde{Q}\overline{\tilde{Q}}e^{-2u} = 0 \iff$$

$$2u_{w\overline{w}} - \frac{1}{2}e^{2u} - \frac{1}{2}\tilde{Q}\overline{\tilde{Q}}e^{-2u} = 0 \tag{3.8}$$

By normalizing $\tilde{Q}$ and away from its zeros the latter equation becomes the **cosh-Gordon equation**

$$2u_{w\overline{w}} - \cosh(2u) = 0 \tag{3.9}$$

which is an elliptic non-linear PDE. The cosh-Gordon equation behaves quite differently compared to the sinh-Gordon equation which appears as the Gauss equation for $|H| > 1$ CMC surfaces in $\mathbb{H}^3$. The consequences will be presented in chapter 5.

### 3.3.2 The Lax pair in terms of $2 \times 2$ matrices

Now we use the Hermitian matrix approach to represent the immersion and the frame. In this section $f$, $f_x$, $f_y$ and $N$ are meant as their images $\psi(f)$, $\psi(f_x)$, $\psi(f_y)$ and $\psi(N)$ under the diffeomorphism $\psi$.

**Proposition 3.3.7.** *There exist a unique matrix $F \in SL_2\mathbb{C}$ in order that*

$$f = FF^*,\ e_1 = \frac{f_x}{\|f_x\|} = F\sigma_1 F^*,\ e_2 = \frac{f_y}{\|f_y\|} = F\sigma_2 F^*,\ N = F\sigma_3 F^*,\ F(z_0) = id$$

*holds.*

*Proof.* As already shown the normalized frame $\mathcal{F}_{on}$ is a positive oriented orthonormal basis of $\mathbb{R}^{3,1}$. Since $\psi$ is a linear isometry $\psi(\mathcal{F}_{on})$ constitutes an orthonormal basis too. Because of this fact there exists a unique rotation $F \in PSL_2\mathbb{C}$ which transforms $\sigma_i$ into $\psi(\mathcal{F}_{on})$, which have then the form stated in the claim. By specifying $F$ to be the identity matrix at a particular point $z_0$ it becomes unique in $SL_2\mathbb{C}$.

■

Because of this proposition it is sufficient to know the matrix $F$ to describe the immersion $f$ and its derivatives as well as the unit normal. Therefore calling $F$ the extended frame in this model of the hyperbolic 3-space is justified.

Define $U := F^{-1}F_z$ and $V := F^{-1}F_{\overline{z}}$. These matrices always exist since $F \in SL_2\mathbb{C}$ and $f$ is $C^\infty$.

**Proposition 3.3.8.** *The Lax pair*

$$F_z = F \cdot U, \quad F_{\overline{z}} = F \cdot V$$

*is characterized by the matrices*

$$U = \frac{1}{2}\begin{pmatrix} -u_z & Qe^{-u} \\ 2(1-H)e^u & u_z \end{pmatrix} \quad \textit{and} \quad V = \frac{1}{2}\begin{pmatrix} u_{\overline{z}} & 2(1+H)e^u \\ -\bar{Q}e^{-u} & -u_{\overline{z}} \end{pmatrix} \tag{3.10}$$

*Proof.* First we have to express the derivatives with respect to $z$ and $\overline{z}$ in the same way like $e_1 = \frac{f_x}{\|f_x\|}$ and $e_2 = \frac{f_y}{\|f_y\|}$. By definition of $\partial_z$ and $\partial_{\overline{z}}$ the following statements hold

$$f_z = \frac{1}{2}(f_x - if_y) = e^u(e_1 - ie_2) = e^u F(\sigma_1 - i\sigma_2)F^* = 2e^u F \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} F^*$$
$$f_{\overline{z}} = \frac{1}{2}(f_x + if_y) = e^u(e_1 + ie_2) = e^u F(\sigma_1 + i\sigma_2)F^* = 2e^u F \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} F^*$$

The general strategy is now to differentiate $f_z$, $f_{\bar z}$ and $N$ and compare it to second derivatives as obtained in the previous section.

$$\begin{aligned}
(f_z)_{\bar z} &= u_{\bar z} f_z + 2e^u \left( F_{\bar z} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} F^* + F \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} (F^*)_{\bar z} \right) \\
&= u_{\bar z} f_z + 2e^u \left( FV \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} F^* + F \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} U^* F^* \right) \\
&= u_{\bar z} f_z + 2e^u \left( F \begin{pmatrix} v_{12} & 0 \\ v_{22} & 0 \end{pmatrix} F^* + F \begin{pmatrix} 0 & 0 \\ \overline{u_{11}} & \overline{u_{21}} \end{pmatrix} (F^*)_{\bar z} \right) \\
&= 2e^u F \begin{pmatrix} \overline{v_{12}} & 0 \\ u_{\bar z} + v_{22} + \overline{u_{11}} & \overline{u_{21}} \end{pmatrix} F^* \qquad (3.11)
\end{aligned}$$

$$\begin{aligned}
(f_{\bar z})_z &= u_z f_{\bar z} + 2e^u \left( F_z \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} F^* + F \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} (F^*)_z \right) \\
&= u_z f_{\bar z} + 2e^u \left( FU \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} F^* + F \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} V^* F^* \right) \\
&= u_z f_{\bar z} + 2e^u \left( F \begin{pmatrix} 0 & u_{11} \\ 0 & u_{21} \end{pmatrix} F^* + F \begin{pmatrix} \overline{v_{12}} & \overline{v_{22}} \\ 0 & 0 \end{pmatrix} (F^*)_z \right) \\
&= 2e^u F \begin{pmatrix} \overline{v_{12}} & u_z + u_{11} + \overline{v_{22}} \\ 0 & u_{21} \end{pmatrix} F^* \qquad (3.12)
\end{aligned}$$

Since second derivatives are symmetric the comparison of 3.11 and 3.12 yields

$$-u_z = u_{11} + \overline{v_{22}} \qquad (3.13)$$
$$-u_{\bar z} = v_{22} + \overline{u_{11}} \qquad (3.14)$$

First recall that $f_{z\bar z} = f_{\bar z z} = 2e^{2u} f + 2He^{2u} N$. By writing it in the same form as above one has

$$f_{z\bar z} = f_{\bar z z} = 2e^u F \begin{pmatrix} e^u + He^u & 0 \\ 0 & e^u - He^u \end{pmatrix} F^*$$

Comparing to 3.11 and 3.12 yields

$$v_{12} = e^u + He^u = (1+H)e^u \qquad (3.15)$$
$$u_{21} = e^u - He^u = (1-H)e^u \qquad (3.16)$$

$$\begin{aligned}
(f_z)_z &= u_z f_z + 2e^u \left( F_z \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} F^* + F \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} (F^*)_z \right) \\
&= u_z f_z + 2e^u \left( F \begin{pmatrix} u_{12} & 0 \\ u_{22} & 0 \end{pmatrix} F^* + F \begin{pmatrix} 0 & 0 \\ \overline{v_{11}} & \overline{v_{21}} \end{pmatrix} F^* \right) \\
&= 2e^u F \begin{pmatrix} u_{12} & 0 \\ u_z + u_{22} & \overline{v_{21}} \end{pmatrix} F^* \qquad (3.17)
\end{aligned}$$

Comparing 3.17 to

$$f_{zz} = 2u_z f_z + QN = 2e^u F \begin{pmatrix} \frac{1}{2}Qe^{-u} & 0 \\ 2u_z & -\frac{1}{2}Qe^{-u} \end{pmatrix} F^*$$

yields

$$u_{12} = \frac{1}{2}Qe^{-u} \tag{3.18}$$

$$\overline{v_{21}} = -\frac{1}{2}Qe^{-u} \iff v_{21} = -\frac{1}{2}\overline{Q}e^{-u} \tag{3.19}$$

The comparison of

$$\begin{aligned} N_z &= F_z\sigma_3 F^* + F\sigma_3(F^*)_z \\ &= FU\sigma_3 F^* + F\sigma_3 V^* F^* \\ &= F\left(\begin{pmatrix} u_{11} & -u_{12} \\ u_{21} & -u_{22} \end{pmatrix} + \begin{pmatrix} \overline{v_{11}} & \overline{v_{21}} \\ -\overline{v_{12}} & -\overline{v_{22}} \end{pmatrix}\right) F^* \end{aligned} \tag{3.20}$$

to the result from the from the previous section

$$N_z = -Hf_z - \frac{1}{2}Qe^{-2u}f_{\overline{z}} = F\begin{pmatrix} 0 & -\frac{1}{2}Qe^{-u} \\ -He^{2u} & 0 \end{pmatrix} \tag{3.21}$$

yields

$$u_{11} = -\overline{v_{11}} \iff \overline{u_{11}} = -v_{11} \tag{3.22}$$

$$-u_{22} = \overline{v_{22}} \tag{3.23}$$

plugging this in 3.13 and 3.14 one gets

$$-u_z = u_{11} + \overline{v_{22}} = -u_{22} + u_{11} \overset{(*)}{=} 2u_{11} \iff u_{11} = -\frac{1}{2}u_z \overset{(*)}{\Rightarrow} u_{22} = \frac{1}{2}u_z \tag{3.24}$$

$$-u_{\overline{z}} = v_{22} + \overline{u_{11}} = v_{22} - v_{11} \overset{(*)}{=} 2v_{22} \iff v_{22} = -\frac{1}{2}u_{\overline{z}} \overset{(*)}{\Rightarrow} v_{11} = \frac{1}{2}u_{\overline{z}} \tag{3.25}$$

By using

$$\frac{d}{dz}\det(F) = \operatorname{tr}(U)\det(F)$$

and because $\det(F) \equiv 1$, $U$ is traceless. Applying the same argument to $V$ shows that $(*)$ holds i.e. $U, V \in \mathfrak{sl}_2(\mathbb{C})$.

By gathering all the single entries of $U$ and $V$ we get the claim.

■

As for the Lax pair in terms of $4 \times 4$ matrices, the compatibility condition and the Gauss-Codazzi equations stay the same as in proposition 3.3.3 resp. equation 3.7. They are the only conditions to be satisfied to obtain a solution of the Lax pair, more precisely there holds

**Proposition 3.3.9.** *Let $O \subset \mathbb{C}$ be an open and simply connected set containing 0 and $U, V : O \to \mathfrak{sl}_2(\mathbb{C})$. The solution $F : O \to SL_2\mathbb{C}$ of the Lax pair*

$$F_z = F \cdot U, \;\; F_{\overline{z}} = F \cdot V$$

*exists for any initial condition $F(0) \in SL_2\mathbb{C}$ if and only if $U$ and $V$ satisfy the Maurer-Cartan equation*

$$U_{\overline{z}} - V_z - [U, V] = 0$$

*Each pair $F$, $\tilde{F}$ of solutions differs only by a multiplication with a constant matrix $G$, i.e. $\tilde{F} = GF$*

A proof can be found in [11], proposition 2.3.1.

# 4 The Sym-Bobenko formula

The aim of this section is to describe a construction method for CMC immersions $f$ with arbitrarily given constant mean curvature $H \in (-1, 1)$. From the last section it is known that the $2 \times 2$ frame $F \in SL_2\mathbb{C}$ describes the surface $f$ entirely. So the approach is to construct such a $F$ using the Lax pair.

Instead of using the two Lax equations there is a different approach using differential forms which allows to perform some computations in a more elegant way. A brief overview is presented in [7]. Let $U, V : O \to \mathfrak{sl}_2(\mathbb{C})$ be the matrices describing the Lax pair and $O$ an open and simply connected subset of $\Sigma$. The Lax pair then transforms into

$$dF = F\alpha$$

where $\alpha = Udz + Vd\overline{z} \in \Omega^1(O, \mathfrak{sl}_2(\mathbb{C}))$. Using the commutator $[\cdot, \cdot]$ of $\mathfrak{sl}_2(\mathbb{C})$ for each two forms $\alpha$ and $\beta$ in $\Omega^1(O, \mathfrak{sl}_2(\mathbb{C}))$ we define

$$[\alpha \wedge \beta](X, Y) = [\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)], \;\; X, Y \in T\Sigma \tag{4.1}$$

The resulting object $[\alpha \wedge \beta]$ is then an element of $\in \Omega^2(O, \mathfrak{sl}_2(\mathbb{C}))$. The Maurer-Cartan equation 3.6 is now transformed into

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0.$$

Now we parameterize the mean curvature $H$ by the so-called spectral parameter $\lambda \in \mathbb{C}^*$ and use the transposes of $U$ respectively $V$, thus $\alpha$ turns into:

$$\alpha_\lambda = \frac{1}{2}\begin{pmatrix} u_z dz - u_{\overline{z}} d\overline{z} & \lambda e^u dz - \overline{Q} e^{-u} d\overline{z} \\ Q e^{-u} dz + \lambda^{-1} e^u d\overline{z} & -u_z dz + u_{\overline{z}} d\overline{z} \end{pmatrix} \tag{4.2}$$

Applying the Maurer-Cartan equation to $\alpha_\lambda$ yields the following reduced Gauss-Codazzi equations for CMC surfaces:

$$u_{z\overline{z}} - \frac{1}{4}e^{2u} - \frac{1}{4}Q\overline{Q}e^{-2u} = 0$$
$$\frac{1}{2}Q_{\overline{z}}e^{-u} = 0$$

Note that they are $\lambda$-independent.

For an arbitrary differential form $\omega \in \Omega^1(O, \mathfrak{sl}_2(\mathbb{C}))$ the splitting into the (1,0) part and (0,1) part is denoted

$$\omega = \omega' + \omega''$$

**Definition 4.1.1.** *The Hodge star operator $*$ for $\omega \in \Omega^1(O, \mathfrak{sl}_2(\mathbb{C}))$ is defined as*

$$*\omega = -i\omega' + i\omega''$$

Now the mean curvature $H$ of a CMC immersion $f$ can be computed using the

**Proposition 4.1.2.** *Setting $\omega := f^{-1}df$ for the mean curvature $H$ there holds*

$$2d * \omega = -iH[\omega \wedge \omega]$$

*whereby $f$ is an immersion $f = FF^* : \Sigma \to \mathbb{H}^3$.*

The proof is done by straight forward calculations on both sides ending with a comparison and can be looked up in [11], lemma 2.3.2.

**Theorem 4.1.3** (Sym-Bobenko formula)**.** *Let $\alpha_\lambda$ be as in 4.2. If $F_\lambda$ is the solution of $dF_\lambda = F_\lambda \alpha_\lambda$ and is defined on an open and simply connected subset $O$ of $\Sigma$ then the mapping $f : O \to \mathbb{H}^3$ defined by*

$$f(z) = F_{\lambda_0}(z)(F_{\lambda_0}(z))^*$$

*with $\lambda_0 = e^{q+2i\psi} \in \mathbb{C}^*$ and $q, \psi \in \mathbb{R}$ is a conformally parameterized immersion and has the constant mean curvature*

$$H = tanh(-q)$$

*Proof.* To be able to apply the previous proposition we first compute $\omega = f^{-1}df$:

$$\begin{aligned}
\omega &= (F_\lambda F_\lambda^*)^{-1} d(F_\lambda F_\lambda^*) = F_\lambda^{*-1} F_\lambda^{-1}(d(F_\lambda)F_\lambda^* + F_\lambda d(F_\lambda^*)) \\
&= F_\lambda^{*-1} F_\lambda^{-1}(F_\lambda \alpha_\lambda F_\lambda^* + F_\lambda (F_\lambda \alpha_\lambda)^*) = F_\lambda^{*-1}(\alpha_\lambda + \alpha_\lambda^*)F_\lambda^* \\
&= \frac{1}{2} F_\lambda^{*-1} \begin{pmatrix} 0 & (\lambda + \overline{\lambda}^{-1})e^u dz \\ (\lambda^{-1} + \overline{\lambda})e^u d\overline{z} & 0 \end{pmatrix} F_\lambda^*
\end{aligned}$$

Decomposing $\omega$ into $dz$-part $\omega'$ and $d\overline{z}$-part $\omega''$ yields

$$\begin{aligned}
\omega' = f^{-1} f_z dz &= \frac{1}{2} F_\lambda^{*-1} \begin{pmatrix} 0 & (\lambda + \overline{\lambda}^{-1})e^u dz \\ 0 & 0 \end{pmatrix} F_\lambda^* \\
\omega'' = f^{-1} f_{\overline{z}} d\overline{z} &= \frac{1}{2} F_\lambda^{*-1} \begin{pmatrix} 0 & 0 \\ (\lambda^{-1} + \overline{\lambda})e^u d\overline{z} & 0 \end{pmatrix} F_\lambda^*
\end{aligned}$$

Using the properties of the metric in the Hermitian matrix model

$$\langle \omega', \omega' \rangle = -\det(\omega') = 0 = -\det(\omega'') = \langle \omega'', \omega'' \rangle$$

and the metric's left-invariance you can see the conformality of $f$:

$$\begin{aligned} 0 &= \langle \omega', \omega' \rangle = \langle f^{-1} f_z dz, f^{-1} f_z dz \rangle \\ &= \langle f_z dz, f_z dz \rangle = \langle f_z, f_z \rangle dz dz \end{aligned}$$

A similar computation with $\omega'$ replaced by $\omega''$ shows $\langle f_{\overline{z}}, f_{\overline{z}} \rangle = 0$. To compute the conformal factor we set

$$F_\lambda = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

then only some matrix multiplications are needed:

$$\begin{aligned} \langle f_z, f_{\overline{z}} \rangle dz d\overline{z} &= \langle \omega', \omega'' \rangle = -\frac{1}{2} \mathrm{tr}(\omega' \sigma_2 (\omega'')^T \sigma_2) \\ &= -\frac{1}{8}(\lambda + \overline{\lambda}^{-1})(\lambda^{-1} + \overline{\lambda})\ e^{2u}\ \mathrm{tr} \left( i^2 \cdot F_\lambda^{*-1} \begin{pmatrix} 0 & dz \\ 0 & 0 \end{pmatrix} F_\lambda^* \right. \\ &\qquad \left. \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} F_\lambda^T \begin{pmatrix} 0 & d\overline{z} \\ 0 & 0 \end{pmatrix} (F_\lambda^{*-1})^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) \\ &= \frac{e^{2u}}{8}(|\lambda| + \frac{1}{|\lambda|})^2 \mathrm{tr} \left( \begin{pmatrix} 0 & f_{22} dz \\ 0 & -f_{21} dz \end{pmatrix} \begin{pmatrix} f_{12} & -f_{11} \\ f_{22} & -f_{21} \end{pmatrix} \begin{pmatrix} 0 & f_{11} d\overline{z} \\ 0 & f_{12} d\overline{z} \end{pmatrix} \begin{pmatrix} -f_{21} & -f_{22} \\ f_{11} & f_{12} \end{pmatrix} \right) \\ &= \frac{e^{2u}}{8}(|\lambda| + \frac{1}{|\lambda|})^2\ \mathrm{tr} \left( \begin{pmatrix} 0 & (-f_{21} f_{22} + f_{22} f_{21}) dz \\ 0 & (f_{11} f_{22} - f_{12} f_{21}) dz \end{pmatrix} \begin{pmatrix} 0 & (f_{12} f_{11} - f_{11} f_{12}) d\overline{z} \\ 0 & (f_{22} f_{11} - f_{21} f_{12}) d\overline{z} \end{pmatrix} \right) \\ &= \frac{e^{2u}}{8}(|\lambda| + \frac{1}{|\lambda|})^2\ \mathrm{tr} \left( \begin{pmatrix} 0 & * \\ 0 & (\det F_\lambda)^2 dz d\overline{z} \end{pmatrix} \right) \\ &\iff \\ \langle f_z, f_{\overline{z}} \rangle &= (|\lambda| + |\lambda|^{-1})^2 \frac{e^{2u}}{8} \end{aligned}$$

By definition 4.1 of the wedge product one gains

$$\begin{aligned} [\omega \wedge \omega]\,(X, Y) &= [\omega(X), \omega(Y)] - [\omega(Y), \omega(X)] \\ &= 2\,[\omega(X), \omega(Y)] \end{aligned}$$

For abbreviation purposes we define:

$$a := \frac{1}{2}(\lambda + \overline{\lambda}^{-1}) e^u,\ \ b := \frac{1}{2}(\lambda^{-1} + \overline{\lambda}) e^u \tag{4.3}$$

$$\frac{1}{2}\,[\omega \wedge \omega] = [\omega(X), \omega(Y)] = \omega(X)\omega(Y) - \omega(Y)\omega(X)$$

$$
\begin{aligned}
&= F_\lambda^{*-1}\left(\begin{pmatrix} 0 & adz(X) \\ bd\overline{z}(X) & \end{pmatrix}\begin{pmatrix} 0 & adz(Y) \\ bd\overline{z}(Y) & 0 \end{pmatrix}\right. \\
&- \left.\begin{pmatrix} 0 & adz(Y) \\ bd\overline{z}(Y) & 0 \end{pmatrix}\begin{pmatrix} 0 & adz(X) \\ bd\overline{z}(X) & \end{pmatrix}\right) F_\lambda^* \\
&= F_\lambda^{*-1}\begin{pmatrix} ab(dz(X)d\overline{z}(Y) - dz(Y)d\overline{z}(X)) & 0 \\ 0 & ab(d\overline{z}(x)dz(Y) - d\overline{z}(Y)dz(X)) \end{pmatrix} F_\lambda^* \\
&= abF_\lambda^{*-1}\begin{pmatrix} (dz \wedge d\overline{z})(X,Y) & 0 \\ 0 & (d\overline{z} \wedge dz)(X,Y) \end{pmatrix} F_\lambda^* \\
&\Longleftrightarrow \\
&[\omega \wedge \omega] = \frac{1}{2}(|\lambda| + |\lambda|^{-1})^2 e^{2u} F_\lambda^{*-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} F_\lambda dz \wedge d\overline{z}
\end{aligned}
$$

$$
\begin{aligned}
d(*\omega) &= d(-i\omega' + i\omega'') \\
&= i \cdot d\left(F_\lambda^{*-1}\underbrace{\begin{pmatrix} 0 & -adz \\ bd\overline{z} & 0 \end{pmatrix}}_{=:M} F_\lambda^*\right) \\
&= i\left(d(F_\lambda^{*-1}) \wedge MF_\lambda^* + F_\lambda^{*-1}(d(M)F_\lambda^* + (-1)^{degM} M \wedge d(F_\lambda^*)\right) \\
&= i\left(-F_\lambda^{*-1}(F_\lambda\alpha_\lambda)^* F_\lambda^{*-1} \wedge MF_\lambda^* + F_\lambda^{*-1}d(M)F_\lambda^* - F_\lambda^{*-1}M \wedge (F_\lambda\alpha_\lambda)^*\right) \\
&= iF_\lambda^{*-1}(-\alpha_\lambda^* \wedge M + dM - M \wedge \alpha_\lambda^*)F_\lambda^* \\
&= iF_\lambda^{*-1}\left(-\frac{1}{2}\begin{pmatrix} 0 + \overline{\lambda}^{-1}be^u dz \wedge d\overline{z} & -au_z d\overline{z} \wedge dz + 0 \\ 0 + bu_z dz \wedge d\overline{z} & -\overline{\lambda}ae^u d\overline{z} \wedge dz + 0 \end{pmatrix}\right. \\
&+ \left.\begin{pmatrix} 0 & -au_{\overline{z}} d\overline{z} \wedge dz \\ bu_z dz \wedge d\overline{z} & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -a\overline{\lambda}e^u dz \wedge d\overline{z} + 0 & au_{\overline{z}} dz \wedge d\overline{z} + 0 \\ 0 - bu_z d\overline{z} \wedge dz & 0 + b\overline{\lambda}^{-1}e^u d\overline{z} \wedge dz \end{pmatrix}\right) F_\lambda^* \\
&= -\frac{i}{2}e^u F_\lambda^{*-1}\begin{pmatrix} \overline{\lambda}^{-1}b - a\overline{\lambda} & 0 \\ 0 & \overline{\lambda}a - b\overline{\lambda}^{-1} \end{pmatrix} F_\lambda^* dz \wedge d\overline{z} \\
&\overset{4.3}{=} -\frac{i}{4}e^{2u}(|\lambda|^{-2} - |\lambda|^2)F_\lambda^{*-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} F_\lambda^* dz \wedge d\overline{z}
\end{aligned}
$$

Setting $\lambda = e^{q+2i\psi}$ the mean curvature can be computed as

$$
\begin{aligned}
H &= \frac{|\lambda|^{-2} - |\lambda|^2}{(|\lambda| + |\lambda|^{-1})^2} \\
&= \frac{e^{-2q} - e^{2q}}{e^{2q} + 2e^q e^{-q} + e^{-2q}} = \frac{(e^{-q} + e^q)(e^{-q} - e^q)}{(e^{-q} + e^q)^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{e^{-q} - e^{q}}{e^{-q} + e^{q}} = \frac{-\sinh(q)}{\cosh(q)} \\
&= -\tanh(q) = \tanh(-q)
\end{aligned}
$$

■

**Remark 4.1.4.** *Unlike the case $|H| > 1$, here $\lambda$ can be chosen arbitrarily in $\mathbb{C}^*$. Especially there are no problems with $\mathbb{H}^3$ degenerating to $\mathbb{R}^3$ if $\lambda \in \mathbb{S}^1$.*

There also exist other construction methods for CMC surfaces in the three standard space forms and particularly the hyperbolic three-space. One approach using Gauss maps to construct surfaces with $H \in (-1, 1)$ is presented in [1].

# 5 Non-existence of compact surfaces

In this chapter a proof for the non-existence of compact surfaces is presented. This fact distinguishes the surfaces in $\mathbb{H}^3$ with $|H| < 1$ completely from surfaces in other space forms or the other settings in $\mathbb{H}^3$. A proof for that can be given using theory of elliptic partial differential equations. Therefore we first need some preparations.

**Proposition 5.1.1.** *The conformal factor $u$ coming from an immersion $f : \Sigma \to \mathbb{H}^3$ with $H$ taking only values in $(-1, 1)$ is a strictly subharmonic map, i.e. $-\Delta u < 0$.*

*Proof.* First recall the Gauss equation from section 3.3 in the general form:

$$\begin{aligned} &2u_{z\bar{z}} + 2(H^2 - 1)e^{2u} - \frac{1}{2}Q\overline{Q}e^{-2u} = 0 \iff \\ &4u_{z\bar{z}} = -4(H^2 - 1)e^{2u} + |Q|^2 e^{-2u} \end{aligned}$$

Since $|H| < 1$ we have $H^2 - 1 < 0$ which leads to

$$4u_{z\bar{z}} = 4(1 - H^2)e^{2u} + |Q|^2 e^{-2u} > 0$$

Use of coordinates $x$ and $y$ again and $u_{z\bar{z}} = \frac{1}{4}\Delta u$ yields

$$\Delta u > 0 \tag{5.1}$$

■

**Corollary 5.1.2.** *The conformal factor $u$ cannot be a constant map.*

*Proof.* Assume $u$ is constant and the conformal factor of an immersion $f$. As for any constant map $\Delta u$ vanishes. Like any conformal factor $u$ satisfies the Gauss equation and therefore is strictly subharmonic, leading to a contradiction.

■

**Lemma 5.1.3.** *Let $O$ be an open subset of $\mathbb{R}^n$. A twice-differentiable map*

$$w : O \to \mathbb{R} \text{ with } \Delta w > 0$$

*cannot have its maximum in $O$.*

*Proof.* We assume $w$ has a local maximum at a point $x_0 \in O$.

The Hessian matrix $H(w, x_0)$ at $x_0$ is then negative semidefinite. According to Schwartz's theorem $H(w, x_0)$ is symmetric and therefore it has only non-positive eigenvalues. Since a matrix' trace is the sum of its eigenvalues $\lambda_i$ we have

$$0 \geq \sum_{i=1}^{n} \lambda_i = \operatorname{tr}(H(w, x_0)) = \Delta w$$

which contradicts with the premise $\Delta w > 0$.

■

**Theorem 5.1.4** (Non-existence of compact surfaces)**.** *Let $\Sigma$ be a closed Riemann surface. Then immersions*

$$f : \Sigma \to \mathbb{H}^3,$$

*modelling a surface, cannot exist.*

*Proof.* We assume the existence of such a surface $f$. The mappings $u, H, Q$ are then defined on $\Sigma$ too. $\Sigma$ is by definition compact and has no boundary. For that reason $u$ has a local maximum at a point $p \in \Sigma$.

Consider $U$ being the neighborhood of $p$ in $\Sigma$ with a conformal chart $z$. The conformal factor $\tilde{u} = u \circ z^{-1}$ is then a real-valued map on a open subset $z(U)$ of $\mathbb{R}^2$ and has a local maximum at $x_0 = z(p)$. According to the proposition 5.1.1 $\tilde{u}$ is subharmonic und because of the previous lemma cannot have a maximum in $z(U)$. In total one gets a contradiction to the existence of $f$.

■

**Remark 5.1.5.** *Consequently there are also no CMC tori, which is a different situation compared to other space forms.*

# 6 Conclusions and outlook

At this point we want to present a summary of the work done so far. First we introduced two different models for the hyperbolic 3-space: the Minkowski model and the Hermitian matrix model. Then a brief overview about surface theory was given including the derivation of the moving frame within the two models. Afterwards the frames' behavior towards differentiation was determined using the Lax pairs. The conditions imposed on the matrices $U$ and $V$ describing the Lax pairs led to the Gauss-Codazzi equations which are crucial for the existence of a specific surface due to the fundamental theorem of surface theory. At this point the differences between the case $|H| > 1$ and the considered setting $|H| < 1$ start to take effect. The Sym-Bobenko formula presented in chapter 4 looks the same as in the other setting but it is now defined for every $\lambda \in \mathbb{C}^*$. The most important consequence is the non-existence of compact surfaces as presented in the previous chapter, which is a completely different situation compared to other settings.

One starting point for future investigations could be the consideration of CMC cylinders. Therefore it is necessary to start with single periodic solutions $u$ and $Q$ of the Gauss equation. Then the frame $F$ is periodic with the same period $\tau \in \mathbb{C}^*$. Its behavior along a period is described with help of the monodromy

$$M : \mathbb{C}^* \to SL_2\mathbb{C}$$

by the equation $F_\lambda(z + \tau) = M(\lambda)F_\lambda(z)$. With the help of this holomorphic map the spectral curve

$$\Gamma = \{(\lambda, \mu) \in \mathbb{C}^2 : \det(\mu \cdot id - M(\lambda)) = 0\}$$

which is a hyperelliptic Riemann surface can be described. Results derived by considering $\Gamma$ then allows conclusions for the original surface. One interesting question is then to determine conditions for $\Gamma$ having finite genus.

# 7 Bibliography


[1] R. Aiyama and K. Akutagawa. Kenmotsu type representation formula for sufaces with prescribed mean curvature in the hyperbolic 3-space. *Journal of Mathematical Society of Japan*, 52:877–898, 2000.

[2] M. Babich and A. Bobenko. Willmore tori with umbilic lines and minimal surfaces in hyperbolic space. *Duke Mathematical Journal*, 72(1):151–185, 1993.

[3] M. Carmeli. *Group Theory and General Relativity.* Imperial College Press, London, 2000.

[4] S. Fujimori, S. Kobayashi, and W. Rossman. Loop group methods for constant mean curvature surfaces, arXiv:math/0602570v1 [math.DG], 2005.

[5] J. Jost. *Partial Differential Equations.* Springer Verlag, Heidelberg, 2002.

[6] W. Kewlin. Deformation of constant mean curvature tori in a three sphere. Master's thesis, Universität Mannheim, 2008.

[7] M. Kilian, S. P. Kobayashi, W. Rossman, and N. Schmitt. Constant mean curvature surfaces in 3-dimensional space forms. preprint.

[8] M. Knopf. Compact surfaces of constant mean curvature in the 3-sphere. Master's thesis, Universität Mannheim, 2008.

[9] A. I. Kostrikin and J. I. Manin. *Linear algebra and geometry.* Gordon and Breach Science Publ., New York [u.a.], 1989.

[10] J. M. Lee. *Introduction to smooth manifolds.* Graduate texts in mathematics ; 218. Springer, New York ; Heidelberg [u.a.], 2003.

[11] M. Leimeister. Deformation of CMC surfaces in $\mathbb{H}^3$. Master's thesis, Universität Heidelberg, 2009.

[12] G. L. Naber. *The geometry of Minkowski spacetime.* Springer Verlag, New York, 2003.

[13] V. Neugebauer. Constant mean curvature tori in the 3-hyperbolic space. Master's thesis, Universität Mannheim, 2008.

[14] W. Rossman. Imperfect draft of a supplement to “Loop group methods for constant mean curvature surfaces”. preprint.