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Diplomarbeit

Deformation of CMC surfaces in $\mathbb{H}^3$

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Contents

1 Introduction

Surfaces of constant mean curvature are a classical field of differential geometry. Apart from their rich mathematical properties they can also be used as models for certain natural phenomenona. For example, the shape of a soap film spanned into a given boundary will form a minimal surface, i.e. a zero mean curvature surface. Minimizing the area of a surface enclosing a given volume yields surfaces of non-zero constant mean curvature. For a long time, the round sphere was the only known compact example of a CMC surface in $\mathbb{R}^3$. Due to the construction of a CMC torus by Wente in [18], the conjecture that the sphere is the only example of that kind could be proved false. After this discovery, new effort took place in studying these surfaces. In the late eighties, Pinkall and Sterling [14] and independently Hitchin [6] classified all CMC tori in $\mathbb{R}^3$. This work was the starting point for applying methods classically rooted in integrable systems theory for geometric problems. The link between these two areas of mathematics is given by the sinh-Gordon equation $2u_{z\bar{z}} + \sinh(2u) = 0$ for a function $u : U \subset \mathbb{C} \to \mathbb{R}$. The function involved describes the conformal factor of a CMC immersion $f : U \to \mathbb{R}^3$ of a torus. On the other hand, the sinh-Gordon equation leads to a completely integrable system that can be solved by means of Riemann surface theory, spectral curves and Theta functions (carried out in [1]).

The goal of this thesis is to study CMC surfaces in hyperbolic three-space $\mathbb{H}^3$. We first describe some of the classical differential geometry of surfaces and introduce moving frames. Then we give a matrix representation of moving frames and the ambient space $\mathbb{H}^3$ which gives an effective way to do computations and relates our theory to the one in the other space forms $\mathbb{R}^3$ and $\mathbb{S}^3$. The frame $F \in SU_2$ satisfies a system of ordinary differential equations, the so called Lax pair $F_z = FU,\ F_{\bar{z}} = FV$ for certain matrices $U, V \in SL_2(\mathbb{C})$. We introduce the Gauss-Codazzi equations as the compatibility condition of the Lax pair. Conversely, for any solution of the Lax pair we prove the Sym-Bobenko formula, which reconstructs the immersion f from the moving frame F. In the next section we show how to study CMC immersions of cylinders and tori via the concept of spectral curves. For a frame F and a period of the metric, we can introduce a monodromy describing how the frame changes after traversing this period once. The eigenvalue curve of the monodromy is a two-sheeted covering of $\mathbb{CP}^1$ and thus a hyperelliptic Riemann surface. After studying the properties of the spectral curve of a CMC immersion of a torus we give conditions a Riemann surface has to fulfill in order to be the spectral curve of such an immersion.

In the third chapter we compute explicitly some examples of CMC cylinders in $\mathbb{H}^3$ and show that for this ambient space, there exist no CMC tori of spectral genus $g = 0$ or $g = 1$. This is a difference compared to the case of $\mathbb{S}^3$ as ambient space, which is caused by the different Sym-Bobenko formula. The case of $\mathbb{H}^3$ is similar to $\mathbb{R}^3$, as there are also no CMC tori of spectral genus $g = 0$ and $g = 1$.

In the fourth chapter, we show how to encode the information of an immersed CMC torus in three polynomials, called the spectral data. Then we introduce a deformation theory for this spectral data, leading to ordinary differential equations that deform the spectral data such that the corresponding surface stays in the set of CMC cylinders and tori respectively. By means of this deformation theory we want to construct a CMC torus in $\mathbb{H}^3$. Since there are no such tori of spectral genus $g = 0$ we take a CMC cylinder with $g = 0$ as starting point of the deformation and describe the spectral data corresponding to a class of cylinders having certain symmetries on the spectral curve. In a next step, the deformation opens double points on the spectral curve, which results in increasing the genus to $g = 2$. Finally we study the deformation equations to determine possible endpoints of the deformation. It is shown how the deformation takes place in the moduli space of doubly periodic solutions of the sinh-Gordon equation and we discuss whether it is possible to deform a cylinder into a torus.

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2 Differential geometry of surfaces in $\mathbb{H}^3$

2.1 Basic surface theory and the $\sinh$-Gordon equation

We use the Minkowski model for the hyperbolic 3-space and define

$$\mathbb{H}^3 = \left\{ (x_0, x_1, x_2, x_3) \in \mathbb{R}^{3,1} \;\middle|\; \sum_{j=1}^{3} x_j^2 - x_0^2 = -1 \,, x_0 > 0 \right\} \tag{2.1}$$

where $\mathbb{R}^{3,1}$ is the 4-dimensional Minkowski space, i.e. $\mathbb{R}^4$ with the standard Minkowski metric

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_0 y_0.$$

Definition 2.1.1. *Let $U \subset \mathbb{R}^2$ be a simply connected open set and $f : U \to \mathbb{H}^3$ be a differentiable map. Then f is called an immersion, if for all $p \in U$ the differential $d_p f : \mathbb{R}^2 \to T_{f(p)}\mathbb{H}^3$ is injective.*

An immersion $f : U \to \mathbb{H}^3$ is called conformal if the derivatives satisfy

$$\langle f_x, f_y \rangle = 0, \quad \langle f_x, f_x \rangle = \langle f_y, f_y \rangle = 4e^{2u}, \tag{2.2}$$

for a function $u : U \to \mathbb{R}$, called the conformal factor.

We identify $\mathbb{R}^2 \cong \mathbb{C}$ and use the Wirtinger operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Denote $f_z = \frac{\partial f}{\partial z}, f_{\bar{z}} = \frac{\partial f}{\partial \bar{z}}$. Then $\{f_z(p), f_{\bar{z}}(p)\}$ form a basis of $T_{f(p)}f(U) \subset T_{f(p)}\mathbb{H}^3$, where $f(U)$ is a conformally immersed surface in $\mathbb{H}^3$. Conformality in this setting reads $\langle f_z, f_z \rangle = \langle f_{\bar{z}}, f_{\bar{z}} \rangle = 0, \; \langle f_z, f_{\bar{z}} \rangle = 2e^{2u}$.

Because the immersed surface lies in $\mathbb{H}^3$ we have $\langle f, f \rangle = -1$ with the induced scalar product from $\mathbb{R}^{3,1}$. Therefore

$$\langle f, f_x \rangle = \frac{\partial}{\partial x}\langle f, f \rangle - \langle f_x, f \rangle = -\langle f_x, f \rangle \quad \Rightarrow \langle f, f_x \rangle = 0$$

and similarly $\langle f, f_y \rangle = 0$. So we see that $\{f, f_x, f_y\}$ is an orthogonal system of vectors. Define the **normal vector** N to the surface $f(U)$ as the unique vector of length 1 such that $\{f, f_x, f_y, N\}$ is a positively oriented basis of $\mathbb{R}^{3,1}$.

Definition 2.1.2. *The collection of vectors $\{f, f_x, f_y, N\}$ is called the extended frame of the surface $f(U)$.*

Proposition 2.1.3. *With $F = (f, f_z, f_{\bar{z}}, N)$ there holds*

$$F_z = FU, \qquad F_{\bar{z}} = FV \tag{2.3}$$

where the Lax pair U, V is given by

$$U = \begin{pmatrix} 0 & 0 & 2e^{2u} & 0 \\ 1 & 2u_z & 0 & -H \\ 0 & 0 & 0 & -\frac{1}{2}Qe^{-2u} \\ 0 & Q & 2He^{2u} & 0 \end{pmatrix} \tag{2.4}$$

$$V = \begin{pmatrix} 0 & 2e^{2u} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2}\bar{Q}e^{-2u} \\ 1 & 0 & 2u_{\bar{z}} & -H \\ 0 & 2He^{2u} & \bar{Q} & 0 \end{pmatrix} \tag{2.5}$$

For a derivation of the above equations, see e.g. [13]. The quantity $H = \frac{1}{2}e^{-2u}\langle f_{z\bar{z}}, N\rangle$ is the mean curvature, and $Q = \langle f_{zz}, N\rangle$ is called the **Hopf differential**.

Equation (2.3) is a system of ordinary differential equations. The compatibility condition reads

$$F_{z\bar{z}} = F_{\bar{z}z} \qquad \Leftrightarrow \qquad U_{\bar{z}} - V_z - [U, V] = 0. \tag{2.6}$$

The second equation is called the **Maurer-Cartan equation** for the Lax pair U, V. The explicit computation with U and V as given in the proposition leads to the **Gauss-Codazzi equations** for the immersion f.

$$u_{z\bar{z}} + e^{2u}(H^2 - 1) - \frac{1}{4}Q\bar{Q}e^{-2u} = 0, \qquad Q_{\bar{z}} = 2H_z e^{2u}. \tag{2.7}$$

Now suppose we have constant mean curvature H with $|H| > 1$. Then $Q_{\bar{z}} = 0$, so Q is holomorphic. With the coordinate transformation $z \mapsto (2\sqrt{H^2-1})^{-1}z$ and $Q \mapsto (2\sqrt{H^2-1})^{-1}e^{-2i\psi}Q$ for a fixed $\psi \in \mathbb{R}$, we get

$$2u_{z\bar{z}} - \frac{1}{2}Q\bar{Q}e^{-2u} + e^{2u} = 0$$

If we further normalize the Hopf differential to have unit modulus, this yields the elliptic sinh-Gordon equation.

$$2u_{z\bar{z}} + \sinh(2u) = 0. \tag{2.8}$$

This nonlinear PDE has been studied extensively in the field of integrable systems. Therefore, when looking for metrics of CMC surfaces, one can use these methods to construct solutions. Note that in the case of CMC tori, the assumption of constant Hopf differential is no loss of generality, since doubly periodic holomorphic functions (i.e. elliptic functions without poles) are constant.

Having introduced the above quantities, one can show that they determine a conformally immersed surface up to isometries. This is the **fundamental theorem of surface theory**.

Theorem 2.1.4. *Let U be a simply connected 2-dimensional manifold and $u, H : U \to \mathbb{R}$ and $Q : U \to \mathbb{C}$ be functions satisfying the Gauss-Codazzi equations. Then there exists a conformal immersion $f : U \to \mathbb{H}^3$ with metric $g = 4e^{2u}(dx^2 + dy^2)$, mean curvature H and Hopf differential Q. This immersion is unique up to rigid motions in $\mathbb{H}^3$.*

2.2 The Lax pair in terms of 2×2 matrices

In order to be able to work in a simpler setting we use a representation of $\mathbb{H}^3$ in terms of 2×2 matrices and then derive the Lax pair and Gauss-Codazzi equations in this notation. We first introduce the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Identify $\mathbb{H}^3$ with $SL_2\mathbb{C}/SU_2$ via

$$(x_0, x_1, x_2, x_3) \mapsto x_0 Id + \sum_{j=1}^{3} x_j \sigma_j = \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix}. \tag{2.9}$$

This mapping is well-defined and an isomorphism since each symmetric hermitian matrix with determinant one (as is the left-hand side) can be written as $F\bar{F}^t, F \in SL_2\mathbb{C}$ and this identification is unique up to multiplication by elements of SU_2 (see [15] for further details). The norm on $\mathbb{H}^3$ in this matrix representation is given by $\langle x, x \rangle = -det(X)$ and the Minkowski scalar product by $\langle x, y \rangle = -\frac{1}{2} tr[X\sigma_2 Y^t \sigma_2]$, where X, Y are the matrices according to the above isomorphism.

For some point $X \in \mathbb{R}^{3,1}$ in matrix representation and an element $A \in SL_2\mathbb{C}$, the mapping

$$X \mapsto AX\bar{A}^t \tag{2.10}$$

is a rotation in $\mathbb{R}^{3,1}$ corresponding to $X \mapsto RX,\ R \in SO(3,1)$. Taking a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$$

the mapping (2.10) corresponds to the rotation in $\mathbb{R}^{3,1}$ given by

$$R = \begin{pmatrix} Re(\bar{a}c + \bar{b}c) & Im(\bar{b}c - \bar{a}d) & Re(\bar{a}c - \bar{b}d) & Re(\bar{a}c + \bar{b}d) \\ Im(\bar{b}c + \bar{a}d) & Re(\bar{a}d - \bar{b}c) & Im(\bar{a}c - \bar{b}d) & Im(\bar{a}c + \bar{b}d) \\ Re(\bar{a}b - \bar{c}d) & Im(a\bar{b} + \bar{c}d) & (a\bar{a} - b\bar{b} - c\bar{c} + d\bar{d})/2 & (a\bar{a} + b\bar{b} - c\bar{c} - d\bar{d})/2 \\ Re(\bar{a}b + \bar{c}d) & Im(a\bar{b} + c\bar{d}) & (a\bar{a} - b\bar{b} + c\bar{c} - d\bar{d})/2 & (a\bar{a} + b\bar{b} + c\bar{c} - d\bar{d})/2 \end{pmatrix}.$$

Now consider a conformal immersion $f : U \to \mathbb{H}^3$ with extended orthonormal frame $\{f, e_1, e_2, N\}$, where $e_1 = \frac{f_x}{|f_x|}$ and $e_2 = \frac{f_y}{|f_y|}$. Since $\{\mathbb{1}, \sigma_1, \sigma_2, \sigma_3\}$ is an orthonormal basis of the matrix representation $SL_2\mathbb{C}\,/SU_2$ of $\mathbb{H}^3$, we can find $F \in SL_2\mathbb{C}$ that rotates this basis to the matrix representation of the extended frame

$$f = F\bar{F}^t, \quad e_1 = F\sigma_1\bar{F}^t, \quad e_2 = F\sigma_2\bar{F}^t, \quad N = F\sigma_3\bar{F}^t. \tag{2.11}$$

Definition 2.2.1. *In the representation $\mathbb{H}^3 \cong SL_2(\mathbb{C})/SU_2$, the matrix $F \in SL_2(\mathbb{C})$ satisfying* (2.11) *is called the exended frame of the immersion f.*

Since $e_i = \frac{f_i}{|f_i|}$, with $i = x, y$, and $|f_i| = 2e^u$ it follows

$$f_z = \frac{1}{2}(f_x - if_y) = e^u F(\sigma_1 - i\sigma_2)\bar{F}^t = 2e^u F \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \bar{F}^t. \tag{2.12}$$

Similarly we get

$$f_{\bar{z}} = 2e^u F \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \bar{F}^t. \tag{2.13}$$

With these formulas one can determine the Lax pair in terms of 2×2 matrices from the system

$$F_z = FU, \qquad F_{\bar{z}} = FV, \qquad F(0) = \mathbb{1}$$

By writing $U = (u_{ij}), V = (v_{ij})$ and using $f_{z\bar{z}} = f_{\bar{z}z}$ we get

$$2e^u F \begin{pmatrix} 0 & -u_z \\ u_{\bar{z}} & 0 \end{pmatrix} \bar{F}^t = 2e^u F \begin{pmatrix} \bar{v}_{12} - v_{12} & u_{11} + \bar{v}_{22} \\ -v_{22} - \bar{u}_{11} & u_{21} - \bar{u}_{21} \end{pmatrix} \bar{F}^t$$

and therefore $v_{12}, u_{21} \in \mathbb{R},\ -u_{\bar{z}} = v_{22} + \bar{u}_{11},\ -u_z = u_{11} + \bar{v}_{22}$.
From the previous section we know that $f_{\bar{z}z} = 2e^{2u}f + 2He^{2u}N$, where $f = F\bar{F}^t,\ N = F\sigma_3\bar{F}^t$ and hence

$$f_{\bar{z}z} = 2e^{2u} F \begin{pmatrix} 1 + H & 0 \\ 0 & 1 - H \end{pmatrix} \bar{F}^t.$$

Comparing this with our previous formula for $f_{\bar{z}z}$ we get

$$F \begin{pmatrix} e^u(1 + H) & -u_z \\ 0 & e^u(1 - H) \end{pmatrix} \bar{F}^t = F \begin{pmatrix} \bar{v}_{12} & u_{11} + \bar{v}_{22} \\ 0 & u_{21} \end{pmatrix} \bar{F}^t.$$

Therefore $v_{12} = e^u(1 + H),\ u_{21} = e^u(1 - H)$.

In the same manner one can compute

$$f_{zz} = u_z f_z + 2e^u F \begin{pmatrix} u_{12} & 0 \\ u_{22} + \bar{v}_{11} & \bar{v}_{21} \end{pmatrix} \bar{F}^t,$$

and the 4×4 Lax pair yields $f_{zz} = 2u_z f_z + QN$. Putting this together we get

$$F \begin{pmatrix} Q & 0 \\ 2e^u u_z & -Q \end{pmatrix} \bar{F}^t = 2e^u F \begin{pmatrix} u_{12} & 0 \\ u_{22} + \bar{v}_{11} & \bar{v}_{21} \end{pmatrix} \bar{F}^t,$$

giving $u_{12} = \frac{1}{2}e^{-u}Q,\ v_{21} = -\frac{1}{2}e^{-u}\bar{Q},\ u_z = u_{22} + \bar{v}_{11}$.

Since $F^{-1}dF = Udz + Vd\bar{z}$ is an element of $\Omega^1(\mathbb{C}, \mathfrak{sl}_2\mathbb{C})$, U and V have to be trace-free, i.e. $u_{11} = -u_{22},\ v_{11} = -v_{22}$. Taking $u_{22} = \bar{v}_{11} = \frac{1}{2}u_z$, we get the Lax pair in terms of 2×2 matrices.

Proposition 2.2.2. *Let $f : U \to \mathbb{H}^3$ be a conformal immersion and $F \in SL_2\mathbb{C}$ such that $f = F\bar{F}^t$. Then there holds*

$$F_z = FU, \qquad F_{\bar{z}} = FV,$$

where the Lax pair U, V is given by

$$U = \frac{1}{2}\begin{pmatrix} -u_z & e^{-u}Q \\ 2e^u(1-H) & u_z \end{pmatrix}, \quad V = \frac{1}{2}\begin{pmatrix} u_{\bar{z}} & 2e^u(1+H) \\ -e^{-u}\bar{Q} & -u_{\bar{z}} \end{pmatrix} \tag{2.14}$$

and Q is the Hopf differential and H the mean curvature of the immersion.

Again, the Gauss-Codazzi equations read

$$u_{z\bar{z}} + e^{2u}(H^2 - 1) - \frac{1}{4}Q\bar{Q}e^{-2u} = 0, \qquad Q_{\bar{z}} = 2H_z e^{2u} \tag{2.15}$$

From the classical theory of surfaces it is known that up to isometries a surface is determined by the triple (u, H, Q), where e^{2u} is the conformal factor, H the mean curvature and Q the Hopf differential. If H is constant, one can replace Q by $\lambda^{-1}Q$ for some $\lambda \in S^1$, and see that it still satisfies the Gauss-Codazzi equations. Therefore there exists a S^1-family of CMC surfaces to every surface, called the associated family.

2.3 The Sym-Bobenko formula

We now want to derive a procedure to construct explicit surfaces in terms of the data (u, H, Q) introduced in the last section. The idea is to use the Lax pair to construct a moving frame to the surface and then reconstruct the surface itself from the moving frame. This is done by the Sym-Bobenko formula. However, we will see that by this construction, the parameter λ plays a slightly different role than being a reparametrization of the Hopf differential. But first we need to know that we can actually find a moving frame for the given invariants.

Proposition 2.3.1. *Let $\mathcal{U} \subset \mathbb{C}$ be a simply connected open set with $0 \in \mathcal{U}$. For $U, V : \mathcal{U} \to \mathfrak{sl}_n\mathbb{C}$ there exists a solution $F = F(z) : \mathcal{U} \to SL_n\mathbb{C}$ to the Lax pair*

$$F_z = FU, \qquad F_{\bar{z}} = FV$$

with initial value $F(0) \in SL_n\mathbb{C}$ if and only if

$$U_{\bar{z}} - V_z - [U, V] = 0.$$

For two solutions $F, \tilde{F}$, there is a constant matrix $G \in SL_n(\mathbb{C})$ such that $\tilde{F} = GF$.

Proof. We have already seen that if there exists a solution to the Lax pair, the compatibility condition leads to the Maurer-Cartan equation for U, V.
Conversely, identify $\mathbb{C} \cong \mathbb{R}^2$ and write $z = x + iy$. The above system of equations is then equivalent to

$$F_x = F(U + V) =: FA, \qquad F_y = F(iU - iV) =: FB.$$

In a first step, solve the linear ODE $F_x(x, 0) = F(x, 0)A(x, 0)$ with initial value $F(0)$. Then, for a fixed x_0, solve the equation $F_y(x_0, y) = F(x_0, 0)B(x_0, 0)$. By this algorithm, there holds $F_y = FB\ \forall(x, y)$. To show that F is well-defined and $F_x = FA$, define

$$G = F_x - FA$$

We have

$$G_y = F_{xy} - F_yA - FA_y = (FB)_x - F_yA - FA_y = F_xB + FB_x - FBA - FA_y + FAB - FAB$$

$$= (F_x - FA)B + F(B_x - A_y - BA + AB)$$

The second part vanishes exactly if U and V satisfy the Maurer-Cartan equation. So there holds $G_y = GB$ with initial value $G(0) = 0$, and hence $(F + G)_y = (F + G)B$ with initial value $F(0)$. Due to the uniqueness of solutions for a given inital value we get $G \equiv 0$, giving $F_x = FA$. So F is well-defined and independent of the integration path.

The formula for the derivative of the determinant gives

$$det(F)_z = det(F)\, tr(F^{-1}F_z) = det(F)\, tr(U) = 0$$

since $U \in \mathfrak{sl}_n\mathbb{C}$. Similary we get $det(F)_{\bar{z}} = 0$. Hence $det(F) = const.$ and since $F(0) \in SL_n\mathbb{C}$ we have $det(F(z)) = 1\ \forall z$ and $F \in SL_n\mathbb{C}$.

Now suppose that F and $\tilde{F}$ are both solutions to the Lax pair. Define $G := \tilde{F}F^{-1} \in SL_n(\mathbb{C})$ and compute

$$G_z = \tilde{F}_zF^{-1} - \tilde{F}F^{-1}F_zF^{-1} = \tilde{F}UF^{-1} - \tilde{F}UF^{-1} = 0,$$

and by a similar computation $G_{\bar{z}} = 0$. Hence G is constant. □

Now if there is given u, Q and H solving the Gauss-Codazzi equations, the corresponding Lax pair U, V satisfies the Maurer-Cartan equation. Therefore we can integrate to get a frame $F \in SL_2\mathbb{C}$. By introducting a spectral parameter λ we get a whole family of frames $F_\lambda(z)$ and we will see how to produce a family of CMC surfaces from these frames by the Sym-Bobenko formula.

For computational reasons, we express the quantities introduced so far by Lie algebra valued differential forms. For the Lax pair U, V given as above, the form $\alpha = Udz + Vd\bar{z}$ takes values in $\mathfrak{sl}_2(\mathbb{C})$. For a general differential form $\omega \in \Omega^1(U, \mathfrak{sl}_2(\mathbb{C}))$ we denote by

$$\omega = \omega' + \omega''$$

the decomposition into the $(1,0)$- and the $(0,1)$-part according to $d = \partial + \bar{\partial}$ in $T\mathbb{C}$. The Hodge star operator on $\Omega^1(U, \mathfrak{sl}_2(\mathbb{C}))$ is defined by

$$*\omega = -i\omega' + i\omega''.$$

For $\mathfrak{sl}_2(\mathbb{C})$-valued one forms we define the element $[\alpha \wedge \beta] \in \Omega^2(U, \mathfrak{sl}_2(\mathbb{C}))$ by

$$[\alpha \wedge \beta](X, Y) = [\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)],$$

where $[A, B] = AB - BA$ is the commutator.

Lemma 2.3.2. *Let $U \subset \mathbb{C}$ and $f : U \to \mathbb{H}^3$ be a conformal immersion. With $\omega = f^{-1}df$ there holds*

$$2\, d * \omega = -iH[\omega \wedge \omega]. \tag{2.16}$$

Proof. By definition of the extended frame $F \in SL_2(\mathbb{C})$ there holds

$$f = F\bar{F}^t = FF^*$$

and $F_z = FU$, $F_{\bar{z}} = FV$ for the Lax pair U, V. In terms of forms this yields $dF = F\alpha$, where

$$\alpha = \frac{1}{2} \begin{pmatrix} -u_z dz + u_{\bar{z}} d\bar{z} & e^{-u} Q dz + 2(1+H) e^u d\bar{z} \\ 2(1-H) e^u dz - e^{-u} \bar{Q} d\bar{z} & u_z dz - u_{\bar{z}} d\bar{z} \end{pmatrix}.$$

We compute

$$\begin{aligned} \omega = f^{-1}df &= F^{*-1}F^{-1}(dFF^* + Fd(F^*)) = F^{*-1}(\alpha + \alpha^*)F^* \\ &= F^{*-1} \begin{pmatrix} 0 & 2e^u d\bar{z} \\ 2e^u dz & 0 \end{pmatrix} F^*. \end{aligned}$$

With this we get

$$\begin{aligned} [\omega \wedge \omega] &= F^{*-1} 4e^{2u} \left[\begin{pmatrix} 0 & d\bar{z} \\ dz & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & d\bar{z} \\ dz & 0 \end{pmatrix} \right] F^* \\ &= 8e^{2u} F^{*-1} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} F^* dz \wedge d\bar{z}. \end{aligned}$$

Using the definition of the Hodge star operator we have

$$*\omega = 2ie^u F^{*-1} \begin{pmatrix} 0 & d\bar{z} \\ -dz & 0 \end{pmatrix} F^*,$$

$$\Rightarrow \quad d * \omega = F^{*-1} \begin{pmatrix} 0 & 2ie^u u_z dz \wedge d\bar{z} \\ 2ie^u u_{\bar{z}} dz \wedge d\bar{z} & 0 \end{pmatrix} F^*$$

$$-F^{*-1}d(F^*)F^{*-1} \begin{pmatrix} 0 & 2ie^u d\bar{z} \\ -2ie^u dz & 0 \end{pmatrix} F^* - F^{*-1} \begin{pmatrix} 0 & 2ie^u d\bar{z} \\ -2ie^u dz & 0 \end{pmatrix} d(F^*)$$

$$= F^{*-1} \left[\begin{pmatrix} 0 & 2ie^u u_z \\ 2ie^u u_{\bar{z}} & 0 \end{pmatrix} dz \wedge d\bar{z} - \alpha^* \begin{pmatrix} 0 & 2ie^u d\bar{z} \\ -2ie^u dz & 0 \end{pmatrix} - \begin{pmatrix} 0 & 2ie^u d\bar{z} \\ -2ie^u dz & 0 \end{pmatrix} \alpha^* \right] F^*$$

$$= -4ie^{2u} H F^{*-1} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} F^* dz \wedge d\bar{z}.$$

Hence $2\, d * \omega = -iH[\omega \wedge \omega]$. □

For the next proposition we want to recover the Gauss-Codazzi equations in this notation. For this we note that writing $\alpha = Udz + Vd\bar{z}$, the Maurer-Cartan equation for the Lax pair is given by

$$U_{\bar{z}} - V_z - [U, V] = 0 \quad \Leftrightarrow \quad d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0. \tag{2.17}$$

By introducing a spectral parameter $\lambda \in \mathbb{C}^*$ in the Lax pair we can produce a whole family of frames $F_\lambda \in SL_2(\mathbb{C})$ leading to CMC immersions in $\mathbb{H}^3$, where the mean curvature is given in terms of the spectral parameter. For that purpose we work with a slightly different version of the Lax pair than given before.

Proposition 2.3.3. *Let $U \subset \mathbb{C}$ and $u : U \to \mathbb{R}$ and $Q : U \to \mathbb{C}$ be smooth functions. Define*

$$\alpha_\lambda = \frac{1}{2}\begin{pmatrix} u_z dz - u_{\bar{z}} d\bar{z} & -\lambda^{-1} e^u dz - \bar{Q} e^{-u} d\bar{z} \\ Q e^{-u} dz + \lambda e^u d\bar{z} & -u_z dz + u_{\bar{z}} d\bar{z} \end{pmatrix}. \tag{2.18}$$

Then $d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$ if and only if u and Q are a solution of

$$u_{z\bar{z}} + \frac{1}{4}e^{2u} - \frac{1}{4}Q\bar{Q}e^{-2u} = 0, \quad Q_{\bar{z}} = 0. \tag{2.19}$$

Proof. As we mentioned before, $d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$ is equivalent to

$$U(\lambda)_{\bar{z}} - V(\lambda)_z - [U(\lambda), V(\lambda)] = 0,$$

where $\alpha_\lambda = U(\lambda)dz + V(\lambda)d\bar{z}$. Using the decomposition of α_λ into the $(1,0)$- and $(0,1)$-part we compute

$$U(\lambda)_{\bar{z}} - V(\lambda)_z - [U(\lambda), V(\lambda)] = \begin{pmatrix} u_{z\bar{z}} + \frac{1}{4}e^{2u} - \frac{1}{4}Q\bar{Q}e^{-2u} & \frac{1}{2}\bar{Q}_z e^{-u} \\ \frac{1}{2}Q_{\bar{z}}e^{-u} & -u_{z\bar{z}} + \frac{1}{4}Q\bar{Q}e^{-2u} - \frac{1}{4}e^{2u} \end{pmatrix}$$

proving the claim. □

If we normalize Q by $|Q| = 1$ we recover the sinh-Gordon equation

$$2u_{z\bar{z}} + \frac{e^{2u} - e^{-2u}}{2} = 2u_{z\bar{z}} + \sinh(2u) = 0.$$

In the next proposition we will see that by integrating the Lax pair $dF_\lambda = F_\lambda \alpha_\lambda$, we can produce a family of moving frames that correspond to CMC immersions. In later applications, we will be interested in CMC tori and cylinders with constant Hopf differential. Therefore, for the Maurer-Cartan equation to be satisfied, it is sufficient that u is a solution of the sinh-Gordon equation. The properties of the resulting surface are as follows.

Proposition 2.3.4. *Let $U \subset \mathbb{C}$ be open and simply connected and $u : U \to \mathbb{R}$ be a solution of the* sinh-*Gordon equation. Let $F_\lambda \in SL_2(\mathbb{C})$ be a solution of $dF_\lambda = F_\lambda \alpha_\lambda$ with α_λ given by* (2.18)*. Then for $\lambda_0 \in \mathbb{C}^* \setminus S^1$, i.e. $\lambda_0 = e^{q+i\psi}$, $q \neq 0$, the map $f : U \to SL_2(\mathbb{C})$ defined by the **Sym-Bobenko formula***

$$f(z) = F_{\lambda_0}(z)\overline{F_{\lambda_0}(z)}^t \tag{2.20}$$

is a conformal CMC immersion in $\mathbb{H}^3$ with mean curvature $H = \coth(q)$.

Proof. First we note that the defined map actually takes values in $\mathbb{H}^3 \cong \{FF^* \mid F \in SL_2(\mathbb{C})\}$. To proof conformality we compute

$$df = dFF^* + Fd(F^*) = F\alpha F^* + F\alpha^* F^* = F(\alpha+\alpha^*)F^*,$$

$$\omega = f^{-1}df = F^{*-1}(\alpha+\alpha^*)F^* = \frac{1}{2}F^{*-1}\begin{pmatrix} 0 & (\bar\lambda-\lambda^{-1})e^u dz \\ (\lambda-\bar\lambda^{-1})e^u d\bar z & 0 \end{pmatrix}F^*.$$

From this we get the decomposition $\omega = \omega' + \omega''$. Using the formula $\langle X, X\rangle = -det(X)$ for the metric in the hermitian matrix model for $\mathbb{H}^3$ we compute

$$\langle \omega', \omega'\rangle = \langle \omega'', \omega''\rangle = 0,$$

and since the metric is left invariant on $SL_2(\mathbb{C})/SU_2 \cong \mathbb{H}^3$ we get

$$\langle f_z, f_z\rangle = \langle f_{\bar z}, f_{\bar z}\rangle = 0,$$

proving conformality of f. The conformal factor can be computed using $\langle X, Y\rangle = -\frac{1}{2}tr[X\sigma_2 Y^t \sigma_2]$ and the left invariance of the metric as

$$\langle f_z, f_{\bar z}\rangle = \langle \omega', \omega''\rangle =$$

$$-\frac{1}{2}tr\left[\frac{1}{4}e^{2u}(\bar\lambda-\lambda^{-1})(\lambda-\bar\lambda^{-1})\begin{pmatrix} 0 & dz \\ 0 & 0\end{pmatrix}\sigma_2\begin{pmatrix} 0 & d\bar z \\ 0 & 0\end{pmatrix}\sigma_2\right]$$

$$= \frac{1}{8}(\bar\lambda-\lambda^{-1})(\lambda-\bar\lambda^{-1})e^{2u}dz\, d\bar z.$$

To prove the formula for the mean curvature we use

$$2\, d*\omega = -iH[\omega\wedge\omega]$$

and compute

$$[\omega\wedge\omega] = \frac{1}{2}e^{2u}(\bar\lambda-\lambda^{-1})(\lambda-\bar\lambda^{-1})F^{*-1}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}F^* dz\wedge d\bar z,$$

$$d*\omega = -\frac{1}{4}ie^{2u}(\lambda\bar\lambda-\lambda^{-1}\bar\lambda^{-1})F^{*-1}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}F^* dz\wedge d\bar z.$$

With $\lambda = e^{q+i\psi}$ we get

$$H = \frac{(\lambda\bar\lambda-\lambda^{-1}\bar\lambda^{-1})}{(\bar\lambda-\lambda^{-1})(\lambda-\bar\lambda^{-1})} = \frac{\lambda\bar\lambda-\lambda^{-1}\bar\lambda^{-1}}{\lambda\bar\lambda+\lambda^{-1}\bar\lambda^{-1}-2} = \frac{e^{2q}-e^{-2q}}{e^{2q}+e^{-2q}-2e^q e^{-q}},$$

$$= \frac{(e^q+e^{-q})(e^q-e^{-q})}{(e^q-e^{-q})^2} = \frac{e^q+e^{-q}}{e^q-e^{-q}}.$$

This gives

$$H = \frac{\cosh(q)}{\sinh(q)} = \coth(q)$$

proving the claim. □

This proposition gives a method to construct families of CMC surfaces parametrized by the spectral parameter λ. The difference in the two versions of the Lax pair is that in the first case, we can replace $Q \mapsto \lambda^{-1}Q$, where $|\lambda| = 1$, and see that the Gauss-Codazzi equations still hold. Hence introducing a spectral parameter amounts to a reparametrization of the Hopf differential. In the second case, introducing a spectral parameter $\lambda \in \mathbb{C}^* \setminus S^1$ means that we get Lax pairs of a family of CMC surfaces, where the mean curvature of the resulting surface is encoded by the spectral parameter. In this notation, $\lambda \in S^1$ corresponds to $H = \infty$, meaning that we get in fact a surface in $\mathbb{R}^3$. However, by using this method, the resulting surface does not have the invariants u and Q but constant multiples of these as conformal factor and Hopf differential.

2.4 Spectral curves and CMC tori

In this section we describe how to associate a hyperelliptic Riemann surface to a solution of the sinh-Gordon equation, called the spectral curve. By studying properties of the spectral curve, one can gain information about the CMC surface corresponding to this solution.

Definition 2.4.1. *Let $U \subset \mathbb{C}$ be open and simply-connected and $u : U \to \mathbb{R}$ a periodic real solution of the* sinh-*Gordon equation with period $\tau \in \mathbb{C}^*$, i.e.*

$$u(z + \tau) = u(z) \quad \forall z \in U \textit{ with } z + \tau \in U.$$

Let $F_\lambda : U \to SL_2(\mathbb{C})$ be a solution of the Lax pair

$$F_z = FU, \quad F_{\bar{z}} = FV,$$

and define $M(\lambda) \in SL_2(\mathbb{C})$ by

$$F_\lambda(z + \tau) = M(\lambda)F_\lambda(z). \tag{2.21}$$

Then $M(\lambda)$ is called the monodromy of the moving frame F_λ corresponding to the period $\tau \in \mathbb{C}^*$.

Since $\alpha = Udz + Vd\bar{z}$ is holomorphic in λ for $\lambda \in \mathbb{C}^*$ and the frame was constructed by integrating $dF_\lambda = F_\lambda\alpha$, F_λ and hence also the monodromy $M(\lambda)$ is a holomorphic map $\mathbb{C}^* \to SL_2(\mathbb{C})$ with essential singularities at $\lambda = 0, \infty$.

Proposition 2.4.2. *The monodromy satisfies*

$$M(\bar{\lambda}^{-1}) = \bar{M}^t(\lambda)^{-1}. \tag{2.22}$$

Proof. Writing $\alpha_\lambda = U_\lambda dz + V_\lambda d\bar{z}$ and using the expressions for U and V we can check that there holds

$$\alpha_{\bar{\lambda}^{-1}} = -\bar{\alpha}^t_\lambda.$$

For the moving frame $F_\lambda \in SL_2(\mathbb{C})$ we compute

$$d\left((\bar{F}_\lambda^t)^{-1}\right) = -(\bar{F}_\lambda^t)^{-1}\, d\bar{F}_\lambda^t\, (\bar{F}_\lambda^t)^{-1} = -(\bar{F}_\lambda^t)^{-1}\, \bar{\alpha}_\lambda^t \bar{F}_\lambda^t\, (\bar{F}_\lambda^t)^{-1}$$
$$= -(\bar{F}_\lambda^t)^{-1}\, \bar{\alpha}_\lambda^t = (\bar{F}_\lambda^t)^{-1}\, \alpha_{\bar{\lambda}^{-1}}$$

Hence $(\bar{F}_\lambda^t)^{-1}$ as well as $F_{\bar{\lambda}^{-1}}$ are a solution of

$$dF = F\alpha_{\bar{\lambda}^{-1}}$$

with the same initial value $F(0) = \mathbb{1}$. Therefore

$$(\bar{F}_\lambda^t)^{-1} = F_{\bar{\lambda}^{-1}}$$

and by the definition of the monodromy matrix it follows that

$$M(\bar{\lambda}^{-1}) = \bar{M}^t(\lambda)^{-1}.$$

□

The monodromy encodes how the moving frame F varies when traversing a period of u. We have seen before that for a solution of the sinh-Gordon equation and a corresponding frame F we can describe a conformal CMC immersion $f : U \to \mathbb{H}^3$ by

$$f(z) = F(z)\overline{F(z)}^t.$$

Therefore we can characterize the periods of the surface given by the Sym-Bobenko formula by investigating the behaviour of the monodromy matrix.

Proposition 2.4.3. *Let $f : U \to \mathbb{H}^3$ be a conformal CMC immersion given by*

$$f(z) = F_{\lambda_0}(z)\overline{F_{\lambda_0}(z)}^t$$

with moving frame $F_{\lambda_0} : U \to SL_2(\mathbb{C})$ where $\lambda_0 \in \mathbb{C}^* \setminus S^1$. Let $u : U \to \mathbb{R}$ be the corresponding solution of the sinh-*Gordon equation and $\tau \in \mathbb{C}^*$ be a period of u. Then the immersion is periodic with period τ, i.e.*

$$f(z + \tau) = f(z)$$

if and only if the monodromy matrix satisfies

$$M(\lambda_0) = M(\bar{\lambda}_0^{-1}) = \pm\mathbb{1}. \tag{2.23}$$

Proof. From the Sym-Bobenko formula we know that

$$f(z) = F_{\lambda_0}(z)\overline{F_{\lambda_0}(z)}^t = F_{\lambda_0}(z)F^{-1}_{\bar{\lambda}_0^{-1}}(z).$$

The condition that the immersion $f(z)$ is periodic with period τ is equivalent to

$$f(z+\tau) = M(\lambda_0)F_{\lambda_0}F^{-1}_{\bar{\lambda}_0^{-1}}M(\bar{\lambda}_0^{-1})^{-1} = f(z) = F_{\lambda_0}F^{-1}_{\bar{\lambda}_0^{-1}}$$
$$\Leftrightarrow \quad M(\lambda_0) = M(\bar{\lambda}_0^{-1}) = \pm\mathbb{1}.$$

□

Equation (2.23) is called the **closing condition** for the monodromy. We can use this proposition to determine if a CMC immersion with periodic metric u is periodic itself. If there is one period τ and the closing condition is satisfied, the resulting immersion is a CMC cylinder in $\mathbb{H}^3$. If there are two linearly independent periods τ_1, τ_2 of the metric and the corresponding monodromy matrices satisfy the closing condition for some λ_0, the immersion f_{λ_0} closes to a CMC torus. This information can be encoded in an algebraic construction, called the spectral curve.

Definition 2.4.4. *Let τ be a period of u, F_λ the moving frame and $M(\lambda)$ the corresponding monodromy matrix. Then the spectral curve of the immersion $f = F_\lambda \bar{F}_\lambda^t$ is given by*

$$\Gamma = \left\{(\lambda, \mu) \in \mathbb{C}^2 \mid det(\mu \mathbb{1} - M(\lambda)) = 0\right\}. \tag{2.24}$$

Hence the spectral curve is by definition the eigenvalue curve of the mondromy matrix. In fact, the spectral curve is associated to the whole family f_λ and therefore depends only on the solution u of the sinh-Gordon equation. If we are working with CMC tori there are two periods τ_1, τ_2 corresponding to two generators of the fundamental group $\mathbb{Z} \oplus \mathbb{Z}$. In this case we can consider the monodromy as a representation of the fundamental group on $SL_2(\mathbb{C})$ or, equivalently, investigate the two commuting monodromy matrices $M_{\tau_1}(\lambda)$ and $M_{\tau_2}(\lambda)$.

Since $det(M_{\tau_i}(\lambda)) = 1$, the eigenvalues are of the form μ_i, μ_i^{-1} and satisfy

$$\mu^2 - tr(M(\lambda))\mu + 1 = 0. \tag{2.25}$$

Using the well-kown formula for the zeroes of a quadratic equation one computes

$$\mu = \frac{1}{2}\left(tr(M(\lambda)) \pm \sqrt{tr(M(\lambda))^2 - 4}\right). \tag{2.26}$$

Compactifying the spectral curve over the points with $\lambda = 0, \infty$ and assuming that there are no singularities we get a two-sheeted covering of $\mathbb{CP}^1$ with branch points at $\lambda = 0, \infty$ and the odd roots of $tr(M(\lambda))^2 - 4$. In general, such a two sheeted covering is called a hyperelliptic Riemann surface and can be written as

$$\{(\lambda, \nu) \in \mathbb{C}^2 \mid \nu^2 = \lambda \prod_{i=1}^{N} (\lambda - \alpha_i)\}, \quad N \in \{2g, 2g+1\},$$

where α_i are distinct branch points and g is the genus of the algebraic curve. A good reference for the basic theory of hyperelliptic Riemann surfaces is [3].

One can show that for CMC tori, there are only finitely many odd roots of $tr(M(\lambda))^2 - 4$ in $\mathbb{C}^*$, hence the spectral curve is of finite genus. Furthermore, for the two generators τ_1, τ_2 of the fundamental group of a torus, the odd roots of $tr(M_{\tau_1}(\lambda))^2 - 4$ and $tr(M_{\tau_2}(\lambda))^2 - 4$ coincide. Therefore both monodromy matrices lead to the same spectral curve (see [6] and [10] for further details). Real solutions of the sinh-Gordon equation leading to spectral curves with finite genus are called finite type solutions. Pinkall and Sterling and independently Hitchin proved that all CMC tori are of finite type ([6], [14]). The genus g of the hyperelliptic Riemann surface Γ is called the **spectral genus** of the solution u of the sinh-Gordon equation.

The eigenvalues μ_i of the monodromy matrices $M_{\tau_i}(\lambda)$ are functions on the spectral curve. From the properties of M one can show the following behaviour (see [13]).

Proposition 2.4.5. *Let Γ be the spectral curve of a conformal CMC immersion $f : U \subset \mathbb{C} \to \mathbb{H}^3$ of a torus and μ_i the eigenvalues of the monodromy corresponding to two periods τ_1, τ_2. Then the following holds:*

1. *There is a holomorphic involution σ and two anti-holomorphic involutions η and ρ, where η has no fixed points on Γ, satisfying*

$$\begin{aligned} \sigma :\ (\lambda, \mu_i) &\mapsto (\lambda, \frac{1}{\bar{\mu}_i}), \\ \eta :\ (\lambda, \mu_i) &\mapsto (\frac{1}{\bar{\lambda}}, \bar{\mu}_i), \\ \rho :\ (\lambda, \mu_i) &\mapsto (\frac{1}{\bar{\lambda}}, \frac{1}{\bar{\mu}_i}). \end{aligned} \tag{2.27}$$

2. *The functions μ_i are holomorphic and non-zero on $\Gamma \setminus \{y^+, y^-\}$, where y^+, y^- are the branch points corresponding to $\lambda = 0$ and $\lambda = \infty$ respectively.*
3. *The logarithmic derivatives $d \ln \mu_i$ are meromorphic differentials of the second kind (i.e. without residues) with double poles at $\lambda = 0$ and $\lambda = \infty$. The singular parts at the poles are linearly independent.*

Recall the closing condition on the monodromy which ensured that a given immersion closes to a torus. It was given by

$$M(\lambda_0) = M(\bar{\lambda}_0^{-1}) = \pm\mathbb{1}$$

for a point $\lambda_0 \in \mathbb{C}^*$. In terms of the functions μ_i on the spectral curve, this condition is given by $\mu_i(\lambda_0) = \mu_i(\bar{\lambda}_0^{-1}) = \pm 1$. The point λ_0, where the frame F_λ is evaluated to produce the immersion f via the Sym-Bobenko formula, is called the **Sym point**.

Knowing the properties of the spectral curve of a torus, we can determine sufficient conditions when a given hyperelliptic Riemann surface is the spectral curve of a conformal CMC immersion of a torus. This leads to the theory of integrable systems, where algebro-geometric methods are used to construct solutions pf the sinh-Gordon equation corresponding to a given hyperelliptic curve. The data which is sufficient for a solution to exist is exactly the one stated in the proposition before.

Proposition 2.4.6. *Let Γ be a hyperelliptic Riemann surface with branch points over $\lambda = 0$ and $\lambda = \infty$ and let there be given two functions μ_i fulfilling properties (1)-(3). If there are points y_1, y_2, y_3, y_4 on Γ, where $y_2 = \sigma(y_1)$, $y_4 = \sigma(y_3)$ and $y_4 = \rho(y_1)$, $y_3 = \rho(y_2)$, such that $\mu_i(y_j) = \pm 1$, then Γ is the spectral curve of a conformal CMC immersion of a torus in $\mathbb{H}^3$.*

We only sketch the proof which makes heavy use of Riemann surface theory. In a first step one can show that the conditions in the proposition enable one to write down an explicit solution of the sinh-Gordon equation $2u_{z\bar{z}} + \sinh(2u) = 0$ in terms of Theta functions. Furthermore, this solution is real and doubly periodic.
Using this solution, one can write down the Lax pair U, V as introduced before. Because the Maurer-Cartan equation is satisfied, one can integrate $F_z = FU,\ F_{\bar{z}} = FV$ to get a frame $F_\lambda \in SL_2(\mathbb{C})$.
Finally, the Sym-Bobenko formula gives the formula for the immersion. The existence of the points y_i which correspond to the Sym points λ_0 and $\bar{\lambda}_0^{-1}$ ensure that the eigenvalues of the monodromy satisfy the closing condition. Hence the immersion is also doubly periodic, i.e. a torus. A detailed proof of these facts can be found in [13].

3 CMC Cylinders in $\mathbb{H}^3$

3.1 Spectral genus $g = 0$

3.1.1 CMC cylinders and tori with $g = 0$

To describe CMC cylinders with spectral genus $g = 0$, we first compute the moving frame.

Proposition 3.1.1. *Every conformally immersed CMC cylinder $f : U \to \mathbb{H}^3$ with constant mean curvature H and spectral genus $g = 0$ can be described as*

$$f = F_{\lambda_0}\overline{F}^t_{\lambda_0}, \quad F_\lambda(z) = exp\left(\frac{i}{2}\begin{pmatrix} 0 & z\lambda^{-1} + \bar{z} \\ z + \bar{z}\lambda & 0 \end{pmatrix}\right) \tag{3.1}$$

where $\lambda_0 = e^{q+i\psi}$ and $H = \coth q$.

Proof. For $g = 0$, there is only the spectral curve given by $\nu^2 = \lambda$. The only solution of the sinh-Gordon equation corresponding to this spectral curve is the flat solution $u \equiv 0$. Integrating $dF_\lambda = F_\lambda\alpha_\lambda$ with $u = 0$ gives the frame as stated in the proposition. By Proposition 2.3.4, the surface $f : U \to \mathbb{H}^3$ can be described by $f = F_{\lambda_0}\overline{F}^t_{\lambda_0}$, proving the claim. □

Taking the initial value $F_\lambda(0) = \mathbb{1}\ \forall\lambda$, the monodromy is given by

$$M_\tau(\lambda) = exp\left(\frac{i}{2}\begin{pmatrix} 0 & \tau\lambda^{-1} + \bar{\tau} \\ \tau + \bar{\tau}\lambda & 0 \end{pmatrix}\right).$$

Define the matrix A by $M(\lambda) = exp(A(\lambda))$, i.e. $A = \frac{i}{2}\begin{pmatrix} 0 & z\lambda^{-1} + \bar{z} \\ z + \bar{z}\lambda & 0 \end{pmatrix}$.

Finding periods for which the immersion corresponding to $F_\lambda(z)$ closes to a cylinder amounts to finding λ_0 and τ for which $M_\tau(\lambda_0) = \pm\mathbb{1}$. Since the matrix A is trace-free, the eigenvalues come in pairs $\pm k$. If we diagonalize A by D, i.e. $diag(...) = D^{-1}AD$, we get $exp(A) = Dexp(D^{-1}AD)D^{-1}$, where the exponential on the right-hand side has $e^{\pm k}$ on the diagonal. So for $M(\lambda)$ to be the identity, we need $\pm k = \pi i N, N \in \mathbb{Z}$, for the eigenvalues $\pm k$ of A.

The eigenvalues of A are given by

$$\ln\mu(\lambda, \tau) = \frac{i}{2}(\tau\sqrt{\lambda}^{-1} + \bar{\tau}\sqrt{\lambda})$$

Set $\lambda = e^{q+i\psi}$, $\tau = e^{s+it}$. We compute

$$\ln \mu = \pi i N \quad \Leftrightarrow \quad e^s(e^{-\frac{q}{2}+i(t-i\frac{\psi}{2})} + e^{\frac{q}{2}-i(t-i\frac{\psi}{2})}) = 2\pi N \quad \Leftrightarrow \quad e^s \cosh(\frac{q}{2} - i(t - \frac{\psi}{2})) = \pi N$$

Therefore a necessary condition is

$$\cosh(\frac{q}{2} - i(t - \frac{\psi}{2})) \in \mathbb{R} \qquad \Leftrightarrow \qquad \frac{q}{2} = 0 \quad or \quad t - \frac{\psi}{2} = \pi k$$

Theorem 3.1.2. *There are no conformally immersed CMC tori of spectral genus $g = 0$ in $\mathbb{H}^3$.*

Proof. Assume that there is such a torus given by $f = F_{\lambda_0}\bar{F}^t_{\lambda_0}$, then there are two linearly independent periods $\tau_1,\ \tau_2$ and λ_0 for which

$$M_{\tau_i}(\lambda_0) = \pm\mathbb{1}.$$

From the discussion of the Sym-Bobenko formula we know that $|\lambda_0| \neq 1$. So $q \neq 0$, giving $t - \frac{\psi}{2} = \pi k$. Therefore $2t - 2\pi k = \psi$.
If the $\tau_i = e^{s_i+it_i}$ are linearly independent, $t_1 \neq t_2 \ mod\ \pi$. Therefore $\psi_1 \ mod\ 2\pi = 2t_1 \neq 2t_2 \ mod\ 2\pi = \psi_2$. Therefore the τ_i cannot belong to the same Sym point λ_0. Hence there cannot be two linearly independent periods. □

Nevertheless we can find periods leading to CMC cylinders. As shown above, for $\ln\mu = \pi i N$ we need $t - \frac{\psi}{2} = \pi k$. For a fixed $\lambda_0 = e^{q+i\psi}$ we can find periods by choosing some $k \in \mathbb{Z}$ and solving $t - \frac{\psi}{2} = \pi k$ for t. Note that this way, the argument of τ is determined up to πk, therefore all periods corresponding to one λ_0 are linearly dependent, as we have just proved above. Then computing

$$\cosh(\frac{q}{2} - i(t - \frac{\psi}{2})),$$

we can solve for a chosen $N \in \mathbb{Z}$

$$e^s \cosh(\frac{q}{2} - i(t - \frac{\psi}{2})) = \pi N,$$

finding a unique $s \in \mathbb{R}$ for which $\tau = e^{s+it}$ is a period. With these choices of λ_0 and τ, we get a CMC immersion which closes to a cylinder in $\mathbb{H}^3$.

3.1.2 Formula for the immersion

Finally, we get the immersion corresponding to the frame $F_\lambda(z)$ by plugging it into the Sym-Bobenko formula

$$f(z) = F_\lambda(z)\overline{F}^t_\lambda(z)$$

$$= exp\left(\frac{i}{2}\begin{pmatrix} 0 & z\lambda^{-1}+\bar z \\ z+\bar z\lambda & 0\end{pmatrix}\right)\overline{exp\left(\frac{i}{2}\begin{pmatrix} 0 & z\lambda^{-1}+\bar z \\ z+\bar z\lambda & 0\end{pmatrix}\right)}^t$$

By the Sym-Bobenko formula, this gives an immersion of a CMC cylinder with mean curvature $H = \frac{1+e^{-2q}}{1-e^{-2q}} = \coth(q)$, if $\lambda = e^{q+i\psi}$. We first compute the frame via the exponential.

$$exp\left(\frac{i}{2}\begin{pmatrix} 0 & z\lambda^{-1}+\bar z \\ z+\bar z\lambda & 0\end{pmatrix}\right) = \mathbb{1} + \frac{i}{2}\begin{pmatrix} 0 & z\lambda^{-1}+\bar z \\ z+\bar z\lambda & 0\end{pmatrix}$$

$$-\frac{1}{8}\begin{pmatrix} (z\lambda^{-1}+\bar z)(z+\bar z\lambda) & 0 \\ 0 & (z\lambda^{-1}+\bar z)(z+\bar z\lambda)\end{pmatrix}$$

$$-\frac{i}{48}\begin{pmatrix} 0 & (z\lambda^{-1}+\bar z)^2(z+\bar z\lambda) \\ (z+\bar z\lambda)^2(z\lambda^{-1}+\bar z) & 0\end{pmatrix} + \dots$$

$$= \begin{pmatrix} \cosh(\frac{i}{2}\sqrt{(z\lambda^{-1}+\bar z)(z+\bar z\lambda)}) & \sqrt{\frac{z\lambda^{-1}+\bar z}{z+\bar z\lambda}}\sinh(\frac{i}{2}\sqrt{(z\lambda^{-1}+\bar z)(z+\bar z\lambda)}) \\ \sqrt{\frac{z+\bar z\lambda}{z\lambda^{-1}+\bar z}}\sinh(\frac{i}{2}\sqrt{(z\lambda^{-1}+\bar z)(z+\bar z\lambda)}) & \cosh(\frac{i}{2}\sqrt{(z\lambda^{-1}+\bar z)(z+\bar z\lambda)})\end{pmatrix}$$

where we used the power series expansion of cosh and sinh. If we set $a = \sqrt{\frac{z\lambda^{-1}+\bar z}{z+\bar z\lambda}}$ and omit the arguments of cosh and sinh then we get

$$F\bar F^t = \begin{pmatrix} |\cosh|^2 + |a\sinh|^2 & \frac{1}{\bar a}\cosh\overline{\sinh} + a\overline{\cosh}\sinh \\ \frac{1}{a}\overline{\cosh}\sinh + \bar a\cosh\overline{\sinh} & |\cosh|^2 + |\frac{1}{a}\sinh|^2\end{pmatrix}$$

For a concrete immersion we have to pick some $\lambda \in \mathbb{C}^* \setminus S^1$ and use the representation $\mathbb{H}^3 \cong SL_2\mathbb{C}/SU_2$ via

$$(x_0, x_1, x_2, x_3) \mapsto \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3\end{pmatrix} \tag{3.2}$$

From the diagonal entries we get

$$x_0 + x_3 = |\cosh|^2 + a^2|\sinh|^2, \qquad x_0 - x_3 = |\cosh|^2 + \frac{1}{a^2}|\sinh|^2$$

which yields

$$x_0 = |\cosh|^2 + \frac{1}{2}(a^2 + \frac{1}{a^2})|\sinh|^2, \qquad x_3 = \frac{1}{2}(a^2 - \frac{1}{a^2})|\sinh|^2$$

From the off-diagonal entries it follows that

$$x_1 = \frac{1}{2}(\frac{1}{\bar a}\cosh\overline{\sinh} + \frac{1}{a}\overline{\cosh}\sinh + a\overline{\cosh}\sinh + \bar a\cosh\overline{\sinh}) = Re(\frac{1}{\bar a}\cosh\overline{\sinh} + a\overline{\cosh}\sinh),$$

$$x_2 = -\frac{i}{2}(\frac{1}{\bar a}\cosh\overline{\sinh} - \frac{1}{a}\overline{\cosh}\sinh + a\overline{\cosh}\sinh - \bar a\cosh\overline{\sinh}) = Im(\frac{1}{\bar a}\cosh\overline{\sinh} + a\overline{\cosh}\sinh).$$

Next we choose some λ out of the admissible set, i.e. $\lambda \notin S^1$, and compute the missing expressions. We choose $\lambda = \frac{1}{2}$. This gives

$$a = \sqrt{\frac{z\lambda^{-1} + \bar{z}}{z + \bar{z}\lambda}} = \sqrt{\frac{2z + \bar{z}}{z + \frac{\bar{z}}{2}}} = \sqrt{\frac{3x + iy}{\frac{3}{2}x - i\frac{y}{2}}} = \sqrt{2},$$

where we set $z = x + iy$. For the argument of the hyperbolic functions we get

$$\frac{i}{2}\sqrt{(z\lambda^{-1} + \bar{z})(z + \bar{z}\lambda)} = \frac{i}{2}\sqrt{(3x + iy)(\frac{3}{2}x - i\frac{y}{2})} = \frac{i}{2}\sqrt{\frac{1}{2}(3x + iy)^2} = \frac{3ix - y}{\sqrt{8}}.$$

Now we just need to plug this into the above formula for the frame. We use the identities

$$\cosh(ix) = \cos(x), \qquad \sinh(ix) = i\sin(x)$$

and

$$|\cosh(w)|^2 = \frac{1}{2}(\cosh(w + \bar{w}) + \cosh(w - \bar{w})), \quad |\sinh|^2 = \frac{1}{2}(\cosh(w + \bar{w}) - \cosh(w - \bar{w}))$$

where in our case $w = \frac{3ix - y}{\sqrt{8}}$. This gives

$$x_0 = \frac{9}{8}\cosh(-\frac{y}{\sqrt{2}}) - \frac{1}{8}\cos(\frac{6x}{\sqrt{8}}),$$
$$x_3 = \frac{3}{8}\cosh(-\frac{y}{\sqrt{2}}) - \frac{3}{8}\cos(\frac{6x}{\sqrt{8}}),$$
$$x_1 = Re(\frac{3}{2\sqrt{2}}\sinh(-\frac{y}{\sqrt{2}}) + \frac{1}{2\sqrt{2}}\sinh(\frac{6ix}{\sqrt{8}})) = \frac{3}{2\sqrt{2}}\sinh(-\frac{y}{\sqrt{2}}),$$
$$x_2 = Im(\frac{3}{2\sqrt{2}}\sinh(-\frac{y}{\sqrt{2}}) + \frac{1}{2\sqrt{2}}\sinh(\frac{6ix}{\sqrt{8}})) = \frac{1}{2\sqrt{2}}\sin(\frac{6x}{\sqrt{8}}).$$

Like described above, we can find out which simple periods belong to our chosen λ by solving

$$e^s\cosh(\frac{q}{2} - i(t - \frac{\psi}{2})) = \pi,$$

where $t - \frac{\psi}{2} \in \pi\mathbb{Z}$. For $\lambda = \frac{1}{2} = e^{-\ln 2}$ this gives

$$s = \ln\frac{\pi}{\cosh(-\ln\sqrt{2})} \quad \Rightarrow \quad \tau = e^s = \frac{\pi}{\cosh(-\ln\sqrt{2})}.$$

This gives the following example of a CMC cylinder in $\mathbb{H}^3$.

Proposition 3.1.3. *The mapping* $f : \mathbb{R}^2 \to \mathbb{H}^3$,

$$f(x, y) = (\frac{9}{8}\cosh(-\frac{y}{\sqrt{2}}) - \frac{1}{8}\cos(\frac{6x}{\sqrt{8}}), \frac{3}{2\sqrt{2}}\sinh(-\frac{y}{\sqrt{2}}), \frac{1}{2\sqrt{2}}\sin(\frac{6x}{\sqrt{8}}), \frac{3}{8}\cosh(-\frac{y}{\sqrt{2}}) - \frac{3}{8}\cos(\frac{6x}{\sqrt{8}}))$$

defines an immersion of a cylinder in $\mathbb{H}^3$ *with constant mean curvature* $H = \coth(-\ln 2)$ *and simple period* $\tau = \frac{\pi}{\cosh(-\ln\sqrt{2})}$.

3.2 Spectral genus $g = 1$

3.2.1 The spectral curve

Let Γ be a hyperelliptic Riemann surface of genus one with branch points at $\lambda = 0, \infty, \alpha, \frac{1}{\alpha}$, $\alpha \in \mathbb{R}$, given as compactification of the algebraic curve

$$\nu^2 = \lambda(\lambda - \alpha)(\lambda - \frac{1}{\alpha}). \tag{3.3}$$

In general, the two branch points are of the form $\alpha, \frac{1}{\bar{\alpha}}$. However, we can always move them to the real axis by performing a Möbius transformation in the λ-plane.

On Γ there is the hyperelliptic involution $\sigma : (\lambda, \nu) \mapsto (\lambda, -\nu)$, and since $\alpha \in \mathbb{R}$, there is the additional involution $(\lambda, \nu) \mapsto (\bar{\lambda}, \bar{\nu})$, . Furthermore the following involutions exist.

$$\begin{aligned} &\rho : (\lambda, \nu) \mapsto (\tfrac{1}{\bar{\lambda}}, -\tfrac{\bar{\nu}}{\bar{\lambda}^2}), \quad \eta : (\lambda, \nu) \mapsto (\tfrac{1}{\bar{\lambda}}, \tfrac{\bar{\nu}}{\bar{\lambda}^2}) \\ &\theta : (\lambda, \nu) \mapsto (\tfrac{1}{\lambda}, -\tfrac{\nu}{\lambda^2}), \quad \chi : (\lambda, \nu) \mapsto (\tfrac{1}{\lambda}, \tfrac{\nu}{\lambda^2}) \end{aligned} \tag{3.4}$$

where η is chosen so that there exist no fixed points (since $\nu^2 < 0$ for $\lambda = \pm 1$, we have $-\bar{\nu} = \nu$).

Let $A, B \in H_1(\Gamma, \mathbb{Z})$ be a canonical basis of the first homology group of Γ, where the cycle A surrounds the branch points $0, \alpha$ in positive direction and the cycle B surrounds $\alpha, \frac{1}{\alpha}$ in positive direction. A model for Γ is given by gluing together two copies of $\mathbb{C}\,\mathbb{P}^1$ along the cuts $[0, \alpha]$, $[\frac{1}{\alpha}, \infty]$. By this construction, ν is a well-defined, single-valued function on Γ (see [3]).

If Γ is the spectral curve of a conformal CMC immersion of a torus in $\mathbb{H}^3$, there exist two linearly independent differentials of the second kind, $d\ln\mu_1$ and $d\ln\mu_2$, with poles of second order at $\lambda = 0, \infty$. They are the logarithmic derivatives of functions μ_i which are the eigenvalues of the monodromy of the extended frame of the torus along two linearly independent periods. Regarding the transformation of μ_i under the involutions, we have

$$\sigma^* d\ln\mu_i = -d\,\ln\mu_i, \quad \rho^* d\,\ln\mu_i = -d\,\ln\bar{\mu}_i, \quad \eta^* d\,\ln\mu_i = d\ln\bar{\mu}_i, \quad i = 1, 2 \tag{3.5}$$

Given the spectral curve, the differentials $d\ln\mu_i$ can uniquely be chosen such that

$$\int_B d\ln\mu_i \in 2\pi i\mathbb{Z}, \quad i = 1, 2. \tag{3.6}$$

The operation of the transformation η on the cycle $B \in H_1(\Gamma, \mathbb{Z})$ is as follows. In the λ-plane, the transformation $\lambda \mapsto \frac{1}{\bar{\lambda}}$ has fixed points in $B \cap S^1$ and the cycle B is traversed in opposite direction. However, η is without fixed points. Hence these points in $B \cap S^1$ are mapped into the other sheet of Γ. So the cycle $\eta \circ B$ is the same as $\sigma \circ -B$. As the hyperelliptic involution acts as $-Id$ on $H_1(\Gamma, \mathbb{Z})$, this yields $\eta \circ B = B$. By the homology invariance of the complex line integral it follows that

$$\int_B d\ln\mu_i = \int_{\eta\circ B} d\ln\mu_i = \int_B \eta^* d\ln\mu_i = \int_B d\ln\bar{\mu}_i.$$

$$\Rightarrow \int_B d\ln\mu_i - d\ln\bar{\mu}_i = \int_B 2i\, Im\,(d\ln\mu_i) = 0.$$

Thus the integral along B has to be real. This can only hold if

$$\int_B d\ln\mu_i = 0, \quad i = 1, 2. \tag{3.7}$$

3.2.2 Periods of CMC cylinders for $g = 1$

In order to determine the explicit form of the first eigenvalue, we need the following

Lemma 3.2.1. *Every meromorphic function f on a hyperelliptic Riemann surface Γ can uniquely be written as*

$$f = r(\lambda)\nu + s(\lambda), \tag{3.8}$$

where r and s are rational functions of λ.

Proof. Let Γ be defined by $\nu^2 = R(\lambda)$ where R is a polynomial in λ of degree $2g+1$ or $2g+2$. The resulting algebraic curve is a two sheeted covering of $\mathbb{CP}^1$ possesing the involution $\sigma : (\lambda, \nu) \mapsto (\lambda, -\nu)$. Splitting f in symmetric and antisymmetric parts with regard to the hyperelliptic involution

$$f = \frac{1}{2}(f + \sigma^* f) + \frac{1}{2}(f - \sigma^* f),$$

we can write the symmetric part as pullback of a meromorphic function s on $\mathbb{C}\,\mathbb{P}^1$. Due to the symmetry $\nu \mapsto -\nu$, s is well-defined and unique. In addition $\frac{1}{\nu}(f - \sigma^* f)$ is symmetric with regard to σ, and therefore the pullback of a meromorphic funtion r on $\mathbb{C}\,\mathbb{P}^1$. The meromorphic functions on $\mathbb{C}\,\mathbb{P}^1$ are rational functions of λ. □

From the previous sections we know the properties which have to be satisfied by the eigenfunction μ of the monodromy if Γ is the spectral curve of a solution of the sinh-Gordon equation. In general, $\ln\mu$ is multi-valued. However, on a genus $g = 1$ spectral curve, there exists a function which has the same properties than $\ln\mu$ has. To determine the explicit form of that function we are looking for a meromorphic function f having simple poles at $\lambda = 0, \infty$. From the lemma we know that $f = r(\lambda)\nu + s(\lambda)$. Since $\sigma^*\mu = \mu^{-1}$ this yields $\sigma^* \ln\mu = -\ln\mu$. Hence $s \equiv 0$. We use the following Ansatz (higher powers of λ cannot come up because of the determined pole order).

$$f = \frac{a\lambda + b}{c\lambda + d}\,\nu$$

Around the branch points $\lambda = 0, \infty$, local charts for Γ are given by $\sqrt{\lambda}$ and $\frac{1}{\sqrt{\lambda}}$ respectively. Plugging in the local paramter in a neighbourhood of these branch points we get

$$\lambda \in U_\epsilon(0),\, z = \sqrt{\lambda} \;\Rightarrow\; f = \frac{az^2+b}{cz^2+d}\sqrt{z^2(z^2-\alpha)(z^2-\frac{1}{\alpha})}$$

$$= \frac{az^2+b}{cz+dz^{-1}}\sqrt{(z^2-\alpha)(z^2-\frac{1}{\alpha})}$$

The expression in the square root is non-singular at $z = 0$, hence we must have $d = 0$, for $\ln\mu$ to have a pole of order one.

$$\lambda \in U_\epsilon(\infty),\, z = \frac{1}{\sqrt{\lambda}} \;\Rightarrow f = \frac{az^{-2}+b}{cz^{-2}}\sqrt{z^{-2}(z^{-2}-\alpha)(z^{-2}-\frac{1}{\alpha})}$$

$$= \frac{az^{-2}+b}{cz^{-2}}z^{-3}\sqrt{1-(\alpha+\frac{1}{\alpha})z^2+z^4} = (\frac{a}{c}z^{-3}+\frac{b}{c}z^{-1})\sqrt{\dots}$$

The square root is non-singular at $z = 0$, hence $a = 0$.

$$\Rightarrow \quad f = \frac{b}{c}\frac{\nu}{\lambda}$$

The constant factor can be determined by the relation $\rho^*\mu = \bar{\mu}^{-1}$ (The functional equations for ln only hold $mod\, 2\pi i$. However, this does not affect our arguments since we are looking for purely imaginary values).

$$\rho^*\mu = \bar{\mu}^{-1} \;\Rightarrow\; \rho^* \ln\mu = -\ln\bar{\mu}$$

$$\rho^* f = c\frac{-\bar{\nu}\bar{\lambda}}{\bar{\lambda}^2} = -c\frac{\bar{\nu}}{\bar{\lambda}}, \quad -\bar{f} = -\bar{c}\frac{\bar{\nu}}{\bar{\lambda}}$$

$$\Rightarrow\; c = \bar{c} \;\Leftrightarrow\; c \in \mathbb{R}$$

The differential reads

$$d\,(c\,\frac{\nu}{\lambda}) = c\,\frac{\dot{\nu}\lambda-\nu}{\lambda^2}d\lambda = \frac{c}{2}\frac{\lambda^2-1}{\lambda\nu}d\lambda,$$

where we use the explicit form of ν. Altogether we have

Proposition 3.2.2. *On the hyperelliptic curve Γ of genus one, there is exactly one function f up to real factors with simple poles at $\lambda = 0, \infty$ satisfying the transformation rules of the mondromy eigenfunction. It is given by*

$$f = c\,\frac{\nu}{\lambda}, \qquad df = \frac{c}{2}\frac{\lambda^2-1}{\lambda\nu}d\lambda, \quad c \in \mathbb{R}. \tag{3.9}$$

It is known that on the spectral curve of genus $g = 1$, there is one direction for the period, in which the metric is constant. Choosing this period, the logarithm of the eigenvalue of the monodromy corresponds to the function derived above.

A necessary condition for the immersion to close to a cylinder is

$$\mu = \pm 1 \ \Leftrightarrow \ln\mu = \pi i N,\ N \in \mathbb{Z} \ \Rightarrow \ln\mu \in i\mathbb{R}$$

With the explicit form of $\ln\mu$ for the first period we get

$$\ln\mu = c\frac{\nu}{\lambda} \in i\mathbb{R},\ c \in \mathbb{R} \quad \Leftrightarrow \quad \frac{\nu}{\lambda} \in i\mathbb{R} \quad \Leftrightarrow \quad \frac{\bar{\nu}}{\bar{\lambda}} = -\frac{\nu}{\lambda}$$

$$\Leftrightarrow \ \lambda^2\bar{\nu}^2 = \nu^2\bar{\lambda}^2\ (\lambda\, mod\ \pm 1) \ \Leftrightarrow \ \lambda(\bar{\lambda}^2 - (\alpha + \frac{1}{\alpha})\bar{\lambda} + 1) = \bar{\lambda}(\lambda^2 - (\alpha + \frac{1}{\alpha})\lambda + 1)$$

$$\Leftrightarrow \ \bar{\lambda}(\lambda^2 - (\alpha + \frac{1}{\alpha})\lambda + 1) \in \mathbb{R} \quad \Leftrightarrow \quad |\lambda|^2\lambda + \bar{\lambda} \in \mathbb{R} \quad \Leftrightarrow \quad \lambda + \frac{1}{\lambda} \in \mathbb{R}$$

Setting $\lambda = e^{q+i\psi}$, this is the case iff $\cosh(q + i\psi) \in \mathbb{R} \ \Leftrightarrow \ q = 0$ or $\psi = \pi k \ \Leftrightarrow \ \lambda \in S^1 \cup \mathbb{R}$. However λ is only determined up to sign. For $\lambda \in \mathbb{R}$ the formula for ν yields:

$$\frac{\nu}{\lambda} \in i\mathbb{R}, \lambda \in \mathbb{R} \ \Leftrightarrow \ \nu \in i\mathbb{R} \ \Leftrightarrow \ \nu^2 < 0 \ \Leftrightarrow \ \lambda \in (-\infty, 0) \cup (\alpha, \frac{1}{\alpha})$$

Altogether we have:

$$\ln\mu \in i\mathbb{R} \quad \Leftrightarrow \quad \lambda \in S^1 \cup (-\infty, 0) \cup (\alpha, \frac{1}{\alpha}) \tag{3.10}$$

By choosing λ from this set not having unit modulus, we get an immersion of a CMC cylinder in $\mathbb{H}^3$ by integrating the Lax pair corresponding to the periodic solution of the sinh-Gordon equation which is determined by the spectral curve and plugging the resulting frame into the Sym-Bobenko formula.

3.2.3 CMC tori with $g = 1$

If we want the immersion to close to a torus, we need two linearly independent periods. The integral along B vanishes for both differentials. However, this can't be true for A. Suppose there holds

$$\int_A d\ln\mu_i = 0, \quad i = 1, 2.$$

Then both differentials are exact, i.e. exterior derivatives of a meromorphic function obeying the transformation rules. Since on a genus one Riemann surface, there is only one such function up to real factors, the differentials are of the form

$$\ln\mu_1 = c_1\frac{\nu}{\lambda}, \quad \ln\mu_2 = c_2\frac{\nu}{\lambda}, \quad c_i \in \mathbb{R}.$$

This is a contradiction to $d\ln\mu_1$ and $d\ln\mu_2$ being linearly independent. So there must hold

$$\int_A d\ln\mu_2 \neq 0, \tag{3.11}$$

i.e. $\ln\mu_2$ is a multi valued function on Γ.

$d\ln\mu_1$ is symmetric under the involution $\chi : (\lambda, \nu) \mapsto (\frac{1}{\lambda}, \frac{\nu}{\lambda^2})$ and satisfies all required conditions. Ansatz:

$$d\ln\mu_2 = d\,\omega + c\,d\ln\mu_1,$$

where $d\,\omega$ is χ-antisymmetric. Due to the prescribed pole order, $d\omega$ is of the form

$$\frac{a\lambda^2 + b\lambda + c}{f\lambda}\frac{d\lambda}{\nu}.$$

The antisymmetry $\chi^* d\omega = -d\omega$ leads to

$$\frac{a\lambda^{-2} + b\lambda^{-1} + c}{f\lambda^{-1}}(-\lambda^{-2})\frac{\lambda^2}{\nu}d\lambda = -\frac{a + b\lambda + c\lambda^2}{f\lambda}\frac{d\lambda}{\nu} = -\frac{a\lambda^2 + b\lambda + c}{f\lambda}\frac{d\lambda}{\nu},$$

hence there must hold $a = c$. Therefore $d\,\omega$ is of the form

$$d\,\omega = \frac{a}{f}\frac{\lambda^2 + 1}{\lambda\nu}d\lambda + \frac{b}{f}\frac{d\lambda}{\nu}.$$

The factor $\frac{a}{f}$ can be determined by the relation $\rho^* d\,\omega = -d\,\bar{\omega}$.

$$\rho^* \frac{a}{f}\frac{\lambda^2 + 1}{\lambda\nu}d\lambda = \frac{a}{f}\frac{\bar{\lambda}^{-2} + 1}{\bar{\lambda}^{-1}\bar{\lambda}^{-2}(-\bar{\nu})}(-\bar{\lambda}^{-2})d\bar{\lambda} = \frac{a}{f}\frac{\bar{\lambda}^2 + 1}{\bar{\lambda}\bar{\nu}}d\bar{\lambda}.$$

Hence we need $-\frac{\bar{a}}{\bar{f}} = \frac{a}{f}$, i.e. $\frac{a}{f} \in i\mathbb{R}$. In the same way, we get $\frac{b}{f} \in i\mathbb{R}$. Thus

$$d\,\omega = ic_1\frac{\lambda^2 + 1}{\lambda\nu}d\lambda + ic_2\frac{d\lambda}{\nu}, \quad c_i \in \mathbb{R}. \tag{3.12}$$

In a next step we want to rule out the existence of tori in $\mathbb{H}^3$ with a genus 1 spectral curve. If Γ is the spectral curve of a torus, there exists λ_0 satisfying $\ln\mu_1(\lambda_0) \in i\mathbb{R}$ and $\ln\mu_2(\lambda_0) \in i\mathbb{R}$. Since $\int_A d\ln\mu_2 \neq 0$ and therefore $\int_A d\,\omega \neq 0$, ω is multi valued along A. We make ω unique by cutting Γ into different branches and investigating each. We have shown earlier that

$$\ln\mu_1 \in i\mathbb{R} \quad \Leftrightarrow \quad \lambda \in S^1 \cup (-\infty, 0) \cup (\alpha, \frac{1}{\alpha}).$$

First case

Cut along $[-\infty, 0]$ and normalize ω by $\omega(\lambda = 1) = 0$. Because of $\chi^* d\,\omega = -d\,\omega$ and the normalization we have $\chi^*\omega = -\omega$ on this branch. Since for $\lambda \in (\alpha, \frac{1}{\alpha})$ we have $\nu^2 < 0$, it follows that $-\bar{\nu} = \nu$ in that interval. Therefore $(\frac{1}{\lambda}, \frac{\nu}{\lambda^2}) = (\frac{1}{\bar{\lambda}}, \frac{-\bar{\nu}}{\bar{\lambda}^2})$, i.e.

$$\rho|_{(\alpha,\frac{1}{\alpha})} = \chi|_{(\alpha,\frac{1}{\alpha})} \tag{3.13}$$

Beacause of $\rho^*\ln\mu_2 = -\ln\bar{\mu}_2$, there holds $\rho^*\omega = -\bar{\omega}$.

Let $\lambda_0 \in (\alpha, \frac{1}{\alpha})$. Suppose that $\omega(\lambda_0) \in i\mathbb{R}$. Then we have

$$\rho^*\omega = -\bar{\omega} = \omega.$$

However, from (3.13) it follows that

$$\rho^*\omega = \chi^*\omega = -\omega \quad \forall \lambda \in (\alpha, \frac{1}{\alpha})$$

Therefore ω cannot be purely imaginary in $(\alpha, \frac{1}{\alpha})$, unless $\lambda_0 = 1$.

Second case
Cut along $[\alpha, \frac{1}{\alpha}]$ and normalize by $\omega(\lambda = -1) = 0$. Then ω fulfills the same transformation rules as in the first case. Due to $\nu^2 < 0$ for $\lambda \in (-\infty, 0)$, again there holds $\rho = \chi$ along that line. Therefore ω cannot be purely imaginary in $(-\infty, 0)$ unless $\lambda_0 = -1$.

Altogether we get

$$\{\lambda \mid \omega \in i\mathbb{R}\} \ \cap \ \{\lambda \mid \ \ln \mu_1 \in i\mathbb{R}\} \subset S^1 \tag{3.14}$$

Since ω is the $\frac{1}{\lambda}$-antisymmetric factor of $\ln \mu_2$, the same holds for the logarithms of the eigenvalues $\ln \mu_i$ of the monodromy for two linearly independent periods. However, due to the Sym-Bobenko formula, $\lambda \in S^1$ is not admissble in the case of $\mathbb{H}^3$. So we have shown

Theorem 3.2.3. *There are no conformally immersed CMC tori of spectral genus $g = 1$ in $\mathbb{H}^3$.*

4 Deformation of spectral curves

4.1 Spectral data and deformation theory

In this section, we study deformations of spectral data corresponding to finite type solutions of the sinh-Gordon equation leading to CMC tori in $\mathbb{H}^3$. First, we define the spectral data of a CMC torus by polynomials describing a hyperelliptic Riemann surface and two meromorphic differentials. Then we derive differential equations describing deformations of this data that leave invariant the closing condition imposed on the deformed meromorphic differentials and therefore conserve the property of the solution to be the metric of a CMC torus. This deformation theory was introduced in [8] and used in [7] to study deformations of CMC tori in $\mathbb{S}^3$.

4.1.1 Spectral data

In the previous sections, when we considered spectral curves of CMC surfaces, we showed that a sufficient condition for the surface to close to a torus is the existence of two meromorphic differentials

$$d\ln\mu_1, \quad d\ln\mu_2$$

with double poles at $\lambda = 0, \infty$ and without residues, that are the logarithmic derivatives of functions μ_i that satisfy $\mu_i(\lambda_0) = \pm 1$ for some λ_0 and

$$\sigma^*\mu_i = \mu_i^{-1}, \quad \rho^*\mu_i = \bar{\mu}_i^{-1}, \quad \eta^*\mu_i = \bar{\mu}_i$$

for the involutions defined on the spectral curve. For the purpose of this section, we use a transformed spectral parameter

$$\kappa = i\frac{\lambda - 1}{\lambda + 1}. \tag{4.1}$$

The points lying over $\lambda = 0, \infty$ correspond to $\kappa = \pm i$ and the points with $|\lambda| = 1$ are mapped to $\kappa \in \mathbb{R}$. By defining κ in that way there is a freedom of taking a Möbius transformation

$$\lambda \mapsto e^{2i\psi}\lambda \quad \Leftrightarrow \quad \kappa \mapsto \frac{\sin(\psi) + \kappa\cos(\psi)}{\cos(\psi) - \kappa\sin(\psi)},$$

which amounts to a rotation in the κ plane by angle ψ having fixed points at $\kappa = \pm i$. This transformation changes the spectral data, but leaves invariant the corresponding solution of the sinh-Gordon equation. In our further investigations, we fix this Möbius transformation such that for the Sym point there holds $\lambda_0 \in \mathbb{R}$, respectively $\kappa_0 \in i\mathbb{R}$.

For a set of pairwise distinct branch points $\alpha_i \in \mathbb{C}^*$, $i = 1, .., g$, with $Im(\alpha_i) < 0$ define the polynomial

$$a(\kappa) = \prod_{i=1}^{g} (\kappa - \alpha_i)(\kappa - \bar{\alpha}_i) \tag{4.2}$$

and the hyperelliptic curve Γ by

$$\nu^2 = (\kappa^2 + 1) a(\kappa). \tag{4.3}$$

Γ is a two-sheeted covering of the Riemann sphere with branch points at $\kappa = \pm i, \alpha_i, \bar{\alpha}_i$. Note that since $a(\kappa)$ does not have any real roots, we have $a(\kappa) \geq 0 \ \forall \kappa \in \mathbb{R}$. On Γ there exist the following involutions

$$\sigma : (\kappa, \nu) \mapsto (\kappa, -\nu), \quad \rho : (\kappa, \nu) \mapsto (\bar{\kappa}, \bar{\nu}), \quad \eta : (\kappa, \nu) \mapsto (\bar{\kappa}, -\bar{\nu}). \tag{4.4}$$

The fixed point set of ρ is called the real part of Γ.

Define $b_i(\kappa) = \frac{1}{\pi i} \partial_\kappa \ln \mu_i (\kappa^2 + 1)\nu$. Because of the freedom of Möbius transformations we can assume that $d \ln \mu_i$ is holomorphic at $\kappa = \infty$. Comparing powers of κ we see that $b_i(\kappa)$ is a polynomial of degree $g + 1$. Because of $\eta^* d \ln \mu_i = d \ln \bar{\mu}_i$ we have

$$\eta^* d \ln \mu_i = \pi i \frac{b_i(\bar{\kappa})}{-\bar{\nu}(\bar{\kappa}^2 + 1)} d\bar{\kappa} = d \ln \bar{\mu}_i = -\pi i \frac{\overline{b_i(\kappa)}}{\bar{\nu}(\bar{\kappa}^2 + 1)} d\bar{\kappa},$$

$$\Leftrightarrow \qquad b_i(\bar{\kappa}) = \overline{b_i(\kappa)},$$

so b_i has real coefficients. The two differentials $d \ln \mu_i$ can then be written as

$$d \ln \mu_i = \pi i \frac{b_i(\kappa)}{(\kappa^2 + 1)\nu} d\kappa. \tag{4.5}$$

In general, there is a whole family of polynomials b_i corresponding to a solution of the sinh-Gordon equation. Fixing a Möbius transformation like described above then determines the parameter κ and hence the polynomials b_i uniquely.

Let the closing condition hold for λ_0. Then by the transformation rules obeyed by μ_i, the closing condition must also hold for $\frac{1}{\lambda_0}$. In the κ-plane, this corresponds to two points κ_0, κ_1 with

$$\kappa_0 = i \frac{\lambda_0 - 1}{\lambda_0 + 1},$$

$$\kappa_1 = i \frac{\bar{\lambda}_0^{-1} - 1}{\bar{\lambda}_0^{-1} + 1} = i \frac{1 - \bar{\lambda}_0}{1 + \bar{\lambda}_0} = \bar{\kappa}_0.$$

With this information, we can formulate the definition of the spectral data of a CMC torus.

Definition 4.1.1. *Let a be a real polynomial of degree $2g$ with highest coefficient equal to one, and let b_i be two real polynomials of degree $g + 1$, and $\kappa_0 \in \mathbb{C} \setminus \mathbb{R}$ a marked point. The spectral data of a CMC torus of finite type in $\mathbb{H}^3$ with mean curvature*

$$H = \frac{1 + \kappa_0 \bar{\kappa}_0}{2\, Im(\kappa_0)} \tag{4.6}$$

consists of (a, b_1, b_2, κ_0) *with the properties*

(A) $a(\kappa) \geq 0$ *for* $\kappa \in \mathbb{R}$.

(B) On the hyperelliptic curve Γ *defined by* $a(\kappa)$ *via* (4.3) *exist two functions* μ_i *with essential singularities at* $\kappa = \pm i$ *and otherwise holomorphic with logarithmic derivatives* (4.5)*, that transform under the involutions* (4.4) *as* $\sigma^* \mu_i = \mu_i^{-1}$, $\rho^* \mu_i = \bar{\mu}_i^{-1}$, $\eta^* \mu_i = \bar{\mu}_i$.

(C) The differentials $d \ln \mu_i$ *are meromorphic differentials of the second kind with double poles at* $\kappa = \pm i$ *and linearly independent principal parts.*

(D) $\mu_i(\kappa_0) = \mu_i(\bar{\kappa}_0) = \pm 1$.

We computed the formula for H using $H = \coth(q)$ with $\lambda_0 = e^{q+i\psi}$ and the parameter transformation $\lambda \mapsto \kappa$. Note that choosing the Sym point $\kappa_0 = 0$ corresponds to $H = \infty$, i.e. a cylinder in $\mathbb{R}^3$. $\kappa_0 = \pm i$ corresponds to $|H| = 1$, which is also a boundary of the domain where we can choose κ_0 since we used $|H| > 1$ to introduce the Lax pair and the Sym-Bobenko formula.

4.1.2 Deformations

Now suppose we have an open set of spectral data $\{(a, b_1, b_2, \kappa_0)\}$ which is parametrized by a real parameter $t \subset (t_0, t_1)$, meaning that all four quantities depend on κ and t and the corresponding family of spectral curves is defined by

$$\Gamma(t): \quad \nu^2 = (\kappa^2 + 1) a(\kappa, t). \tag{4.7}$$

We want to derive conditions on the deformation in order to leave invariant the above properties of the spectral data. From condition (B) we know that $\partial_t \ln \mu_i$ is meromorphic on the family of spectral curves and can only have simple poles at the branch points, i.e. $\pm i$ and the roots of $a(\kappa)$. Therefore we can write

$$\partial_t \ln \mu_i = \pi i \frac{c_i(\kappa)}{\nu} d\kappa. \tag{4.8}$$

The condition of being holomorphic in $\kappa = \infty$ again shows that c_i is of degree at most $g + 1$. From the transformation rules of μ_i it follows that c_i is a real polynomial. To find deformations such that the polynomials a and b_i stay in the set of spectral data of CMC tori, we look at the second derivatives and derive the integrability condition for the vector field corresponding to the polynomial c_i.

$$\partial^2_{t\kappa} \ln \mu_i = \pi i \frac{2a\, \partial_t b_i - \partial_t a\, b_i}{2\nu^3}, \quad \partial^2_{\kappa t} \ln \mu_i = \pi i \frac{2(\kappa^2+1) a\, \partial_\kappa c_i - 2\kappa a c_i - (\kappa^2+1) \partial_\kappa a\, c_i}{2\nu^3}.$$

Therefore we get

$$\partial^2_{t\kappa} \ln \mu_i \;=\; \partial^2_{\kappa t} \ln \mu_i \quad \Leftrightarrow$$

$$2a\, \partial_t b_i - \partial_t a\, b_i \;=\; 2(\kappa^2+1)a\, \partial_\kappa c_i - 2\kappa a c - (\kappa^2+1)\partial_\kappa a\, c_i. \tag{4.9}$$

If we want the solution to stay doubly periodic, we need to specify the relation in which the two linearly independent differentials $d \ln \mu_i$ should be deformed, i.e. a condition on the polynomials c_i. For this, we build the differential

$$\omega = \partial_t \ln \mu_1 \; d \ln \mu_2 - \partial_t \ln \mu_2 \; d \ln \mu_1 = -\pi^2 \frac{c_1 b_2 - c_2 b_1}{\nu^2(\kappa^2+1)} d\kappa.$$

Since the $d \ln \mu_i$ have poles of second order at $\kappa = \pm i$ and $\partial_t \ln \mu_i$ at most simple poles, ω has at most poles of order three. Furthermore $\ln \mu_i$ is constant along the curves $\kappa_0(t), \bar{\kappa}_0(t)$ of the marked points. Therefore $\partial_t \ln \mu_i$ must have zeroes at κ_0 and $\bar{\kappa}_0$. These conditions are satisfied by the differential

$$\frac{(\kappa - \kappa_0)(\kappa - \bar{\kappa}_0)}{(\kappa^2+1)^2} d\kappa.$$

Therefore if we require ω to be of that form up to factors of proportionality, the deformation conserves the properties of $d \ln \mu_i$ and $\partial_t \ln \mu_i$. This is the case iff

$$c_1 b_2 - c_2 b_1 = -\frac{1}{\pi^2}(\kappa - \kappa_0)(\kappa - \bar{\kappa}_0) a(\kappa). \tag{4.10}$$

For the deformation to be well-defined, we need to show that for given initial data, the above equations define unique tangent vectors $\partial_t a$ and $\partial_t b_i$ and hence vector fields on the set of spectral data.

Proposition 4.1.2. *Let the spectral data of a CMC torus be given. If the differentials $d \ln \mu_i$ have no common roots, then the equations* (4.9) *and* (4.10) *describe a well-defined deformation of spectral data and the initial data is contained in an open set of spectral data of CMC tori.*

Proof. If $d \ln \mu_i$ do not have commen roots, neither do b_i. Fixing the parameter κ such that $\kappa_0 \in i\mathbb{R}$, the b_i are uniquely determined. Evaluating equation (4.10) at the $2g+2$ roots of b_1 and b_2 then uniquely determines c_1 and c_2. Now let α_i, $i = 1, ...2g$, be the roots of a. Evaluating equation (4.9) at these roots yields

$$b_i(\alpha_j)\partial_t a(\alpha_j) = (\alpha_j^2+1) c_i(\alpha_j) \partial_\kappa a(\alpha_j) \tag{4.11}$$

whereas equation (4.10) at these roots gives

$$c_1(\alpha_j) b_2(\alpha_j) = c_2(\alpha_j) b_1(\alpha_j). \tag{4.12}$$

Since the b_i have no common roots, one of the ratios $\frac{c_i}{b_i}$ is well-defined. But then the other one also must exist and they coincide. Putting this into (4.11), we get $\partial_t a(\alpha_j)$ for the $2g$ roots and hence a unique $\partial_t a$. But given $\partial_t a$, $\partial_t b_1$ and $\partial_t b_2$ are uniquely determined by (4.9). Therefore, given our initial data, the tangent vectors on the set of spectral data are uniquely determined. Since the derivatives are rational functions in κ and continuous in t, a solution to the system of ODEs exists on some open interval. Hence the initial spectral data lies in an open set. □

If we want to deform spectral data of CMC tori, we also need the closing condition to be invariant under the deformation. Therefore we have to pose a condition on the Sym point κ_0 to fulfill the closing condition for every t, meaning that

$$\ln \mu_i(\kappa_0(t), t) = \pi i N = const. \quad \forall t \in \mathbb{R}$$

This holds if and only if

$$\partial_t \ln \mu_i(\kappa_0(t), t) = \partial_\kappa \ln \mu_i(\kappa_0(t), t) \partial_t \kappa_0(t) + \partial_t \ln \mu_i(\kappa_0(t), t) = 0.$$

Solving for $\partial_t \kappa_0$ and using the description of $\partial \ln \mu_i$ in terms of b_i and c_i we get

$$\partial_t \kappa_0 = -\frac{\partial_t \ln \mu_i}{\partial_\kappa \ln \mu_i} = -(\kappa_0^2 + 1)\frac{c_i(\kappa_0)}{b_i(\kappa_0)}. \tag{4.13}$$

Again the ratio is well-defined at the roots of $a(\kappa)$ and therefore yields a unique solution $\partial_t \kappa_0$.

4.2 Spectral genus $g = 0$ data

In this section we describe the spectral data of flat cylinders with spectral genus $g = 0$. These will be used as initial data for a deformation of spectral data. As there are no CMC tori of spectral genus $g = 0$ and 1 in $\mathbb{H}^3$, we start with cylinders but want to recover tori of higher spectral genus in the course of the deformation.

4.2.1 Moving frame and spectral data

We have already shown the following.

Proposition 4.2.1. *Let $f : \mathbb{R}^2 \to \mathbb{H}^3$ be a conformal immersion of a flat cylinder of spectral genus $g = 0$ and constant mean curvature H. Then there exists $\lambda_0 = e^{q+i\psi} \in \mathbb{C}$ with $|\lambda_0| \neq 1$ such that $H = \coth(q)$ and*

$$f(z) = F_{\lambda_0} \bar{F}_{\lambda_0}^t, \quad \text{where} \quad F_\lambda(z) = exp\left(\frac{i}{2}\begin{pmatrix} 0 & z\lambda^{-1} + \bar{z} \\ z + \bar{z}\lambda & 0 \end{pmatrix}\right).$$

From this formula, we want to derive the spectral data of the cylinder in terms of the polynomials $a(\kappa), b(\kappa)$ and the Sym point κ_0 introduced in the previous chapter. Note that because we are working with cylinders, there is a priori only one period and hence only one eigenfunction μ with corresponding polynomial $b(\kappa)$. First of all we need the eigenvalues of the monodromy. Since $F_\lambda(0) = \mathbb{1}$ we have

$$M_\tau(\lambda) = F_\lambda(\tau) \quad \forall \lambda \in \mathbb{C}^*$$

and a period $\tau \in \mathbb{C}^*$. So the eigenvalues of the monodromy are simply the eigenvalues of F. For the function μ this yields

$$\mu(z, \lambda) = exp\left(\pm\frac{i}{2}\sqrt{(z\lambda^{-1} + \bar{z})(z + \bar{z}\lambda)}\right)$$

and so the logarithm is

$$\ln \mu(z,\lambda) = \pm\frac{i}{2}(z\sqrt{\lambda}^{-1} + \bar{z}\sqrt{\lambda}). \tag{4.14}$$

Next we look at the closing condition, which ensures that the immersion closes the period and becomes a cylinder. For a marked point λ_0 and a period τ it is given by

$$M_\tau(\lambda_0) = \overline{M_\tau(\lambda_0)}^t = \pm\mathbb{1}$$

and since the eigenvalues of the transposed matrix are the same as the original ones, we get

$$\mu(\tau,\lambda_0) = \pm 1.$$

This is the case if there exists an integer $K \in \mathbb{Z}$ with

$$\ln \mu(\tau,\lambda_0) = \pi i K. \tag{4.15}$$

K is called the **wrapping number** and tells how many times the frame F returns to its initial position while traversing one period.

Later we want to parametrize the $g = 0$ cylinders by the mean curvature H. Because H is independent of the argument of λ_0, we can fix $\lambda_0 \in \mathbb{R}$ by a Möbius transformation. Given λ_0, we now compute the periods satisfying the closing condition.

$$\ln \mu(\tau,\lambda_0) = \pi i K \quad \Leftrightarrow \quad \tau\sqrt{\lambda_0}^{-1} + \bar{\tau}\sqrt{\lambda_0} = 2\pi K$$

$$\Leftrightarrow \quad a(\sqrt{\lambda_0}^{-1} + \sqrt{\lambda_0}) + ib(\sqrt{\lambda_0}^{-1} - \sqrt{\lambda_0}) = 2\pi K, \qquad \tau = a + ib$$

If $\lambda_0 > 0$ the roots are real. Since the right hand side is real, b has to vanish and we get

$$\tau = a = 2\pi K \frac{1}{\sqrt{\lambda_0}^{-1} + \sqrt{\lambda_0}} = 2\pi K \frac{\sqrt{\lambda_0}}{|\lambda_0| + 1}.$$

If $\lambda_0 < 0$ the roots are purely imaginary, so a has to vanish, leading to

$$b = -i2\pi K \frac{1}{\sqrt{\lambda_0}^{-1} - \sqrt{\lambda_0}} = -i2\pi K \frac{1}{-i\sqrt{|\lambda_0|}^{-1} - i\sqrt{|\lambda_0|}} = 2\pi K \frac{\sqrt{|\lambda_0|}}{|\lambda_0| + 1}$$

$$\Leftrightarrow \quad \tau = ib = 2\pi K \frac{i\sqrt{|\lambda_0|}}{|\lambda_0| + 1} = 2\pi K \frac{\sqrt{\lambda_0}}{|\lambda_0| + 1}.$$

So we see that in both cases we can write

$$\tau = 2\pi K \frac{\sqrt{\lambda_0}}{|\lambda_0| + 1} \tag{4.16}$$

From this forumla we can see that if we choose $\lambda_0 \in \mathbb{R}_+$, then $\tau \in \mathbb{R}$, whereas $\lambda_0 \in \mathbb{R}_-$ leads to $\tau \in i\mathbb{R}$.

The polynomial b was defined by

$$b(\kappa) = \frac{1}{\pi i}\nu(\kappa^2+1)\partial_\kappa \ln\mu(\kappa).$$

Since we are on a genus zero Riemann surface, the defining equation $\nu^2 = (\kappa^2+1)a(\kappa)$ breaks down to $\nu^2 = (\kappa^2+1)$ and hence $a(\kappa) = 1$. Therefore we have

$$b(\kappa) = \frac{1}{\pi i}(\kappa^2+1)^{\frac{3}{2}}\partial_\kappa \ln\mu(\tau,\kappa). \tag{4.17}$$

Using our formula for $\ln\mu$ in terms of λ and the transformed spectral parameter $\kappa = i\frac{\lambda-1}{\lambda+1}$ we have

$$\begin{aligned}\ln\mu(\tau,\kappa) &= \frac{i}{2}\left(2\pi K\frac{\sqrt{\lambda_0}}{|\lambda_0|+1}\sqrt{\frac{i+\kappa}{i-\kappa}}^{-1} + 2\pi K\frac{\overline{\sqrt{\lambda_0}}}{|\lambda_0|+1}\sqrt{\frac{i+\kappa}{i-\kappa}}\right)\\ &= \frac{2\pi iK}{|\lambda_0|+1}\frac{Re(\sqrt{\lambda_0}) - Im(\sqrt{\lambda_0})\kappa}{\sqrt{\kappa^2+1}}.\end{aligned}$$

Note that the different branches of the root $\sqrt{\kappa^2+1}$ correspond to different branches of ν, so $\ln\mu$ is only well-defined on the Riemann surface.

Taking the derivative with respect to κ gives

$$\partial_\kappa \ln\mu = \frac{-2\pi iK}{|\lambda_0|+1}\frac{Re(\sqrt{\lambda_0})\kappa + Im(\sqrt{\lambda_0})}{(\kappa^2+1)^{\frac{3}{2}}}.$$

Pluggin this into (4.17) gives

$$b(\kappa) = -\frac{2K}{|\lambda_0|+1}\left(Im(\sqrt{\lambda_0}) + Re(\sqrt{\lambda_0})\kappa\right).$$

Finally we express the Sym point λ_0 in terms of κ via the above transformation, i.e. $\lambda_0 = \frac{i+\kappa_0}{i-\kappa_0}$, which eventually gives the explicit formula for b

$$b(\kappa) = -\frac{2K}{|\frac{i+\kappa_0}{i-\kappa_0}|+1}\left(Im(\sqrt{\frac{i+\kappa_0}{i-\kappa_0}}) + Re(\sqrt{\frac{i+\kappa_0}{i-\kappa_0}})\kappa\right)$$

As we have seen above, the Sym point is given by $\kappa_0 = i\frac{\lambda_0-1}{\lambda_0+1}$. This gives the following characterization of spectral genus zero cylinders.

Proposition 4.2.2. *The spectral data (a, b, κ_0) of a flat cylinder of spetral genus $g = 0$ in $\mathbb{H}^3$ with wrapping number K is given by the polynomials*

$$a(\kappa) = 1, \qquad b(\kappa) = -\frac{2K}{|\frac{i+\kappa_0}{i-\kappa_0}|+1}\left(Im(\sqrt{\frac{i+\kappa_0}{i-\kappa_0}}) + Re(\sqrt{\frac{i+\kappa_0}{i-\kappa_0}})\kappa\right) \tag{4.18}$$

and a marked point κ_0 with $\kappa_0 \notin \mathbb{R}$.

We can simplify computations by considering the special property of λ_0 being real. For positive λ_0 we get

$$b(\kappa) = -\frac{2K}{\frac{i+\kappa_0}{i-\kappa_0}+1}\sqrt{\frac{i+\kappa_0}{i-\kappa_0}}\kappa = -K\sqrt{\kappa_0^2+1}\kappa,$$

so in this case, b is an odd polynomial in κ. On the other hand, if we choose $\lambda_0 \in \mathbb{R}_-$ we have a purely imaginary period and the polynomial is given by

$$b(\kappa) = -\frac{K\sqrt{\kappa_0^2+1}}{\kappa_0},$$

which is even in κ. Regarding this, it seems that the two situations are crucially different, but later we are going to describe a transformation which interchanges the two cases, allowing us to refer only to one of them in our further considerations.

4.2.2 Double points on the spectral curve

In the last section, we have seen how to obtain the spectral data of a spectral genus $g = 0$ cylinder. Now we want to use a deformation of this data in order to obtain CMC tori. Since there are no tori with spectral genus $g = 0$ in $\mathbb{H}^3$ we have to change the spetral genus during the deformation. As the branch points of the spectral curve can be characterized by the trace of the monodromy being ± 2, this can be done if there are two additional points where the eigenvalues of the monodromy are ± 1. By interpreting these two points as a pair of branch points that have fallen together, we can deform them into four distinct branch points and obtain a Riemann surface of genus $g = 2$.

Definition 4.2.3. *Let (a, b, κ_0) be the spectral data of a CMC cylinder with spectral genus $g = 0$, μ the eigenvalue of the monodromy and τ a simple period. A double point is given by $\kappa_d \in \Gamma$, where Γ is defined by $\nu^2 = (\kappa^2 + 1)a(\kappa)$, such that $\kappa_d \neq \kappa_0$ and*

$$\ln \mu(\tau, \kappa_d) = \pi i L, \quad L \in \mathbb{Z}. \tag{4.19}$$

Using the formula for the eigenvalues of the monodromy in the case $g = 0$, we can compute the possible double points of the spectral curve. For this, let $\lambda_0 \in \mathbb{R}$ be a Sym point with period $\tau = 2\pi K \frac{\sqrt{\lambda_0}}{|\lambda_0|+1}$ and wrapping number K. We are looking for another point λ_d which fulfills the closing condition with the same period τ and some wrapping number $L \in \mathbb{Z}$:

$$\ln \mu(\tau, \lambda_d) = \pi i L.$$

Since we want to deform a cylinder into a torus we need a second period during the deformation. Since the values of $\ln \mu_i$ at the branch points of the spectral curve stay constant during the deformation, we need to make sure that $\ln \mu_i(\lambda_d) \in \pi i \mathbb{Z}$ at the beginning. From the discussion of CMC cylinders of spectral genus $g = 0$ we know that for two linearly independent periods τ_1, τ_2 this can only hold if the point satisfies $\lambda_d \in S^1$ or equivalently $\kappa_d \in \mathbb{R}$. So without loss of generality we can set $\lambda_d = e^{i\psi}$. Then the above condition reads

$$\frac{i}{2}(\tau\sqrt{\lambda_d}^{-1} + \bar{\tau}\sqrt{\lambda_d}) = \pi i L \quad \Leftrightarrow \quad \tau e^{-i\frac{\psi}{2}} + \bar{\tau} e^{i\frac{\psi}{2}} = 2\pi L.$$

As we said before, the choice of $\lambda_0 \in \mathbb{R}$ being positive or negative affects the shape of the polynomials $b(\kappa)$. We also need to distinguish these cases when looking for double points. From now on we refer to $\lambda_0 > 0$ as the **first case** and $\lambda_0 < 0$ as the **second case** (of course $\lambda_0 \neq \pm 1$ in order to obtain a well-defined cylinder in $\mathbb{H}^3$).

In the first case the condition on $\lambda_d = e^{i\psi}$ is

$$\tau(e^{-i\frac{\psi}{2}} + e^{i\frac{\psi}{2}}) = 2\pi L \quad \Leftrightarrow \quad 2\pi K \frac{\sqrt{\lambda_0}}{1+\lambda_0} 2\cos(\frac{\psi}{2}) = 2\pi L$$

$$\Leftrightarrow \quad \psi = 2\arccos(\frac{L}{2K}\frac{1+\lambda_0}{\sqrt{\lambda_0}}).$$

In the second case the period corresponding to λ_0 is purely imaginary. Therefore we get

$$\tau(e^{-i\frac{\psi}{2}} - e^{i\frac{\psi}{2}}) = 2\pi L \quad \Leftrightarrow \quad 2\pi K\, i \frac{\sqrt{|\lambda_0|}}{1+|\lambda_0|} 2\sinh(-i\frac{\psi}{2}) = 2\pi L$$

$$\Leftrightarrow \quad \sin(\frac{\psi}{2}) = \frac{L}{2K}\frac{1+|\lambda_0|}{\sqrt{|\lambda_0|}} \quad \Leftrightarrow \quad \psi = 2\arcsin(\frac{L}{2K}\frac{1+|\lambda_0|}{\sqrt{|\lambda_0|}}).$$

The arguments of arccos and arcsin are real. Since these functions are only defined on $[-1, 1]$ we get an additional restriction for the $L \in \mathbb{Z}$ we can choose when constructing a double point. It is given by

$$|\frac{L}{2K}\frac{1+|\lambda_0|}{\sqrt{|\lambda_0|}}| \leq 1 \quad \Leftrightarrow \quad |L| \leq \lfloor \frac{2K\sqrt{|\lambda_0|}}{1+|\lambda_0|} \rfloor,$$

where $\lfloor\ \rfloor$ denotes the Gauss bracket. So we see that the choice of the Sym point λ_0 and a wrapping number $K \in \mathbb{Z}$ determines a finite set of numbers $L \in \mathbb{Z}$ which themselves determine possible double points on the spectral curve. We sum up the previous discussion in a proposition.

Proposition 4.2.4. *Let Γ be the spectral curve of a CMC cylinder with spectral genus $g = 0$ and $\lambda_0 \in \mathbb{R}$ a Sym point with period $\tau = 2\pi K \frac{\sqrt{\lambda_0}}{1+|\lambda_0|}$ and wrapping number K. If we choose an additional number L satisfying*

$$|L| \leq \lfloor \frac{2K\sqrt{|\lambda_0|}}{1+|\lambda_0|} \rfloor \tag{4.20}$$

then there is a double point $\lambda_d = e^{i\psi}$ on Γ, i.e. there holds $\ln\mu(\tau, \lambda_d) = \pi i L$, where

$$\psi = 2\arccos(\frac{L}{2K}\frac{1+\lambda_0}{\sqrt{\lambda_0}}) \quad if\ \lambda_0 > 0, \ and \ \ \psi = 2\arcsin(\frac{L}{2K}\frac{1+|\lambda_0|}{\sqrt{|\lambda_0|}}) \quad if\ \lambda_0 < 0. \tag{4.21}$$

We express the transformed double point in terms of the Sym point

$$\kappa_d = i\frac{e^{i\psi}-1}{e^{i\psi}+1} = i\frac{e^{i\psi/2}-e^{-i\psi/2}}{e^{i\psi/2}+e^{-i\psi/2}} = -\frac{\sin(\frac{\psi}{2})}{\cos(\frac{\psi}{2})} = -\tan(\arccos(\frac{L}{K}\frac{1}{\sqrt{\kappa_0^2+1}}))$$
$$= -\sqrt{\frac{K^2}{L^2}(\kappa_0^2+1)-1}. \tag{4.22}$$

Vice versa, we could also choose $K, L \in \mathbb{Z}$ first. These two constants uniquely determine the spectral curve. Then the above formula tells us for which Sym points $\kappa_0 \in i\mathbb{R}$ there exist double points on the spectral curve. They are given by

$$|Im(\kappa_0)| \leq \sqrt{1-\frac{L^2}{K^2}}.$$

Therefore, for each $K, L \in \mathbb{Z}$ with $L < K$, there exist Sym points κ_0 on the spectral curve to which we can associate double points according to the above algorithm.

Because we want to branch from spectral genus $g = 0$ to spectral genus $g = 2$ we need to choose two such double points. A special case which omits an additional symmetry on the spectral curve is provided by the following

Proposition 4.2.5. *Let a spectral curve of a genus $g = 0$ CMC cylinder be given such that for the Sym point there holds $\lambda_0 \in \mathbb{R}$. If λ_d is a double point on the spectral curve, then λ_d^{-1} is also a double point. In the κ-plane this corresponds to the two points $\kappa_d, -\kappa_d$, where $\kappa_d = i\frac{\lambda_d-1}{\lambda_d+1}$.*

Proof. We use the expression for $\ln\mu$ in the case $g = 0$ to check $\ln\mu(\tau, \lambda_d^{-1}) \in \pi i\mathbb{Z}$. Note that the choice of $\lambda_0 \in \mathbb{R}$ leads the two cases for the period τ we described above. In the first case we have

$$\ln\mu(\tau,\lambda_d^{-1}) = \frac{i}{2}\tau(\sqrt{\lambda_d^{-1}}^{-1} + \sqrt{\lambda_d^{-1}}) = \frac{i}{2}\tau(\sqrt{\lambda_d} + \sqrt{\lambda_d}^{-1}) = \ln\mu(\tau,\lambda_d) = \pi iL.$$

In the second case

$$\ln\mu(\tau,\lambda_d^{-1}) = \frac{i}{2}(\tau\sqrt{\lambda_d^{-1}}^{-1} + \bar{\tau}\sqrt{\lambda_d^{-1}}) = \frac{i}{2}(-\bar{\tau}\sqrt{\lambda_d} - \tau\sqrt{\lambda_d}^{-1}) = -\ln\mu(\tau,\lambda_d) = -\pi iL.$$

Hence λ_d^{-1} is also a double point. □

4.2.3 The period lattice

After having chosen two double points on the spectral curve, we want to deform the spectral data in such a way that the double points open to four distinct branch points on the family of spectral curves. However, so far we know only one period τ_1 (the period of the cylinder) which is deformed such that the corresponding monodromy satisfies the closing condition at the Sym point λ_0. To obtain a CMC torus at the endpoint of the deformation we need to specify a second period τ_2, for which there is a doubly periodic solution u of the sinh-Gordon equation belonging to this spectral curve. Since the values of $\ln \mu_i$ at the branch points must be in $\pi i\mathbb{Z}$, we can compute which vectors we are allowed to choose as periods, so that we can construct the solution u from the resulting spectral cuve.

Since the CMC cylinder at the beginning of the deformation is of spectral genus $g = 0$, its metric is flat. Therefore, a priori any $\tau \in \mathbb{C}^*$ is a period of the metric. After having chosen two double points, we want the period to satisfy

$$\ln \mu(\tau, \lambda_d) \in \pi i\mathbb{Z}.$$

This yields a lattice of admissible periods. Set $\tau = a + ib$. In the first case we get

$$\ln \mu(\tau, \lambda_d) \in \pi i\mathbb{Z} \quad \Leftrightarrow \quad \tau\sqrt{\lambda_d}^{-1} + \bar{\tau}\sqrt{\lambda_d} \in 2\pi\mathbb{Z}$$

$$\Leftrightarrow \quad a(\sqrt{\lambda_d}^{-1} + \sqrt{\lambda_d}) + ib(\sqrt{\lambda_d}^{-1} - \sqrt{\lambda_d}) = 2a\cos(\frac{\psi}{2}) + 2ib\sinh(-i\frac{\psi}{2}) \in 2\pi\mathbb{Z}$$

$$\Leftrightarrow \quad a\frac{L}{K}\frac{1+\lambda_0}{\sqrt{\lambda_0}} + 2b\sin(\frac{\psi}{2}) \in 2\pi\mathbb{Z} \quad \Leftrightarrow \quad a \in \frac{2\pi K}{L}\frac{\sqrt{\lambda_0}}{1+\lambda_0}\mathbb{Z}, \quad b \in \frac{\pi}{\sin\frac{\psi}{2}}\mathbb{Z}.$$

So we see that in the first case, the lattice of admissible periods to define a second monodromy is given by

$$\Delta_1 = \frac{2\pi K}{L}\frac{\sqrt{\lambda_0}}{1+\lambda_0}\mathbb{Z} + i\frac{\pi}{\sin(\frac{\psi}{2})}\mathbb{Z}. \tag{4.23}$$

In the second case, we obtained another formula for the double point λ_d. Hence the lattice is given by

$$\ln \mu(\tau, \lambda_d) \in \pi i\mathbb{Z} \quad \Leftrightarrow \quad \tau\sqrt{\lambda_d}^{-1} + \bar{\tau}\sqrt{\lambda_d} \in 2\pi\mathbb{Z}$$

$$\Leftrightarrow \quad a(\sqrt{\lambda_d}^{-1} + \sqrt{\lambda_d}) + ib(\sqrt{\lambda_d}^{-1} - \sqrt{\lambda_d}) = 2a\cos(\frac{\psi}{2}) + 2ib\sinh(-i\frac{\psi}{2}) \in 2\pi\mathbb{Z}$$

$$\Leftrightarrow \quad 2a\cos(\frac{\psi}{2}) + b\frac{L}{K}\frac{1+|\lambda_0|}{\sqrt{|\lambda_0|}} \in 2\pi\mathbb{Z} \quad \Leftrightarrow \quad a \in \frac{\pi}{\cos(\frac{\psi}{2})}\mathbb{Z}, \quad b \in \frac{2\pi K}{L}\frac{\sqrt{|\lambda_0|}}{1+|\lambda_0|}.$$

So we get the lattice in the second case as

$$\Delta_2 = \frac{\pi}{\cos(\frac{\psi}{2})}\mathbb{Z} + i\frac{2\pi K}{L}\frac{\sqrt{|\lambda_0|}}{1+|\lambda_0|}. \tag{4.24}$$

Knowing these lattices, we now choose a second period which is linearly independent to τ_1. Again we have to be careful about the two different cases. As there are no spectral genus $g = 0$ cylinders in $\mathbb{H}^3$, the eigenvalues of the second monodromy at the Sym point λ_0 must be different from ± 1. In order to keep the symmetry imposed on the spectral curve, we choose a homogenous lattice, i.e. for $\tau_1 \in \mathbb{R}$ we take $\tau_2 \in i\mathbb{R}$.

For later purposes we compute everything in terms of the transformed spectral parameter κ. For both cases we have with $\tau = a + ib$

$$\ln \mu(\tau, \kappa) = \frac{i}{2}\left(\tau\sqrt{\frac{i-\kappa}{i+\kappa}} + \bar{\tau}\sqrt{\frac{i+\kappa}{i-\kappa}}\right) = \frac{ia}{\sqrt{\kappa^2+1}} - \frac{ib\kappa}{\sqrt{\kappa^2+1}}. \tag{4.25}$$

So in the first case, where $\tau_1 = 2\pi K \frac{\sqrt{\lambda_0}}{1+\lambda_0}$ and $\tau_2 = \frac{\pi i}{\sin(\frac{\psi}{2})}$ we get

$$\ln \mu_2(\kappa) = -\frac{\pi i \kappa}{\sin(\frac{\psi}{2})\sqrt{\kappa^2+1}} = -\frac{\pi i \kappa}{\sqrt{\kappa^2+1}} \frac{1}{\sqrt{1 - \frac{L^2}{K^2}\frac{1}{\kappa_0^2+1}}}.$$

For the second case, where $\tau_1 = 2\pi i K \frac{\sqrt{|\lambda_0|}}{1+|\lambda_0|}$ and $\tau_2 = \frac{\pi}{\cos(\frac{\psi}{2})}$ we have

$$\ln \mu_2(\kappa) = \frac{\pi i}{\cos(\frac{\psi}{2})\sqrt{\kappa^2+1}} = \frac{\pi i}{\sqrt{\kappa^2+1}} \frac{1}{\sqrt{1 - \frac{L^2}{K^2}\frac{\kappa_0^2}{\kappa_0^2+1}}}.$$

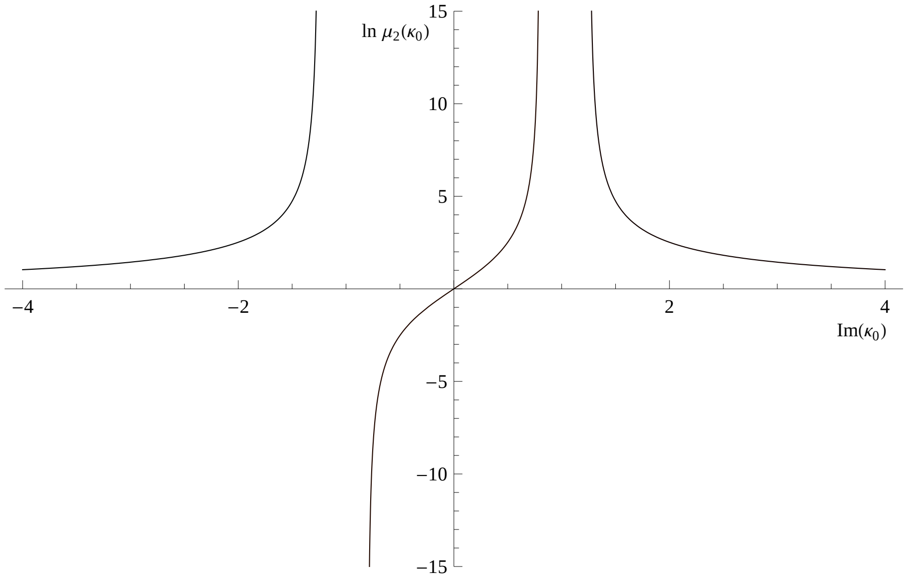
In both computations we used the identities $\sin(\arccos(x)) = \cos(\arcsin(x)) = \sqrt{1-x^2}$.

Now as the second monodromy is defined we are interested in the value of $\ln \mu_2$ at the Sym point. As we have chosen $\lambda_0 \in \mathbb{R}$, the corresponding κ_0 is purely imaginary. Therefore in both of the above cases, $\ln \mu_2(\kappa_0) \in \mathbb{R}$. For the specific values $K = 5$ and $L = 3$ we compute

$$\ln \mu_2(\kappa_0) = -\frac{\pi i \kappa_0}{\sqrt{\kappa_0^2 + \frac{16}{25}}} = \frac{\pi Im(\kappa_0)}{\sqrt{-Im(\kappa_0)^2 + \frac{16}{25}}}, \qquad |\kappa_0| < 1, \tag{4.26}$$

$$\ln \mu_2(\kappa_0) = \frac{\pi i}{\sqrt{\frac{16}{25}\kappa_0^2 + 1}} = \frac{\pi i}{\sqrt{-\frac{16}{25}Im(\kappa_0)^2 + 1}}, \qquad |\kappa_0| > 1. \tag{4.27}$$

As we saw before, for a fixed wrapping number $K \in \mathbb{Z}$ the CMC cylinders of spectral genus $g = 0$ are sufficiently determined by the value of the Sym point. Therefore we get a one parameter family of CMC cylinders parametrized by $Im(\kappa_0)$. After choosing one branch of the root, we can picture the values of $\ln \mu_2(\kappa_0)$ and see that for $\kappa_0 \neq 0$ this function is non zero. However, since it is real for all $\kappa_0 \in i\mathbb{R}$ we have $\ln \mu_2(\kappa_0) \notin \pi i \mathbb{Z}$. This means that in the family of CMC cylinders with wrapping number K and the chosen double points there are no tori of spectral genus $g = 0$ in $\mathbb{H}^3$.



We also want to describe the boundaries of the $g = 0$ family. We see that for $|\kappa_0| < 1$, the value of $\ln \mu_2(\kappa_0)$ is only well-defined for $|\kappa_0| < \frac{\sqrt{K^2-L^2}}{K}$. To describe what happens when we reach this boundary, we look at the period τ_2. In the λ-plane, for a double point

$$\lambda_d = e^{i\psi}, \quad \psi = 2\arccos(\frac{L}{2K}\frac{1+\lambda_0}{\sqrt{\lambda_0}}),$$

it was given by $\tau_2 = \frac{\pi i}{\sin(\frac{\psi}{2})}$. The limit $\kappa_0 \to \pm i\frac{\sqrt{K^2-L^2}}{K}$ corresponds to $\psi \to 0$. For the period this means $\tau_2 \to \infty$, i.e. the conformal class of the metric degenerates. Looking at the formula for the two double points $\lambda_d, \lambda_d^{-1}$ we have chosen, we see that these fall together to $\lambda = 1$. In the κ-plane this corresponds to the two double points $\pm\kappa_d$ falling together to $\kappa = 0$.

Another boundary is $\kappa_0 = 0$, since choosing this Sym point corresponds to mean curvature $H = \infty$. Hence the resulting cylinder in fact lies in $\mathbb{R}^3$. This point will also be dealt with later when investigating the deformation of cylinders.

4.2.4 Branching from spectral genus $g = 0$ to $g = 2$

After having chosen double points and a second period, we can write down the following spectral data in the case $g = 0$. We use tildes to distinguish between the $g = 0$ case and the $g = 2$ case with double points.

In the first case, we have already seen that $\tilde{b}_1(\kappa) = -K\sqrt{\kappa_0^2+1}\kappa$, where we used the first period $\tau_1 = \pi K\sqrt{\kappa_0^2+1}$. For the second period $\tau_2 = \frac{\pi i}{\sin(\frac{\psi}{2})}$ we compute

$$\ln\tilde{\mu}_2(\kappa) = -\frac{\pi i}{\sin(\frac{\psi}{2})}\frac{\kappa}{\sqrt{\kappa^2+1}} \quad \Rightarrow \quad \partial_\kappa \ln\tilde{\mu}_2(\kappa) = -\frac{\pi i}{\sin(\frac{\psi}{2})}\frac{1}{(\kappa^2+1)^{\frac{3}{2}}},$$

and hence

$$\tilde{b}_2(\kappa) = \frac{1}{\pi i}(\kappa^2+1)^{\frac{3}{2}}\partial_\kappa \ln \mu_2(\kappa) = -\frac{1}{\sin(\frac{\psi}{2})}.$$

Because $|Im(\kappa_0)| < 1$ in the first case, the roots involving κ_0 are real. Therefore we can write the spectral data in the form

$$\tilde{a}(\kappa) = 1, \quad \tilde{b}_1(\kappa) = f_1\kappa, \quad \tilde{b}_2(\kappa) = f_2, \qquad f_1, f_2 \in \mathbb{R}. \tag{4.28}$$

In the second case we have $\tilde{b}_1(\kappa) = -\frac{K\sqrt{\kappa_0^2+1}}{\kappa_0}$ for the first period $\tau_1 = \pi i K\frac{\sqrt{\kappa_0^2+1}}{\kappa_0}$. For the second period $\tau_2 = \frac{\pi}{\cos(\frac{\psi}{2})}$ we compute

$$\ln\tilde{\mu}_2(\kappa) = \frac{\pi i}{\cos(\frac{\psi}{2})}\frac{1}{\sqrt{\kappa^2+1}} \quad \Rightarrow \quad \partial_\kappa \ln\tilde{\mu}_2(\kappa) = -\frac{\pi i}{\cos(\frac{\psi}{2})}\frac{\kappa}{(\kappa^2+1)^{\frac{3}{2}}},$$

giving

$$\tilde{b}_2(\kappa) = -\frac{1}{\cos(\frac{\psi}{2})}\kappa.$$

So in this case we can write the spectral data as

$$\tilde{a}(\kappa) = 1, \quad \tilde{b}_1(\kappa) = f_1, \quad \tilde{b}_2(\kappa) = f_2\kappa, \qquad f_1, f_2 \in \mathbb{R}. \tag{4.29}$$

This shows that compared to the case $|Im(\kappa_0)| < 1$, the roles of the polynomials $\tilde{b}_1, \tilde{b}_2$ are changed when we consider $|Im(\kappa_0)| > 1$.

We now want to derive the form of the spectral data if we consider a spectral curve with double points on it. So suppose we have chosen a pair of double point $\pm\kappa_d$ according to the introduced algorithm. Then the polynomial $a(\kappa)$ must have double zeroes at these double points. This means that two branch points (which would be simple zeroes of a) have fallen together, so that they don't contribute to the geometric genus of the resulting curve. Having leading coefficient one, the polynomial is

$$a(\kappa) = (\kappa-\kappa_d)^2(\kappa+\kappa_d)^2 = \kappa^4 - 2\kappa_d^2\kappa^2 + \kappa_d^4.$$

Because $d\ln\mu_i = \pi i\frac{b_i}{(\kappa^2+1)\nu}d\kappa$ and the additional zeroes of $a(\kappa)$ result in simple zeroes of ν at $\pm\kappa_d$ we have to factor out these zeroes in order for $d\ln\mu_i$ to have poles only at $\pm i$. Therefore we have

$$b_i(\kappa) = \tilde{b}_i(\kappa)(\kappa-\kappa_d)(\kappa+\kappa_d) = \tilde{b}_i(\kappa)(\kappa^2-\kappa_d^2).$$

We sum up the above discussion.

Proposition 4.2.6. *The spectral data of a CMC cylinder with Sym point κ_0 at a branch point of the deformation from genus $g=0$ to $g=2$ is given as follows.*

(i) If $|Im(\kappa_0)| < 1$, then

$$a(\kappa) = \kappa^4 + a_2\kappa^2 + a_0, \quad b_1(\kappa) = f_1(\kappa^2-\beta_1)\kappa, \quad b_2(\kappa) = f_2(\kappa^2-\beta_2),$$

$$a_2 = -\kappa_d^2,\ a_0 = \kappa_d^4,\ f_1 = -K\sqrt{\kappa_0^2+1},\ f_2 = -\frac{1}{\sin(\frac{\psi}{2})},\ \beta_1 = \beta_2 = \kappa_d^2.$$

(ii) If $|Im(\kappa_0)| > 1$*, then*

$$a(\kappa) = \kappa^4 + a_2\kappa^2 + a_0, \quad b_1(\kappa) = f_1(\kappa^2 - \beta_1), \quad b_2(\kappa) = f_2(\kappa^2 - \beta_2)\kappa,$$

$$a_2 = -\kappa_d^2,\ a_0 = \kappa_d^4,\ f_1 = \frac{K\sqrt{\kappa_0^2+1}}{\kappa_0},\ f_2 = -\frac{1}{\cos(\frac{\psi}{2})},\ \beta_1 = \beta_2 = \kappa_d^2.$$

Next we have to investigate how the expressions for the eigenvalues of the monodromy change as we change the spectral data from genus $g = 0$ to $g = 2$. We look at the case $|Im(\kappa_0)| < 1$ first. In the $g = 0$ case the differentials $d\ln\mu_i$ were given by

$$\partial_\kappa \ln\tilde{\mu}_1 = \pi i \frac{f_1\kappa}{(\kappa^2+1)\tilde{\nu}},$$

$$\partial_\kappa \ln\tilde{\mu}_2 = \pi i \frac{f_2}{(\kappa^2+1)\tilde{\nu}},$$

where $\tilde{\nu}^2 = \kappa^2 + 1$.

For reasons that will become clear later, we normalize the differentials such that $\ln\mu_1(\infty) = 0$ and $\ln\mu_2(0) = 0$. As we will see, these values do not change during the deformation. This yields

$$\ln\tilde{\mu}_1 = -\pi i\,\frac{f_1}{\tilde{\nu}}, \tag{4.30}$$

$$\ln\tilde{\mu}_2 = \pi i\,\frac{f_2\kappa}{\tilde{\nu}}. \tag{4.31}$$

Now we compute the differentials in the case $g = 2$. We have seen that when branching to higher genus at double points $\pm\kappa_d$ of the spectral curve, the spectral data changes to

$$a(\kappa) = \kappa^2 - 2\kappa_d^2\kappa^2 + \kappa_d^4, \quad b_1(\kappa) = f_1(\kappa^2 - \kappa_d^2)\kappa, \quad b_2(\kappa) = f_2(\kappa^2 - \kappa_d^2).$$

Therefore we get

$$\partial_\kappa \ln\mu_1 = \pi i\,\frac{f_1(\kappa^2-\kappa_d^2)\kappa}{(\kappa^2+1)\nu}, \quad \partial_\kappa \ln\mu_2 = \pi i\,\frac{f_2(\kappa^2-\kappa_d^2)}{(\kappa^2+1)\nu},$$

where now $\nu^2 = (\kappa^2+1)(\kappa^2-\kappa_d^2)^2$.

As we will see later, the Sym point κ_0 stays purely imaginary during the deformation. The value we are interested in is $\ln\mu_2(\kappa_0, \nu)$. With the above formula we can determine it in terms of the genus $g = 0$ data as follows.

For $\kappa \in i\mathbb{R}$ we have $\kappa^2 - \kappa_d^2 < 0$ and $(\kappa^2 - \kappa_d^2)^2 > 0$. Therefore, for purely imaginary κ, the additional factors in the $g = 2$ case are

$$\frac{\kappa^2-\kappa_d^2}{\nu} = \frac{\kappa^2-\kappa_d^2}{|\kappa^2-\kappa_d^2|}\,\frac{1}{\tilde{\nu}}.$$

For the special case of the Sym point and with the normalization $\ln \mu_2(0) = 0$ this yields

$$\ln \mu_2(\kappa_0, \nu) = -\ln \tilde{\mu}_2(\kappa_0, \tilde{\nu}). \tag{4.32}$$

So far we distinguished the two different cases $|Im(\kappa_0)| < 1$ and $|Im(\kappa_0)| > 1$. The special form of the spectral data and hence the deformation ODEs depend on this choice. However, there is a transformation which allows us to transform the one case into the other. By taking the change of variable $\kappa \mapsto -\frac{1}{\kappa}$ which in the λ-plane corresponds to $\lambda \mapsto -\lambda$ the respective cases are interchanged. The transformed spectral parameter is now of the form

$$\kappa = i\frac{\lambda + 1}{\lambda - 1}$$

and for the mean curvature this means that $\kappa_0 = 0$ corresponds to $H = -\infty$. Therefore, for the further investigation, it is sufficient to consider only one of the cases described above. We will focus on $|Im(\kappa_0)| < 1$ for the rest of the thesis.

4.3 Deformation of spectral curves of CMC cylinders

4.3.1 The moduli space

In section 4.1 we introduced a deformation theory for CMC tori. First we represented a CMC torus by its spectral data, i.e. a polynomial $a(\kappa)$ describing a hyperelliptic curve and two polynomials $b_1(\kappa), b_2(\kappa)$ representing two meromorphic forms which are logarithmic derivatives of the eigenvalues of the monodromy corresponding to two linearly independent periods τ_1, τ_2. By choosing some initial data, we could derive differential equations that leave invariant the properties of a CMC torus.

In $\mathbb{H}^3$ the situation is more complicated. As there are no CMC tori of spectral genus $g = 0$ and the spectral data of genus $g = 2$ tori is rather difficult to compute, we start with a $g = 0$ cylinder with double points on the spectral curve, deform this into a $g = 2$ cylinder and look for $g = 2$ tori in the course of the deformation. As CMC tori are of finite type and have constant Hopf differential Q, we also focus on cylinders which share these properties.

Definition 4.3.1. *CMC cylinders with constant Hopf differential, whose metric is a periodic solution of finite type of the* sinh*-Gordon equation, are called CMC cylinders of finite type.*

For spectral genus $g = 2$, the metric of the finite type cyliners is a doubly periodic solution of the sinh-Gordon equation, according to choosing the second period from the lattices described above. In order to deform such finite type cylinders we need some additional data to write down the deformation ODE. We have seen that if we choose the Sym point $\kappa_0 \in i\mathbb{R}$ with $|\kappa_0| < 1$ and a second period $\tau_2 \in i\mathbb{R}$ we have the following spectral data for the double points $\pm\kappa_d$.

$$a(\kappa) = \kappa^4 + a_2\kappa^2 + a_0, \quad b_1(\kappa) = f_1(\kappa^2 - \beta_1)\kappa, \quad b_2 = f_2(\kappa^2 - \beta_2),$$

where $a_2 = -2\kappa_d$, $a_0 = \kappa_d^4$, $\beta_1 = \beta_2 = \kappa_d^2$ and $f_1, f_2 \in \mathbb{R}$. We have also seen that $\ln \mu_2(\kappa_0) \notin \pi i\mathbb{Z}$ at the initial conditions. The deformation ODE for CMC tori was constructed such that $\ln \mu_i(\kappa_0) = const.$ during the whole deformation. Since we want to reach a value $\ln \mu_2(\kappa_0) = \pi i M,\ M \in \mathbb{Z}$, we must change this value during the deformation as well. Define

$$\varphi(t) := \ln \mu_2(\kappa_0(t), t). \tag{4.33}$$

This function will be used later to determine whether a deformation of CMC cylinders leads to a CMC torus. Then we have the necessary data to describe the deformation of a cylinder whose metric has the periods τ_1, τ_2. Remember that we considered the differential form

$$\omega = \partial_t \ln \mu_1 \ d\ln \mu_2 - \partial_t \ln \mu_2 \ d\ln \mu_1 = -\pi^2 \frac{c_1 b_2 - c_2 b_1}{\nu^2(\kappa^2+1)} d\kappa.$$

In the case of tori we found the expression for ω by the fact that the $\partial_t \ln \mu_i$ both have zeroes at $\kappa = \kappa_0, \bar{\kappa}_0$. In the case of cylinders this is not the case as the value of $\ln \mu_2(\kappa_0)$ should be changed. Nevertheless we know that ω is a meromorphic form with only possible poles at $\kappa = \pm i$ and at the branch points. Especially it has to be holomorphic in $\kappa = \infty$. Therefore it has to be of the form

$$\omega = \frac{P(\kappa)}{(\kappa^2+1)^2} d\kappa,$$

where $P(\kappa)$ is a polynomial of degree at most three in κ. Because we have chosen the pair of double points $\pm\kappa_d$, the deformation changes only the real parameters of the polynomial $a(\kappa) = \kappa^4 + a_2\kappa^2 + a_0$. This keeps the symmetry of the branch points of the form $\alpha, -\alpha, \bar{\alpha}, -\bar{\alpha}$ and there is the additional involution $(\kappa, \nu) \mapsto (-\kappa, \nu)$ on each spectral curve. For the $g = 0$ case this yields $\omega(-\kappa, \nu) = -\omega(\kappa, \nu)$, i.e. ω is an odd form in κ. Since the degree of the polynomials involved in the definition of ω does not change during the deformation, ω stays an odd form. Hence $P(\kappa)$ must be symmetric in κ, giving $P(\kappa) = e\kappa^2 + f$. The factors e, f are restricted by the condition

$$\rho^* \ln \mu_i = -\ln \bar{\mu}_i \quad \Rightarrow \quad \rho^* \omega = \bar{\omega}.$$

For this there must hold

$$P(\bar{\kappa}) = \overline{P(\kappa)} \quad \Leftrightarrow \quad e,\ f \in \mathbb{R}.$$

So we see that we can choose two real numbers e, f that describe how our spectral data is deformed. Denote by $\mathcal{M}$ the moduli space of CMC cylinders of finite type corresponding to a doubly periodic solution of the sinh-Gordon equation such that the spectral curve obeys the described symmetry $\kappa \mapsto -\kappa$. In the next section we will see that defining the polynomial $P(\kappa)$ uniquely determines the polynomials $c_i(\kappa)$ and by the introduced algorithm all derivatives of the spectral data from the deformation ODE. Solving the deformation ODE gives curves in $\mathcal{M}$ whose initial tangent vector is determined by the parameters $(e, f) \in \mathbb{R}^2$. This yields

Proposition 4.3.2. *$\mathcal{M}$ is a locally two dimensional set sufficiently described by the parameters of the polynomials a, b_1, b_2 and the Sym point κ_0. Coordinates in the tangent space can be described by the two parameters $(e, f) \in \mathbb{R}^2$ of the polynomial $P(\kappa) = e\kappa^2 + f$.*

Similar to the case of tori we can now write down the deformation ODEs. The integrability condition again reads

$$\partial^2_{t\kappa} \ln \mu_i \;=\; \partial^2_{\kappa t} \ln \mu_i \quad \Leftrightarrow$$

$$2a\, \partial_t b_i - \partial_t a\, b_i \;=\; 2(\kappa^2+1)a\, \partial_\kappa c_i - 2\kappa a c - (\kappa^2+1)\partial_\kappa a\, c_i. \tag{4.34}$$

An algebraic manipulation of

$$\frac{P(\kappa)}{(\kappa^2+1)^2} d\kappa = -\pi^2 \frac{c_1 b_2 - c_2 b_1}{\nu^2(\kappa^2+1)} d\kappa$$

gives the second equation

$$c_1 b_2 - c_2 b_1 = -\frac{1}{\pi^2}(e\kappa^2 + f)a(\kappa). \tag{4.35}$$

Furthermore, since $\ln \mu_1(\kappa_0) = \pi i K$ at the initial conditions, we want this value to stay constant and therefore change the Sym point according to

$$\partial_t \ln \mu_1(\kappa_0(t), t) = 0 \quad \Leftrightarrow \quad \dot{\kappa}_0 = -\frac{\partial_t \ln \mu_1}{\partial_\kappa \ln \mu_1} = -(\kappa_0^2+1)\frac{c_1(\kappa_0)}{b_1(\kappa_0)}. \tag{4.36}$$

4.3.2 The deformation ODE

We have seen that in order to choose a direction in the moduli space, in which we want to deform the spectral data, we have to choose two real parameters e, f. For a fixed choice of these, the polynomials c_1, c_2 in

$$c_1 b_2 - c_2 b_1 = -\frac{1}{\pi^2}(e\kappa^2 + f)a(\kappa)$$

are uniquely determined. Using the initial spectral data, we can compute them as follows.

On the right hand side of the equation involving the c_i, there are only even powers of κ. Since b_1 is an odd polynomial and b_2 is an even polynomial, c_1 has to be even and c_2 has to be odd in κ. With the Ansatz

$$c_1(\kappa) = \gamma_{1,2}\kappa^2 + \gamma_{1,0}, \qquad c_2(\kappa) = \gamma_{2,3}\kappa^3 + \gamma_{2,1}\kappa$$

and comparing different powers of κ in the above equation we get

$$c_1(\kappa) = \frac{(a_0(e\beta_2+f) + a_2\beta_2(e\beta_1+f) + e\beta_1^2\beta_2 + f\beta_1\beta_2)\kappa^2 + f a_0(\beta_2-\beta_1)}{\pi^2 f_2 \beta_2(\beta_2-\beta_1)},$$

$$c_2(\kappa) = \frac{(a_0(f+e\beta_2) + \beta_2(f\beta_2 + e\beta_1\beta_2 + a_2(f+e\beta_2)))\kappa + \beta_2 e(\beta_2-\beta_1)\kappa^3}{\pi^2 f_1 \beta_2(\beta_2-\beta_1)}.$$

With this we can compute the deformation ODE for the rest of the data according to the algorithm described before.

$$\dot{\kappa}_0 = -\frac{(1+\kappa_0^2)(\kappa_0^2(a_2+\beta_1)(f+e\beta_1)\beta_2 + a_0(f\kappa_0^2 - f\beta_1 + (f+e\kappa_0^2)\beta_2))}{\pi^2\kappa_0(\beta_1-\kappa_0^2)(\beta_1-\beta_2)\beta_2 f_1 f_2},$$

$$\dot{a}_0 = \frac{2a_0(-2(f+e\beta_1)\beta_2 + 2a_0(f+e\beta_2) + a_2(-f+(-e+f+e\beta_1)\beta_2))}{\pi^2(\beta_1-\beta_2)\beta_2 f_1 f_2},$$

$$\dot{a}_2 = \frac{2(-1+a_2)a_2(f+e\beta_1)\beta_2 + 2a_0(-2f-2(e+f+e\beta_1)\beta_2 + a_2(f+e\beta_2))}{\pi^2(\beta_1-\beta_2)\beta_2 f_1 f_2},$$

$$\dot{f}_1 = \frac{a_0(e\beta_2+f) + a_2\beta_2(e\beta_1+f) + e\beta_1^2\beta_2 + f\beta_1\beta_2}{\pi^2(\beta_1-\beta_2)\beta_2 f_2}, \tag{4.37}$$

$$\dot{f}_2 = \frac{4a_0(f+e\beta_2) + \beta_2((-3e+4f)\beta_2 + e\beta_1(3+4\beta_2) + a_2(4f-2e\beta_1+6e\beta_2))}{\pi^2(\beta_1-\beta_2)\beta_2 f_1},$$

$$\dot{\beta}_1 = \frac{(f+e\beta_1)(a_2-\beta_1^2)\beta_2 + a_0(2f(1+\beta_1) + (2e-f+e\beta_1)\beta_2)}{\pi^2(\beta_1-\beta_2)\beta_2 f_1 f_2},$$

$$\dot{\beta}_2 = \frac{(-1+a_2)(f+e\beta_1)\beta_2^2 + a_0(1+2\beta_2)(f+e\beta_2)}{\pi^2(\beta_1-\beta_2)\beta_2 f_1 f_2}.$$

4.3.3 Properties of the deformation

Now we derive some properties following directy from the deformation ODEs.

Proposition 4.3.3. *Let the spectral data of a doubly periodic solution of the* sinh-*Gordon equation be given by*

$$a(\kappa) = \kappa^4 + a_2\kappa^2 + a_0, \quad b_1(\kappa) = f_1(\kappa^2-\beta_1)\kappa, \quad b_2(\kappa) = f_2(\kappa^2-\beta_2).$$

Then this form of the spectral data is preserved during the whole deformation and the solution to the ODEs stays in the set of doubly periodic solutions.

Proof. For spectral data of this special form, we can solve the deformation ODE for $\dot{a}, \dot{b}_1$ and $\dot{b}_2$. Then we see that if $a_0, a_2, f_1, f_2, \beta_1, \beta_2$ are real, all vector fields on the right hand side of the deformation ODE are real. Therefore these parameters stay real during the deformation and a, b_1, b_2 keep their special form. Furthermore, the deformation ODE was constructed in such a way that the properties of the spectral curve which are sufficient for being the spectral curve of a doubly periodic solution of the sinh-Gordon equation are preserved. □

With this proposition we have also shown that the data describing the intial spectral data is sufficient to describe all spectral data that occurs during the deformation. Curves in the moduli space $\mathcal{M}$ are obtained by choosing $(e, f) \in \mathbb{R}^2$ describing the initial tangent vector at a point and then solving the system of ODEs from the previous section. As the tangent space $T_x\mathcal{M}$ to a point can be defined as equivalence classes of curves joining this point having the same velocity vector, we see that $T_x\mathcal{M}$ and hence also $\mathcal{M}$ is two dimensional apart from the possible singularities.

Remembering the normalization of the functions $\ln \mu_i$ in the $g = 0$ case, we can see now that these values are preserved.

Proposition 4.3.4. *The values of* $\ln \mu_1(\infty)$ *and* $\ln \mu_2(0)$ *stay constant during the whole deformation.*

Proof. By definition of the polynomials c_i we have

$$\partial_t \ln \mu_i = \pi i \frac{c_i(\kappa)}{\nu} = \pi i \frac{c_i(\kappa)}{\sqrt{(\kappa^2+1)(\kappa^4 + a_2\kappa^2 + a_0)}}.$$

Since c_2 is an odd polynomial in κ we have $c_2(0) = 0$ and therefore $\partial_t \ln \mu_2(0) = 0$. Furthermore, c_1 contains only powers of order 2 and 0 in κ. Because of the power of order 3 in the denominator we get

$$\lim_{\kappa\to\infty} \partial_t \ln \mu_1(\kappa) = \lim_{\kappa\to\infty} \pi i \frac{\gamma_{1,2}\kappa^2 + \gamma_{1,0}}{\sqrt{(\kappa^2+1)(\kappa^4 + a_2\kappa^2 + a_0)}} = 0.$$

So $\partial_t \ln \mu_1(\infty) = 0$ as well. □

To rule out possible singularities during the deformation, we have to determine the domain of definition of a solution to the system of ODEs. One possible boundary are poles of the vector fields describing the system. As we can see from the formulas above, the factor $f_1 f_2$ appears in all denominators. However, the next proposition allows us to rule out possible zeroes of this factor.

Proposition 4.3.5. *Apart from the boundary of the moduli space, there holds* $f_i \neq 0$, $i = 1, 2$.

Proof. The differentials $d \ln \mu_i$ are not identically zero on the spectral curve. Therefore there exists a cyle $a \in H_1(\Gamma, \mathbb{Z})$ such that $\int_a d\ln \mu_i \neq 0$. Since $\int d\ln \mu_i \in \pi i \mathbb{Z}$ and the deformation is continuous in t, the value of this integral is preserved during the deformation. If we assume $f_i = 0$ at some point of the deformation, there holds $\int d\ln \mu_i = 0$ for all cycles, which is a contradiction. □

Apart from the coefficients of the spectral data, we have also a deformation ODE for the Sym point κ_0 and the eigenvalues of the second monodromy at the Sym point $\varphi(t) = \ln \mu_2(\kappa_0(t), t)$. These are given by

$$\dot{\kappa}_0 = -\frac{(1+\kappa_0^2)(\kappa_0^2(a_2+\beta_1)(f+e\beta_1)\beta_2 + a_0(f\kappa_0^2 - f\beta_1 + (f+e\kappa_0^2)\beta_2))}{\pi^2\kappa_0(\beta_1-\kappa_0^2)(\beta_1-\beta_2)\beta_2 f_1 f_2}$$

and

$$\begin{aligned}\dot\varphi(t) &= \partial_\kappa \ln \mu_2(\kappa_0(t),t)\dot\kappa_0(t) + \partial_t \ln \mu_2(\kappa_0(t),t)\\ &= \pi i \left(\frac{b_2(\kappa_0)}{\nu(\kappa_0^2+1)}\frac{-(\kappa_0^2+1)c_1(\kappa_0)}{b_1(\kappa_0)} + \frac{c_2(\kappa_0)}{\nu}\right)\\ &= \pi i\,\frac{-c_1(\kappa_0)b_2(\kappa_0)+c_2(\kappa_0)b_1(\kappa_0)}{\nu b_1(\kappa_0)} = \frac{i}{\pi}\frac{(e\kappa_0^2+f)\sqrt{a(\kappa_0)}}{\sqrt{\kappa_0^2+1}\,f_1(\kappa_0^2-\beta_1)\kappa_0},\end{aligned}$$

where we used (4.35) in the last step. From these ODEs we can determine how the Sym point and φ are changing during the deformation.

Proposition 4.3.6. *If $\kappa_0 \in i\mathbb{R}$ at the beginning of the deformation, then $\kappa_0(t) \in i\mathbb{R}$ during the whole deformation.*

Proof. Writing $\eta_0 = Im(\kappa_0)$, the deformation ODE for κ_0 is

$$\dot\kappa_0 = i\dot\eta_0 = -\frac{(\kappa_0^2+1)c_1(\kappa_0)}{b_1(\kappa_0)} = -\frac{(-\eta_0^2+1)(-\gamma_{1,2}\eta_0^2+\gamma_{1,0})}{f_1(-\eta_0^2-\beta_1)i\eta_0}$$

$$\Leftrightarrow \quad \dot\eta_0 = \frac{(-\eta_0^2+1)(-\gamma_{1,2}\eta_0^2+\gamma_{1,0})}{f_1(-\eta_0^2-\beta_1)\eta_0}.$$

We get an ODE for η_0 with real coefficients and real initial value, so η_0 stays real. Therefore κ_0 keeps the form $i\eta_0$ with $\eta_0 \in \mathbb{R}$ during the whole deformation. ☐

Lemma 4.3.7. *During the deformation there holds $a(\kappa_0) > 0$.*

Proof. Writing $\kappa_0 = i\eta_0$ we have for the initial values

$$a(\kappa_0) = \kappa_0^4 - 2\kappa_d^2\kappa_0^2 + \kappa_d^4 = \eta_0^4 + 2\kappa_d^2\eta_0^2 + \kappa_d^4 > 0$$

since all quantities in the last equation are non-negative real numbers and $\kappa_d \neq 0$. Furthermore the polynomial $a(\kappa_0)$ keeps the form

$$\kappa_0^4 + a_2\kappa_0^2 + a_0 = \eta_0^4 - a_2\eta_0^2 + a_0, \quad a_2, a_0 \in \mathbb{R}$$

and therefore stays real during the deformation. Suppose now that during the deformation there is a point where $a(\kappa_0) = 0$. This means that κ_0 has fallen together with a branch point of the spectral curve. Since the deformation takes place such that the branch points are of the form $\alpha, \bar\alpha, -\alpha, -\bar\alpha$ and $\kappa_0 \in i\mathbb{R}$, the resulting branch point would be a double point on the imaginary axis. But we know from the discussion of the double points that the branch points must fall together on the real axis. This can only happen if $\kappa_0 = 0$. Otherwise $a(\kappa_0)$ does not change its sign and stays positive.

The case $\kappa_0 = 0$ and $a(\kappa_0) = 0$ means that all four branch points fall together to $\alpha = 0$. However, this cannot happen either, since $\ln \mu_2(0,t) = 0\ \forall t$ whereas $\ln \mu_2(\kappa_d, 0) \neq 0$ and this value stays constant for the branch points during the deformation. Hence $a(\kappa_0) > 0$. ☐

This enables us to give a restriction on the values of $\ln \mu_2(\kappa_0(t), t)$. From the formulas for the $g = 0$ data we derived above we get that

$$\varphi(0) = \ln \mu_2(\kappa_0(0), 0) \in \mathbb{R}.$$

Proposition 4.3.8. *For the choice of τ_2 like above, there holds $\varphi(t) \in \mathbb{R}\ \forall t$.*

Proof. We have just seen that there holds $\sqrt{a(\kappa_0)} \in \mathbb{R}$. For $|Im(\kappa_0)| < 1$ we have $\sqrt{\kappa_0^2 + 1} \in \mathbb{R}$. Looking at the deformation ODE for φ the factors i of the enumerator and of $\kappa_0 = i\eta_0$ in the denominator cancel, so we get a real ODE with real initial condition $\varphi(0) = -\ln \tilde{\mu}_2(\kappa_0)$. So $\varphi(t)$ is real for all t. □

If the closing condition should hold for the second period, we need $\ln \mu_2(\kappa_0) \in \pi i\mathbb{Z}$. However, the restriction of being real gives that this can only hold if $\ln \mu_2(\kappa_0) = 0$. So the subset of CMC tori in the moduli space we are looking at is given by $\varphi = 0$.

Another observation is that in the family of $g = 0$ cylinders, there is the point $(\kappa_0, \varphi) = (0, 0)$. However, this point does not correspond to a torus with $H = \infty$ or a torus in $\mathbb{R}^3$ respectively, since for the spectral data there holds $\beta_2 \neq 0$. In $\mathbb{R}^3$ there is the additional closing condition $\partial_\kappa \ln \mu_2(\kappa_0) = 0$ (see [7]). But $\beta_2 \neq 0$ means that $d \ln \mu_2$ does not have a root at $\kappa_0 = 0$. So the $\mathbb{R}^3$ closing condition is not fulfilled for the second period.

4.3.4 Homology of the spectral curve

In this section, we describe invariants of the deformation in terms of homology cycles. As there holds

$$\int_\gamma d \ln \mu_i \in \pi i\mathbb{Z} \quad \forall\, \gamma \in H_1(\Gamma, \mathbb{Z})$$

and the deformation is continuous in t, the value of these line integrals is preserved throughout the whole deformation.

Let Γ be a spectral curve of genus $g = 2$ of a CMC cylinder in $\mathbb{H}^3$ and the branch points $\kappa = \pm i$ and four other points of the form

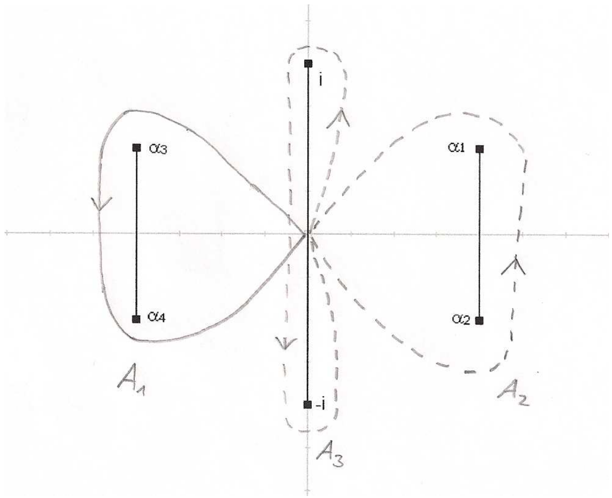
$$\alpha_1 = \alpha,\ \alpha_2 = \bar{\alpha},\ \alpha_3 = -\bar{\alpha},\ \alpha_4 = -\alpha, \tag{4.38}$$

for some α in the quadrant $Re(\alpha) > 0$, $Im(\alpha) > 0$. We have seen that during the deformation, the branch points stay in the respective quadrants. We choose a basis for the first homology group $H_1(\Gamma, \mathbb{Z})$, denoted by

$$A_1,\ A_2,\ B_1,\ B_2\ \in H_1(\Gamma, \mathbb{Z}),$$

where the cycle A_1 surrounds the branch points α_3 and α_4, A_2 surrounds α_1 and α_2, B_1 surrounds α_3 and i and B_2 surrounds α_1 and i.

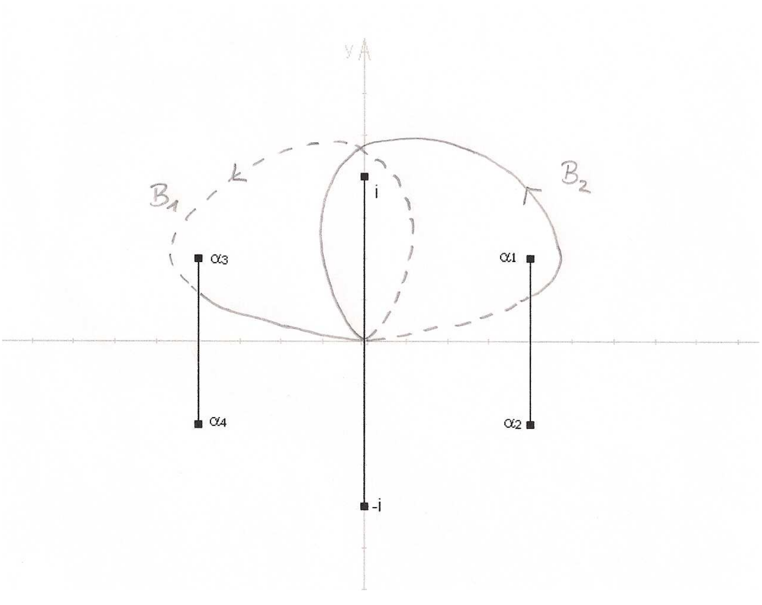
Visualizing the hyperelliptic Riemann surface Γ by cutting two copies of $\mathbb{CP}^1$ along the lines shown below, the cycles A_i look as follows.



Note that the cycles A_1 and A_2 are in different sheets of Γ over $\kappa = \infty$. Hence they only intersect in one point lying over $\kappa = 0$. We see that the following relation holds (the minus sign is because we have to interchange the sheet for A_1 in order to form a closed cycle).

$$-A_1 + A_2 + A_3 = 0. \tag{4.39}$$

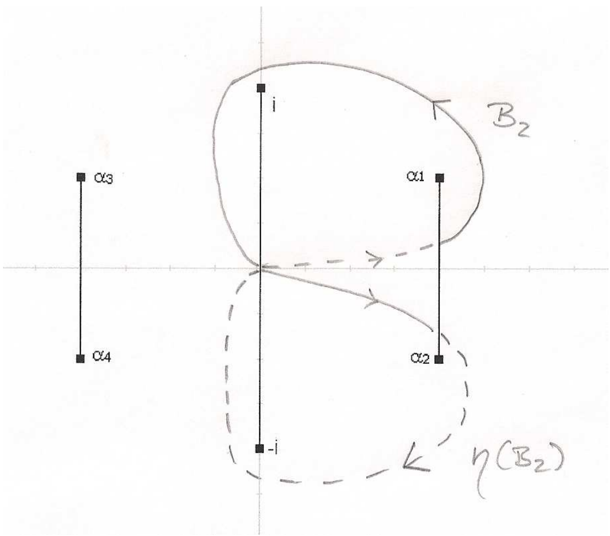
The cycles B_i are as follows.



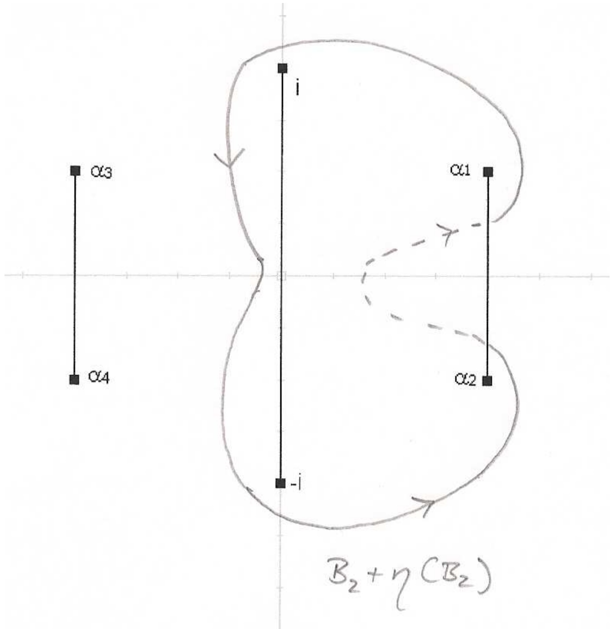
On Γ, there exists the hyperelliptic involution $\sigma : (\kappa, \nu) \mapsto (\kappa, -\nu)$, acting on $H_1(\Gamma, \mathbb{Z})$ as $-Id$. Furthermore, there is the involution $\eta : (\kappa, \nu) \mapsto (\bar{\kappa}, -\bar{\nu})$ which has no fixed points and the involution $\xi : (\kappa, \nu) \mapsto (-\kappa, \nu)$ induced by the symmetry of the branch points. We now derive the operation of the transformations η and ξ on the basis of the first homology.

By complex conjugation $\kappa \mapsto \bar{\kappa}$, the cycles A_i are mapped to themselves, but traversed in opposite direction. Because the involution η has no fixed points, the points on the real axis, and hence the whole cycle, is mapped into the other sheet. But since the sheet interchange operates as $-Id$ on $H_1(\Gamma, \mathbb{Z})$, we have $\eta(A_i) = \sigma(-A_i) = A_i$.

For the cycles B_i, the complex conjugation amounts to a refelction on the real axis. Since η does not have fixed points, $\eta(B_i)$ must be in the other sheet around $\kappa = 0$. For the cycle B_2 this looks like



To determine $\eta(B_2)$ in terms of basis elements we build the cycle $B_2 + \eta(B_2)$. Note that traversing a cylce in opposite direction in the other sheet is homologous to the orinigal one. Hence we get



This gives a cycle that sorrounds the branch points $\alpha_1, \alpha_2, i, -i$. In terms of the cycles A_i this can be written as $-A_3 - A_2$. Using (4.39) we have

$$B_2 + \eta(B_2) = -A_3 - A_2 = -A_1, \qquad \Rightarrow \qquad \eta(B_2) = -B_2 - A_1.$$

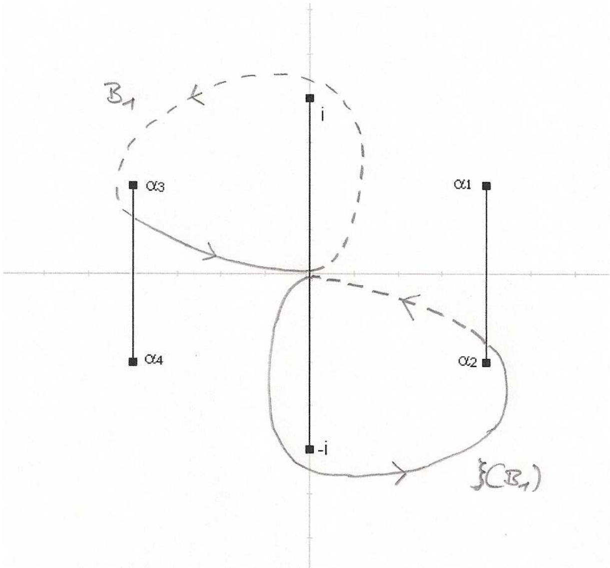
In the same manner one can compute

$$\eta(B_1) = -B_1 - A_2.$$

Regarding the operation of ξ on the cycles A_i, we see that $\kappa \mapsto -\kappa$ amounts to a point refelction at $\kappa = 0$, where $\kappa = 0$ is a fixed point. This give

$$\xi(A_1) = A_2, \qquad \xi(A_2) = A_1.$$

For the cycle B_1 the point reflection in $\kappa = 0$ having a fixed point there leads to



But this is exactly the cycle $\eta(B_2)$ in the above picture traversed in opposite direction in the other sheet, hence homologous to $\eta(B_2)$ itself. So

$$\xi(B_1) = \eta(B_2) = -B_2 - A_1.$$

With the same argument we get

$$\xi(B_2) = \eta(B_1) = -B_1 - A_2.$$

Altogether we have shown the following operations of the involutions on $H_1(\Gamma, \mathbb{Z})$.

$$\eta(A_1) = A_1, \quad \eta(A_2) = A_2, \quad \eta(B_1) = -B_1 - A_2, \quad \eta(B_2) = -B_2 - A_1, \tag{4.40}$$

$$\xi(A_1) = A_2, \quad \xi(A_2) = A_1, \quad \xi(B_1) = -B_2 - A_1, \quad \xi(B_2) = -B_1 - A_2. \tag{4.41}$$

So we see that the η-invariant cycles are precisely the A_i. Using the transformation rules for $d\ln\mu_i$ under the involutions, for the integrals along these cycles there holds

$$\int_{A_i} d\ln\mu_j = \int_{\eta\circ A_i} d\ln\mu_j = \int_{A_i} \eta^* d\ln\mu_j = \int_{A_i} d\ln\bar{\mu}_j.$$

But since $\int_\gamma d\ln\mu_j \in \pi i\mathbb{Z}$ for all cycles $\gamma \in H_1(\Gamma, \mathbb{Z})$, this can only hold if

$$\int_{A_i} d\ln\mu_j = 0 \quad i, j = 1, 2. \tag{4.42}$$

Given the spectral curve with its branch points, this condition determines the differentials $d\ln\mu_i$ up to real factors. In our special case with the additional symmetry of the branch points, we can further reduce the above equation. The differentials are given by

$$d\ln\mu_1 = \pi i\,\frac{f_1(\kappa^2 - \beta_1)\kappa}{(\kappa^2+1)^2\nu}\,d\kappa, \qquad d\ln\mu_2 = \pi i\,\frac{f_2(\kappa^2 - \beta_2)}{(\kappa^2+1)^2\nu}\,d\kappa. \tag{4.43}$$

We see that under the involution $\xi : (\kappa, \nu) \mapsto (-\kappa, \nu)$ they transform as

$$\xi^* d\ln\mu_1 = d\ln\mu_1, \quad \xi^* d\ln\mu_2 = -d\ln\mu_2.$$

Considering the transformation of the cycles A_i we get

$$\int_{A_1} d\ln\mu_1 = \int_{\xi\circ A_2} d\ln\mu_1 = \int_{A_2} \xi^* d\ln\mu_1 = \int_{A_2} d\ln\mu_1,$$

$$\int_{A_1} d\ln\mu_2 = \int_{\xi\circ A_2} d\ln\mu_2 = \int_{A_2} \xi^* d\ln\mu_1 = -\int_{A_2} d\ln\mu_2.$$

So for (4.42) to hold, it is sufficient that the integral over one of the cycles vanishes,

$$\int_{A_2} d\ln\mu_i = 0 \quad i = 1, 2. \tag{4.44}$$

4.3.5 Special vector fields for the deformation

The moduli space $\mathcal{M}$ we are looking at was shown to be sufficiently characterized by the spectral data (a, b_1, b_2, κ_0) of a CMC cylinder with doubly periodic metric possesing a certain symmetry on its spectral curve. We now want to introduce local coordinates on this set and thus be able to deform along the flow of the coordinate vector fields. This is done as follows.

On $\mathcal{M}$, there exist the functions $\kappa_0 : \mathcal{M} \to \mathbb{C}$ and $\varphi : \mathcal{M} \to \mathbb{C}$. We have already seen that the Sym point stays purely imaginary during the deformation. Similarly we proved that φ is real. Hence we get two real functions $Im(\kappa_0)$ and φ on $\mathcal{M}$. We define the map

$$\Phi : \ \mathcal{M} \to \mathbb{R}^2, \qquad (a, b_1, b_2, \kappa_0) \mapsto (Im(\kappa_0), \varphi) \tag{4.45}$$

and want this map to be a local coordinate chart. If the Jacobian of the map is invertible, it is locally a homeomorphism. For this, consider curves in the moduli space

$$\gamma : I \to \mathcal{M}, \qquad t \mapsto (a(t), b_1(t), b_2(t), \kappa_0(t))$$

joining a fixed point $x = (a, b_1, b_2, \kappa_0) \in \mathcal{M}$. With $\Phi \circ \gamma = (Im(\kappa_0), \varphi)$, for the derivative in the chart Φ there holds

$$\frac{d}{dt}(\Phi \circ \gamma) = d\Phi(\gamma(t))\,\dot{\gamma}(t) = (Im(\dot{\kappa}_0), \dot{\varphi}).$$

Hence $d\Phi$ sends tangent vectors of curves in $\mathcal{M}$ to tangent vectors of curves $(Im(\kappa_0), \varphi)$ in $\mathbb{R}^2$. The curves $\gamma(t)$ are determined by choosing parameters $(e, f) \in \mathbb{R}^2$ and solving the system of deformation ODEs (4.37) to get $\gamma(t) = (a(t), b_1(t), b_2(t), \kappa_0(t))$. In order to show that $d\Phi$ is an isomorphism we construct two linearly independent vector fields on $\mathcal{M}$ by choosing parameters

$$X = (X_e, X_f), \quad Y = (Y_e, Y_f),$$

such that these satisfy

$$d\Phi(X_e, X_f) = (0,1), \quad d\Phi(Y_e, Y_f) = (1,0).$$

If these vector fields exist at a point $x = (a, b_1, b_2, \kappa_0) \in \mathcal{M}$ we see that the differential $d\Phi$ is surjective. Since the space of curves in $\mathcal{M}$ is spanned by two real parameters $(e, f) \in \mathbb{R}^2$, the tangent space $T_x\mathcal{M}$ is two-dimensional and $d\Phi$ an isomorphism. By the inverse function theorem, the map Φ can be inverted from an open subset of $\mathbb{R}^2$ to an open neighbourhood of the point $x \in \mathcal{M}$. Hence $(Im(\kappa_0), \varphi)$ are local coordinates and the so constructed vector fields are the usual coordinate fields $X = \frac{\partial}{\partial\varphi}$, $Y = \frac{\partial}{\partial\kappa_0}$ in the corresponding chart.

We see how the values of $d\Phi$ depend on the parameters (e, f) from the expressions for $\dot\varphi$ and $\dot\kappa_0$ we derived previously.

$$\dot\varphi = \frac{i}{\pi}\frac{(e\kappa_0^2 + f)\sqrt{a(\kappa_0)}}{\sqrt{\kappa_0^2+1}\, f_1(\kappa_0^2 - \beta_1)\kappa_0},$$

$$\dot\kappa_0 = -\frac{(1+\kappa_0^2)(\kappa_0^2(a_2+\beta_1)(f+e\beta_1)\beta_2 + a_0(f\kappa_0^2 - f\beta_1 + (f+e\kappa_0^2)\beta_2))}{\pi^2\kappa_0(\beta_1-\kappa_0^2)(\beta_1-\beta_2)\beta_2 f_1 f_2}.$$

We can solve the above equations defining the vector fields X and Y for the coefficients e, f and obtain

$$(X_e, X_f) =$$

$$\left(\frac{\pi\sqrt{\kappa_0^2+1}(\kappa_0^2(a_2+\beta_1)\beta_2 + a_0(\kappa_0^2 - \beta_1 + \beta_2))f_1}{i\kappa_0\sqrt{\kappa_0^4 + a_2\kappa_0^2 + a_0}(a_0 + (a_2+\beta_1)\beta_2)}, -\frac{\pi\kappa_0\sqrt{\kappa_0^2+1}(a_0 + \beta_1(a_2+\beta_1))\beta_2 f_1}{i\sqrt{\kappa_0^4 + a_2\kappa_0^2 + a_0}(a_0 + (a_2+\beta_1)\beta_2)}\right),$$

$$(Y_e, Y_f) = \left(-\frac{i\pi^2(\beta_1-\beta_2)\beta_2 f_1 f_2}{\kappa_0(\kappa_0^2+1)(a_0 + (a_2+\beta_1)\beta_2)}, \frac{i\pi^2\kappa_0(\beta_1-\beta_2)\beta_2 f_1 f_2}{(\kappa_0^2+1)(a_0 + (a_2+\beta_1)\beta_2)}\right).$$

We see that these parameters exist and are well-defined unless the denominators vanish. This is the case if $\kappa_0 = 0$, $\kappa_0 = \pm i$, $a(\kappa_0) = 0$ or $a_0 + (a_2 + \beta_1)\beta_2 = 0$. The first two cases are general boundaries of the moduli space corresponding to $H = \infty$ and $H = 1$. We showed before that there holds $a(\kappa_0) \neq 0$. Hence the only critical points during the deformation are $a_0 + (a_2 + \beta_1)\beta_2 = 0$. Those will be dealt with later when investigating the singularities of the deformation vector fields.

To obtain the local coordinate vector fields in terms of the spectral data, we have to insert the parameters e, f in the deformation ODEs computed before. This yields

Proposition 4.3.9. *The deformation where $\dot\varphi \equiv 1$ and $\dot\kappa_0 \equiv 0$ is given by*

$$\dot a_2 = -\frac{2\sqrt{\kappa_0^2+1}a_0(a_0(2\kappa_0^2 - 2 + a_2 - 2\beta_1) - \kappa_0^2 a_2 + (a_2(\kappa_0^2 - 1 + a_2) - 2\kappa_0^2)\beta_1)}{\pi i\kappa_0\sqrt{\kappa_0^4 + a_2\kappa_0^2 + a_0}(a_0 + (a_2+\beta_1)\beta_2)f_2},$$

$$\dot a_0 = -\frac{2\sqrt{\kappa_0^2+1}a_0(2a_0^2 - \kappa_0^2 a_2(a_2+\beta_1) + a_0(a_2(\kappa_0^2 - 1 + \beta_1) + 2(\kappa_0^2 + (\kappa_0^2 - 1)\beta_1)))}{\pi i\kappa_0\sqrt{\kappa_0^4 + a_2\kappa_0^2 + a_0}(a_0 + (a_2+\beta_1)\beta_2)f_2},$$

$$\dot\beta_1 = -\frac{\sqrt{\kappa_0^2+1}a_0(a_0(2 + \kappa_0^2 + \beta_1) + a_2(\kappa_0^2 + \beta_1 + 2\kappa_0^2\beta_1) + \beta_1(2\kappa_0^2 + 3\kappa_0^2\beta_1 - \beta_1^2))}{\pi i\kappa_0\sqrt{\kappa_0^4 + a_2\kappa_0^2 + a_0}(a_0 + (a_2+\beta_1)\beta_2)f_2},$$

$$\dot\beta_2 = -\frac{\sqrt{\kappa_0^2+1}a_0(a_0+(2a_0+\kappa_0^2a_2+(2\kappa_0^2-1+a_2)\beta_1)\beta_2+\kappa_0^2(a_2+\beta_1+\beta_2))}{\pi i\kappa_0\sqrt{\kappa_0^4+a_2\kappa_0^2+a_0}(a_0+(a_2+\beta_1)\beta_2)f_2}.$$

The deformation where $\dot\varphi \equiv 0$ and $\dot\kappa_0 \equiv i$ is given by

$$\dot a_2 = \frac{2((-1+a_2)a_2(\kappa_0^2-\beta_1)\beta_2+a_0(\kappa_0^2(-2+a_2)-(-2+2\kappa_0^2+a_2-2\beta_1)\beta_2))}{-i\kappa_0(\kappa_0^2+1)(a_0+(a_2+\beta_1)\beta_2)},$$

$$\dot a_0 = \frac{2a_0(\kappa_0^2(2a_0-a_2)+(-2\kappa_0^2-2a_0+a_2(1+\kappa_0^2-\beta_1)+2\beta_1)\beta_2)}{-i\kappa_0(\kappa_0^2+1)(a_0+(a_2+\beta_1)\beta_2)},$$

$$\dot\beta_1 = \frac{(\kappa_0^2-\beta_1)(a_2-\beta_1^2)\beta_2+a_0(2\kappa_0^2(1+\beta_1)-(2+\kappa_0^2+\beta_1)\beta_2)}{-i\kappa_0(\kappa_0^2+1)(a_0+(a_2+\beta_1)\beta_2)},$$

$$\dot\beta_2 = \frac{(a_2-1)(\kappa_0^2-\beta_1)\beta_2^2+a_0(\kappa_0^2-\beta_2)(1+2\beta_2)}{-i\kappa_0(\kappa_0^2+1)(a_0+(a_2+\beta_1)\beta_2)}.$$

Each of the vector fields is a system of ordinary differential equations. From the theory of ODEs we use the following (see [17])

Theorem 4.3.10. *(Global existence and uniqueness) Let $(V, ||.||)$ be a Banach space, $O \subset \mathbb{R}\times V$ and $f : O \to V$ a continuous map which is locally Lipschitz continuous in the second variable. Then for every $(t_0, u_0) \in O$ there exists a maximal interval $(a, b) \subset \mathbb{R}$ containing t_0 where the initial value problem*

$$\dot u(t) = f(u(t), t), \quad u(0) = u_0$$

has a unique solution. At the boundary of the interval holds one of the following

1. *$a = -\infty$ ($b = \infty$ respectively),*
2. *$t \mapsto ||f(u(t), t)||$ is unbounded on $(a, a+\epsilon)$ ($(b-\epsilon, b)$ resp.) for all $\epsilon > 0$,*
3. *The solution is not contained in O for $t \to a$ and $t \to b$ respectively.*

Apart from the zeroes of the denominators, the expressions on the right hand side of the vector fields are continuously differentiable and locally Lipschitz in the coefficients of the moduli space. Hence the conditions in the theorem are satisfied. Taking V as the space of C^1 functions in a neighbourhood of a nonsingular point of the vector field, the solution leaves this set only if the vector field has a pole at the boundary. So either the solution can be extended to $t = \infty$ or the vector field on the right hand side has a singularity. This is the case if either the enumerator becomes infinitely large or the denominator vanishes.

The first case can be treated as follows. When introducing the spectral parameter κ, we saw that we can apply a Möbius transformation without changing the solution of the sinh-Gordon equation corresponding to the spectral curve. Suppose now that one of the vector fields becomes infinitely large for $t \to b$. Assume that this happens because the enumerator becomes infinitely large, i.e. one of the parameters $a_0, a_2, \beta_1, \beta_2$ becomes ∞. This is equivalent to either a branch point of the spectral curve or a root of the differentials $d\ln\mu_i$ being deformed to $\kappa = \infty$. But in that case we can perform a Möbius transformation which changes $\kappa = \infty$ to some other point and carry on the deformation in the transformed κ-plane. Hence this case does not contribute to singularities of the deformation.

For the second case, we can directly see at which points the denominators vanish and get the following

Proposition 4.3.11. *If the maximal interval is not $[0, \infty)$, then the deformation is well-defined unless one of the following singularities occurs:*

$$\kappa_0 = 0, \quad \kappa_0 = \pm i, \quad a_0 + (a_2 + \beta_1)\beta_2 = 0.$$

We have already said that $\kappa_0 = 0$ and $\kappa_0 = \pm i$ are boundaries of the moduli space. The third case can be characterized as follows.

Proposition 4.3.12. *During the deformation there holds*

$$a_0 + (a_2 + \beta_1)\beta_2 = 0 \quad \Leftrightarrow \quad a(\kappa) = u\, b_2(\kappa) + v\, \kappa\, b_1(\kappa), \quad u, v \in \mathbb{R}$$

Proof. Looking at the right hand side, inserting the special form of the spectral data yields

$$\kappa^4 + a_2\kappa^2 + a_0 = uf_2(\kappa^2 - \beta_2) + vf_1\kappa^2(\kappa^2 - \beta_1).$$

Comparing the highest powers of κ yields $vf_1 = 1$. Using this and comparing the other powers of κ, u must be determined by the equations

$$a_2 = uf_2 - \beta_1, \qquad a_0 = -uf_2\beta_2.$$

Defining u by these equations, both yield the same result if and only if

$$uf_2 = a_2 + \beta_1 = -a_0/\beta_2 = uf_2 \quad \Leftrightarrow \quad a_0 + (a_2 + \beta_1)\beta_2 = 0.$$

If $\beta_2 = 0$ we see that there has to hold $a_0 = 0$. In this case we can determine u solely by $a_2 = uf_2 - \beta_1$ and the condition $a_0 + (a_2 + \beta_1)\beta_2 = 0$ is fulfilled as well.

□

To interpret what it means if the deformation hits a point like this outside the $g = 0$ family, we look at the polynomials $c_i(\kappa)$. Simplifying these under the assumption $a_0 + (a_2 + \beta_1)\beta_2 = 0$ we get

$$c_1(\kappa) = \frac{a_0}{\pi^2\beta_2 f_2}(e\kappa^2 + f), \quad c_2(\kappa) = \frac{\kappa}{\pi^2 f_1}(e\kappa^2 + f).$$

So both c_1 and c_2 have roots at $\kappa = \pm i\sqrt{\frac{f}{e}}$, where e, f are the parameters of the polynomial $P(\kappa)$ determining the direction of the deformation. With the vector fields X and Y introduced before one can compte that in both cases there holds

$$\kappa = \pm i\sqrt{\frac{f}{e}} = \pm\kappa_0,$$

so the polynomials have zeroes at $\pm\kappa_0$. For the deformation of the Sym point this means that

$$\dot{\kappa}_0 = -\frac{(\kappa_0^2+1)c_i(\kappa_0)}{b_i(\kappa_0)} = 0.$$

Since this holds for two linearly independent vector fields on the two dimensional space $\mathcal{M}$, we have $\dot{\kappa}_0 = 0$ for all directions $(e, f) \in \mathbb{R}^2$. But since the mean curvature only depends on the Sym point we see that the points where $a_0 + (a_2 + \beta_1)\beta_2 = 0$ are critical points of the mean curvature.

Proposition 4.3.13.

$$a_0 + (a_2+\beta_1)\beta_2 = 0 \quad \Leftrightarrow \quad \dot{H} = 0 \quad \forall\ (e,f) \in \mathbb{R}^2.$$

A direct computation then shows that at these points we can write

$$a(\kappa) = -\frac{a_0}{\beta_2 f_2} b_2(\kappa) + \frac{1}{f_1}\kappa\, b_1(\kappa).$$

4.3.6 Singular initial conditions

We proved that apart from the boundary points $\kappa_0 = 0, \pm i$, the deformation along the flow of the introduced vector fields is well-defined unless $a_0 + (a_2 + \beta_1)\beta_2 = 0$. However, looking at the spectral genus $g = 0$ data we compute with the double point κ_d

$$a_0 + (a_2 + \beta_1)\beta_2 = \kappa_d^4 + (-2\kappa_d^2 + \kappa_d^2)\kappa_d^2 = 0.$$

So the entire family of $g = 0$ cylinders is a singularity of the deformation. Furthermore using the $g = 0$ data gives that all spectral data becomes $\frac{0}{0}$ in the limit $t \to 0$. Therefore we need to make sure that we can define tangent vectors that enable us to leave the initial conditions and carry on the deformation along the vector fields outside the $g = 0$ family. This will be done by using a power series expansion of the spectral data and solving the resulting system of equations successively.

Using the Ansatz

$$\kappa_0(t) = \kappa_0(0) + \dot{\kappa}_0(0)t,$$
$$a_0(t) = a_0(0) + \dot{a}_0(0)t, \quad a_2(t) = a_2(0) + \dot{a}_2(0)t,$$
$$\beta_1(t) = \beta_1(0) + \dot{\beta}_1(0)t, \quad \beta_2(t) = \beta_2(0) + \dot{\beta}_2(0)t.$$

and putting this into the deformation ODE gives a system of equations of the form

$$(\dot{a}_0, \dot{a}_2, \dot{\beta}_1, \dot{\beta}_2) = \Theta\,(a_0, a_2, \beta_1, \beta_2)$$

which can be solved by computer algebraic operations using *Mathematica* ([11]). There are two solutions to the system. The first one is given by

$$\dot{a}_0(0) = -\frac{4(f\kappa_d^2 + f\kappa_d^4)}{\pi^2},\ \dot{a}_2(0) = \frac{4(f + f\kappa_d^2)}{\pi^2},$$

$$\dot{\beta}_1(0) = -\frac{2(f + f\kappa_d^2)}{\pi^2}, \ \dot{\beta}_2(0) = -\frac{2f + e\kappa_d^2 + 2f\kappa_d^2}{\pi^2},$$

and the second one by

$$\dot{a}_0(0) = -\frac{e\kappa_d^6(3 + 4\kappa_d^2) + 4f\kappa_d^2(-1 + \kappa_d^2 + 2\kappa_d^4)}{(\kappa_d^2 - 1)\pi^2},$$

$$\dot{a}_2(0) = \frac{e\kappa_d^2(3 + 4\kappa_d^2)(4 + 5\kappa_d^2) + 12f(1 + 3\kappa_d^2 + 2\kappa_d^4)}{(\kappa_d^2 - 1)\pi^2},$$

$$\dot{\beta}_1(0) = -\frac{12f + 12(e + f)\kappa_d^2 + 13e\kappa_d^4 - 4e\kappa_d^6}{2(\kappa_d^2 - 1)\pi^2},$$

$$\dot{\beta}_2(0) = -\frac{4f + 4(e + 3f)\kappa_d^2 + (13e + 8f)\kappa_d^4 + 4e\kappa_d^6}{2(\kappa_d^2 - 1)\pi^2}.$$

When starting the deformation, we want the two double points $\pm\kappa_d$ on the $g = 0$ spectral curve to open to four distinct nonreal branch points and hence get a spectral curve of genus $g = 2$. We will see that this happens only in one direction of the deformation. The condition on the spectral data for this to happen is given by

Proposition 4.3.14. *Let $(a_0, a_2, \beta_1, \beta_2, \kappa_0)$ be the spectral data of a $g = 0$ cylinder with double points $\pm\kappa_d$ on its spectral curve. For a deformation of spectral data, the double points open in positive direction, i.e. with increasing $t \in \mathbb{R}$ if and only if*

$$-\kappa_d^2\dot{a}_2(0) - \dot{a}_0(0) < 0.$$

Proof. The branch points of the spectral curve are given by $\kappa = \pm i$ and the roots of $a(\kappa)$. For the $g = 0$ case we have $a(\kappa) = \kappa^4 - 2\kappa_d^2 + \kappa_d^4$, so $\pm\kappa_d$ are double roots of a. Increasing the genus of the algebraic curve

$$\nu^2 = (\kappa^2 + 1)a(\kappa)$$

is equivalent to $a(\kappa)$ having four distinct roots. For the roots we compute

$$\alpha_i = \pm\sqrt{\frac{-a_2 \pm \sqrt{a_2^2 - 4a_0}}{2}}.$$

If $a_2^2 - 4a_0 > 0$, there are four distinct roots on the real axis, whereas for $a_2^2 - 4a_0 < 0$, there are four distinct roots of the form $\alpha, -\alpha, \bar{\alpha}, -\bar{\alpha}$ off the real axis. From the conditions on the spectral curve of a real and doubly periodic solution of the sinh-Gordon equation, we know that the involution $\eta : (\kappa, \nu) \mapsto (\bar{\kappa}, -\bar{\nu})$ must not have fixed points. If the roots of $a(\kappa)$ lie on the real axis, i.e.

$$a(\kappa) = (\kappa - \alpha_1)(\kappa - \alpha_2)(\kappa - \alpha_3)(\kappa - \alpha_4), \quad \alpha_1 < \alpha_2 < 0 < \alpha_3 < \alpha_4,$$

we have $\nu = \sqrt{(\kappa^2 + 1)a(\kappa)} \in i\mathbb{R}$ for $\alpha_1 < \kappa < \alpha_2$ and $\alpha_3 < \kappa < \alpha_4$ and therefore $-\bar{\nu} = \nu$. So in this case, the involution η has fixed points on the real axis, which means that the spectral curve does not correspond to a real solution of the sinh-Gordon equation. Therefore we need

$a_2^2 - 4a_0 < 0$. As there holds $a_2^2 - 4a_0 = 0$ in the $g = 0$ case, the discriminant becomes negative if at the initial values the deformation satisfies

$$\frac{\partial}{\partial t}(a_2^2 - 4a_0)(0) = 2a_2(0)\dot{a}_2(0) - 4\dot{a}_0(0) = -4\kappa_d^2\dot{a}_2(0) - 4\dot{a}_0(0) < 0,$$

proving the claim. □

We see that the first solution satisfies $-\kappa_d^2\dot{a}_2(0) - \dot{a}_0(0) = 0$, so the double points do not open for this deformation. To discuss the second solution, we need to specify the parameters e, f prescribing the vector fields X and Y in the $g = 0$ case. We do this again by using power series expressions of the spectral data and solving the resulting system of equations. For $\dot{\kappa}_0$ and $\dot{\varphi}$ we get

$$\dot{\kappa}_0(0) = \lim_{t\to 0}\dot{\kappa}_0(t) = \frac{4f(-28 + 97\kappa_0^2 + 125\kappa_0^4) + e\kappa_0^2(1456 + 3875\kappa_0^2 + 2500\kappa_0^4)}{16\pi^2\kappa_0(7 + 25\kappa_0^2)},$$

$$\dot{\varphi}(0) = \lim_{t\to 0}\dot{\varphi}(t) = \frac{i(f + e\kappa_0^2)f_2}{\pi\kappa_0\sqrt{1 + \kappa_0^2}}.$$

The condition $\dot{\varphi} = 1$, $\dot{\kappa}_0 = 0$ gives

$$(e, f) = \left(\frac{4\pi i(\kappa_0^2 + 1)^{\frac{3}{2}}(-28 + 125\kappa_0^2)}{\kappa_0(1568 + 34878\kappa_0^2 + 2000\kappa_0^4)f_2}, -\frac{\pi i\kappa_0\sqrt{1 + \kappa_0^2}(16 + 25\kappa_0^2)(91 + 100\kappa_0^2)}{(1568 + 3487\kappa_0^2 + 2000\kappa_0^4)f_2}\right),$$

while condition $\dot{\kappa}_0 = i$, $\dot{\varphi} = 0$ leads to

$$(e, f) = \left(\frac{16i\pi^2(7 + 25\kappa_0^2)}{\kappa_0(1568 + 3487\kappa_0^2 + 2000\kappa_0^4)}, -\frac{16i\pi^2\kappa_0(7 + 25\kappa_0^2)}{1568 + 3487\kappa_0^2 + 2000\kappa_0^4}\right).$$

The denominator of the parameters has no zeroes on the imaginary axis. Therefore, for $\kappa_0 \in i\mathbb{R}$ both parameters are well-defined.

Now we can compute the vector fields X and Y by inserting these parameters into the second solution. We choose again $K = 5$ and $L = 3$ as fixed constants. This yields tangent vectors on the set of spectral data along which the deformation will leave the subset of $g = 0$ cylinders.

Proposition 4.3.15. *Let $(a_0, a_2, \beta_1, \beta_2, \kappa_0)$ be initial values of spectral data lying in the family of $g = 0$ cylinders. Then the vector field X leaving κ_0 constant and changing φ is in the $g = 0$ case given by*

$$\dot{a}_0 = \frac{4i(1 + \kappa_0^2)^{\frac{3}{2}}(16 + 25\kappa_0^2)^2(91 + 100\kappa_0^2)(64 + 325\kappa_0^2)}{729\pi\kappa_0(1568 + 3487\kappa_0^2 + 2000\kappa_0^4)f_2}$$

$$\dot{a}_2 = -\frac{116i(1 + \kappa_0^2)^{\frac{3}{2}}(16 + 25\kappa_0^2)^2(91 + 100\kappa_0^2)}{81\pi\kappa_0(1568 + 3487\kappa_0^2 + 2000\kappa_0^4)f_2},$$

$$\dot{\beta}_1 = \frac{10i(1 + \kappa_0^2)^{\frac{3}{2}}(16 + 25\kappa_0^2)^2(91 + 100\kappa_0^2)}{81\pi\kappa_0(1568 + 3487\kappa_0^2 + 2000\kappa_0^4)f_2},$$

$$\dot{\beta}_2 = \frac{10i(1+\kappa_0^2)^{\frac{3}{2}}(16+25\kappa_0^2)^2(161+260\kappa_0^2)}{81\pi\kappa_0(1568+3487\kappa_0^2+2000\kappa_0^4)f_2}.$$

The vector field Y changing κ_0 and leaving φ constant is given by

$$\dot{a}_0 = -\frac{16i(16+25\kappa_0^2)(23296+77700\kappa_0^2 71175\kappa_0^4+17500\kappa_0^6)}{729\kappa_0(1568+3487\kappa_0^2+2000\kappa_0^4)},$$

$$\dot{a}_2 = \frac{16i(168896+520800\kappa_0^2-528675\kappa_0^4+177500\kappa_0^6)}{81\kappa_0(1568+3487\kappa_0^2+2000\kappa_0^4)},$$

$$\dot{\beta}_1 = \frac{40i(-5824-3360\kappa_0^2+14235\kappa_0^4+12500\kappa_0^6)}{81\kappa_0(1568+3487\kappa_0^2+2000\kappa_0^4)},$$

$$\dot{\beta}_2 = -\frac{40i(10304+28320\kappa_0^2+22245\kappa_0^4+3500\kappa_0^6)}{81\kappa_0(1568+3487\kappa_0^2+2000\kappa_0^4)}.$$

.

Again we see that the denominator does not vanish except for $\kappa_0 = 0$. This means that we can start a deformation of cylinders for arbitrary Sym points in the domain of definition of the $g = 0$ family.

To investigate how the double points open for the vector fields X and Y it is sufficient to consider only one of them. The other one can be written as a linear combination of the first one and the direction along the curve of $g = 0$ cylinders. Since the discriminant vanishes constantly along this set, adding this vector field does not contribute to the sign of the discriminant along X and Y.

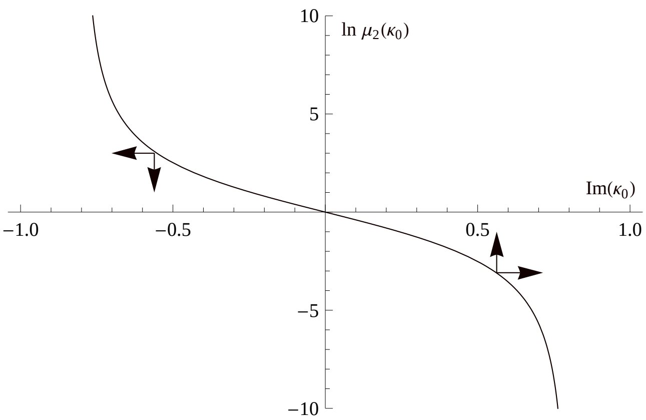
For the vector field with $Im(\dot{\kappa}_0) = 1$, the condition that the double points open

$$-\kappa_d^2\dot{a}_2(0) - \dot{a}_0(0) < 0$$

is satisfied for

$$-1 < Im(\kappa_0) < -\frac{4}{5}, \quad 0 < Im(\kappa_0) < \frac{4}{5}, \quad 1 < Im(\kappa_0).$$

The first and the third intervals are not contained in the set of Sym points for which the $g = 0$ family of cylinders is defined. Hence the deformation opens the double points in positive direction, i.e. with increasing κ_0 for $Im(\kappa_0) > 0$ and in negative direction, i.e. decreasing $Im(\kappa_0)$, for $Im(\kappa_0) < 0$. To see how these directions lie in the moduli space of cylinders we have to look at the values of $\ln\mu_2(\kappa_0)$. We already pictured one branch of the curve corresponding to the $g = 0$ family before. We saw that when we consider the $g = 0$ family with double points as a subset of the moduli space of $g = 2$ cylinders, we get an additional minus sign for $\ln\mu_2(\kappa_0, \nu)$. Therefore the family of $g = 0$ cylinders lies in the moduli space of $g = 2$ cylinders as the curve $(Im(\kappa_0), -\ln\tilde{\mu}_2(\kappa_0))$, where $\ln\tilde{\mu}_2$ denotes the eigenvalues in the $g = 0$ case, and looks like



So we see that for $Im(\kappa_0) > 0$, the double points open in positive direction and hence the spectral genus $g = 2$ cylinders lie between the $g = 0$ family and the axis $\varphi = 0$. The same holds for the vector fields with $\dot{\varphi} = 1$. Since traversing the $g = 0$ family gives a vector field that increases κ_0 while it decreases φ, we need a linear combination of the vector field along the $g = 0$ curve with a positive multible of Y to get X. Therefore, if Y opens the double points in positive direction, then X does as well and vice versa.

4.3.7 Simulation of the deformation

To know where the deformation can take place in the moduli space, we need to determine where possible singularities can occur. Apart from the boundaries $\kappa_0 = 0, \pm i$, the singularities were given by

$$a_0 + (a_2 + \beta_1)\beta_2 = 0$$

and this equation is satisfied for the whole $g = 0$ family.

From the spectral data of the $g = 0$ family one obtains the following observation. Starting a deformation from the family of $g = 0$ cylinders that opens the double points, one reaches values in the $(Im(\kappa_0), \varphi)$-plane that lie between the $g = 0$ family and the axis $\varphi = 0$, that corresponds to CMC tori. If we assume that there is no singularity in this quadrant of the $(Im(\kappa_0), \varphi)$-plane, one can deform the initial cylinder into a CMC torus in $\mathbb{H}^3$. But then one could deform the spectral data within the class of CMC tori, as it was shown in [7] and reach a CMC torus in $\mathbb{R}^3$, corresponding to $(Im(\kappa_0), \varphi) = (0, 0)$. However, as we saw before, the point $(0, 0)$ in the moduli space $\mathcal{M}$ does not belong to a CMC torus, since the $\mathbb{R}^3$ closing condition is not satisfied. So if we start the deformation in a local neighbourhood of $(0, 0)$, reaching the axis $\varphi = 0$ in this neighbourhood would be a contradiction. Therefore, we assume that there is a singularity of the deformation between the $g = 0$ family and the axis $\varphi = 0$ around the point $(0, 0)$.

To see whether two singularities intersect on the $g = 0$ family, consider the derivative of $a_0 + (a_2 + \beta_1)\beta_2$ in the $g = 0$ case.

$$\dot{a}_0 + (\dot{a}_2 + \dot{\beta}_1)\beta_2 + (a_2 + \beta_1)\dot{\beta}_2 =$$

$$\frac{100(1+\kappa_0^2)(16+25\kappa_0^2)(e(16+25\kappa_0^2)(98+125\kappa_0^2)+9f(107+125\kappa_0^2))}{729\pi^2(7+25\kappa_0^2)}$$

It vanishes if and only if

$$\kappa_0 = \pm i, \ \ or \ \kappa_0 = \pm\frac{4}{5}i, \ \ or \ e = -\frac{9(107f+125f\kappa_0^2)}{1568+4450\kappa_0^2+3125\kappa_0^4}.$$

If there was another curve satisfying $a_0+(a_2+\beta_1)\beta_2 = 0$ which intersects the $g = 0$ family, the derivative had to vanish for all directions $(e, f) \in \mathbb{R}^2$. The third of the above relations gives a unique direction $(e, f) \in \mathbb{R}^2$ which corresponds to the tangent vector field of the $g = 0$ family, whereas at the boundary $\kappa_0 = \pm\frac{4}{5}i$ the derivative vanishes for all directions. Furthermore, at $\kappa_0 = 0$, the deformation has a general singularity. This means that we can rule out all points but $\kappa_0 = 0, \pm i\frac{4}{5}$ as intersections of the $g = 0$ family with a curve along which there holds $a_0 + (a_2 + \beta_1)\beta_2 = 0$.

A numerical simulation of the deformation shows that there are indeed singularities of the form we expected. To simulate the vector fields X and Y we choose some initial data on the $g = 0$ family, as described above. Then we leave the $g = 0$ family with a vector field changing only κ_0 according to the Ansatz

$$\kappa_0(t) = \kappa_0(0) + \dot\kappa_0(0)t, \quad a_0(t) = a_0(0) + \dot a_0(0)t \quad etc.$$

with a very small parameter $t \in \mathbb{R}$. By this, we get spectral data of genus $g = 2$ CMC cylinders, which implies that $\beta_1 \neq \beta_2$ and $a_0 + (a_2 + \beta_1)\beta_2 \neq 0$, so there is no singularity at this point. Eventually we start the deformation along the vector field X, which corrsponds to $\dot\kappa_0 = 0$ and $\dot\varphi = 1$ to see whether we can reach the axis $\varphi = 0$ in the course of the deformation.

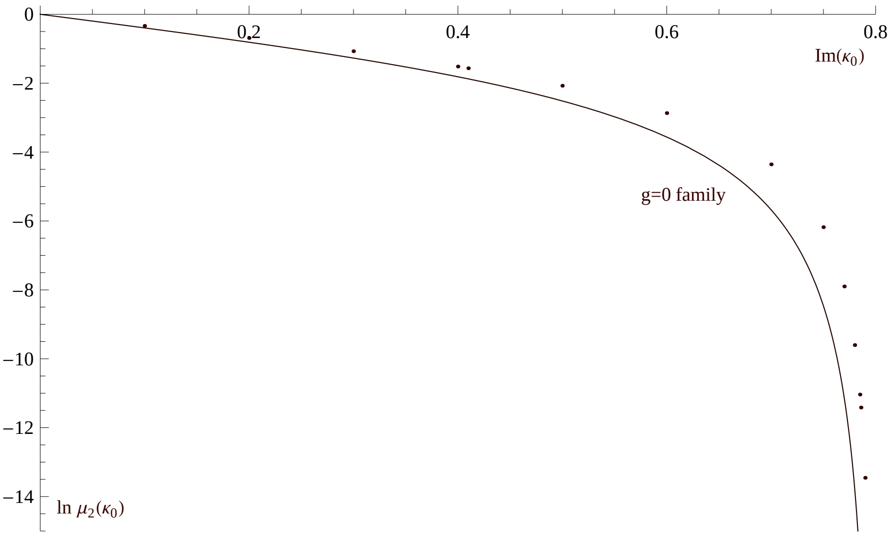
For the concrete values of the winding numbers $K = 5$ and $L = 3$, we choose different points in the corresponding family. The simulation with *Mathematica* ([11]) then gives a maximal interval (t_{min}, t_{max}) on which the deformation ODE can be solved numerically. For the endpoints there holds

$$\lim_{t\to t_{min}} (a_0 + (a_2 + \beta_1)\beta_2)(t) = 0,$$

which corresponds to the initial values in the $g = 0$ family, and

$$\lim_{t\to t_{max}} (a_0 + (a_2 + \beta_1)\beta_2)(t) = 0.$$

The second limit means that at the boundary of the maximal interval, the deformation along X reaches a singularity and we can compute the corresponding points in the $(Im(\kappa_0), \varphi)$-plane to get the following graph.



This suggests that starting from the family with $K = 5$ and $L = 3$, the deformation hits a singularity before reaching the subset of CMC tori. Investigating the spectral data at the endpoints of the simulation of the deformation gives

$$\lim_{t\to t_{max}} a_0(t) = 1, \quad \lim_{t\to t_{max}} a_2(t) = 2, \quad \lim_{t\to t_{max}} \beta_1(t) = \lim_{t\to t_{max}} \beta_2(t) = -1.$$

This data describes a spectral curve given by $a(\kappa) = (\kappa^2 + 1)^2$, meaning that the branch points have converged to $\pm i$. Due to the differentials $d \ln \mu_i$ both having roots at $\pm i$, the additional poles cancel and this is again a family of spectral genus $g = 0$ cylinders. This data can be interpreted as follows.

If the branch points on a spectral curve of a solution u of the sinh-Gordon equation are deformed to $\pm i$, respectively $0, \infty$ in the λ-plane, the solution u becomes singular. By a coordinate transformation this corresponds to the Hopf differential Q vanishing identically. $Q \equiv 0$ means that all points on the resulting surface are umbilic points. The only CMC surfaces which satisfy this condition are spheres. Therefore, the spectral data at the end of the simulation corresponds to cylinders which are chains of spheres, i.e. spheres glued to each other in only one point.

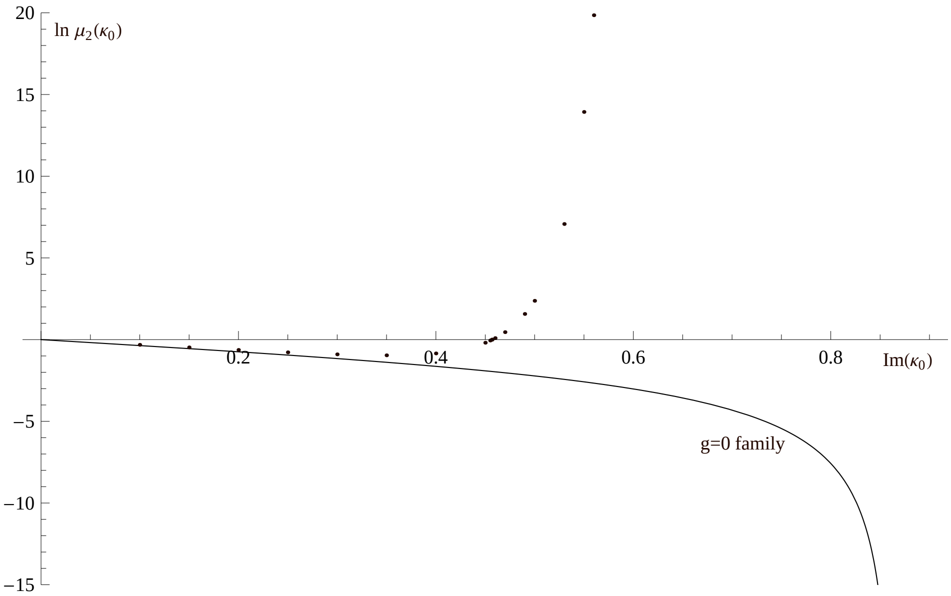
However, the above simulation is only valid for the concrete choice of the parameters $K = 5$, $L = 3$. For other combinations of $K, L \in \mathbb{Z}$ with $L < K$ we have to change the initial values of the deformation by using the formula for the double point

$$\kappa_d^2 = \frac{K^2}{L^2}(\kappa_0^2 + 1) - 1$$

and hence change the vector field that leaves the $g = 0$ family. For the choice of $K = 2$ and $L = 1$ we get the following picture. Starting from the family of $g = 0$ cylinders, which in this case is defined for $-\frac{\sqrt{3}}{2} < Im(\kappa_0) < \frac{\sqrt{3}}{2}$, and deforming along the vector field X, there is again a singularity of the form

$$\lim_{t\to t_{max}} (a_0 + (a_2 + \beta_1)\beta_2)(t) = 0.$$

In a neighbourhood of the point $(Im(\kappa_0), \varphi) = (0,0)$, the points where this happens lie between the initial points and the axis $\varphi = 0$ passing the $g = 0$ family in $(0,0)$. But for growing $\kappa_0(0)$, the singular curve crosses the axis $\varphi = 0$ and converges to $\varphi = \infty$ for $\kappa_0(0) \to \frac{\sqrt{3}}{2}$.



This simulation suggests that we can reach the subset of $g = 2$ CMC tori by starting the deformation at the $g = 0$ family of cylinders for certain initial values. The spectral data at the endpoints is again of the form

$$a_0(t_{max}) = 1, \quad a_2(t_{max}) = 2, \quad \beta_1(t_{max}) = \beta_2(t_{max}) = -1,$$

which again corresponds to chains of spheres.

Altogether the numerical simulation suggests that starting with an initial value on the $g = 0$ family with $0.456899 < Im(\kappa_0(0)) < \frac{\sqrt{3}}{2}$ and then running the deformation along X, the flow passes a point for which $\varphi = \ln \mu_2(\kappa_0) = 0$ and hence the closing condition on the second monodromy is satisfied.

By this, it would be possible to construct the spectral data of a family of $g = 2$ CMC tori in $\mathbb{H}^3$ from the initial data of a family of $g = 0$ CMC cylinders with two double points on the spectral curve. Using the formula $H = \frac{1+\kappa_0\bar{\kappa}_0}{2Im(\kappa_0)}$, the mean curvature of the resulting family of CMC tori is bounded by $\frac{7}{4\sqrt{3}} < H < 1.32278$.

By varying the constants $K, L \in \mathbb{Z}$, the numerical simulation resulted in the same spectral data at the endpoints, i.e.

$$a_0 = 1, \quad a_2 = 2, \quad \beta_1 = \beta_2 = -1.$$

However, the case $K = 2$, $L = 1$ and common multiples of them were the only examples where the deformation passed the axis $\varphi = 0$.

To prove that by this procedure we can actually construct CMC tori in $\mathbb{H}^3$ out of CMC cylinders, one has to determine the endpoints of the deformation analytically. We assume that these can be computed using the restrictions on the differentials $d\ln\mu_i$ in terms of line integrals described in section 4.3.4. First, one should show that if the branch points are deformed to $\pm i$, the roots of the differentials $d\ln\mu_i$ converge to $\pm i$ as well. Furthermore, in the simulation, the only points where a singularity occured was in the case $g = 0$. Therefore we give the following conjecture, of which an analytical prove would be desireable.

Conjecture 4.3.16. *Let $(a, b_1, b_2, \kappa_0) \in \mathcal{M}$ be the spectral data of a spectral genus $g = 0$ CMC cylinder in $\mathbb{H}^3$ with two double points on its spectral curve and winding numbers $K, L \in \mathbb{Z}$, where $L < K$. Then the deformation along the vector field X corresponding to $\dot{\varphi} = 1$ and $\dot{\kappa}_0 = 0$ ends in a chain of spheres. For certain choices of K and L the deformation passes the subset of spectral genus $g = 2$ CMC tori in $\mathbb{H}^3$. Furthermore, for the singularities of the deformation there holds*

$$a_0 + (a_2 + \beta_1)\beta_2 = 0 \quad \Leftrightarrow \quad g = 0.$$

5 Conclusion and open questions

To study CMC surfaces in $\mathbb{H}^3$ we have first introduced the classic concepts of moving frames and Lax pairs. We showed how the integrability condition of the Lax pair is related to the sinh-Gordon equation, an equation studied in the theory of integrable systems. By that means, one is able to use the concept of spectral curves to produce solutions and hence metrics of CMC surfaces.
After investigating the properties of spectral curves of finite type solutions, we gave some explicit examples of cylinders of low spectral genus and showed that in the special case of $\mathbb{H}^3$ there are no CMC tori of spectral genus $g = 0$ and $g = 1$. This is different to the situation of CMC tori immersed in $\mathbb{S}^3$.
The main part of the thesis focused on a deformation theory of spectral data. The aim was to show the existence of CMC tori of spectral genus $g = 2$ by starting a deformation with a $g = 0$ CMC cylinder and changing the spectral genus in the course of the deformation. By computing different examples of families of $g = 0$ CMC cylinders as initial values and simulating the deformation numerically we could find an example of a family for which the simulation reached the subset of CMC tori of spectral genus $g = 2$ in $\mathbb{H}^3$. Furthermore, the simulation suggested that at the endpoints of the deformation, the resulting surfaces are chains of spheres, a very special limit case of CMC cylinders of spectral genus $g = 0$.
An interesting question would be if every deformation starting from a $g = 0$ cylinder in $\mathbb{H}^3$ ends in a family of chains of spheres. The simulations showed that not every deformation of that kind passes the subset of CMC tori. Finding an analytic description of the dependence of this fact on the winding numbers $K, L \in \mathbb{Z}$ as well as determining the endpoints for arbitrary combinations of these would be a good aim for further research.

Bibliography

[1] A.I. Bobenko. *All constant mean curvature tori in $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$ in terms of theta-functions.* Math. Ann., 290(2), pp. 209-245, 1991

[2] A.I. Bobenko. *Surfaces in terms of 2 by 2 matrices. Old and new integrable cases.* In: *Harmonic maps and integrable systems*, Aspects of Math., pp. 83-127, Vieweg, Braunschweig, 1994

[3] H.M. Farkas and I. Kra. *Riemann Surfaces.* Graduate Texts in Mathematics 71, Springer Verlag, New York, 1992

[4] S. Fujimori, S. Kobayashi and W. Rossman. *Loop group methods for constant mean curvature surfaces.* Rokko lectures in mathematics, October 2005, arXiv:math/0602570v1 [math.DG]

[5] F. Hélein. *Constant mean curvature surfaces, harmonic maps and integrable systems.* ETH Zürich Lectures in Mathematics, Birkhäuser, Basel Boston Berlin, 2000

[6] N.J. Hitchin. *Harmonic maps from a 2-torus to the 3-sphere.* J. Differential Geom., 31(3), pp. 627-710, 1991

[7] W. Kewlin. *Deformation of constant mean curvature tori in a three sphere.* Diplomarbeit, Universtität Mannheim, 2008

[8] M. Kilian and M.U. Schmidt. *On the moduli of constant mean curvature cylinders of finite type in the 3-sphere.* preprint arXiv:0712.0108v2 [math.DG]

[9] M. Kilian, S-P. Kobayashi, W. Rossman and N. Schmitt. *Constant mean curvature surfaces in 3-dimenstional space forms.* Preprint, 2002

[10] M. Knopf. *Konforme CMC-Immersionen von Tori nach* $\mathbb{S}^3$. Seminar Geometrische Analysis FSS 09, Universität Mannheim, 2009

[11] *Mathematica for Students.* Version 7.0.1.0. Wolfram Research, www.wolfram.com

[12] R. Miranda. *Algebraic curves and Riemann surfaces.* Graduate Studies in Mathematics 5, Amer. Math. Soc., Providence, 1995

[13] V. Neugebauer. *Constant mean curvature tori in the 3-hyperbolic space.* Diplomarbeit, Universität Mannheim, 2008

[14] U. Pinkall and I. Sterling. *On the classification of constant mean curvature tori.* Ann. of Math. 130(2), pp. 407-451, 1989

[15] W. Rossman. *Supplement to 'Loop group methods for constant mean curvature surfaces'*, preprint

[16] N. Schmitt, M. Kilian, S.-P. Kobayashi and W. Rossman. *Unitarization of monodormy representations and constant mean curvature trinoids in 3-dimensional space forms.* J. London Math. Soc., 75(3), pp. 563-581, 2007, arXiv:math/0403366v3 [math.DG]

[17] W. Walter. *Gewöhnliche Differentialgleichungen. Eine Einführung.* Springer Verlag, Berlin, 2007

[18] H.C. Wente. *Counterexample to a conjecture of H. Hopf.* Pacific J. Math., 121(1), pp. 193-243, 1986

Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit ohne die Hilfe Dritter und ohne Benutzung anderer als der angegebenen Quellen und Hilfsmittel angefertigt und die den benutzten Quellen wörtlich oder inhaltlich entnommenen Stellen als solche kenntlich gemacht habe. Diese Arbeit hat in gleicher oder ähnlicher Form noch keiner Prüfungsbehörde vorgelegen.

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