Master's Thesis

A new parametrization of the spectral curves of the sinh-Gordon equation of spectral genus two

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Abstract

The goal of this Master's thesis is to discuss the one-dimensional manifolds $T^{-1}(\tau_a)$ and characterize their boundary points. We will see that they have two types of boundary points: Those where $a \in \mathcal{M}_2^2 \cup \mathcal{M}_2^3$ holds and those where coefficients are unbounded. In the first case it becomes clear that these are true boundary points through which we can flow smoothly. In the second case it is shown that we can extend continuously to the limit and established a biholomorphic relationship between elliptic curves. In the second part of the thesis we proved that one can solve the Whitham equations inductively for all the coefficients. We also established properties of $V(q_1, q_2)$.

Contents

1 Introduction

In differential geometry the construction of tori with constant mean curvature is a topic of research. The solutions of the elliptic sinh-Gordon equation

$$\Delta u + 2\sinh(2u) = 0$$

for twice differentiable functions $u\colon \mathbb{R}^2 \to \mathbb{R}$ describe such tori. Here we can distinguish between finite type solutions and infinite type solutions. In this thesis we only consider the class of finite type solutions whose spectral genus $g = 2$. These solutions can be described through the space of polynomials which is a space of complex polynomials with matrix valued coefficients. This space gives rise to a certain family of polynomials of degree four. The goal of this work is to establish certain properties of this space and a mapping that leaves the conformal class of these polynomials constant.
In chapter two we will introduce several concepts that will be used in this thesis such as the classification of one-manifolds or the homology on Riemann surfaces.
In chapter three we will introduce the most important concepts of the theory of CMC tori that will be used later in the thesis.
In chapter four we will introduce the most important aspects of CMC tori and the solutions of the sinh-Gordon equation.
In chapter four we will examine the boundary points of $T^{-1}(\tau_a)$ and consider two distinct cases: the case $a \in \mathcal{M}_2^2 \cup \mathcal{M}_2^3$ and the case where coefficients go to infinity. In the first case we will prove that the boundary points are true boundary points by examining a certain condition. In the second case we will use the blow-up technique to prove that each connected component is biholomorphic to the elliptic curve defined by the limits of our polynomials. Further we use the Whitham equation to try to prove that this blow-up is also a one-dimensional manifold.
In chapter five we will construct a specific curve in the plane $\mathbb{R}^2$ $V(q_1, q_2)$ defined by the imaginary parts of q_1 and q_2 restricted to $\mathbb{S}^1$. We will prove properties of this curve. Then we will prove that we can solve the Whitham equations for the linear and constant coefficients of the Taylor series expansions of our polynomials. We try to use this to prove that the sequence of curves has a cusp when intersecting $\mathcal{S}^2$.

2 Preliminaries

2.1 The classification of 1-manifolds

The following proof is from Milnors Topology from the Differentiable Viewpoint (1965).

Classification of 1-manifolds 2.1. *Any smooth, connected one dimensional manifold is diffeomorphic either to* $\mathbb{S}^1$ *or some interval in* $\mathbb{R}$*.*

We will prove this using the concept of parametrization by arc length, which we will first define.

Definition 2.2. *A map* $f : I \to M$ *where* I *is an interval and* M *is a manifold, is called parametrization by arc length if* f *maps* I *diffeomorphically onto an open subset of* M *and if* $\mathrm{d}f_s(1) \in T_{f(s)}M$ *has unit length for each* $s \in I$*.*

We note that any given local parametrization, a change of variables can be used to transform our parametrization into a parametrization by arc length.

Definition 2.3. *Let* X *be a topological space and* $A \subset X$ *a subset of* X*. We say that a set* U_A *is relatively open in* A *if there exists an open set* $U \subset X$ *such that* $U_A = U \cap A$*.*

In the following we will consider M to be a connected 1-manifold.

Lemma 2.4. *Let* $f : I \to M$*,* $g : J \to M$ *be two parametrizations by arc length. Then* $f[I] \cap g[J]$ *has at most two connected components. If it has only one connected component, then* f *can be extended to a parametrization by arc length on the union* $f[I] \cup g[J]$*. If it has two connected components, it must be diffeomorphic to* $\mathbb{S}^1$*.*

Proof: $g^{-1} \circ f$ maps a relatively open subset of I to a relatively open subset of J. By construction, the derivative of $g^{-1} \circ f$ has to have the absolute value 1, so it has to be equal to ± 1 everywhere. Now let $\Gamma \subset I \times J$ be the graph of all $(s, t) \in I \times J$ where $f(s) = g(t)$. So by definition, Γ is a closed subset of $I \times J$ which is made up of line segments that have a slope of ± 1 because we are considering the one dimensional case. Since $g^{-1} \circ f$ is diffeomorphic as the composition of diffeomorphic maps and since we can consider Γ as the graph of $g^{-1} \circ f$ we can extend it to the boundary of $I \times J$. Yet because our map is bijective and single-valued it can only take at most one value on each line of $\partial(I \times J)$. So that makes at most 4 boundary values. These have to be connected by line segments, and since the map is again single valued and

bijective, no boundary point can be reached by more than one line. That makes at most two connected components, which proves the first part of our claim.
Now let Γ have only one connected component. Therefore, Γ consists of exactly one line segment that goes through the whole of $I \times J$. It follows that $g^{-1} \circ f$ can be extended to a translation of a linear map $L : \mathbb{R} \to \mathbb{R}$. We will now use L to extend f to a larger map in the following way

$$F : I \cup L^{-1}[J] \to f[I] \cup g[J]$$
$$x \mapsto \begin{cases} f, & x \in I \\ g \circ L, & x \in L^{-1}[J]. \end{cases}$$

We can easily see that both maps agree on any overlap of I and $L^{-1}[J]$ since $g \circ L\Big|_{I \cap J} = g \circ g^{-1} \circ f = f$. So this is again diffeomorphic as a composition of diffeomorphic maps and also since L needs to have a slope of ± 1, F is a parametrization by arc length.
To finish the proof, we now consider the case where our map has two connected components. We will only consider the case where the derivative of both f and g is 1 since all other cases can be done in a similar way, because if the slopes have different sign we can just multiply them with (-1) in order to get to the case where they are parallel. First, we will name the 4 boundary values that Γ assumes on $I \times J$. Let $a < b \leq c < d \in I$, $\gamma < \delta \leq \alpha < \beta \in J$ and consider the boundary points $(a, \alpha), (b, \beta), (c, \gamma)$ and (d, δ). Then both of the connected components connect two of these points. Without loss of generality we assume that the first line connects (a, α) with (b, β) and the second one connects (c, γ) with (d, δ). Since both components have the same slope, they are parallel. Now we can translate one of our intervals to get that $\gamma = c$ and $\delta = d$. From that follows that

$$a < b \leq c < d \leq \alpha < \beta.$$

Further we set $\theta = \frac{2\pi}{\alpha - a}$ and use polar coordinates to define a diffeomorphism

$$h : \mathbb{S}^1 \to M$$

by setting

$$h(\cos(\theta), \sin(\theta)) = \begin{cases} f(t), & a < t < d \\ g(t), & c < t < \beta \end{cases}$$

where f and g agree on (c, d) by the construction of Γ. So h is a diffeomorhpism from $\mathbb{S}^1$ to $f[I] \cup g[J]$. By definition, $\mathbb{S}^1$ is compact and open and since

h is a diffeomorphism, $h[\mathbb{S}^1] \subset M$ is compact and open as well. So, because M as a manifold is a topological Hausdorff space, any compact set in M is also closed. Therefore, $h[\mathbb{S}^1]$ is both closed and open in M, meaning it is the whole M because M is connected. **q.e.d.**

Now we will use this to prove the classification theorem.
Proof: First we see that as mentioned before we know the existence of a parametrization by arc length because we can transform any local parametrization into one. We only need to consider the case where M is not diffeomorphic to $\mathbb{S}^1$ because that case was already considered in the lemma before. Using the aforementioned emma, we see that every parametrization by arc length can be extended to a maximal parametrization by arc length

$$f : I \to M$$

which can't be extended over any larger interval than I. We can do this by finding other parametrizations by arc length where the intersection of the images only has one connected component and extend the parametrization over the overlap on each side of the interval until we can not go further. So now we consider a maximal parametrization by arc length and we want to show that $f[I] = M$. Assume that this does not hold. Then, because $f[I]$ is open in M, we get that $M \setminus f[I]$ has to contain a limit point x. So then there needs to exist a neighborhood $U \ni x$ with $U \subset M \setminus f[I]$. So then we can find a local parametrization for x and transform it into a parametrization by arc length. Now if M is not diffeomorphic to $\mathbb{S}^1$ then there can't be two parametrizations by arc length where the intersection of they images has two connected components. So the intersection has only one connected components and therefore, we can extend f to a larger parametrization by arc length. That is a contradiction to f being maximal so therefore, it follows that $f[I] = M$. Yet by definition, f is a diffeomorphism, so we get $I \cong M$.
q.e.d.

2.2 Homology on compact Riemann surfaces

The following introduction into homology theory on compact Riemann surfaces is based on the book Computational Approach to Riemann Surfaces by Bobenko (2013).

Definition 2.5. *Let X be a Riemann surface with a triangulation $\mathcal{T}$. We define formal sums of points $P = \sum_i n_i P_i$ as 0-chains, formal sums of oriented edges γ_i as $\gamma = \sum n_i \gamma_i$ 1-chains and formal sums of oriented triangles*

D_i as $D = \sum n_i D_i$ 2-chains. We denote these sets with C_0, C_1, C_2 which are all abelian groups with respect to the addition.

Definition 2.6. *We define (P_1, P_2) as the oriented edge from P_1 to P_2 and $D_0 = (P_1, P_2, P_3)$ as the oriented triangle bounded by the oriented edges $(P_1, P_2), (P_1, P_3)$ and (P_2, P_3). Now on these edges and triangles we define the boundary operator δ as*

$$\delta(P_1, P_2) = P_2 - P_1 \qquad \delta D_0 = (P_1, P_2) + (P_1, P_3) + (P_2, P_3).$$

We extend the boundary operator to C_1 and C_2 by linearity $\delta D = \sum k_i \delta D_i$, $\delta\gamma = \sum n_i \delta\gamma_i$ and define the group homomorphisms $\delta\colon C_2 \to 1, \delta\colon C_1 \to C_0$.

Definition 2.7. *A $\gamma \in C_1$ is called a cycle if $\delta\gamma = 0$ and a $\gamma \in C_1$ is called a boundary if there exists a $D \in C_2$ such that $\gamma = \delta D$. We denote these sets by*

$$Z = \ker\{\delta : C_1 \to C_0\}, \qquad B = \delta C_2.$$

Since by definition $\delta \circ \delta = 0$, every boundary is a cycle which means $Z \subset B \subset C_1$. We define two 1-chains to be homologous if their difference is a boundary.

Definition 2.8. *The factor group*

$$H_1(X, \mathbb{Z}) = Z/B$$

is called the first homology group of X.

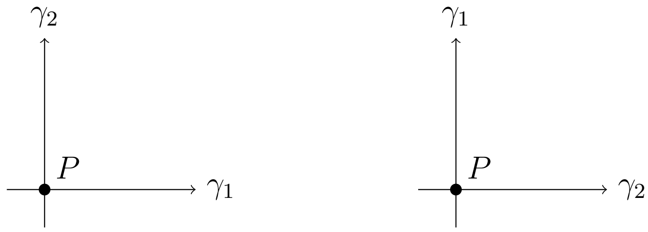
This is also an abelian group which is described by the following equivalence classes

$$[\gamma] = \frac{\{\text{1-cycles}\}}{\{\text{1-dimensional boundaries}\}}.$$

Any closed oriented continuous curve $\tilde{\gamma}$ can be deformed homotopically into a 1-cycle in the triangulation $\mathcal{T}$. Homotopical simplicial 1-cycles are homologous to each other so we can now define the homology group as a homology group of cycles composed of arbitrary closed curves. The definition of homologous continuous cycles is independent of $\mathcal{T}$.
One can represent elements of the first homology group by smooth cycles. Moreover, given two elements of $H_1(X, \mathbb{Z})$ we can represent them by smooth cycles intersecting in finitely many points.

Definition 2.9. *Let γ_1, γ_2 be two curves intersecting transversally at the point P. Then we associate to this point the intersection number $(\gamma_1 \circ \gamma_2)_P = \pm 1$ where the sign is determined by the orientation of the basis $\gamma_1'(P), \gamma_2'(P)$ as shown below where γ_i' is the curve with the inverse orientation.*

Figure 2.1: Intersection number at a point



(a) $(\gamma_1 \circ \gamma_2)_P = 1$ (b) $(\gamma_1 \circ \gamma_2)_P = -1$

Definition 2.10. *Let γ_1, γ_2 be two smooth cycles intersecting transversally at the finite set of their intersection points. The intersection number of γ_1 and γ_2 is defined as*

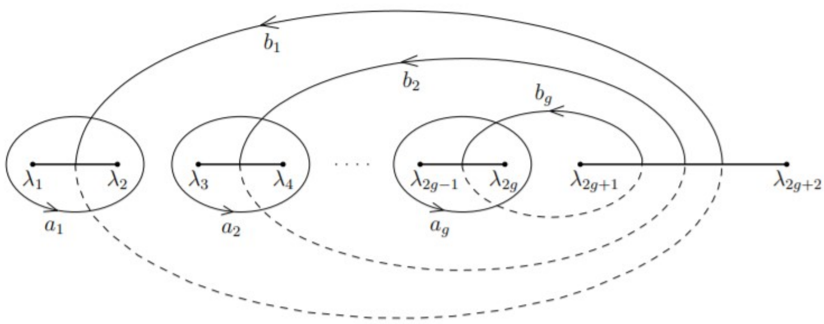
$$\gamma_1 \circ \gamma_2 = \sum_{P \in \gamma_1 \cap \gamma_2} (\gamma_1 \circ \gamma_2)_P.$$

Definition 2.11. *A set of cycles $A_1, B_1, \ldots, A_g, B_g$ such that every cycle on the Riemann surface X of genus g is homologous to a sum of these cycles is called a homology basis. A homology basis $A_1, B_1, \ldots, A_g, B_g$ with the intersection numbers*

$$A_i \circ B_j = \delta_{ij}, \qquad A_i \circ A_j = B_i \circ B_j = 0$$

is called canonical homology basis.

Figure 2.2: Homology basis of a hyperelliptic surface of genus g from Bobenko (2013)



Now we will define certain types of differentials on Riemann surfaces

Definition 2.12. *A differential ω on a Riemann surface X is called an Abelian differential of the first kind if in any local chart it is represented as*

$$\omega = h(z)\mathrm{d}z$$

where $h(z)$ is a holomorphic map.

Definition 2.13. *A meromorphic differential with singularities is called an Abelian differential of the second kind if the residues are equal to zero at all singular points. A meromorphic differential with non-zero residues is called an Abelian differential of the third kind.*

Definition 2.14. *Let $(\gamma_i)_{i\in I}$ be a homology basis of the Riemann surface X and ω a closed differential. Then the integrals*

$$\Lambda_i = \int_{\gamma_i} \omega$$

are called periods.

3 Spectral curves of CMC tori

The following section is based on the paper Solutions of the Sinh-Gordon Equation of Spectral Genus Two and constrained Willmore Tori I by Knopf, Peña Hoepner and M.U. Schmidt as well as the Master Thesis Solutions of the Sinh-Gordon Equation of Spectral Genus Two by Peña Hoepner. We will establish important definitions and theorems which will be used later in this thesis. We will describe the relation between the sinh-Gordon eqation and the Tori of constant mean curvature which we will call CMC tori in the following.

Definition 3.1. *The equation*

$$\Delta u + 2\sinh(2u) = 0$$

is called the elliptic sinh-Gordon equation. Here, $u\colon \mathbb{R}^2 \to \mathbb{R}$ is a real-valued twice partial differentiable function.

Another important space is the space of potentials, which we now define as well.

Definition 3.2. *The set of potentials is a set of cubic polynomials with matrix valued coefficients which we define as follows*

$$\mathcal{P}_2 = \left\{ \zeta_\lambda = \begin{pmatrix} \alpha\lambda - \overline{\alpha}\lambda^2 & -\gamma^{-1} + \beta\lambda - \gamma\lambda^2 \\ \gamma\lambda - \overline{\beta}\lambda^2 + \gamma^{-1}\lambda^3 & -\alpha\lambda + \overline{\alpha}\lambda^2 \end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{C}, \gamma \in \mathbb{R}_+ \right\}$$

Definition 3.3. *Polynomial Killing fields are maps $\zeta_\lambda\colon \mathbb{R}^2 \to \mathcal{P}_2$, $(x, y) \mapsto \zeta_\lambda(x, y)$ which solve the Lax equations*

$$\frac{\partial \zeta_\lambda}{\partial x} = [\zeta_\lambda, U(\zeta_\lambda], \qquad \frac{\partial \zeta_\lambda}{\partial y} = [\zeta_\lambda, V(\zeta_\lambda]$$

with $\zeta_\lambda(0) = \zeta_\lambda^0 \in \mathcal{P}_2$ and

$$U(\zeta_\lambda) = \begin{pmatrix} \frac{\alpha - \overline{\alpha}}{2} & -\gamma^{-1}\lambda^{-1} - \gamma \\ \gamma + \gamma^{-1}\lambda & \frac{\overline{\alpha} - \alpha}{2} \end{pmatrix}, V(\zeta_\lambda) = i \begin{pmatrix} \frac{\alpha + \overline{\alpha}}{2} & -\gamma^{-1}\lambda^{-1} + \gamma \\ \gamma - \gamma^{-1}\lambda & -\frac{\alpha + \overline{\alpha}}{2} \end{pmatrix}.$$

The corresponding function $u(x, y) := \ln\gamma(x, y)$ solves the sinh-Gordon equation.

Definition 3.4. *The space of potentials defines the following set*

$$\mathcal{M}_2 = \{a \in \mathbb{C}^4[\lambda] \mid \lambda a(\lambda) = \det(\zeta_\lambda) \textit{ for a } \zeta_\lambda \in \mathcal{P}_2\}$$

which we will in turn divide into the following sets

$\mathcal{M}_2^1 = \{a \in \mathcal{M}_2 \mid a \text{ has four pairwise distinct roots absent } \mathbb{S}^1\},$
$\mathcal{M}_2^2 = \{a \in \mathcal{M}_2 \mid a \text{ has one double root on } \mathbb{S}^1 \text{ and two simple roots absent } \mathbb{S}^1\},$
$\mathcal{M}_2^3 = \{a \in \mathcal{M}_2 \mid a \text{ has two distinct double roots on } \mathbb{S}^1\},$
$\mathcal{M}_2^4 = \{a \in \mathcal{M}_2 \mid a \text{ has a fourth order root on } \mathbb{S}^1\},$
$\mathcal{M}_2^5 = \{a \in \mathcal{M}_2 \mid a \text{ has two distinct double roots absent } \mathbb{S}^1\}.$

One can easily see that $\mathcal{M}_2$ is the disjoint union of these five sets.

In Peña Hoepner (2015) Theorem 4.3 states the following result about $\mathcal{M}_2$

Theorem 3.5. *The following holds true for $\mathcal{M}_2$*

$$\mathcal{M}_2 = \{a(\lambda) \in \mathbb{C}^4[\lambda] \mid a(0) = 1, \lambda^4 \overline{a(\bar{\lambda}^{-1})} = a(\lambda), \lambda^{-2} a(\lambda) \geq 0 \text{ for } \lambda \in \mathbb{S}^1\}.$$

Definition 3.6. *The condition $p(\lambda) = \overline{p(\bar{\lambda}^{-1})}$ is called the reality condition. We will denote the space of all polynomials of degree d to satisfy the reality condition with $P_{\mathbb{R}}^d$.*

Definition 3.7. *For $a \in \mathcal{M}_2$ we define the level set*

$$I(a) = \{\zeta_\lambda \in \mathcal{P}_2 \mid \det(\zeta_\lambda) = \lambda a(\lambda)\}$$

to be the isospectral set of $a(\lambda)$.

Definition 3.8. *The polynomial Killing fields induce an action*

$$\phi\colon (x, y) \mapsto \phi(x, y), \qquad \phi(x, y) : \mathcal{P}_2 \to \mathcal{P}_2,\ \zeta_\lambda \mapsto \phi(x, y)\zeta_\lambda.$$

The isospectral sets are invariant with respect to this action and decompose in either one or two orbits.

Definition 3.9. *We define*

$$\mathcal{F} = \{\tau \in \mathbb{C} \mid \Im(\tau) > 0, |\Re(\tau)| \leq 1/2, ||\tau|| \geq 1\}/\sim.$$

Here the equivalence relation $\sim$ identifies the points τ of the boundary of the region above with $-\bar{\tau}$. Note that this is the fundamental domain for the action of the modular group on the upper half-plane.

Definition 3.10. *For $a \in \mathcal{M}_2 \setminus \mathcal{M}_2^5$ we define*

$$\Gamma_a = \{x + iy \in \mathbb{C} \mid \forall \zeta_\lambda \in I(a) : \phi(x, y)(\zeta_\lambda) = \zeta_\lambda\}$$

which is an abelian and normal subgroup of $\mathbb{C}$ with the quotient group $\mathbb{C}/\Gamma_a$. For $a \in \mathcal{M}_2^1$ it is proven in Knopf et al. (2018) that for $a \in \mathcal{M}_2^1$, Γ_a is a discrete subgroup with a compact quotient. That means, that there exist $\mathbb{R}$-linearly independent $\omega_1, \omega_2 \in \mathbb{C}$ such that

$$\Gamma_a = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z},$$

so Γ_a is a lattice. Now this is isomorphic up to a rotation-dilation to Γ_τ where $\tau \in \mathcal{F}$ holds.

Corollary 3.11. *There exists a unique map*

$$T : \mathcal{M}_2^1 \to \mathcal{F}, \qquad a \mapsto \tau_a$$

such that Γ_a is isomorphic to Γ_{τ_a}.

Definition 3.12. *Let $\zeta_\lambda \colon \mathbb{R}^2 \to \mathcal{P}_2$ be a polynomial Killing field with initial potential $\zeta_0 \in \mathcal{P}_2$. We define the fundamental solution of the system of ODEs*

$$\frac{\partial F}{\partial x} = FU(\zeta_\lambda), \qquad \frac{\partial F}{\partial y} = FV(\zeta_\lambda), \qquad F(0,0) = \mathbb{1}$$

which we will call frame in the following. Now identify $(x, y) \in \mathbb{R}^2$ with $z = x + iy \in \mathbb{C}$ and consider F to be a function on $\mathbb{C}$. For $\omega \in \Gamma_a$ we detone $M_\omega = F(\omega)$ as a monodromy. M_ω commutes with ζ_0 and maps the eigenspaces of ζ_0 onto themselves.

Definition 3.13. *For every polynomial $a \in \mathcal{M}_2^1$ we define the Riemann surface*

$$\Sigma^* = \{(\lambda, \nu) \in \mathbb{C}^\times \times \mathbb{C} \mid \det(\nu\mathbb{1} - \zeta_0) = \nu^2 + \lambda a(\lambda) = 0\}.$$

Let $\bar{\Sigma}$ be the two-sheeted covering of $\mathbb{CP}^1$ branched at the four roots of $a(\lambda)$, $\alpha_1, \ldots, \alpha_4$ as well as $\lambda = 0$ and $\lambda = \infty$.

Definition 3.14. *On the Riemann surface Σ^* we now define two involutions*

$$\sigma : (\lambda, \nu) \to (\lambda, -\nu), \qquad \rho : (\lambda, \nu) \to (\overline{\lambda}^{-1}, -\overline{\lambda}^{-3}\overline{\nu})$$

where ρ is an involution because $a(\lambda)$ satisfies the reality condition.

Definition 3.15. *The monodromies M_ω act on the one-dimensional eigenspaces of ζ_0 as the multiplication with a function $\mu_\omega \colon \Sigma^* \to \mathbb{C}^\times$ which satisfies*

$$\sigma^* \mu_\omega = \mu_\omega^{-1}, \qquad \rho^* \mu_\omega = \overline{\mu}_\omega^{-1}.$$

In the paper by Knopf et al. (2018) they then have proven the following properties of such μ_ω.

Lemma 3.16. *For all $a \in \mathcal{M}_2^1$ the elements of Γ_a are characterized as those $\omega \in \mathbb{C}$ such that the function $\exp(\omega\lambda^{-1}\nu)$ on Σ^* factorizes into the product of a holomorphic function μ_ω obeying the equations above with a holomorphic function on Σ^* that holomorphically extends to $\lambda = 0$ and takes the value 1 there.*

Definition 3.17. *The logarithmic derivative of μ_ω is a meromorphic function of the second kind with second order poles at $\lambda = 0$ and $\lambda = \infty$. It takes the form*

$$\mathrm{d}\ln \mu_\omega = \frac{b_\omega(\lambda)}{2\nu}\,\mathrm{d}\ln\lambda$$

where $b_\omega(\lambda) \in \mathrm{P}^3_{\mathbb{R}}$.

Definition 3.18. *For any $b \in \mathrm{P}^3_{\mathbb{R}}$ we define the meromorphic differential*

$$\Theta_b = \frac{b(\lambda)}{\nu}\,\mathrm{d}\ln\lambda.$$

For any $a \in \mathcal{M}_2$ we define $\mathcal{B}_a$ to be space of all $b \in \mathrm{P}^3_{\mathbb{R}}$ such that Θ_b has purely imaginary periods. This vector space has real dimension two.

Now we want to establish a connection between τ_a and the space $\mathcal{B}_a$. Theorem 3.5 by Knopf et al. (2018) states that for all $a \in \mathcal{M}_2^1 \cup \mathcal{M}_2^2 \cup \mathcal{M}_2^3$ the values $b_\omega(0)$ fulfilling the equality in Definition 3.17 build a lattice $\tilde{\Gamma}_a \subset \mathbb{C}$ if they define $\mathrm{d}\ln\mu_\omega$ of a function μ_ω on Σ^* that satisfies the equalities in Defintion 3.15 as well as

$$\mu_\omega = f_\omega + g_\omega \nu, \qquad f_\omega, g_\omega \in \mathcal{O}(\mathbb{C}^\times)$$

(see proof of Lemma 4.1).

Definition 3.19. *Let (b_1, b_2) be a base of $\mathcal{B}_a$. Then, $(b_1(0), b_2(0))$ are $\mathbb{R}$-linearly independent and therefore, build a lattice. In the Master thesis A New Parametrization of the Solutions of the sinh-Gordon Equation of Spectral*

Genus Two by B. Schmidt (2020) the following extension of the map T is defined: Let $\hat{T}$ be the map

$$\hat{T} : \mathcal{M}_2^1 \times \mathcal{B}_a \times \mathcal{B}_a \to \mathcal{F}$$
$$(a, b_1, b_2) \mapsto \frac{b_1(0)}{b_2(0)} = \tau_a$$

We will use this extension in our further examinations of the level sets $T^{-1}(\tau_a)$. In the later parts of this thesis the concept of the Willmore energy will be important so we define it here.

Definition 3.20. *Let Σ be a Riemann surface and H the mean curvature of Σ. Then we define the Willmore energy to be*

$$W(\Sigma) = \int_\Sigma H^2 \, \mathrm{d}A$$

From Knopf et al. (2018) we also see the following remark.

Remark 3.21. *If we take $p = (\lambda, \nu) \in \Sigma^*$ then every $\zeta_0 \in I(a)$ has a non-trivial eigenspace at λ with eigenvalue ν. Away from the roots of $a(\lambda)$ the eigenspace is one-dimensional. At simple roots of $a(\lambda)$ ν vanishes, ζ_0 is nilpotent and the eigenspace is one-dimensional as well.*

Definition 3.22. *We define the eigenfunction ψ of ζ for $p = (\lambda, \nu) \in \Sigma^*$ in terms of the fundamental solution F in the following way*

$$\psi(z) = F|_\lambda^{-1}(z)\chi, \quad z \in \mathbb{C}, \chi \in \mathbb{C}^2 \setminus \{(0,0)\}, \quad s.t. \; \zeta_0|_\lambda \chi = \nu\chi.$$

Definition 3.23. *Let $j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ be the matrix representation of the quaternion $j \in \mathbb{H}$. Then we can define the space of quaternions to be*

$$\mathbb{H} = \left\{ A \in \mathbb{C}^{2\times 2} \;\middle|\; jA = \overline{A}j \right\}$$
$$= \left\{ (\chi, -j\overline{\chi}) \;\middle|\; \chi \in \mathbb{C}^2 \right\}.$$

Lemma 3.24. *The involution $\eta = \sigma \circ \rho$ acts on ζ_0 and F as*

$$\eta^*\zeta_0 = -j\overline{\zeta_0}j, \quad \eta^*F(z) = -j\overline{F}(z)j.$$

That means that $-j\chi$ is the eigenvector of $\zeta_0|_{\eta(\lambda)}$ with eigenvalue $\eta(\nu)$ if χ is the eigenvector of $\zeta_0|_\lambda$ with eigenvalues ν and $F|_{\eta(\lambda)}^{-1}(z)(-j\overline{\chi}) = -j\overline{\psi}(z)$.

Definition 3.25. *For two points $p_1, p_2 \in \Sigma^*$ and non-trivial eigenvectors $\chi_1, \chi_2 \in \mathbb{C}^2 \setminus \{(0,0)\}$ of ζ_0 at the two points before the vectors*

$$\chi_3 = -j\overline{\chi_1}, \qquad \chi_4 = -j\overline{\chi_4}$$

are eigenvectors of ζ_0 at the new points $p_3 = \eta(p_1)$, $p_4 = \eta(p_2)$. Now we consider the corresponding functions $\psi_1, \dots, \psi_4$ defined as in Definition 3.22 and define the following maps $s_1, s_2, f_a \colon \mathbb{C} \to \mathbb{H}$ to be

$$s_1 = (\psi_1, \psi_3),\ s_2 = (\psi_2, \psi_4),\ f_a = s_1^{-1} s_2.$$

In the paper by Knopf et al. (2018) the following theorem is proven

Theorem 3.26. *For all $a \in \mathcal{M}_1^2 \cup \mathcal{M}_2^2 \cup \mathcal{M}_2^3$ the Willmore energy of f_a is equal to*

$$W(a) = \int_{\mathbb{C}/\hat{\Gamma}_a} 4\gamma^2 \, \mathrm{d}x \wedge \mathrm{d}y = \int_{\mathbb{C}/\tilde{\Gamma}_a} 8\gamma^2 \, \mathrm{d}x \wedge \mathrm{d}y = 4iRes_{\lambda=0} \log(\mu_2) \mathrm{d} \log(\mu_1).$$

Now from B. Schmidt (2020) Chapter 6 we get the following result

$$\dot{W}(a) = 4i \mathrm{Res}_{\lambda=0} t \frac{\mathrm{d}\lambda}{\lambda}.$$

Definition 3.27. *We define $\mathcal{H}^2 = \{a \in P_{\mathbb{R}}^4 \mid$ a describes the spectral curve of a CMC torus of finite type$\}$. These polynomials can be described by the following four conditions (see The prevalence of tori amongst constant mean curvature planes in $\mathbb{R}^3$ by Carberry and M. U. Schmidt (2016))*

- *(i) $a(\lambda)$ satisfies the reality condition*
- *(ii) $\frac{a(\lambda)}{\lambda^2} > 0$ for all $\lambda \in \mathbb{S}^1$.*
- *(iii) the highest coefficient of $a(\lambda)$ has absolute value 1*
- *(iv) $a(\lambda)$ has pairwise distinct roots.*

$a(\lambda)$ therefore defines a smooth Riemann surface X_a defined by pairs $(\lambda, \nu) \in \mathbb{C}^\times \times \mathbb{C}$ fulfilling

$$\nu^2 = \lambda a(\lambda) = \lambda \prod_{j=1}^{2} \frac{\overline{\eta_j}}{|\eta_j|} (\lambda - \eta_j)(\lambda - \overline{\eta_j}^{-1}).$$

Definition 3.28. *For $\lambda_0 \in \mathbb{S}^1$ we define*

$$\mathcal{S}_{\lambda_0}^2 = \{a \in \mathcal{H}^2 \mid b(\lambda_0) = 0 \text{ for all } b \in \mathcal{B}_a\}.$$

Further we define

$$\mathcal{S}^2 = \bigcup_{\lambda_0 \in \mathbb{S}^1} \mathcal{S}_{\lambda_0}^2.$$

Our next goal is to define the concept of the winding number which is defined in the paper by Carberry and Schmidt (2016). In order to do so we will first define

Definition 3.29. *Let $b_1, b_2 \in \mathcal{B}_a$ be linearly independent. Then we define*

$$f = \frac{b_1}{b_2} \colon \mathbb{C}\mathbf{P}^1 \to \mathbb{C}\mathbf{P}^1$$

which is defined by $a(\lambda)$ up to a Möbius transformation. Therefore, we can define $\deg(f)$ independently from our choice of b_1, b_2.

Definition 3.30. *Let $b_1, b_2 \in \mathcal{B}_a$ be the unique pair s.t. $b_1(0) = 1$ and $b_2(0) = i$. Then we define the map*

$$\tilde{f} \colon \mathbb{S}^1 \to \mathbb{S}^1$$

where

$$\tilde{f} = \frac{b_1 + ib_2}{b_1 - ib_2} = \frac{f + i}{f - i}$$

holds. We will then define the winding number to be $n(\tilde{f}) = \deg(\tilde{f})$.

In the paper from Carberry and Schmidt (2016), Lemma 3.1 states

Lemma 3.31. *The degree $\deg(f)$ and the winding number $n(\tilde{f})$ of f satisfy*

$$n(\tilde{f}) \equiv \deg(f) \mod 2 \qquad \textit{and} \qquad -\deg(f) < n(\tilde{f}) \leq \deg(f).$$

4 The boundary of $T^{-1}(\tau_a)$

In the Master's thesis by B. Schmidt (2020), Corollary 4.3 states that the level sets $T^{-1}(\tau_a)$ are submanifolds of dimension one for any $a \in \mathcal{M}_2^1$. By our Classification of 1-manifolds we see that each connected component of these submanifolds is either diffeomorphic to $\mathbb{S}^1$ or an interval in $\mathbb{R}$. In this section we discuss the case of those components diffeomorphic to an interval and we will examine the boundary points of these components. We can identify each $a \in \mathcal{M}_2^1$ with its two roots $\alpha_1, \alpha_2 \in B(0,1) \subset \mathbb{C}\mathbf{P}^1$. Therefore, every sequence $(a_n)_{n\in\mathbb{N}}$ in such a component has a convergent subsequence in the sense that the roots converge in the projective space. If we consider a sequence that converges to a boundary point of one of these intervals we can assign the limit to a spectral curve we will examine. There, we only need to consider two cases: either the limit spectral curve is now an element $a \in \bigcup_{i=2}^{5} \mathcal{M}_2^i$ or the spectral curve has roots of higher order at $\lambda = 0$ and $\lambda = \infty$. The Theorem 3.5 of the paper by Knopf et al. (2018) states that in the case of $a \in \mathcal{M}_2^4 \cup \mathcal{M}_2^5$ T takes the value ∞ on these sets, meaning that the corresponding lattice goes to infinity. Since that is not the case here, we only need to consider $a \in \mathcal{M}_2^2 \cup \mathcal{M}_2^3$.

4.1 The case $a \in \mathcal{M}_2^2 \cup \mathcal{M}_2^3$

In this case every spectral curve $a(\lambda)$ we consider all have in common that they have a double root $\lambda_0 \in \mathbb{S}^1$. We will use the fifth chapter of B. Schmidt (2020) to examine the condition $\frac{a(\lambda)}{\lambda^2} \geq 0$ at the double root λ_0 because at every other value of $\lambda \in \mathbb{S}^1$ beside a potential second double root it needs to hold that the expression is greater than zero because otherwise $a(\lambda)$ would have to vanish at that value which is a contradiction to λ_0 being the double root (and λ_1 being the second one for which the proof would go similarly). To examine the condition we will derive it by t

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{a(\lambda_0)}{(\lambda_0)^2} = \frac{\dot{a}(\lambda_0)}{(\lambda_0)^2}.$$

Our goal is to prove that this expression doesn't vanish so that we know that the expression $\frac{a(\lambda)}{\lambda^2}$ is not staying at 0 so we know that we can flow through the boundary point smoothly. Therefore, the value we need to look at is $\dot{a}(\lambda_0)$. In section 5.2 of B.Schmidt (2020) it is shown that this expression is given by

$$\dot{a}(\lambda_0) = \frac{\lambda_0 a''(\lambda_0) c_k(\lambda_0) i}{b_k'(\lambda_0)}$$

for $k = 1, 2$ such that $b_k'(\lambda_0) \neq 0$. Since $a(\lambda)$ only has a root of order two, we know that $a''(\lambda_0)$ can't vanish. So we need to look at the values $c_k(\lambda_0)$ for both $k = 1, 2$. Here, we also get that the value of $c_2(\lambda_0)$ can be expressed as follows

$$c_2(\lambda_0) = \frac{2\lambda_0^2 Q_1 a''(\lambda_0) b_2'(\lambda_0)}{b_2'(\lambda_0)b_1''(\lambda_0) - b_1'(\lambda_0)b_2''(\lambda_0) + 2\lambda_0 b_1'(\lambda_0)b_2''(\lambda_0) - 2\lambda_0 b_2'(\lambda_0)b_1''(\lambda_0)}$$

Again we see that $a''(\lambda_0)$ doesn't vanish and we can assume that $b_2'(\lambda_0)$ also is not zero if c_2 is used in the equation for $\dot{a}$ above which holds for both $c_1(\lambda)$ and $c_2(\lambda)$. So the only value that could vanish is Q_1. If we look at the other case, $b_1'(\lambda_0) \neq 0$ we get

$$c_1(\lambda_0) = \frac{c_2(\lambda_0)b_1'(\lambda_0)}{b_2'(\lambda_0)}$$

so we see that in the case of $b_1'(\lambda_0)$ being finite we have the same value for $\dot{a}(\lambda_0)$ as for $c_2(\lambda_0)$ with the only difference being $b_1'(\lambda_0)$ in the fraction instead of $b_2'(\lambda_0)$. Therefore, it remains to examine first the case that both $b_k'(\lambda_0)$ vanish and second the value Q_1. To examine the former we first establish a fact concerning the roots of the polynomials $b(\lambda) \in \mathcal{B}_a$.

Lemma 4.1. *If $\lambda_0 \in \mathbb{C}$ is a root of $a \in \mathcal{M}_2$ of even order $2k$, $k = 1, 2$ then every $b \in \mathcal{B}_a$ has a root of order k at λ_0.*

Proof: We know from the paper from Knopf et al. (2018) that μ_ω satisfies the condition

$$\mu_\omega = f_\omega + g_\omega \nu \quad \text{with} \quad f_\omega, g_\omega \in \mathcal{O}(\mathbb{C}^\times)$$

Therefore, we can use this representation to compare with Θ_{b_ω} since

$$\mathrm{d} \ln \mu_\omega = \Theta_{b_\omega} = \frac{b_\omega(\lambda)}{2\nu} \mathrm{d} \ln \lambda$$

Now we use our representation formula to differentiate the left hand side of the formula above

$$\begin{aligned}
\mathrm{d}\ln\mu_\omega &= \mathrm{d}\ln(f_\omega + g_\omega\nu)\\
&= \mathrm{d}\ln(f_\omega(\lambda) + g_\omega(\lambda)\sqrt{\lambda a(\lambda)})\\
&= \frac{f'_\omega(\lambda) + g'_\omega(\lambda)\sqrt{\lambda a(\lambda)}) + \frac{g_\omega(\lambda)(\lambda a'(\lambda)+a(\lambda)}{2\sqrt{\lambda a(\lambda)}}}{f_\omega + g_\omega\nu}\mathrm{d}\lambda\\
&= \frac{\lambda g_\omega(\lambda)a'(\lambda) + 2\sqrt{\lambda a(\lambda)}f'_\omega(\lambda) + 2\lambda a(\lambda)g'_\omega(\lambda) + a(\lambda)g_\omega(\lambda)}{2\sqrt{\lambda a(\lambda)}\left(\sqrt{\lambda a(\lambda)}g_\omega(\lambda) + f_\omega(\lambda)\right)}\mathrm{d}\lambda\\
&= \frac{b_\omega(\lambda)}{2\sqrt{\lambda a(\lambda)\lambda}}\mathrm{d}\lambda.
\end{aligned}$$

Now comparing coefficients and rationalizing results in the following equation we obtain

$$b_\omega(\lambda)(f_\omega(\lambda) + g_\omega(\lambda)\nu) = \lambda^2 g_\omega(\lambda)a'(\lambda) + 2\lambda\nu f'_\omega(\lambda) + 2\lambda^2 a(\lambda)g'_\omega(\lambda) + \lambda a(\lambda)g_\omega(\lambda).$$

Next, if we examine both sides at a root of order $2k$ λ_0 of $a(\lambda)$ we see that ν has a root of order k there and a' has a root of order $k-1$. Since every term on the right side includes at least one of a, ν or a' we see that the right side has a root of order k at every root of order $2k$ of $a(\lambda)$. Then, if we evaluate the left hand side of the equation above we see that at such a root the expression takes the form $b_\omega(\lambda_0)f_\omega(\lambda_0)$. Now since $f_\omega(\lambda)$ is holomorphic on $\mathbb{C}^\times$ we see that $b_\omega(\lambda_0)$ needs to have a root of order k at λ_0, thus completing the proof.

q.e.d.

Considering that we are in the case that $a(\lambda)$ has at least one double root we see that b_1 and b_2 already share one root and therefore, $\deg(f) \leq 2$. Now if it were to be true that both $b'_1(\lambda_0)$ and $b'_2(\lambda_0)$ vanish, λ_0 would be a root of order 2 for both b_k meaning that $\deg(f) \leq 1$. Although Theorem 3.2 from the paper by Carberry and M.U. Schmidt (2016) states that the condition $\deg(f) = 1$ is equivalent to $g(X_a) = 0$. Since that is not true for the surfaces we consider, we know that this case won't occur. Therefore, we only need to look at Q_1 now. Now in B. Schmidt (2020) section 5.2 we see that for $a \in \mathcal{M}_2^2 \cup \mathcal{M}_2^3$ the values for $\dot{a}, \dot{b}_1, \dot{b}_2, c_1, c_2$ all depend on $Q_1 \in \mathbb{R}$ in the way that if $Q_1 = 0$ all other values vanish as well, so we get the trivial zero solution. So what we know is first that $\frac{a(\lambda)}{\lambda^2} = 0$ for any double root $\lambda \in \mathbb{S}^1$ as well as $\frac{a(\lambda)}{\lambda^2} > 0$ for all other values $\lambda \in \mathbb{S}^1$. So since we have proven that

$$\frac{\dot{a}(\lambda_0)}{(\lambda_0)^2} \neq 0 \qquad \text{for any double root } \lambda_0 \in \mathbb{S}^1$$

holds we see that the sign of this condition needs to change if we go through this boundary point $a \in \mathcal{M}_2^2 \cup \mathcal{M}_2^3$ so then it holds that $\frac{a(\lambda)}{\lambda^2} < 0$ for some $\lambda \in \mathbb{S}^1$. However we know from Theorem 3.5 that the condition $\frac{a(\lambda)}{\lambda^2} > 0$ for all $\lambda \in \mathbb{S}^1$ needs to hold for $a \in \mathcal{M}_2$. So we see that this boundary point is a true boundary point of our set because any further value is not an element of $\mathcal{M}_2$.

4.2 The case of unbounded coefficients

4.2.1 Blowing up $\mathcal{M}_1^2$

Now we will consider the second boundary case of the components of $T^{-1}(\tau_a)$ which is that at least one of the coefficients of $a(\lambda)$ goes to infinity which because of the reality condition implies that another coefficient goes to zero. The same things then happens to at least one pair of roots $\alpha, \overline{\alpha}^{-1}$ of $a(\lambda)$. In this case the limits are not part in $\mathcal{M}_2$ or any other set that can a priori be considered a manifold because some of the coefficients form singularities. This means that we can't use the implicit function theorem to prove that these limits of $T^{-1}(\tau_a)$ form a one-dimensional manifold as well. So in this case we will construct a blow up in which the coefficients of the limit should still be finite. In order to do so, we first need to consider a new parametrization for $a(\lambda) \in \mathcal{M}_2$. Therefore, let a^+ and a^- be the polynomials of degree two where a^+ has the roots of $a(\lambda)$ that lie in $B(0,1) \cup \mathbb{S}^1$ and a^- has the roots of $a(\lambda)$ in $\mathbb{C} \setminus \overline{B(0,1)} \cup \mathbb{S}^1$ with the new coefficients $\lambda_t^+ = \frac{t}{\lambda}, \lambda_t^- = \lambda t$ where $\lambda \in \mathbb{C}, t \in \mathbb{R}_+$. The new parametrization will have the following form

$$a(\lambda) = (\lambda_t^+)^{-2} \cdot a^+(\lambda_t^+) \cdot a^-(\lambda_t^-). \tag{1}$$

To include the case where $a(\lambda)$ has roots on $\mathbb{S}^1$ we establish the following fact

Proposition 4.2. *Any polynomial $a \in \mathcal{M}_2$ has only roots of even order on $\mathbb{S}^1$.*

Proof: Any $a \in \mathcal{M}_2$ has to satisfy the condition

$$\frac{a(\lambda)}{\lambda^2} \geq 0, \quad \text{for } \lambda \in \mathbb{S}^1.$$

Therefore, we also know that $\lambda \mapsto \frac{a(\lambda)}{\lambda^2}$ is a map into the real line if we restrict λ to the unit sphere. However we can also then replace the complex λ with real polar coordinates and therefore, we can express our fraction as a polynomial in the real numbers

$$\begin{aligned}\phi : \mathbb{R} &\to \mathbb{R},\\ \varphi &\mapsto \frac{a(\lambda(\varphi))}{\lambda(\varphi)^2}.\end{aligned}$$

Now if $\phi(\varphi)$ were to have a root of uneven order that would mean that it would change the sign at some point of the real numbers. This, however is a contradicition to the condition we stated in the beginning. Although if $\frac{a(\lambda)}{\lambda^2}$ does not have a root of uneven order then the same has to hold for $a(\lambda)$ which completes the proof. **q.e.d.**

In our construction we will need an extra condition on $\mathcal{M}_2$ for our decomposition to work. Therefore, we define the following equivalence relation

Definition 4.3. *We identify those $a \in \mathcal{M}_2$ to be equivalent where the roots of $a_1, a_2 \in \mathcal{M}_2$ can be transformed from one to the other by the multiplication with i. In the following we consider those representatives of $\mathcal{M}_2/\sim$ that have the property for those two roots $\alpha_1, \alpha_2 \in B(0,1)$ that the product $\alpha_1\alpha_2 > 0$ holds.*

Remark 4.4. *Note that because of $a(0) = 1$ it holds that $\alpha_1\alpha_2 = \overline{\alpha_1\alpha_2}$ which in turn means that the product is real. Then, since roots at zero do not occur in $P^d_{\mathbb{R}}$ the product is either greater than zero or smaller. Multiplying all roots with i turns every $a(\lambda)$ where the product is smaller than zero into a polynomial where it is greater than zero.*

Lemma 4.5. *Any polynomial $a \in \mathcal{M}_2$ has a unique decomposition of type (1) where a^+ is the polynomial of degree two with all of the roots inside $B(0,1)$ and half of each root on $\mathbb{S}^1$ (of even order, as seen in the previous proof) and a^- is the polynomial of degree two with all of the roots outside of $\overline{B(0,1)}$ and half of each root on $\mathbb{S}^1$. The new polynomials satisfy the following conditions*

(i) $a^+_k = \overline{a^-_k}$ *for all* $k = 0, 1, 2$

(ii) $(\lambda^+_t)^{-2}a^+(\lambda^+_t) = \lambda^2 t^{-2} + \mathfrak{a}\lambda t^{-1} + 1$

where $\mathfrak{a} \in \mathbb{C}$ and $t \in \mathbb{R}$ are the two parameters describing $a(\lambda)$ following the new parametrization.

Proof: The first observation we make is that by Proposition 3.2 and the fact that roots of $a(\lambda)$ are invariant under $\lambda \to \overline{\lambda}^{-1}$ the polynomials a^+ and

a^- have the same degree $\deg(a^+) = \deg(a^-) = 2$. Now we also know that $a(0) = 1$ holds and that the polynomials $a^\pm$ have the following form

$$a^\pm(\lambda_t^\pm) = a_2^\pm \prod_{k=1}^{2} (\lambda_t^\pm - \varrho_{t,k}^\pm)$$

where we denote the roots of $a^\pm$ by $\varrho^\pm$. Since λ_t^+ and λ_-^t can be transformed into each other by the mapping $\lambda \to \lambda^{-1}$ and the roots of a^+ and a^- can be mapped to each other by $\lambda \to \overline{\lambda}^{-1}$ because of the reality condition we can number the roots of $a^\pm$ in a way that $\varrho_{t,k}^+ = \overline{\varrho_{t,k}^-}$ and therefore, the claim for the coefficients holds as well.
Now we consider the second condition and calculate the left hand side of (1) which yields

$$\begin{aligned}(\lambda_t^+)^{-2} a^+(\lambda_t^+) a^-(\lambda_t^-) &= \frac{\lambda^2}{t^2}\Big(\frac{t^2}{\lambda^2} + \mathfrak{a}\frac{t}{\lambda} + 1\Big)\Big(\lambda^2 t^2 + \overline{\mathfrak{a}}\lambda t + 1\Big) \\ &= \Big(\lambda^2 t^{-2} + \mathfrak{a}\lambda t^{-1} + 1\Big)\Big(\lambda^2 t^2 + \overline{\mathfrak{a}}\lambda t + 1\Big) \\ &= t^2 t^{-2}(\lambda - \alpha_1)(\lambda - \alpha_2)(\lambda - \overline{\alpha_1}^{-1})(\lambda - \overline{\alpha_2}^{-1}) \\ &= t^{-2}(\lambda - \alpha_1)(\lambda - \alpha_2) t^2 (\lambda - \overline{\alpha_1}^{-1})(\lambda - \overline{\alpha_2}^{-1}).\end{aligned}$$

We use the first part of both sides to calculate conditions on t and $a(\lambda)$. That yields

$$\Big(\lambda^2 t^{-2} + \mathfrak{a}\lambda t^{-1} + 1\Big) = t^{-2}(\lambda - \alpha_1)(\lambda - \alpha_2)$$

which is equivalent to

$$t^{-2}(\lambda^2 + \mathfrak{a} t\lambda + t^2) = t^{-2}(\lambda^2 + (\alpha_1 + \alpha_2)\lambda + \alpha_1\alpha_2).$$

By comparing coefficients we get two equations to solve

$$\begin{aligned} t^2 &= \alpha_1\alpha_2 \\ \mathfrak{a} t &= \alpha_1 + \alpha_2 \end{aligned}$$

which results in the unique solution

$$\begin{aligned} t &= \sqrt{\alpha_1\alpha_2} \\ \mathfrak{a} &= \frac{\alpha_1 + \alpha_2}{\sqrt{\alpha_1\alpha_2}} \end{aligned}$$

because we forced $\alpha_1\alpha_2 > 0$. Since we know that every $a \in \mathcal{M}_2/\sim$ is uniquely defined by the two roots $\alpha_1, \alpha_2 \in B(0,1)$ s.t. $\alpha_1\alpha_2 > 0$ holds, the two equations we were able to solve uniquely define us a decomposition for each $a \in \mathcal{M}_2$. **q.e.d.**

Remark 4.6. *The composition described in Equation one can now be calculated a bit further. We get*

$$\begin{aligned} a(\lambda) &= \frac{\lambda^2}{t^2} \cdot \Big(\frac{t^2}{\lambda^2} + \mathfrak{a}\frac{t}{\lambda} + 1\Big) \cdot \Big(t^2\lambda^2 + \bar{\mathfrak{a}}t\lambda + 1\Big) \\ &= \Big(t^{-2}\lambda^2 + \mathfrak{a}t^{-1}\lambda + 1\Big) \cdot \Big(t^2\lambda^2 + \bar{\mathfrak{a}}t\lambda + 1\Big) \end{aligned}$$

Definition 4.7. *We will denote the space of pairs $(t, \mathfrak{a}) \in \mathbb{R}_+ \times \mathbb{C}$ that correspond to an $a \in \mathcal{M}_2$ with $\mathcal{A}$.*

Now in the next step we also want to decompose the elements of $\mathcal{B}_a$ in a similar way. Here we will use the t that is uniquely defined for every $a \in \mathcal{M}_2/\sim$ in our new decomposition. We will start with a decomposition of the real 4-dimensional space $P^3_{\mathbb{R}}$. To do so, we show that such a decomposition works for all $t \in \mathbb{R}$ and for arbitrary $p(\lambda) \in P^d_{\mathbb{R}}$. For that, again we decompose p into three factors, p^+, p^- and p^0 where p^+ contains all roots in $B(0,1)$, p^0 contains all the roots on $\mathbb{S}^1$, and p^- contains all the roots on $\mathbb{C} \setminus \overline{B(0,1)}$. Then we consider for $t \in \mathbb{R}^\times$ the following decomposition

$$p(\lambda) = (\lambda_t^+)^{-\deg(p^+)} \cdot p^+(\lambda_t^+) \cdot p^0(\lambda) \cdot p^-(\lambda_t^-). \tag{2}$$

The following lemma and proof are from the unpublished paper The boundary of the space of spectral curves of constant mean curvature tori with spectral genus two by Carberry, Kilian, Klein and M.U. Schmidt (2020)

Lemma 4.8. *Let $\varphi \in \mathbb{R}$. Then each $p \in P^d_{\mathbb{R}}$ has a unique decomposition as in (2) with the following conditions*

(i) $|p^0(0)| = 1$ *and* $p^0 \in P^{d^0}_{\mathbb{R}}$ *where* d^0 *is the number of roots of* p *on* $\mathbb{S}^1$.

(ii) The coefficients $p^\pm_k$ *of* $p^\pm(\lambda^\pm_t) = \sum_{k=0}^{d'} p^\pm_k (\lambda^\pm_t)^k$ *obey* $p^-_k = \overline{p^+_k}$ *where* d' *is the degree of* p^+ *and* p^-

(iii) $p^-_{d'} \in e^{i\varphi} \cdot \mathbb{R}_+$

Proof: For given $t \in \mathbb{R}$ we can easily construct polynomials $p^\pm, p^0$ that satisfy the condition on the roots. In a similar way as in Lemma 3.5 we consider the map $\lambda \to \bar{\lambda}^{-1}$ which assigns each root inside $B(0,1)$ exactly one root on $\mathbb{C} \setminus \overline{B(0,1)}$ which in turn means that $\deg(p^+) = \deg(p^-)$. Since each root on $\mathbb{S}^1$ is of the form $e^{i\varphi}$ the mapping assigns such a root itself. Therefore, it preserves the roots of p^0 which we then can choose to be in $P^{d^0}_{\mathbb{R}}$. Since every polynomial is uniquely determined by its roots and the leading coefficient, the condition $|p^0(0)| = 1$ determines p^0 uniquely up to sign, since the leading coefficient and lowest coefficient are complex conjugates for polynomials satisfying the reality condition. So now the first part is proven. Now we take a look at

$$p^{\pm}(\lambda_t^{\pm}) = p_{d'}^{\pm} \cdot \prod_{k=1}^{d'} (\lambda_t^{\pm} - \varrho_{t,k}^{\pm}),$$

where $\varrho_{t,k}^{\pm}$, $k = 1, \ldots, d'$ are the roots of $p^{\pm}$.
We already established that $\lambda \to 1/\overline{\lambda}$ maps the roots of p^+ to the roots of p^-. Since λ_t^+ can be transformed into λ_t^- by the mapping $\lambda \to 1/\lambda$ we see that the roots $\varrho_{t,k}^{\pm}$ can be numbered such that the corresponding pairs satisfy $\varrho_{t,k}^{+} = \overline{\varrho_{t,k}^{-}}$ for all $k = 1, \ldots, d'$. That in turn yields the same relation for the coefficients of $p^{\pm}$ if $p_{d'}^{+} = \overline{p_{d'}^{-}}$. The highest and lowest coefficients of the right hand side of equation (2) are

$$p_{d'}^{+} \cdot p_{d'}^{-} \cdot p_{d^0}^{0} \cdot \prod_{k=1}^{d'} (-\varrho_{t,k}^{+}) \quad \text{and} \quad p_{d'}^{+} \cdot p_{d'}^{-} \cdot p_{0}^{0} \cdot \prod_{k=1}^{d'} (-\varrho_{t,k}^{-}).$$

Now if we impose the condition on the highest coefficients discussed before as well as our knowledge of the roots, which yields the following for the highest and lowest coefficient

$$\overline{p_{d'}^{-}} \cdot p_{d'}^{-} \cdot p_{d^0}^{0} \prod_{k=1}^{d'} (-\overline{\varrho_{t,k}^{-}}) \text{ and } \overline{p_{d'}^{-}} \cdot p_{d'}^{-} \cdot p_{0}^{0} \cdot \prod_{k=1}^{d'} (-\varrho_{t,k}^{-}).$$

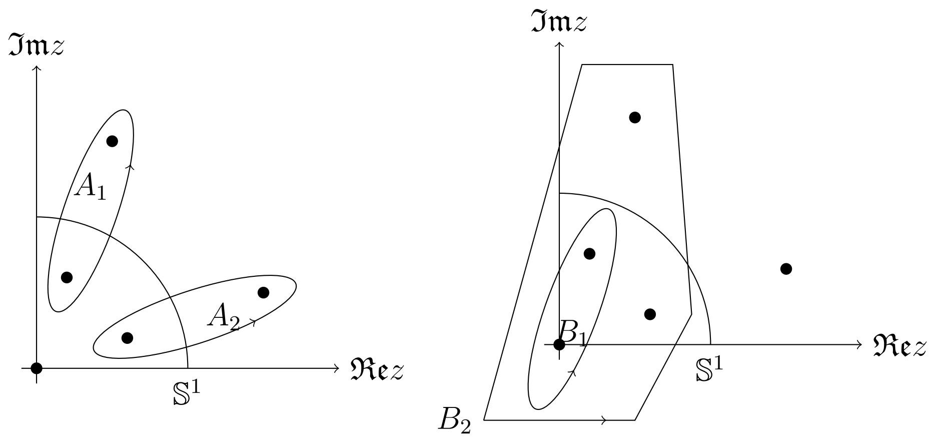
Because of the first condition of the Lemma, $p_{d^0}^{0} = \overline{p_0^0}$ holds. Then the highest coefficient is the complex conjugate of the lowest for any choice of $p_{d'}^{-} \in \mathbb{C}^{\times}$. By construction, both sides of (2) have the same roots and we know that the left hand side is a polynomial in $P_{\mathbb{R}}^{d}$. Therefore, the right hand side is also a polynomial satisfying the reality condition. Now the right hand side equals the left hand side if the leading coefficient of sides is equal. Since we chose p^0 uniquely up to sign and the roots of $p^{\pm}$ are also already determined the now determining factor is $|p_{d'}^{-}|$. Therefore, we can choose a unique representative of the form $p_{d'}^{-} \in e^{i\varphi} \cdot \mathbb{R}_{+}$. **q.e.d.**

Now let $b(\lambda) \in P_{\mathbb{R}}^{3}$. Then we consider the same decomposition as before

$$b(\lambda) = (\lambda_t^{+})^{-\deg(b^{+})} \cdot b^{+}(\lambda_t^{+}) \cdot b^{0}(\lambda) \cdot b^{-}(\lambda_t^{-}). \tag{3}$$

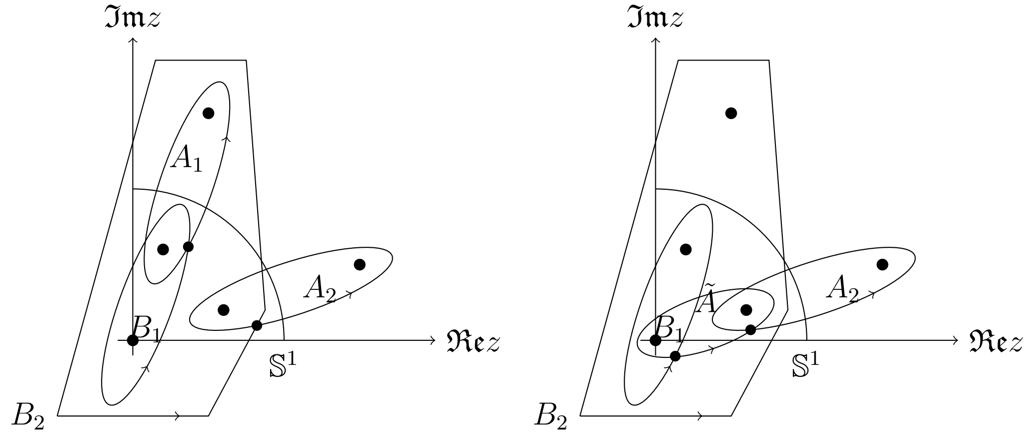
Since we now know how to decompose the space $P_{\mathbb{R}}^{3}$ we want to have a look at $\mathcal{B}_a$ as well. In order to do so we need to discuss the homology basis and how to change it, such that some of the cycles are still defined in the blow up. First, we consider the canonical homology basis of our Riemann surface $\Sigma^* = \{(\lambda, \nu) \in (\mathbb{C}^\times \times \mathbb{C}) \mid \det(\nu\mathbb{1} - \xi_0) = \nu^2 + \lambda a(\lambda)\}$ as described in the paper by Knopf et al. (2018). Those are the cycles A_1, A_2, B_1, B_2 described in the first 3 pictures. We added the orientation and intersection points to make the chosen basis unique. That is because our surface Σ^* is a hyperelliptic surface of genus two and therefore, a two-sheeted covering of $\mathbb{C}\mathbf{P}^1$ and therefore, if we look at the cycles only in the $\mathbb{C}$-plane we need to make clear in which sheet the cycles are at all times and we do so by fixing the intersection points. However in our blow-up, some of our cycles, namely those surrounding more than one root won't be well defined in our blow-up, so we need to consider a new basis. On the other hand we also know that in the limit we will discuss a Riemann surface of lower genus, namely $g = 1$ so it suffices to have two cycles that will remain well-defined in the limit. The first cycle is already a part of our canonical basis, namely B_1. Then we want to exchange the cycle A_2 with a cycle surrounding the root of $a(\lambda)$ inside $B(0, 1)$ not surrounded by B_1 and zero which we will call $\tilde{A}$. We then can use our canonical homological basis to express $\tilde{A}$ with the old homological basis. To do so, we need to fix a orientation for that cycle and see in which sheet the cycle is at each point. Then we can fix an intersection point with the cycle B_1. The next figures will make clear how $\tilde{A}$ is defined.

Figure 4.1: Homology basis of $\bar{\Sigma}$



(a) A cycles (b) B cycles

Figure 4.2: Homology basis of $\overline{\Sigma}$ with intersection points



(a) Homology basis with intersection points (b) $\tilde{A}$ and intersections with other cycles

We see that the new defined cycle has the following intersection numbers with the canonical basis: $(\tilde{A} \circ A_2)_{P_1} = -1$, $(\tilde{A} \circ B_1)_{P_2} = 1$. Since for our canonical basis it holds that $(A_i \circ B_j) = \delta_{ij}$ we know that our cycle $\tilde{A}$ is homological to $B_2 + A_1$ since they have the same intersection numbers with every cycle of our base. So we can now consider our new homological basis $(A_1, \tilde{A}, B_1, B_2)$. We now want to show that the periods of Θ_b uniquely determine the underlying b defining the differential.

Lemma 4.9. *For every $a \in \mathcal{M}_2^1 \cup \mathcal{M}_2^2 \cup \mathcal{M}_2^3$ for every pair of numbers $(\mu_1, \mu_2) \in \mathbb{R}$ there exists a unique $b \in \mathcal{B}_a$ satisfying the following conditions*

(i) $\int_{A_i} \Theta_b$ *vanishes for both* $i = 1, 2$

(ii) $\int_{B_i} \Theta_b = \mu_i i$ *for both* $i = 1, 2$

Proof: At first we prove that the polynomials satisfying the reality condition need to vanish along the A-cycles. In order to do so, we look at how the involutions σ, ρ act on our differential.

$$\begin{aligned} \sigma(b(\lambda)) &= b(\lambda), \sigma(\nu) = -\nu \\ \rho(b(\lambda)) &= b(\overline{\lambda}^{-1}) = \overline{\lambda}^{-3}\overline{b(\lambda)}, \rho(\nu) = \overline{\lambda}^{-3}\overline{\nu} \end{aligned}$$

$$\sigma^* \frac{b(\lambda)}{\nu} \mathrm{d}\ln\lambda = -\frac{b(\lambda)}{\nu}\mathrm{d}\ln\lambda$$

$$\rho^* \frac{b(\lambda)}{\nu} \mathrm{d}\ln\lambda = -\frac{\overline{\lambda}^{-3}\overline{b(\lambda)}}{\overline{\lambda}^{-3}\overline{\nu}}\mathrm{d}\ln\overline{\lambda} = -\frac{\overline{b(\lambda)}}{\overline{\nu}}\mathrm{d}\ln\overline{\lambda}$$

Now since we also know that $\sigma^* A = -A, \rho^* A = -A$, the following calculation for the periods of Θ_b holds true

$$\int_{A_i} \Theta_b = \int_{\rho^* A_i} \rho^* \Theta = \int_{-A_i} -\overline{\Theta} = -\int_{A_i} -\overline{\Theta} = \int_{A_i} \overline{\Theta}, \quad i = 1, 2$$

which in turn means that the A-periods are real. However since $b \in \mathcal{B}_a$ only has purely imaginary periods by definition, the periods need to vanish.
Now we assume that there exist two different polynomial $b, \tilde{b}$ such that $\int_{B_i} \Theta_b = \int_{B_i} \Theta_{\tilde{b}}$. Then the difference of these two differentials is a differential form whose periods all vanish. That means that our Differential form is exact and we can integrate it, because the integral only depends on the endpoints of each path γ and not on the path itself. So then it should exist a meromorphic function with simple poles at $\lambda = 0, \infty$ defining the differential. That function then is a function of degree two. Now since our surface is hyperelliptic we know that there exists up to Möbius transformations only one such meromorphic function of degree 2, namely the function λ. However the poles of λ do not match those of this meromorphic function even with Möbius transformation. So there can't exist such a meromorphic function meaning there can't be two polynomials in $\mathcal{B}_a$ with the same periods. Therefore, we have now proven the uniqueness of such a polynomial but it remains to be shown that such polynomials exist. Now we can consider the identity (ii) from above as a linear mapping $\varphi : \mathcal{B}_a \to \mathbb{R}^2$ that identifies each polynomial $a \in \mathcal{B}_a$ with its B-periods. Now we know that $\mathcal{B}_a$ has real dimension two and the arguments before show that $\ker \varphi = \{0\}$ so the mapping is an isomorphism which completes the proof. **q.e.d.**

Remark 4.10. *Note that the period for $\tilde{A}$ only vanishes if the periods for B_2 vanishes as well since $\tilde{A}$ is homologous to $B_2 + A_1$.*

Definition 4.11. *For a fixed $t \in \mathbb{R}_+$ we define the spectral curve $\Sigma_t = \{(\lambda, \nu_t) \in \mathbb{C}^2 \mid \nu_t^2 = \lambda \cdot a_t(\lambda)\}$ where a_t is a polynomial $a \in \mathcal{M}_2$ whose image in $\mathcal{A}$ is (t, a) with an arbitrary $a \in \mathbb{C}$.*

The following ideas are transferred from the unpublished paper by Carberry et al. (2020) for our case. For a rigorous proof of the original thoughts see the seminar thesis Spektralgeschlecht der sinh-Gordon Gleichung: Der Rand von $\mathcal{S}_1^2$ unter Blow-Up by Hasse (2020). Here we want to consider the limit case $t \to 0$ for which we need to define a singular curve that arises as the limit of Σ_t. We will call the limit curve Σ_0 whose normalization has three connected components, namely $\Sigma^+, \Sigma^0, \Sigma^-$ who are all hyperelliptic. Therefore, we can consider Σ_0 to be a two-sheeted covering over three copies of the projective space $\mathbb{C}\mathbf{P}^1$ which we will call $\mathbb{C}\mathbf{P}^1_{\lambda^+}, \mathbb{C}\mathbf{P}^1_{\lambda}$ and $\mathbb{C}\mathbf{P}^1_{\lambda^-}$. Here each subscript denotes the parameter used in the copy. We consider the following equations

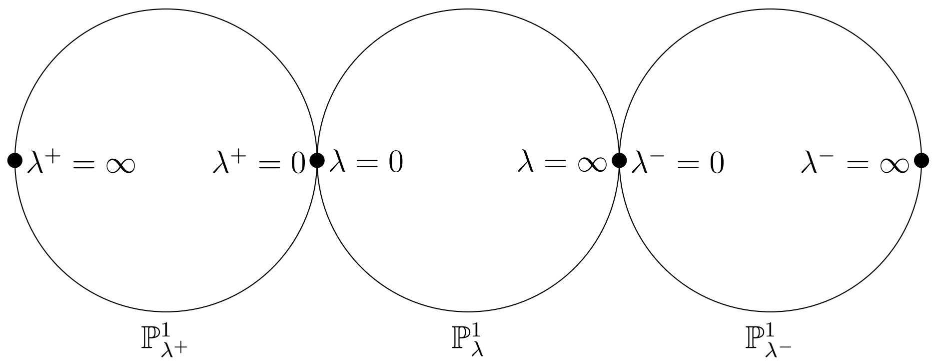
$$\lambda \cdot \lambda_t^+ = t \quad \text{and} \quad \lambda_t^- \cdot \lambda^{-1} = t$$

of the parameters. In the limit $t \to 0$ we use these equations to describe the double point $(\lambda, \lambda^+) = (0, 0)$ at which $\mathbb{C}\mathbf{P}^1_{\lambda^+}$ and $\mathbb{C}\mathbf{P}^1_{\lambda}$ are joined and the second double point $(\lambda, \lambda^-) = (\infty, 0)$ which joins $\mathbb{C}\mathbf{P}^1_{\lambda}$ and $\mathbb{C}\mathbf{P}^1_{\lambda}$. Now the equations to do so are

$$\lambda \cdot \lambda^+ = 0 \quad \text{and} \quad \lambda^- \cdot \lambda^{-1} = 0.$$

We will use a figure to illustrate the procedure we just described.

Figure 4.3: Model for our construction from Carberry et al. (2020)



(a) Three copies of $\mathbb{C}\mathbf{P}^1$ joint by double points as described above

We now examine the connected components of Σ_0. We define them to be the hyperelliptic curves that are the one point compactifications of

$$\{(\lambda^+, \nu^+) \in \mathbb{C}^2 \mid (\nu^+)^2 = \lambda^+((\lambda^+)^2 + \mathfrak{a}\lambda^+ + 1)\} \qquad \text{at } \lambda^+ = \infty$$
$$\{(\lambda, \nu^0) \in \mathbb{C}^2 \mid (\nu^0)^2 = \lambda^3\} \qquad \text{at } \lambda = \infty$$
$$\{(\lambda^-, \nu^-) \in \mathbb{C}^2 \mid (\nu^-)^2 = \lambda^-((\lambda^-)^2 + \bar{\mathfrak{a}}\lambda^- + 1)\} \qquad \text{at } \lambda^- = \infty.$$

These definitions will be motivated by Lemma 4.15 where we show that they match with certain limits of $a_t(\lambda)$ for $t \to 0$.

Definition 4.12. *The curve Σ_0 is defined by identifying $(\lambda^+, \nu^+) = (0,0) \in \Sigma_0^+$ with $(\lambda, \nu^0) = (0,0) \in \Sigma_0^0$, forming an ordinary double point and also identifying $(\lambda, \nu^0) = (\infty, \infty) \in \Sigma_0^0$ with $(\lambda^-, \nu^-) = (0,0) \in \Sigma_0^-$ which again forms an ordinary double point.*

To describe the limit process of Σ_t we will define some subsets of Σ_t, Σ_0

Definition 4.13. *Let $K \subset \mathbb{C}^\times$ be a compact set. Then we define*

$$\begin{aligned} \Sigma_{t,K}^+ &= \{(\lambda,\nu) \in \Sigma_t \mid \lambda_t^+ = t/\lambda \in K\} & \Sigma_{0,K}^+ &= \{(\lambda^+,\nu^+) \in \Sigma_0^+ \mid \lambda^+ \in K\} \\ \Sigma_{t,K}^0 &= \{(\lambda,\nu) \in \Sigma_t \mid \lambda \in K\} & \Sigma_{0,K}^0 &= \{(\lambda,\nu^0) \in \Sigma_0^0 \mid \lambda \in K\} \\ \Sigma_{t,K}^- &= \{(\lambda,\nu) \in \Sigma_t \mid \lambda_t^- = t\lambda \in K\} & \Sigma_{0,K}^- &= \{(\lambda^-,\nu^-) \in \Sigma_0^- \mid \lambda^- \in K\} \end{aligned}$$

Definition 4.14. *For any compact set K there then exists some $\varepsilon > 0$ such that for all $t \in (-\varepsilon, \varepsilon)\backslash\{0\}$ the hyperelliptic Riemann surfaces $\Sigma_{t,K}^+, \Sigma_{t,K}^0, \Sigma_{t,K}^-$ have the same branch points as the connected components of the limit $\Sigma_{0,K}^+, \Sigma_{0,K}^0$ and $\Sigma_{0,K}^-$. By mapping the branch points to each other we construct biholomorphic maps*

$$\Phi_t^+ : \Sigma_{0,K}^+ \to \Sigma_{t,K}^+,\ \Phi_t^0 : \Sigma_{0,K}^0 \to \Sigma_{t,K}^0,\ \Phi_t^- : \Sigma_{0,K}^- \to \Sigma_{t,K}^-$$

such that

$$(\Phi_t^+)^*(\lambda_t^+) = \lambda^+,\ (\Phi_t^0)^*(\lambda_t^0) = \lambda^0,\ (\Phi_t^-)^*(\lambda_t^-) = \lambda^-.$$

We will now prove our version of Lemma 2.9 from Carberry et al. (2020) (or alternatively Lemma 30 of Hasse (2020)).

Lemma 4.15. *Let $K \subset \mathbb{C}^\times \subset \mathbb{CP}^1$ be compact. Then the following limits for $t \to 0$ are uniform:*

$$\begin{aligned} a(t/\lambda^+) &\to (\lambda^+)^{-2} \cdot a^+(\lambda^+) && \text{for } \lambda^+ \in K \\ t^2 \cdot a(\lambda) &\to \lambda^2 && \text{for } \lambda \in K \\ t^4 \cdot a(\lambda^-/t) &\to (\lambda^-)^2 \cdot a^-(\lambda^-) && \text{for } \lambda^- \in K. \end{aligned}$$

Proof: We fix $\lambda, \lambda^\pm \in K$ and because of compactness every limit is then uniform. Now we will use Remark 4.6 to describe the limits. It follows then that

$$\begin{aligned}
a(t/\lambda^+) &= \Big(t^{-2}t^2(\lambda^+)^{-2} + \mathfrak{a}t^{-1}t(\lambda^+)^{-1} + 1\Big) \cdot \Big(t^4(\lambda^+)^{-2} + \bar{\mathfrak{a}}t^2(\lambda^+)^{-1} + 1\Big) \\
&= \Big((\lambda^+)^{-2} + \mathfrak{a}(\lambda^+)^{-1} + 1\Big) \cdot \Big(t^4(\lambda^+)^{-2} + \bar{\mathfrak{a}}t^2(\lambda^+)^{-1} + 1\Big) \\
&\to \Big((\lambda^+)^{-2} + \mathfrak{a}(\lambda^+)^{-1} + 1\Big) \cdot 1 \\
&= (\lambda^+)^{-2}a^+(\lambda^+). \\
t^2 \cdot a(\lambda) &= \Big(t^2t^{-2}\lambda^2 + \mathfrak{a}t^2t^{-1}\lambda + t^2\Big) \cdot \Big(t^2\lambda^2 + \bar{\mathfrak{a}}t\lambda + 1\Big) \\
&= \Big(\lambda^2 + \mathfrak{a}t\lambda + t^2\Big) \cdot \Big(t^2\lambda^2 + \bar{\mathfrak{a}}t\lambda + 1\Big) \\
&\to \lambda^2 \cdot 1 = \lambda^2. \\
t^4 \cdot a(\lambda^-/t) &= t^4 \cdot \Big(t^{-4}(\lambda^-)^2 + \mathfrak{a}t^{-2}\lambda^- + 1\Big) \cdot \Big(t^2t^{-2}(\lambda^-)^2 + \bar{\mathfrak{a}}tt^{-1}\lambda^- + 1\Big) \\
&= \Big((\lambda^-)^2 + \mathfrak{a}t^2\lambda^- + t^4\Big) \cdot \Big((\lambda^-)^2 + \bar{\mathfrak{a}}\lambda^- + 1\Big) \\
&\to (\lambda^-)^2 \cdot a^-(\lambda^-).
\end{aligned}$$

q.e.d.

We want to establish a similar result for the polynomials b_t.

Lemma 4.16. *Let $K \subset \mathbb{C}^\times \subset \mathbb{CP}^1$ be compact. Then the following limits for $t \to 0$ are uniform*

$$\begin{aligned}
b_t(t/\lambda^+) &\to (\lambda^+)^{-1}b^+(\lambda^+), && \text{for } \lambda^+ \in K \\
t \cdot b_t(\lambda) &\to \lambda \cdot b^0(\lambda) && \text{for } \lambda \in K \\
t^3 \cdot b_t(\lambda^-/t) &\to (\lambda^-)^2 \cdot b^-(\lambda^-), && \text{for } \lambda^- \in K.
\end{aligned}$$

Proof: As in the Lemma before, we fix $\lambda, \lambda^\pm \in K$ and since K is compact the limits are uniform. Using our new parameters and Equation (3) we get

$$\begin{aligned}
b(t/\lambda^+) &= (\lambda^+)^{-1} \cdot b_t^+(\lambda^+) \cdot b_t^0(t/\lambda^+) \cdot b_t^-(t/(\lambda^+)^2) \\
&\to (\lambda^+)^{-1} \cdot b^+(\lambda^+) \cdot b^+(0) \cdot b^-(0). \\
t \cdot b_t(\lambda) &= \lambda \cdot b_t^+(t/\lambda) \cdot b_t^0(\lambda) \cdot b_t^-(t\lambda) \\
&\to \lambda \cdot b^+(0) \cdot b^0(\lambda) \cdot b^-(0).
\end{aligned}$$

The same procedure yields the result for the last equation as well. **q.e.d.**

We have described how the limit curve arises. Now we want to define an analogon of $\mathcal{B}_a$ on Σ_0^+ for a^+. The connected component Σ_0^+ has genus $g = 1$ and therefore, any homology basis on this surface consists of two cycles. In our discussion before we already defined two cycles of our homology basis in a way that they are well defined in the limit on Σ_0^+ as well.

Definition 4.17. *For any basis* $(b_1^\pm, b_2^\pm)$ *of our two-dimensional vector spaces* $\mathcal{B}_{a^\pm}$ *we define the rational functions*

$$f^\pm(\lambda^\pm) = \frac{b_1^\pm(\lambda^\pm)}{b_2^\pm(\lambda^\pm)}.$$

Each function is unique up to Möbius transformations.

Definition 4.18. *For any polynomial* $b^\pm \in \mathbb{C}^1[\lambda^\pm]$ *or* $b^0(\lambda) \in \mathbb{C}^1[\lambda]$ *we define the meromorphic differentials*

$$\Theta^\pm_{b^\pm} = \frac{b^\pm(\lambda^\pm)}{\nu^\pm}\mathrm{d}\lambda^\pm \text{ and } \Theta^0_{b^0} = \frac{b^0(\lambda)}{\nu^0}\frac{\mathrm{d}\lambda}{\lambda}$$

on $\Sigma_0^\pm$ *and* Σ_0^0 *respectively.*

Proposition 4.19. *The above defined differential* $\Theta^+_{b^+}$ *is a holomorphic differential at any value not* $\lambda = \infty$ *and has a second order pole at* $\lambda = \infty$

Proof: Since $\lambda^+ \cdot a^+(\lambda^+)$ is zero for $\lambda^+ = 0$, so λ^+ is no local chart in any neighbourhood of zero. We can evaluate our differential in the chart $z^2 = \lambda^+$. The identity $\mathrm{d}\lambda^+ = 2z\mathrm{d}z$ holds then. Now we will look at Θ^+ in the z-coordinates.

$$\begin{aligned}\frac{b(\lambda^+)}{\sqrt{\lambda^+ \cdot a^+(\lambda^+)}}\,\mathrm{d}\lambda^+ &= \frac{b^+(z^2)}{\sqrt{z^2 a^+(z^2)}}\,2z\mathrm{d}z \\ &= \frac{b^+(z^2)}{\sqrt{a^+(z^2)}}\,\mathrm{d}z\end{aligned}$$

$b^+(\lambda^+)$ is a polynomial of degree one so $b^+(z^2)$ is a polynomial in z of degree two. $a^+(\lambda^+)$ is a polynomial of degree two so $\sqrt{a^+(z^2)}$ is a polynomial of degree two as well. Therefore, we see that Θ^+ is the quotient of two polynomials of degree two in any neighbourhood of $\lambda^+ = 0$. So that means that it takes value in $\mathbb{C}^\times$ at $\lambda^+ = 0$ by the rule of L'Hospital which in turn means that $\Theta^+_{b^+}$ is holomorphic at $\lambda^+ = 0$. Now we will look at $\lambda^+ = \infty$. In this case we will use the chart $\lambda^+ = z^{-2}$ which yields the formula $\mathrm{d}\lambda^+ = -2z^{-3}\mathrm{d}z$. Plugging that into $\Theta^+_{b^+}$ then gives us

$$\begin{aligned}\frac{b^+(\lambda^+)}{\sqrt{\lambda^+ \cdot a^+(\lambda^+)}}\mathrm{d}\lambda^+ &= \frac{b^+(z^{-2})}{\sqrt{z^{-2} \cdot a^+(z^{-2})}}(-2z^{-3})\mathrm{d}z \\ &= \frac{-2z^{-2}b^+(z^{-2})}{\sqrt{a^+(z^{-2})}}\mathrm{d}z\end{aligned}$$

Now we again see that b^+ is a polynomial of degree one and $b^+(z^{-2})$ has therefore, degree -2. Evaluating $\sqrt{a^+(z^{-2})}$ gets us a polynomial of degree -2 as well so these two cancel out and the remaining term is z^{-2}. So $\lambda^+ = \infty$ is the same as $z = 0$ so we see that $\Theta^+_{b^+}$ has a second order pole at $\lambda^+ = \infty$. Since the original $\Theta^+_{b^+_t}$ is holomorphic everywhere except at $\lambda^+_t = 0, \infty$ we see that $\Theta^+_{b^+}$ is holomorphic everywhere else too. So then the claim holds and our proof is finished. **q.e.d.**

Definition 4.20. *Consider a^+ as the limit of $a(\lambda) \in \mathcal{M}_2/\sim$ as in Lemma 4.15 and Σ^+_0 as defined before. Then $\mathcal{B}_{a^+} \subset \mathbb{C}^1[\lambda^+]$ denotes the two dimensional vector space of polynomials $b^+(\lambda^+)$ such that Θ_{b^+} has purely imaginary periods. We consider the cycles $\tilde{A}, B_1$ defined in Figure 4.1 as the homology basis of Σ^+_0.*

Now we want to prove an analogon of Lemma 4.9 in the limit case.

Lemma 4.21. *Let $a^+(\lambda^+)$ be the limit of a polynomial $a \in \mathcal{M}^1_2 \cup \mathcal{M}^2_2 \cup \mathcal{M}^3_2$ as in Lemma 3.14. Then every pair of numbers $(\mu_1, \mu_2) \in \mathbb{R}^2$ determines a unique element $b^+(\lambda^+) \in \mathcal{B}_{a^+}$ such that*

(i) $\displaystyle\int_{\tilde{A}} \Theta_{b^+} = \mu_1 i$

(ii) $\displaystyle\int_{B_1} \Theta_{b^+} = \mu_2 i$

Proof: Again we will at first consider uniqueness and then see from the dimension theorem that the existence of such a polynomial follows. Consider two polynomials b and $\tilde{b} \in \mathcal{B}_{a^+}$ such that they have the same periods. Then, $\hat{\Theta} = \Theta_b - \Theta_{\tilde{b}}$ has vanishing periods, which means that every closed integral vanishes. Therefore, $\hat{\Theta}$ is an exact differential form and we can integrate it. That means there needs to exist a meromorphic function f on Σ^+_0 with exactly one first order pole at $\lambda = 0$. Therefore, $\deg(f) = 1$. However since any function of degree one takes every value exactly once and is therefore, a bijective map from $\Sigma^+_0 \to \mathbb{CP}^1$. We also know that f is holomorphic everywhere except at $\lambda = \infty$. Now by mapping $\lambda = \infty$ to $\infty \in \mathbb{CP}^1$ we get that $f\colon \Sigma^+_0 \to \mathbb{CP}^1$ is a biholomorphic map. However since $g(\Sigma^+_0) = 1$ and $g(\mathbb{CP}^1) = 0$ such a map can't exist which yields a contradiction.
Now again we know uniqueness holds and need to prove existence of such polynomials. Consider the linear mapping $\varphi : \mathcal{B}_{a^+} \to \mathbb{R}^2$ that identifies each polynomial $b \in \mathcal{B}_{a^+}$ with its periods along the cycles $\tilde{A}$ and B_1. By the previous considerations, $\ker\varphi = \{0\}$ holds, and because $\dim(\mathcal{B}_{a^+}) = 2$, φ becomes an isomorphism which means there it should exist one polynomial b for each pair of periods in $i\mathbb{R} \times i\mathbb{R}$. **q.e.d.**

Definition 4.22. *Now we define the basis of the real two-dimensional vector space $\mathcal{B}_{a^+}$ as the unique polynomials $b_1^+, b_2^+ \in \mathcal{B}_{a^+}$ such that*

$$\int_{\tilde{A}} \Theta_{b_1^+} = 2\pi i, \int_{B_1} \Theta_{b_1^+} = 0,$$
$$\int_{\tilde{A}} \Theta_{b_2^+} = 0, \int_{B_1} \Theta_{b_2^+} = 2\pi i$$

hold.

Now we know that the periods of the Θ_b determine the unique elements of our spaces $\mathcal{B}_a$ and $\mathcal{B}_{a^\pm}$. The next claim that we want to show is that the periods determine the limits as well as that it is possible extend our lattice in this limit continuously. In order to do so we will prove the following two lemmata where the first one is just our analogon of Lemma 2.8 from Carberry et al. (2020) or alternatively Lemma 12 from Hasse (2020)

Lemma 4.23. *Let $\Theta_{b_t} = \frac{b_t(\lambda)}{\nu_t}\frac{\mathrm{d}\lambda}{\lambda}$ be a meromorphic differential defined on Σ_t with $\beta_t = b_t(0)$. For fixed t the limits for $\lambda \to \infty$*

$$\Theta_{b_t} = \Big(\overline{\beta_t}\lambda^{-1/2} + O(\lambda^{-3/2})\Big)\mathrm{d}\lambda = \Big(t^{-1/2}\overline{\beta_t}(\lambda_t^-)^{-1/2} + O((\lambda_t^-)^{-3/2})\Big)\mathrm{d}\lambda_t^-$$

and $\lambda \to 0$

$$\Theta_{b_t} = \Big(\beta_t\lambda^{-1/2} + O(\lambda^{1/2})\Big)\mathrm{d}\lambda = -\Big(t^{1/2}\beta_t(\lambda_t^+)^{-1/2} + O((\lambda_t^+)^{-3/2}\Big)\mathrm{d}\lambda_t^+$$

hold.

Proof: Considering the case $\lambda \to \infty$ any polynomial is dominated by the highest coefficients which means that our polynomials a_t and b_t become

$$b_t(\lambda) = \overline{\beta_t}\lambda^3 + O(\lambda^2), \quad a_t(\lambda) = \lambda^4 + O(\lambda^3).$$

Now we need to see how ν_t acts in the limit. In order to do so, we first exclude the highest power of λ and put it outside of the brackets and then examine the Laurent series of $\sqrt{1+\lambda^{-1}}$ in the limit $\lambda \to \infty$

$$\sqrt{1+\lambda^{-1}} = 1 + \frac{1}{2}\lambda^{-1} - \frac{1}{8}\lambda^{-2} + \frac{1}{16}\lambda^{-3} + O(\lambda^{-4}).$$

This series is converging for all $|\lambda| > 1$. We use this to calculate

$$\begin{aligned}\nu_t(\lambda) &= \sqrt{\lambda \cdot a_t(\lambda)} = \sqrt{\lambda^5 + O(\lambda^4)} \\ &= \sqrt{\lambda^5(1 + O(\lambda^{-1}))} \\ &= \lambda^{5/2}(1 + O(\lambda^{-1}) = \lambda^{5/2} + O(\lambda^{3/2}).\end{aligned}$$

Using this we get the result

$$\begin{aligned}\Theta_{b_t} &= \Big(\frac{\overline{\beta_t}\lambda^3 + O(\lambda^2)}{\lambda^{5/2} + O(\lambda^{3/2})}\Big)\mathrm{d}\lambda \\ &= \Big(\frac{\overline{\beta_t}\lambda^3}{\lambda^{5/2} + O(\lambda^{3/2})} + \frac{O(\lambda^2)}{\lambda^{5/2} + O(\lambda^{3/2})}\Big)\mathrm{d}\lambda \\ &= \Big(\overline{\beta_t}\lambda^{-1/2} + O(\lambda^{-3/2})\Big)\mathrm{d}\lambda.\end{aligned}$$

Now for the second equality in this limit we calculate $\mathrm{d}\lambda_t^- = t\mathrm{d}\lambda$ and therefore, $\mathrm{d}\lambda = t^{-1}\mathrm{d}\lambda_t^-$. Plugging this into our former result we see that

$$\lambda^{-1/2}\mathrm{d}\lambda = \lambda^{-1/2}t^{-1}\mathrm{d}\lambda_t^- = t^{-1/2}(\lambda_t^-)^{-1/2}\mathrm{d}\lambda_t^-$$
$$\text{as well as } \lambda^{-3/2}\mathrm{d}\lambda = t^{1/2}(t\lambda)^{-3/2}\mathrm{d}\lambda_t^- = O((\lambda_t^-)^{-3/2})\mathrm{d}\lambda_t^-.$$

This proves the first limit case. Considering the second case $\lambda \to 0$ the lowest coefficients are now the ones that dominate the polynomials. Therefore,

$$b_t(\lambda) = \beta_t + O(\lambda), \quad a_t(\lambda) = 1 + O(\lambda)$$

hold. Again we now need to calculate ν_t in the limit. Now we will exclude the lowest power of λ and calculate the Taylor series of $\sqrt{1+\lambda}$. Then it holds that

$$\sqrt{1+\lambda} = 1 + \frac{1}{2}\lambda - \frac{1}{8}\lambda^2 + \frac{1}{16}\lambda^3 + O(\lambda^4).$$

Then ν_t becomes

$$\begin{aligned}\nu_t(\lambda) &= \sqrt{\lambda(1 + O(\lambda)} = \sqrt{\lambda}\cdot\sqrt{1 + O(\lambda)} \\ &= \lambda^{1/2}\cdot(1 + O(\lambda)) = \lambda^{1/2} + O(\lambda^{3/2}).\end{aligned}$$

Calculating the limit of Θ_{b_t} then gets us

$$\begin{aligned}\Theta_{b_t} &= \Big(\frac{b_t}{\nu_t}\Big)\frac{\mathrm{d}\lambda}{\lambda} = \Big(\frac{\beta_t + O(\lambda)}{\lambda^{1/2} + O(\lambda^{3/2})}\Big)\frac{\mathrm{d}\lambda}{\lambda} \\ &= \Big(\frac{\beta_t}{\lambda^{3/2} + O(\lambda^{5/2})} + \frac{O(\lambda)}{\lambda^{3/2} + O(\lambda^{5/2})}\Big)\mathrm{d}\lambda \\ &= \Big(\beta_t\lambda^{-3/2} + O(\lambda^{-1/2})\Big)\mathrm{d}\lambda.\end{aligned}$$

Now we consider the second equation. We calculate the change of coordinates and see that $\mathrm{d}\lambda_t^+ = \mathrm{d}t/\lambda = -t/\lambda^2\mathrm{d}\lambda$. Plugging that in our result yields the final equality.

$$\lambda^{-3/2}\mathrm{d}\lambda = \lambda^{-3/2}\cdot(-\lambda^2/t)\mathrm{d}\lambda_t^+ = -t^{-1/2}(\lambda_t^+)^{-1/2}\mathrm{d}\lambda_t^+$$
$$\lambda^{-1/2}\mathrm{d}\lambda = \lambda^{-1/2}\cdot(-\lambda^2/t)\mathrm{d}\lambda_t^+ = -t^{1/2}(\lambda_t^+)^{-3/2}\mathrm{d}\lambda_t^+ = -O((\lambda_t^+)^{-3/2})\mathrm{d}\lambda_t^+$$

q.e.d.

Lemma 4.24. *Suppose that $f^-(\lambda^- = 0) \in \mathbb{C} \setminus \mathbb{R}$ holds. Let $K \subset \mathbb{C}^\times \subset \mathbb{CP}^1$ be compact. Then for sufficiently small t, the polynomials $b_t^\pm, b_t^0$ have degree 1 and as $t \to 0$ they converge uniformly to polynomials $b^\pm, b^0$ such that the parts of Θ_{b_t} converge to differentials with the same periods.*

Proof: Lemma 3.4 of the paper by Knopf et al. (2018) proves that the coefficients of b_t depend continuously on $t \in (0, \varepsilon)$. Therefore, the same holds true for the polynomials $b_t^\pm(\lambda_t^\pm)$ and $b_t^0(\lambda)$ if the degree of these polynomials doesn't change. It is clear by Lemma 4.8 that there are only two possible cases: either all three polynomials have degree one or b^0 has degree three and the other two polynomials are of degree zero. First, we will prove continuity of b_t^- at $t = 0$. The claim follows then for b_t^+ as well since the coefficients are just the complex conjugates. Now consider a convergent sequence $(t_n, \alpha_n)_{n\in\mathbb{N}} \subset \mathbb{R}_{>0} \times \mathbb{C}$ such that $\lim_{n\to\infty} t_n = 0$ holds. Then we define $a_n = a_{(t_n,\alpha_n)}$ and $b_n^\pm = b^\pm_{(t_n,\alpha_n)}, b_n^0 = b^0_{(t_n,\alpha_n)}$ as well as $\Theta_n^- = \Theta^-_{(t_n,\alpha_n)}$. By Lemma 4.8 (i) $b_n^0(0) \in \mathbb{S}^1$ holds for all $n \in \mathbb{N}$. So we can rename $(t_n)_{n\in\mathbb{N}}$ to a subsequence such that $b_n^0(0)$ converges to a $\delta \in \mathbb{S}^1$.
Since for $n \to \infty$ $\lambda^+_{t_n} = t_n^2/\lambda^-_{t_n}$ and $\lambda^{-1} = t_n/\lambda^-_{t_n}$ hold and the first converges uniformly to zero on K we use Lemma 4.15 to calculate the following limit

$$
\begin{aligned}
\lim_{n\to\infty} t_n^{5/2} \cdot (\Phi_t^-)^* \nu_{t_n} &= \lim_{n\to\infty} t_n^{5/2} \cdot (\Phi_t^-)^* \sqrt{\lambda \cdot a_t(\lambda)} \\
&= \lim_{n\to\infty} (\Phi_t^-)^* \sqrt{t_n^4 \cdot \lambda^-_{t_n} \cdot a_{t_n}(\lambda^-_{t_n}/t_n)} \\
&= \lim_{n\to\infty} \sqrt{\lambda^- \cdot t_n^4 \cdot a_{t_n}(\lambda^-/t)} = \sqrt{\lambda^- \cdot (\lambda^-)^2 \cdot a^-(\lambda^-)} \\
&= \lambda^- \cdot \nu^-
\end{aligned}
\tag{4}
$$

which holds on $\Sigma_{t_n,K}$ and is uniform there.
Now we want to prove that $\deg(b_n^-) = 1$ by contradiction. So we assume that $\deg(b_n^-) = 0$ holds. Then it follows that $b_n^\pm$ are constant polynomials so $|b_n^\pm| \in \mathbb{R}$ holds. Equation (3) then becomes

$$b_n(\lambda) = b_n^+ b_n^0(\lambda) b_n^- = |b_n^-|^2 b_n^0(\lambda) = |b_n^-|^2 b^0(\lambda^-/t_n).$$

The reality condition forces that the highest coefficient of b^0 converges to $\overline{\delta}$. Now by construction we know that the highest coefficient as well as all the roots of b_n^0 are unimodular and therefore, all coefficients of the polynomial of degree three b_n^0 are bounded. So then by the equation above the sequence $(t_n/\lambda^-)^3 \cdot b_n^0(\lambda^-_{t_n})$ converges uniformly on K to $\overline{\delta}$. Comparing the asymptotics for $\lambda \to \infty$ in Lemma 4.21 and for $\lambda^- \to \infty$ from above (equation (4)) we get

$$\Big(\overline{\beta}\frac{1}{\sqrt{\lambda^-}} + O((\lambda^-)^{-3/2})\Big)\mathrm{d}\lambda = \lim_{n\to\infty} (t_n)^{1/2} (\Phi^-_{t_n})^* \Theta_{b_{t_n}} = \lim_{n\to\infty} \frac{|b_n^-|^2 \cdot \overline{\delta} \cdot (\lambda^-)^3}{\lambda^- \cdot \nu^-} \frac{\mathrm{d}\lambda^-}{\lambda^-}.$$

Now the proof of Lemma 4.21 and the fact that a^- has degree two show that for $\lambda \to \infty$ we obtain that $\nu^- = (\lambda^-)^{3/2} + O((\lambda^-)^{1/2})$ the right hand side of the above equation becomes

$$\lim_{n\to\infty} \frac{|b_n^-|^2 \cdot \overline{\delta} \cdot (\lambda^-)^3}{(\lambda^-)^{5/2} + O((\lambda^-)^{3/2})} \frac{\mathrm{d}\lambda^-}{\lambda^-} = \lim_{n\to\infty} \left(|b_n^-|^2 \cdot \overline{\delta} \cdot \frac{1}{\sqrt{\lambda^-}} \right) \mathrm{d}\lambda^-.$$

Then it follows by comparing asymptotics that $\lim_{n\to\infty} |b_n^-|^2 \neq 0$. Also because $\delta \in \mathbb{S}^1$ we see that $\delta = \beta/|\beta|$ needs to hold. Again in all these equations we denote by $\beta = b_t(0)$. Then it follows that $\lim_{n\to\infty} |b_n^-|^2 = |\beta|$ holds true as well. Now if we look at Lemma 4.8 and our choice of $\varphi = 0$ it follows that $b_n^- > 0$ holds which implies that

$$\lim_{n\to\infty} b_n^- = \sqrt{|\beta|}.$$

Now the limit of $t_n^{1/2}(\Phi_{t_n}^-)^*\Theta_{b_{t_n}}$ as seen above is uniform on $\Sigma_{0,K}^-$ and defines a meromorphic differential of the second kind of Σ^-. Now by our considerations of $|b_n^-|^2$ as well as δ we see that the limit is of the form

$$\lim_{n\to\infty} (\Phi_{t_n}^-)^*\Theta_{b_{t_n}} = \overline{\beta} \cdot \frac{\lambda^-}{\nu^-} \mathrm{d}\lambda^-.$$

Now in the same way as in the proof of Proposition 4.18 we use at $\lambda^- = 0$ the chart $\lambda^- = z^2$ and get that the differential in our limit has the form

$$\overline{\beta} \cdot \frac{z^2}{z^2 + O(z^3)} \cdot 2z\mathrm{d}z = \overline{\beta} z \mathrm{d}z$$

which directly implies that our differential has a zero at $\lambda^- = 0$. Now we choose K in a way that all cycles of $\Sigma_{t_n,K}^-$ lie in $\Sigma_{t_n}^-$ for sufficiently large n and therefore, all periods of the differential are purely imaginary. Then the limit of our polynomial b_n^- needs to vanish at $\lambda^- = 0$ so it has no constant term. That means $b^-(\lambda^- = 0) = 0$ which automatically implies $f^-(\lambda^- = 0) \in \mathbb{R}$ because it vanishes or is ∞ for every second polynomial b_2^- as well. So we reach a contradiction and have proven $\deg(b_n^\pm) = \deg(b_n^0) = 1$ for sufficiently large n.

It remains to show that the polynomials $b_n^\pm$ and b_n^0 converge to polynomials $b^\pm$ and b^0 defining $\Theta_{b^\pm}$ and Θ_{b^0} that have the same periods as the differentials defined by $b_t^\pm$ and b_t^0.

At first we prove that every sequence $(t_n, \alpha_n)_{n\in\mathbb{N}}$ has a subsequence $(t_{n_k}, \alpha_{n_k})_{k\in\mathbb{N}}$ such that $(b_{n_k})_{k\in\mathbb{N}}$ converges to a polynomial in $P_{\mathbb{R}}^3$ as well. In order to do so, we prove that all the coefficients of b_n are bounded for all $n \in \mathbb{N}$. We prove this by contradiction, so assume at least one sequence of coefficients converges to ∞. Let $(\beta_n)_{n\in\mathbb{N}}$ be the sequence of coefficients of b_n that converges the fastest to ∞. So now we define $(\tilde{b}_n)_{n\in\mathbb{N}}$ as $\frac{1}{|\beta_n|}b_n(\lambda)$ for all $n \in \mathbb{N}$. By definition, at least one of the coefficients converges to a number unequal to zero, namely one of modulus one. So now $(\tilde{b}_n)_{n\in\mathbb{N}}$ is a sequence of polynomials of degree three where every coefficient is bounded. The differentials $\Theta_{\tilde{b}_n}$ are also renormed. Because all the coefficients of $\tilde{b}_n$ are bounded, we can look at a convergent subsequence $(\tilde{b}_{n_k})_{k\in\mathbb{N}}$. Now we need to consider the renormed differentials $\Theta_{\tilde{b}_{n_k}}$ that converge as well as the polynomials $\tilde{b}_{n_k}$ to a differential $\Theta_{\tilde{b}}$. Therefore, the periods converge to renormed periods that are renormed by $\frac{1}{|\beta|}$ and therefore, converge to zero since they were finite before. Now we know that the differential we consider is one that has vanishing periods everywhere. We will now calculate it's order at $\lambda^+ = 0$. Now because we already have proven that $\deg(b_n^{\pm,0}) = 1$ we know that this holds true in the limit as well. Because in the limit the root that was at infinity is now finite, every root that was finite now goes to zero. That means that b has a second order root at $\lambda^+ = 0$ because only one root remains non-zero. We can now use the same calculations for ν. Because $\deg(a^+) = \deg(a^-) = 2$ we see that $\nu = \sqrt{\lambda \cdot a(\lambda)}$ has a root of order 3/2 there. If we again as in Proposition 4.18 consider the local chart $z^2 = \lambda$ we see that b has now a root of order 4 and ν one of order 3 at $z = 0$.From $\frac{\mathrm{d}\lambda}{\lambda}$ we get a root of order 2 in the denominator. Now if we calculate $\mathrm{d}\lambda = \mathrm{d}z^2 = 2z\mathrm{d}z$ we see that we get another root of order one at $z = 0$. So now we see that in the z-coordinate we have a root of order five both in the nominator and the denominator. So therefore, the differential $\Theta_{\tilde{b}}$ is finite and holomorphic at zero. The same calculations yield that it has a pole of order one at $\lambda = \infty$. Considering this we have a holomorphic differential whose periods all vanish. That means that the differential $\Theta_{\tilde{b}}$ vanishes as well, but that is a contradiction with the way we defined our sequence because at least one of the coefficients converges to a complex number with modulus one. So we know that the sequences of coefficients are all bounded and therefore, we can go to a subsequence $(t_{n_k}, \alpha_{n_k})_{k\in\mathbb{N}}$ such that b_{n_k} converges to a polynomial in $P^3_{\mathbb{R}}$. The last claim that remains to be shown is that every subsequence of $(t_n, \alpha_n)_{n\in\mathbb{N}}$ converges to the same polynomial $b \in P^3_{\mathbb{R}}$. There we use the same discussion as before. Now we again have a convergent subsequence $\Theta_{b_{n_k}}$ that converges to a differential Θ_b that is holomorphic at zero. Then it follows that the limit is unique. So then we have proven the last part of the claim. **q.e.d.**

Now we know that we can extend b_t continuously to $t = 0$. Therefore, we now that we can extend the vector space $\mathcal{B}_{a_t}$ continuously to $t = 0$ and therefore, τ_{a_t} can be extended continuously to $t = 0$ as well. Now since Θ_b is as seen before holomorphic at $\lambda^+ = 0$ we will use instead the values at $\lambda^+ = \infty$ because Θ_b still has a simple pole there. So we define the following.

Definition 4.25. *For $t = 0$ we consider the polynomial a^+ as the limit of a_t and in the same way b^+ as the limit of b_t. Then we define*

$$\tau_{a^+} = \lim_{\lambda^+ \to \infty} \frac{b_1^+(\lambda^+)}{b_2^+(\lambda^+)}$$

where (b_1^+, b_2^+) is the basis of $\mathcal{B}_{a^+}$.

It is easy to see that this is just the quotient of the highest coefficient of the two polynomials, which because of the lemma before depend continuously on t and α. We will now consider the differentials $\Theta_{b_1^+}$, $\Theta_{b_2^+}$ defined by our unique limit polynomials $b_1^+, b_2^+ \in \mathcal{B}_{a^+}$. These differentials have simple poles at $\lambda^+ = \infty$ and we will denote by $\alpha, \beta \in \mathbb{C}$ their share in this pole in the following way: Since both have simple poles there exist complex numbers $\alpha, \beta \in \mathbb{C}$ such that

$$\hat{\Theta} = \alpha \cdot \Theta_{b_1^+} - \beta \cdot \Theta_{b_2^+}$$

is holomorphic at $\lambda^+ = \infty$. However at any other value λ^+ this is the difference of two holomorphic differentials. Therefore, $\hat{\Theta}$ is a holomorphic differential on the elliptic curve Σ_0^+. So since every elliptic curve has genus $g = 1$ we know that the space of holomorphic one forms is one dimensional. So we know that $\hat{\Theta}$ is an element of this one dimensional family and we know that for the representantive Θ of this family the following relation holds

$$\hat{\Theta} = \lambda\Theta, \qquad \lambda \in \mathbb{R}.$$

Then we can calculate the periods of $\hat{\Theta}$ because we know those of $\Theta_{b_1^+}, \Theta_{b_2^+}$. We see that

$$\int_{\tilde{A}} \hat{\Theta} = \alpha \cdot 2\pi i - \beta \cdot 0 = 2\pi\alpha i$$
$$\int_{B_1} \hat{\Theta} = \alpha \cdot 0 - \beta \cdot 2\pi i = -2\pi\beta i$$

hold so now we know the periods of $\hat{\Theta}$. They span the lattice $\Gamma = 2\pi\alpha i\mathbb{Z} - 2\pi\beta i\mathbb{Z} = 2\pi\alpha i\mathbb{Z} + 2\pi\beta\mathbb{Z}$. We also know that our differential is holomorphic and we define the map $\varphi\colon \Sigma \to \Gamma$ as follows: We choose a base point $x \in \Sigma$ and assign to every point $y \in \Sigma$ the value of the curve integral from x to y of the differential Θ. In this way we get a map from our elliptic curve into the periods of Σ. We see that φ is an immersion because Θ is a holomorphic differential without roots. So we can construct a map $\Phi\colon \Sigma \to \mathbb{C}/\Gamma$ that is an immersion. Now we want to prove that such a map is injective. So now we consider $x, x' \in \Sigma$ such that

$$\Phi(x) \equiv \Phi(x') \mod \Gamma$$

holds. We know that Θ is a holomorphic differential and therefore, closed. So it is exact on $\mathbb{C}$. So we know that $\Phi(x) - \Phi(x')$ is the integral from x to x' concerning Θ. That is equivalent to $\Phi(x) - \Phi(x') \in \Gamma$ and therefore, our map is injective. In this way we can define the elliptic curve $\Sigma = \mathbb{C}/(2\pi\alpha i\mathbb{Z} + 2\pi\beta i\mathbb{Z})$. We know that this elliptic curve is biholomorphic to an elliptic curve $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ where $\tau \in \mathcal{F}$ holds. In order to transform Γ into $\tilde{\Gamma} = \mathbb{Z} + \tau\mathbb{Z}$ we need to divide the lattice by $2\pi\alpha i$. Then we get that

$$\tau = -\frac{2\pi\beta i}{2\pi\alpha i} = -\frac{\beta}{\alpha}$$

holds. Now we turn our calculations to τ_{a^+} defined in Definition 4.24. As we know that our polynomials are of degree one and therefore, we see by the rule of L'Hospital that it is the quotient of the two leading coefficients which depend continuously on t and α as seen in Lemma 4.23. Then if we consider our polynomials to be $b_1^+(\lambda^+) = \mathfrak{b}_{1,1}\lambda^+ + \mathfrak{b}_{1,2}, b_2^+(\lambda^+) = \mathfrak{b}_{2,1}\lambda^+ + \mathfrak{b}_{2,2}$ we see that τ_{a^+} has the value $\frac{\mathfrak{b}_{1,1}}{\mathfrak{b}_{2,1}}$. On the other hand we now want to calculate the explicit value of τ. In order to do so, we will need to calculate the residue of $\Theta_{b_1^+}, \Theta_{b_2^+}$ at $\lambda^+ = \infty$. Now we know that $a^+(\lambda^+) = (\lambda^+)^2 + a\lambda^+ + 1$. So now we can calculate

$$\Theta_{b_{1,2}^+} = \frac{b_{1,2}^+(\lambda^+)}{\sqrt{\lambda^+ \cdot ((\lambda^+)^2 + a\lambda^+ + 1)}} \mathrm{d}\lambda^+$$

From the proof of Proposition 4.18 we see that in the local chart $\lambda^+ = z^{-2}$ our differentials take the value

$$\begin{aligned} \frac{-2z^{-2}b_{1,2}^+(z^{-2})}{\sqrt{a^+(z^{-2})}}\mathrm{d}z &= \frac{-2\mathfrak{b}_{(1,2),1}z^{-4} - 2\mathfrak{b}_{(1,2),2}z^{-2}}{\sqrt{z^{-4} + az^{-2} + 1}}\mathrm{d}z \\ &= -2\frac{\mathfrak{b}_{(1,2),1}z^{-4}}{\sqrt{z^{-4} + az^{-2} + 1}} - 2\frac{\mathfrak{b}_{(1,2),2}z^{-2}}{\sqrt{z^{-4} + az^{-2} + 1}} \end{aligned}$$

Now if we look at the asymptotics of both summands for $z \to 0$ we get that the first one has a pole of second order and the second one is finite. So we only need to consider the first one. For $z \to 0$ we exclude the lowest power of z in ν^+ in terms of z and then use the power series expansion of $\sqrt{1+x^2}$ at zero

$$\begin{aligned}-2\frac{\mathfrak{b}_{(1,2),1}z^{-4}}{\sqrt{z^{-4}(1+az^2)}}\mathrm{d}z &= -2\frac{\mathfrak{b}_{(1,2),1}z^{-4}}{z^{-2}(1+O(z^2))}\mathrm{d}z\\ &= -2\frac{\mathfrak{b}_{(1,2),1}z^{-4}}{z^{-2}+C}\mathrm{d}z\\ &= -2\mathfrak{b}_{(1,2),1}z^{-2}\mathrm{d}z\end{aligned}$$

which means that we get for α and β that $\alpha = -2\mathfrak{b}_{1,1}$ and $\beta = -2\mathfrak{b}_{2,1}$ which in turn means that

$$\tau = -\frac{\beta}{\alpha} = -\frac{-2\mathfrak{b}_{2,1}}{-2\mathfrak{b}_{1,1}} = -\frac{\mathfrak{b}_{2,1}}{\mathfrak{b}_{1,1}}.$$

Comparing τ and τ^+ we see that they can be transformed into each other with the mapping $\tau \mapsto -\tau^{-1}$. Since we consider lattices, the sign makes no difference and it remains to consider $\tau \mapsto \tau^{-1}$. Now we consider the lattices $\Gamma = \mathbb{Z}+\tau\mathbb{Z}$ and $\tilde{\Gamma} = \mathbb{Z}+\tau^{-1}\mathbb{Z}$. We can transform $\tilde{\Gamma}$ into Γ using multiplication with τ and an interchange of the arguments which can be achieved with a rotation of the lattice. Therefore, the lattices are biholomorphic thus inducing a biholomorphism between the elliptic curves $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ and $\mathbb{C}/(\mathbb{Z}+\tau^{-1}\mathbb{Z})$.

4.2.2 Level sets in the blow-up

The next thing we need to prove is that in the same way as in section 4.1 the one-dimensional manifold $T^{-1}(\tau_a)$ can be expanded to a manifold with boundary where our $a(\lambda)$ with unbounded coefficients can be made to the boundary point. In order to do so we will need a similar approach as in chapter 4.2 of B.Schmidt (2020). But in this chapter we will make the mistake of first taking the limits $t \to 0$ of the polynomials defined before and then taking the s-derivatives which means that we will not be able to interpret these results. In the end of this chapter the right derivatives are taken. It remains to examine them to see if the correct result can be established. Still, the following calculations and considerations are still included as a guideline for solving the right equations. First of all, we will need to consider the mapping $\hat{T} : \mathcal{M}_2^1 \times \mathcal{B}_a \times \mathcal{B}_a$ from Definition 3.19 and translate it into our blow-up case. In order to do so we need to calculate the limit of the map $\hat{T}$ for $t \to 0$. In order to do so we consider the definition of the map which is $\hat{T}(a, b_1, b_2) = \frac{b_1(0)}{b_2(0)}$. So then we get that

$$\lim_{t\to 0}\frac{b_{1,t}(0)}{b_{t,2}(0)} = \frac{b_1^+(0)\cdot b_1^0(0)\cdot b_1^-(0)}{b_2^+(0)\cdot b_2^0(0)\cdot b_2^-(0)}$$

holds. Our goal is to use the implicit function theorem on $(T^+)^{-1}(\tau_{a^+})$ in our scenario. We want to define conditions on our space to make it a three dimensional space. We can see that if we just consider $\dim(\mathcal{M}_2^1 \times P^3_{\mathbb{R}} \times P^3_{\mathbb{R}}) = 3+4+4 = 11$. Then posing the condition that the periods of all polynomials $b \in \mathcal{B}_a$ need to be purely imaginary reduces the dimension of these spaces to 2. Using now Lemma 4.9 we see that by choosing our periods to be fixed there exist unique representatives reducing the dimension to 3. Then if we force the equation $\tau_a = \frac{b_1(0)}{b_2(0)}$ to hold we again reduce the dimension by 2 and get that the space $T^{-1}(\tau_a)$ with fixed periods for $\mathcal{B}_a$ is a one-dimensional set. Now the goal is to do the same calculations with our blown-up space. It is easy to see that even with our new parametrization of $\mathcal{A}$ we get that $\dim(\mathcal{A}) = 3$. Now we consider $P^3_{\mathbb{R}}$ and the parametrization from Lemma 4.8 for $b \in P^3_{\mathbb{R}}$. Condition one forces $b^0(\lambda)$ to have the form $e^{i\psi}\lambda + e^{-i\psi}$ where $\psi \in [0, 2\pi)$ as well as that $b^+(\lambda_t^+) = \mathfrak{b}_2\lambda_t^+ + \mathfrak{b}_1$ with $\mathfrak{b}_2 \in \mathbb{R}, \mathfrak{b}_1 \in \mathbb{C}$. So we see that b^0 is one-dimensional and (b^+, b^-) is three-dimensional. Now if we impose the same conditions on $b_t \in P^3_{\mathbb{R}}$ as in Lemma 4.20 we get four conditions on the polynomials b_t by splitting the real and imaginary parts of the periods. Now forcing the equation from Definition 4.24 to hold we get another two conditions and all together we have ten conditions to reduce a 11-dimensional space to a one-dimensional space again.
Our goal now is to examine the differential of the map $\hat{T}$ at the level sets $T^{-1}(\tau_a)$ in the limit for $t \to 0$. In order to do so we will consider the limits of our decomposed polynomials in the parameters $\lambda_t^{\pm}$ as well as λ. We see that our conditions force the periods of $(\Theta_{b_1^{\pm,0}}, \Theta_{b_2^{\pm,0}})$ to remain constant on $T^{-1}(\tau_a)$. On the other hand if we consider $\mathrm{d}\hat{T}$ on these level sets in the respective limits it holds that the periods of

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \lim_{t\to 0} \Theta_{b_{t,k}} \quad k = 1, 2,$$

in the parameters $t(\lambda_t^+)^{-1}, \lambda$ or $t^{-1}\lambda_t^-$ vanish since they are the derivative of a constant function. So now we have a differential with vanishing periods, meaning said differential is exact so there exist meromorphic functions $\dot{q}_{1,2}$ such that

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \lim_{t\to 0} \Theta_{b_{t,k}} = \mathrm{d}\dot{q}_k^{\pm,0}, \qquad k = 1, 2 \tag{5}$$

holds. In the next step we need to calculate the poles of the left hand side to see what kind of functions $\dot{q}^{\pm,0}$ consist of. In order to do so, we need to calculate

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \lim_{t\to 0} \frac{b}{\nu}\frac{\mathrm{d}\lambda}{\lambda}$$

using the parameters $\lambda_t^+, \lambda_t^-, \lambda$ and calculate what order this function has at $\lambda^\pm, \lambda = 0$. We start with λ_t^+ and transform λ and λ_t^- in terms of λ_t^+: $\lambda = t\cdot(\lambda_t^+)^{-1}$, $\lambda_t^- = t^2\cdot(\lambda_t^+)^{-2}$. We will now calculate limits of the polynomials for $t \to 0$ in the same way as in Lemma 4.15. So we first see

$$\begin{aligned} a(t\cdot(\lambda_t^+)^{-1}) &= (\lambda_t^+)^{-2}((\lambda_t^+)^2 + \mathfrak{a}(\lambda_t^+) + 1)(t^4(\lambda_t^+)^2 + t^2\overline{\mathfrak{a}}(\lambda_t^+) + 1) \\ &\to (\lambda^+)^{-2}((\lambda^+)^2 + \mathfrak{a}\lambda^+ + 1)\cdot 1 = (\lambda^+)^{-2}a^+(\lambda^+). \end{aligned}$$

Now we will consider the same reasoning for $b(\lambda_t^+)$

$$\begin{aligned} b(t\cdot(\lambda_t^+)^{-1}) &= (\lambda_t^+)^{-1}(b_1^+\lambda_t^+ + b_2^+)(tb_1^0(\lambda_t^+)^{-1} + b_2^0)(\overline{b_1^+}t^2(\lambda_t^+)^{-1} + \overline{b_2^+}) \\ &\to (\lambda^+)^{-1}b^+(\lambda^+)b^0(0)b^-(0). \end{aligned}$$

We calculate the same way as in Lemma 4.22

$$\begin{aligned} \mathrm{d}\frac{t}{\lambda} &= -\frac{t}{\lambda^2}\mathrm{d}\lambda \\ \mathrm{d}\lambda &= -\frac{\lambda^2}{t}\mathrm{d}\lambda_t^+ = -\frac{(t(\lambda_t^+)^{-1})^2}{t}\mathrm{d}\lambda_t^+ = -t(\lambda_t^+)^{-2}\mathrm{d}\lambda_t^+. \end{aligned}$$

Now using that as well as the transformation of the differential to calculate the limit for Θ_b yields

$$\begin{aligned} \Theta_{b_t} &= -\frac{t}{(\lambda_t^+)^2}\frac{b(\lambda_t^+)}{t(\lambda_t^+)^{-1}\sqrt{t(\lambda_t^+)^{-1}a(\lambda_t^+)}}\mathrm{d}(\lambda_t^+) \\ &= -\frac{(\lambda_t^+)^{-1}b^+(\lambda_t^+)\cdot b^0(t(\lambda_t^+)^{-1})\cdot b^-(t^2(\lambda_t^+)^{-1})}{(\lambda_t^+)\sqrt{t(\lambda_t^+)^{-1}a(\lambda_t^+)}}d(\lambda_t^+) \\ &= -\frac{b^+(\lambda_t^+)\cdot b^0(t(\lambda_t^+)^{-1})\cdot b^-(t^2(\lambda_t^+)^{-1})}{(\lambda_t^+)^2\sqrt{t(\lambda_t^+)^{-1}\cdot(\lambda_t^+)^{-2}\cdot a^+(\lambda_t^+)\cdot a^-(t^2(\lambda_t^+)^{-1})}}\mathrm{d}(\lambda_t^+) \\ &= -t^{-1/2}(\lambda_t^+)^{-2}\frac{b^+(\lambda_t^+)\cdot b^0(t(\lambda_t^+)^{-1})\cdot b^-(t^2(\lambda_t^+)^{-1})}{\sqrt{(\lambda_t^+)^{-3}\cdot a^+(\lambda_t^+)\cdot a^-(t^2(\lambda_t^+)^{-1})}}\mathrm{d}(\lambda_t^+) \\ &= -t^{-1/2}(\lambda_t^+)^{-1/2}\frac{b^+(\lambda_t^+)\cdot b^0(t(\lambda_t^+)^{-1})\cdot b^-(t^2(\lambda_t^+)^{-1})}{\sqrt{a^+(\lambda_t^+)\cdot a^-(t^2(\lambda_t^+)^{-1})}}\mathrm{d}(\lambda_t^+). \end{aligned}$$

So we see that the differential admits a factor $t^{-1/2}$ but we also know that the cycles defined earlier converge and $a(\lambda)$ as well as $b(\lambda)$ converge in a way that the differential Θ_{b_t} converges in the right way if we were to assume that the limit is finite and non-zero. So therefore, the rest of the differential must admit a factor $\sqrt{t}$ such that the whole limit is well defined. We see that the coefficients of $b^+(\lambda_t^+)$ are depending on t and from Lemma 4.8 we know that the coefficients from b^+ and b^- are complex conjugates so if one of these admits some kind of power of t the other one does as well which then means, that the coefficients of b^+ and b^- both contain $\sqrt[4]{t}$ so we will extract them and go over to denote by $\hat{b}^\pm$ the polynomials without $\sqrt[4]{t}$. Therefore, we can calculate

$$\begin{aligned}\lim_{t\to 0}\Theta_{b_t} &= \lim_{t\to 0}\Big(-t^{-1/2}(\lambda_t^+)^{-1/2}\frac{t^{1/4}\hat{b}^+(\lambda_t^+)\cdot b^0(t(\lambda_t^+)^{-1})\cdot t^{1/4}\hat{b}^-(t^2(\lambda_t^+)^{-1})}{\sqrt{a^+(\lambda_t^+)\cdot a^-(t^2(\lambda_t^+)^{-1})}}\mathrm{d}(\lambda_t^+)\Big)\\ &= \lim_{t\to 0}\Big(-(\lambda_t^+)^{-1/2}\frac{\hat{b}^+(\lambda_t^+)\cdot b^0(t(\lambda_t^+)^{-1})\cdot \hat{b}^-(t^2(\lambda_t^+)^{-1})}{\sqrt{a^+(\lambda_t^+)\cdot a^-(t^2(\lambda_t^+)^{-1})}}\mathrm{d}(\lambda_t^+)\Big)\\ &= -(\lambda^+)^{-1/2}\frac{b^+(\lambda^+)\cdot b^0(0)\cdot b^-(0)}{\sqrt{a^+(\lambda^+)\cdot a^-(0)}}\mathrm{d}\lambda^+\\ &= -(\lambda^+)^{-1/2}\frac{b^+(\lambda^+)\cdot b^0(0)\cdot b^-(0)}{\sqrt{a^+(\lambda^+)}}\mathrm{d}\lambda^+.\end{aligned}$$

Now if we evaluate the differential at $\lambda^+ = 0$ we evaluate it in the chart $z^2 = \lambda^+$ we see that

$$\begin{aligned}-z^{-1}\frac{b^+(z^2)\cdot b^0(0)\cdot b^-(0)}{\sqrt{a^+(z^2)}}\mathrm{d}z^2 &= -z^{-1}\frac{b^+(z^2)\cdot b^0(0)\cdot b^-(0)}{\sqrt{a^+(z^2)}}2z\mathrm{d}z\\ &= -\frac{(\mathfrak{b}_1^+z^2+\mathfrak{b}_2^+)\cdot b^0(0)\cdot b^-(0)}{\sqrt{z^4+\mathfrak{a}z^2+1}}\mathrm{d}z\end{aligned}$$

the differential is the quotient of two functions of degree two which means that it doesn't have a pole there meaning Θ_{b^+} is holomorphic at $\lambda^+ = 0$. The only other value in question is $\lambda^+ = \infty$ which we will now evaluate as well using now the chart $z^{-2} = \lambda^+$

$$\begin{aligned}-z\frac{b^+(z^{-2})\cdot b^0(0)\cdot b^+(0)}{\sqrt{a^+(z^{-2})}}\mathrm{d}z^{-2} &= -z\frac{(\mathfrak{b}_1^+z^{-2}+\mathfrak{b}_2^+)\cdot b^0(0)\cdot b^-(0)}{\sqrt{z^{-4}+\mathfrak{a}z^{-2}+1}}\cdot\Big(-\frac{2}{z^3}\Big)\mathrm{d}z\\ &= -2z^{-2}\frac{(\mathfrak{b}_1^+z^{-2}+\mathfrak{b}_2^+)\cdot b^0(0)\cdot b^-(0)}{\sqrt{z^{-4}+\mathfrak{a}z^{-2}+1}}\mathrm{d}z\end{aligned}$$

we see that our differential has a pole of order two at $\lambda^+ = \infty$. That means that our differential Θ_{b_t} converges to the right differential which justifies our reasoning concerning the limit process. Now we go back to arguing about $\frac{\mathrm{d}}{\mathrm{d}s}\Theta_{b^+}|_{s=0}$ which can only have a pole at the values where Θ_{b^+} has a pole as well. So that means that the derivative can only have a pole at $\lambda^+ = \infty$. In the following we will denote derivatives $\frac{\mathrm{d}}{\mathrm{d}s}f$ by a dot $\dot{f}$ and derivatives $\frac{\mathrm{d}}{\mathrm{d}\lambda^+}f$ by f'. Now from our knowledge of the poles of $\Theta_{b_k^+}$ we see how $\dot{q}_k$ needs to look like. In order to do so we define $\nu^+ = \sqrt{\lambda^+ \cdot a^+(\lambda^+)}$. Recalling Equation (5) we see that $\dot{q}_k$ needs to be of the following form

$$\dot{q}_k^+ = \frac{ic_k^+(\lambda^+)\lambda^+}{\nu^+}, \qquad k = 1, 2$$

where $c_k^+(\lambda^+) \in \mathbb{C}^1[\lambda^+]$. Now calculating Equation (5) we get

$$\frac{\partial}{\partial\lambda^+}\frac{ic^+(\lambda^+)\lambda^+}{\nu^+} = \frac{\mathrm{d}}{\mathrm{d}s}\frac{b_k^+(\lambda^+)\cdot b_k^0(0)\cdot b_k^-(0)}{\nu^+}\bigg|_{s=0}, \qquad k = 1, 2.$$

Now we will do the same as in the bachelor The Closure of Spectral Curves of Constant Mean Curvature Tori of Spectral Genus 2 by B. Schmidt (2017) by using chain rule to extract the Whitham equations. For simplicity we will define $\beta_k = b_k^0(0) \cdot b_k^-(0)$. Then we get

$$\frac{i((c_k^+)'(\lambda^+)\lambda^+ + c^+(\lambda^+))\nu^+ - ic_k^+(\lambda^+)\lambda^+(\nu^+)'}{(\nu^+)^2}$$
$$= \frac{(\dot{b}_k^+(\lambda^+)\cdot\beta_k + b_k^+(\lambda^+)\cdot\dot{\beta}_k)\cdot\nu^+ - b_k^+(\lambda^+)\cdot\beta_k\cdot\dot{\nu}^+}{(\nu^+)^2}.$$

Simplifying each sides yields

$$\frac{i(c_k^+)'(\lambda^+)\lambda^+\sqrt{\lambda^+a^+(\lambda^+)} + c^+(\lambda^+)\nu^+ - ic_k^+(\lambda^+)\lambda^+\frac{a^+(\lambda^+)+\lambda^+(a^+)'(\lambda^+)}{2\sqrt{\lambda^+a^+(\lambda^+)}}}{\lambda^+a^+(\lambda^+)}$$
$$= \frac{i(c_k^+)'(\lambda^+)\lambda^+}{\nu^+} + \frac{ic_k^+(\lambda^+))}{\nu^+} - \frac{ic_k^+(\lambda^+)\lambda^+(a^+(\lambda^+)+\lambda^+(a^+)'(\lambda^+)}{2(\nu^+)^3}$$

as well as

$$\frac{\dot{b}_k^+(\lambda^+)\cdot\beta_k}{\nu^+} + \frac{b_k^+(\lambda^+)\cdot\dot{\beta}_k}{\nu^+} - \frac{b_k^+(\lambda^+)\cdot\beta_k\cdot\frac{\lambda^+\dot{a}^+(\lambda^+)}{2\nu^+}}{(\nu^+)^2}$$
$$= \frac{\dot{b}_k^+(\lambda^+)\cdot\beta_k}{\nu^+} + \frac{b_k^+(\lambda^+)\cdot\dot{\beta}_k}{\nu^+} - \frac{b_k^+(\lambda^+)\cdot\beta_k\cdot\lambda^+\dot{a}^+(\lambda^+)}{2(\nu^+)^3}.$$

Now multiplying both sides of Equation (5) by $2(\nu^+)^3$ and dividing by λ^+ yields

$$i(2\lambda^+(c_k^+)'a^+ + 2c_k^+a^+ - c_k^+a^+ - \lambda^+c_k(a^+)') = 2\dot{b}^+\beta a^+ + 2b^+\dot{\beta}a^+ - b^+\beta\dot{a}^+.$$

Plugging in $k = 1$ yields

$$\begin{aligned} &i(2\lambda^+(c_1^+)'a^+ + c_1^+a^+ - \lambda^+c_1^+(a^+)') \\ =&2\dot{b}_1^+\beta_1 a^+ + 2b_1^+\dot{\beta}_1 a^+ - b_1^+\beta_1\dot{a}^+ \end{aligned} \tag{6}$$

and plugging in $k = 2$ yields

$$\begin{aligned} &i(2\lambda^+(c_2^+)'a^+ + c_2^+a^+ - \lambda^+c_2^+(a^+)') \\ =&2\dot{b}_2^+\beta_2 a^+ + 2b_2^+\dot{\beta}_2 a^+ - b_2^+\beta_2\dot{a}^+. \end{aligned} \tag{7}$$

Now we want to combine these two equations above by calculating $(c_2^+)\cdot(6) - (c_1^+)\cdot(7)$ to get

$$\begin{aligned} &i(2\lambda^+(c_1^+)'a^+c_2^+ + c_1^+a^+c_2^+ - \lambda^+c_1^+(a^+)'c_2^+ \\ &- 2\lambda^+(c_2^+)'a^+c_1^+ - c_2^+a^+c_1^+ + \lambda^+c_2^+(a^+)'c_1^+) \\ =&2\lambda^+\dot{b}_1^+\beta_1 a^+c_2^+ + 2b_1^+\dot{\beta}_1 a^+c_2^+ - b_1^+\beta_1\dot{a}^+c_2^+ \\ &- 2\dot{b}_2^+\beta_2 a^+c_1^+ - 2b_2^+\dot{\beta}_2 a^+c_1^+ + b_2^+\beta_2\dot{a}^+c_1^+. \end{aligned}$$

Now simplifying this yields us

$$\begin{aligned} &2a^+((\lambda^+)i((c_1^+)'c_2^+ - (c_2^+)'c_1^+) - \dot{b}_1^+\beta_1c_2^+ - b_1^+\dot{\beta}_1c_2^+ \\ &-\dot{b}_2^+\beta_2c_1^+ + b_2^+\dot{\beta}_2c_1^+ + \dot{b}_2^+\beta_2c_1^+ + b_2^+\dot{\beta}_2c_1^+) \\ &=\dot{a}^+(b_2^+\beta_2c_1^+ - b_1^+\beta_1c_2^+). \end{aligned}$$

Now we see again as in B. Schmidt (2017) that both sides of this equation need to vanish at all roots of a^+ which means that if $\dot{a}^+$ doesn't vanish at all roots of a^+, $b_2^+\beta_2c_1^+ - b_1^+\beta_1c_2^+$ needs to vanish at the remaining roots of a^+. Further we see that if we consider the equations (6) and (7) we see that c_1^+ and c_2^+ need to vanish at all roots that a^+ and $\dot{a}^+$ have in common since the equations would then reduce to $\lambda^+c_k^+(a^+)' = 0$. So that means that the expression $b_2^+\beta_2c_1^+ - b_1^+\beta_1c_2^+$ vanishes at every root of a^+ yielding us the following equation

$$Q^+a^+ = b_2^+\beta_2c_1^+ - b_1^+\beta_1c_2^+ \tag{8}$$

where $Q^+ \in \mathbb{C}$ holds. Now since we are interested in the kernel of $\mathrm{d}\hat{T}^+$ we consider those triples $(\dot{a}^+, \dot{b_1^+\beta_1}, \dot{b_2^+\beta_2})$ that leave τ_{a^+} constant. However since we define τ_{a^+} as $\lim_{t\to 0}\frac{b_{1,t}(0)}{b_{2,t}(0)} = \frac{b_1^+(0)\cdot\beta_1}{b_2^+(0)\cdot\beta_2}$ we see that this yields the following condition

$$\mathrm{d}\Big(\frac{b_1^+(0)\cdot\beta_1}{b_2^+(0)\cdot\beta_2}\Big)\Big|_{s=0} = 0 \tag{9}$$

which we calculate to be

$$\begin{aligned}&\frac{\dot b_1^+(0)\cdot\beta_1\cdot b_2^+(0)\cdot\beta_2+b_1^+(0)\cdot\dot\beta_1\cdot b_2^+(0)\cdot\beta_2}{(b_2^+(0)\cdot\beta_2)^2}\\ +&\frac{-b_1^+(0)\cdot\beta_1\cdot\dot b_2^+(0)\cdot\beta_2-b_1^+(0)\cdot\beta_1\cdot b_2^+(0)\cdot\dot\beta_2}{(b_2^+(0)\cdot\beta_2)^2}\\ =&\ 0\ .\end{aligned}$$

So then we arrive at the condition

$$\begin{aligned}\dot b_1^+(0)\cdot\beta_1\cdot b_2^+(0)\cdot\beta_2+b_1^+(0)\cdot\dot\beta_1\cdot b_2^+(0)\cdot\beta_2\\ -b_1^+(0)\cdot\beta_1\cdot\dot b_2^+(0)\cdot\beta_2-b_1^+(0)\cdot\beta_1\cdot b_2^+(0)\cdot\dot\beta_2=0.\end{aligned}\tag{10}$$

Now we consider the polynomials involved in these calculations in order to solve these equations

$$\begin{aligned}a^+(\lambda^+)&=(\lambda^+)^2+\mathfrak{a}\lambda^++1\\ \dot a^+(\lambda^+)&=\dot{\mathfrak{a}}\lambda^+,\\ b_1^+(\lambda^+)&=\mathfrak{b}_{1,1}^+\lambda^++\mathfrak{b}_{2,1}^+,\\ \dot b_1^+(\lambda^+)&=\dot{\mathfrak{b}}_{1,1}^+\lambda^++\dot{\mathfrak{b}}_{2,1}^+,\\ b_2^+(\lambda^+)&=\mathfrak{b}_{1,2}^+\lambda^++\mathfrak{b}_{2,2}^+,\\ \dot b_2^+(\lambda^+)&=\dot{\mathfrak{b}}_{1,2}^+\lambda^++\dot{\mathfrak{b}}_{2,2}^+,\\ c_1^+(\lambda^+)&=c_{1,1}^+\lambda^++c_{2,1}^+,\\ c_2^+(\lambda^+)&=c_{1,2}^+\lambda^++c_{2,2}^+.\end{aligned}$$

Using this for equation (10) we get

$$\dot{\mathfrak{b}}_{2,1}^+\mathfrak{b}_{2,2}^+\beta_1\beta_2+\mathfrak{b}_{2,1}^+\mathfrak{b}_{2,2}^+\dot\beta_1\beta_2-\mathfrak{b}_{2,1}^+\dot{\mathfrak{b}}_{2,2}^+\beta_1\beta_2-\mathfrak{b}_{2,1}^+\mathfrak{b}_{2,2}^+\beta_1\dot\beta_2=0.\tag{11}$$

We now evaluate Equation (8) at $\lambda^+=0$

$$Q^+=\mathfrak{b}_{2,2}^+\beta_2c_{2,1}^+-\mathfrak{b}_{2,1}^+\beta_1c_{2,2}^+.\tag{12}$$

Now we want to evaluate Equation (6) as well as equation (7) at $\lambda^+=0$ which yields

$$ic_{2,1}^+=2\dot{\mathfrak{b}}_{2,1}^+\beta_1+2\mathfrak{b}_{2,1}^+\dot\beta_1$$

and

$$ic_{2,2}^+=2\dot{\mathfrak{b}}_{2,2}^+\beta_2+2\mathfrak{b}_{2,2}^+\dot\beta_2.$$

So if we now use that and plug it into Equation (12) we obtain that

$$\begin{aligned} Q^+ &= -i(2\dot{\mathfrak{b}}_{2,1}^+\beta_1 + 2\mathfrak{b}_{2,1}\dot{\beta}_1)\mathfrak{b}_{2,2}^+\beta_2 + i(2\dot{\mathfrak{b}}_{2,2}^+\beta_2 + 2\mathfrak{b}_{2,2}\dot{\beta}_2)\mathfrak{b}_{2,1}^+\beta_1 \\ &= -2i(\dot{\mathfrak{b}}_{2,1}^+\mathfrak{b}_{2,2}^+\beta_1\beta_2 + \mathfrak{b}_{2,1}^+\mathfrak{b}_{2,2}^+\dot{\beta}_1\beta_2 - \mathfrak{b}_{2,1}^+\dot{\mathfrak{b}}_{2,2}^+\beta_1\beta_2 - \mathfrak{b}_{2,1}^+\mathfrak{b}_{2,2}^+\beta_1\dot{\beta}_2) \\ &= 0. \end{aligned}$$

Since this is just a multiple of Equation (11) we see that Q^+ vanishes. We can further assume $\gcd(b_1^+, b_2^+) = 0$ because otherwise b_1^- and b_2^- would have a common root as well, which then would mean that $\deg(b_1/b_2) \leq 1$ which can't hold. We consider λ_1^+ to be the root of b_1^+ and λ_2^+ to be the root of b_2^+. We now evaluate Equation (8) at λ_1^+ and get

$$0 = b_2^+(\lambda_1^+)\beta_2 c_1^+(\lambda_1^+)$$

and from evaluating it at λ_2^+

$$0 = b_1^+(\lambda_2^+)\beta_1 c_2^+(\lambda_2^+).$$

So we see from our assumption that c_1^+ needs to vanish at λ_2^+ and c_2^+ vanishes at λ_2^+. Therefore, we obtain the following conditions

$$\begin{aligned} c_1^+(\lambda_1^+) &= 0 \\ c_2^+(\lambda_2^+) &= 0. \end{aligned}$$

So that means that our polynomials c_k^+ are multiples of b_k^+ which we will write in the following way

$$c_k^+(\lambda^+) = \mu_k \cdot b_k^+(\lambda^+),\ k = 1, 2$$

with $\mu_k \in \mathbb{C}^\times$. The next step is to evaluate the equations Equation (6) as well as Equation (7) at the two roots of a^+ which we will denote by $\lambda_{a,k}^+, k = 1, 2$.

$$-i\lambda_{a,k}^+\mu_k(a^+)'(\lambda_{a,k}^+)b_k^+(\lambda_{a,k}^+) = -b_k^+(\lambda_{a,k}^+)\beta_k\dot{a}^+(\lambda_{a,k}^+),\ k = 1, 2.$$

We divide both sides of the equation $b_k^+(\lambda_{a,k}^+)$ and get

$$i\lambda_{a,k}^+\mu_k(a^+)'(\lambda_{a,k}^+) = \beta_k\dot{a}^+(\lambda_{a,k}^+),\ k = 1, 2.$$

If we plug in the definition of $\dot{a}^+$, we can divide both sides by $\lambda_{a,k}^+$, evaluate at $\lambda_{a,1}^+$ and get for $\dot{\mathfrak{a}}$

$$\dot{\mathfrak{a}} = i\mu_k(a^+)'(\lambda_{a,1}^+)\beta_k^{-1},\ k = 1, 2.$$

Now since we have determined $a^+(\lambda^+)$ fully by its coefficients it follows that there is an underlying relation between the roots of a^+ and its coefficients. Therefore, we need to express $\lambda^+_{a,1}$ by the coefficients of a^+. However since we consider a polynomial of degree two, elementary algebra tells us that these two invariants of a^+ satisfy

$$\lambda^+_{a,1} = -\frac{1}{2}\Big(\mathfrak{a} - \sqrt{\mathfrak{a}^2 - 4}\Big).$$

Using this for our solution we get

$$\begin{aligned}\dot{\mathfrak{a}} &= \frac{i\mu_k(2 \cdot -\frac{1}{2}\Big(\mathfrak{a} - \sqrt{\mathfrak{a}^2 - 4}\Big) + \mathfrak{a})}{\beta_k} \\ &= \frac{i\mu_k(-\mathfrak{a} - \sqrt{\mathfrak{a}^2 - 4} + \mathfrak{a})}{\beta_k} \\ &= \frac{i\mu_k\sqrt{\mathfrak{a}^2 - 4}}{\beta_k}, \qquad k = 1, 2.\end{aligned}$$

Since we get two results for $\dot{\mathfrak{a}}$ we can equal them and try to solve for μ_1

$$i\mu_1(a^+)'(\lambda^+_{a,1})\beta_1^{-1} = i\mu_2(a^+)'(\lambda^+_{a,1})\beta_2^{-1}.$$

Dividing the two sides by $i(a^+)'(\lambda^+_{a,1})$ and solving for μ_1 we obtain

$$\mu_1 = \frac{\beta_1\mu_2}{\beta_2}.$$

We now evaluate the equations Equation (6) and Equation (7) at the roots of b_k^+ λ_k^+ and get

$$2i\lambda_k^+\mu_k(b_k^+)'(\lambda_k^+)a^+(\lambda_k^+) = 2\dot{b}_k^+(\lambda_k^+)\beta_k a^+(\lambda_k^+),\ k = 1, 2.$$

This formula depends on λ_k^+ which is already determined by the two coefficients $\mathfrak{b}^+_{k,j},\ k, j = 1, 2$. In order to do so we can write down the following formula

$$\mathfrak{b}_k^+(\lambda^+) = \mathfrak{b}^+_{1,k}\lambda_k^+ + \mathfrak{b}^+_{2,k} = \mathfrak{b}^+_{1,k}(\lambda^+ - \lambda_k^+), \qquad k = 1, 2.$$

So now that means that

$$\mathfrak{b}^+_{2,k} = -\mathfrak{b}^+_{1,k}\lambda_k^+, \qquad k = 1, 2,$$

holds. We can also use this to eliminate the further use of λ_k^+ by writing it as

$$\lambda_k^+ = -\frac{\mathfrak{b}^+_{2,k}}{\mathfrak{b}^+_{1,k}}$$

We solve the equations before for $\dot{b}_k^+$ and get

$$\begin{aligned}\dot{b}_k^+(\lambda_k^+) &= \frac{i\lambda_k^+\mu_k(b_k^+)'(\lambda_k^+)}{\beta_k}\\ &= \frac{i\mu_k\lambda_k^+\mathfrak{b}_{1,k}^+}{\beta_k}\\ &= -\frac{i\mu_k\mathfrak{b}_{2,k}^+}{\beta_k},\ k=1,2.\end{aligned}$$

Next, using our knowledge of c_k^+ we can newly consider Equation (6) and Equation (7) at $\lambda^+ = 0$ to get

$$i\mu_k b_k^+(0) = 2\dot{b}_k^+(0)\beta_k + 2b_k^+(0)\dot{\beta}_k,\ k=1,2.$$

We solve these two equations for $\dot{b}_k^+(0)$ and get

$$\dot{b}_k^+(0) = b_k^+(0)\frac{i\mu_k + 2\dot{\beta}_k}{2\beta_k},\ k=1,2.$$

So we now fully know the coefficients of $\dot{b}_k^+$. Finally, we can use our knowledge in Equation (10) which then results in

$$\begin{aligned}&b_1^+(0)\frac{i\mu_1+2\dot{\beta}_1}{2\beta_1}\cdot\beta_1\cdot b_2^+(0)\cdot\beta_2 + b_1^+(0)\cdot\dot{\beta}_1\cdot b_2^+(0)\cdot\beta_2\\ &-b_1^+(0)\cdot\beta_1\cdot b_2^+(0)\frac{i\mu_2+2\dot{\beta}_2}{2\beta_2}\cdot\beta_2 - b_1^+(0)\cdot\beta_1\cdot b_2^+(0)\cdot\dot{\beta}_2\\ =&b_1^+(0)\cdot b_2^+(0)\cdot\beta_2\cdot\frac{i\mu_1+2\dot{\beta}_1}{2} + b_1^+(0)\cdot\dot{\beta}_1\cdot b_2^+(0)\cdot\beta_2\\ &-b_1^+(0)\cdot b_2^+(0)\cdot\beta_1\cdot\frac{i\mu_2+2\dot{\beta}_2}{2} - b_1^+(0)\cdot\beta_1\cdot b_2^+(0)\cdot\dot{\beta}_2\\ =&b_1^+(0)\cdot b_2^+(0)\cdot\left(\beta_2\frac{i\mu_1+2\dot{\beta}_1}{2} + \dot{\beta}_1\cdot\beta_2 - \beta_1\cdot\frac{i\mu_2+2\dot{\beta}_2}{2} - \beta_1\cdot\dot{\beta}_2\right)\\ =&0.\end{aligned}$$

We see that it is enough to evaluate the right bracket which gets us

$$2\dot{\beta}_1\beta_2 + \frac{i\mu_1\beta_2}{2} = 2\dot{\beta}_2\beta_1 + \frac{i\mu_2\beta_1}{2}.$$

Solving this for $\dot{\beta}_1$ leads us to

$$\dot{\beta}_1 = \dot{\beta}_2\frac{\beta_1}{\beta_2} + \frac{i(\mu_2\beta_1 - \mu_1\beta_2)}{4\beta_2}.$$

Now we will calculate the $\frac{\mathrm{d}}{\mathrm{d}s}$ derivative of Equation (8) at $s = 0$ to get to

$$\dot{Q}^+ a^+ + Q^+ \dot{a}^+ = \dot{b}_2^+ \beta_2 c_1^+ + b_2^+ \dot{\beta}_2 c_1^+ + b_2^+ \beta_2 \dot{c}_1^+ - \dot{b}_1^+ \beta_1 c_2^+ - b_1^+ \dot{\beta}_1 c_2^+ - b_1^+ \beta_1 \dot{c}_2^+.$$

Next, we will use that $Q^+ = 0, c_k^+ = \mu_k \cdot b_k^+$

$$\dot{Q}^+ a^+ = \mu_1 b_1^+ (\dot{b}_2^+ \beta_2 + b_2^+ \dot{\beta}_2) + b_2^+ \beta_2 \dot{c}_1^+ - \mu_2 b_2^+ (\dot{b}_1^+ \beta_1 + b_1^+ \dot{\beta}_1) - b_1^+ \beta_1 \dot{c}_2^+.$$

We will evaluate this equation now at the roots of b_k^+, λ_k^+, $k = 1, 2$

$$\dot{Q}^+ a^+(\lambda_1^+) = b_2^+(\lambda_1^+) \beta_2 \dot{c}_1^+(\lambda_1^+) - \mu_2 b_2^+(\lambda_1^+) \dot{b}_1^+(\lambda_1^+) \beta_1.$$

We can easily solve this for Q^+ to get

$$\dot{Q}^+ = \frac{b_2^+(\lambda_1^+) \beta_2 \dot{c}_1^+(\lambda_1^+) - \mu_2 b_2^+(\lambda_1^+) \dot{b}_1^+(\lambda_1^+) \beta_1}{a^+(\lambda_1^+)}.$$

The same procedure for λ_2^+ yields

$$\dot{Q}^+ a^+(\lambda_2^+) = \mu_1 b_1^+(\lambda_2^+) \dot{b}_2^+(\lambda_2^+) - b_1^+(\lambda_2^+) \beta_1 \dot{c}_2^+(\lambda_2^+)$$

as well as

$$\dot{Q}^+ = \frac{\mu_1 b_1^+(\lambda_2^+) \dot{b}_2^+(\lambda_2^+) - b_1^+(\lambda_2^+) \beta_1 \dot{c}_2^+(\lambda_2^+)}{a^+(\lambda_2^+)}.$$

Now we equal our two results for Q^+ and arrive at

$$\frac{b_2^+(\lambda_1^+) \beta_2 \dot{c}_1^+(\lambda_1^+) - \mu_2 b_2^+(\lambda_1^+) \dot{b}_1^+(\lambda_1^+) \beta_1}{a^+(\lambda_1^+)} = \frac{\mu_1 b_1^+(\lambda_2^+) \dot{b}_2^+(\lambda_2^+) - b_1^+(\lambda_2^+) \beta_1 \dot{c}_2^+(\lambda_2^+)}{a^+(\lambda_2^+)}.$$

We can solve this for $\dot{c}_1^+(\lambda_1^+)$ and get

$$\dot{c}_1^+(\lambda^+) = \frac{\mu_2 b_2^+(\lambda_1^+) \dot{b}_1^+(\lambda_1^+) \beta_1}{b_2^+(\lambda_1^+) \beta_2} + \frac{a^+(\lambda_1^+)}{a^+(\lambda_2^+)} \frac{\mu_1 b_1^+(\lambda_2^+) \dot{b}_2^+(\lambda_2^+) - b_1^+(\lambda_2^+) \beta_1 \dot{c}_2^+(\lambda_2^+)}{b_2^+(\lambda_1^+) \beta_2}.$$

Next, we evaluate the equation at $\lambda^+ = 0$ to get

$$\begin{aligned}\dot{Q}^+ = {} & \mu_1 b_1^+(0)(\dot{b}_2^+(0) \beta_2 + b_2^+(0) \dot{\beta}_2) + b_2^+(0) \beta_2 \dot{c}_1^+(0) \\ & - \mu_2 b_2^+(0)(\dot{b}_1^+(0) \beta_1 + b_1^+(0) \dot{\beta}_1) - b_1^+(0) \beta_1 \dot{c}_2^+(0).\end{aligned}$$

We can solve this for $\dot{c}_1^+(0)$ and arrive at

$$\dot{c}_1^+(0) = \frac{\dot{Q}^+ + \mu_2 b_2^+(0)(\dot{b}_1^+(0) \beta_1 + b_1^+(0) \dot{\beta}_1) + b_1^+(0) \beta_1 \dot{c}_2^+(0) - \mu_1 b_1^+(0)(\dot{b}_2^+(0) \beta_2 + b_2^+(0) \dot{\beta}_2)}{b_2^+(0) \beta_2}.$$

That fully determines $\dot{c}_1^+$. However now we only have two more conditions we could use, namely evaluating the equation at the roots of a^+ and that only suffices to determine $\dot{c}_2^+$ but not $\dot{\beta}_2$. Therefore, we didn't gain anything from this approach.
Now we will turn to the limit of our polynomials in the λ parameter as opposed to the λ^+ parameter we used before. We know from Lemma 4.15 as well as Lemma 4.16 that the limit of Θ_{b_t} in this particular parameter has the following form

$$\begin{aligned}\Theta_{b_t} &= \frac{b(\lambda)}{\lambda \cdot \sqrt{\lambda \cdot a(\lambda)}} \mathrm{d}\lambda \\ &= \frac{t \cdot b(\lambda)}{\lambda \cdot \sqrt{\lambda \cdot t^2 \cdot a(\lambda)}} \mathrm{d}\lambda \\ &\to \frac{\lambda \cdot b^0(\lambda) \cdot b^+(0) \cdot b^-(0)}{\lambda \cdot \sqrt{\lambda \cdot \lambda^2}} \mathrm{d}\lambda ast \to 0.\end{aligned}$$

Using Lemma 4.8 (ii) we also know that $b^+(0) = \overline{b^-(0)}$ so we can simplify our limit to be

$$\lim_{t \to 0} \Theta_{b_t} = \frac{b^0(\lambda) \cdot |b^+(0)|^2}{(\nu^0)} \mathrm{d}\lambda$$

We will now continue with the same procedure as before in the case of λ^+ parameters to introduce the limit case of Whitham equations. We consider the limit functions ν^0, b_1^0, b_2^0 as well as the periods of our limit of Θ_{b_t} which we will call Θ_{b^0} to be dependent on s, so we have a family $(a^0, b_1^0, b_2^0)(s)$ which defines the level set $\lim_{t \to 0} T^{-1}(\tau_a)$. So we see that if we derive Θ_{b^0} by s, we will get that they vanish since the periods are constant with respect to s. That in turn means that $\dot{\Theta}_{b^0}$ is again an exact polynomial, and we can get new Whitham equations. In order to do so we need to calculate the pole orders of Θ_{b^0} at $\lambda = 0$ as well as $\lambda = \infty$. We start with the first one. Since $\lambda = 0$ is a double point, we will use the chart $\lambda = z^2$ here. So then we get

$$\begin{aligned}\Theta_{b^0} &= \frac{b^0(\lambda) \cdot |b^+(0)|^2}{\lambda^{3/2}} \mathrm{d}\lambda \\ &= \frac{b^0(z^2) \cdot |b^+(0)|^2}{z^3} \mathrm{d}z^2 \\ &= \frac{b^0(z^2) \cdot |b^+(0)|^2}{z^3} \, 2z \cdot \mathrm{d}z.\end{aligned}$$

It is now easy to see that both nominator and denominator are functions of degree three, which in turn means that this is finite at $\lambda = 0$. Calculating the order at $\lambda = \infty$ we need the map $\lambda = z^{-2}$ since this is a double point as well. Then we get

$$\begin{aligned}\Theta_{b^0} &= \frac{b^0(z^{-2}) \cdot |b^+(0)|^2}{z^{-3}} \mathrm{d}z^{-2} \\ &= \frac{b^0(z^{-2}) \cdot |b^+(0)|^2}{z^{-3}} \cdot \Big(- \frac{2}{z^3} \Big) \mathrm{d}z \\ &= -2b^0(z^{-2}) \cdot |b^+(0)|^2 \mathrm{d}z.\end{aligned}$$

That means that Θ_{b^0} has a pole of order two at $\lambda = \infty$. This is the same situation as in the λ^+ coordinate. So we get similar equations. Using the ansatz

$$\mathrm{d}\dot{q}_k^0 = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} \Theta_{b_k^0}, \; k = 1,2.$$

we get that

$$\dot{q}_k^0 = \frac{ic^0(\lambda) \cdot \lambda}{(\nu^0)}$$

has to hold where $c_k^0 \in \mathbb{C}^1[\lambda]$. That means we get to the following equation

$$\frac{\partial}{\partial \lambda} \frac{ic_k^0(\lambda) \cdot \lambda}{\nu^0} = \frac{\mathrm{d}}{\mathrm{d}s} \frac{b_k^0(\lambda) \cdot |b_k^+(0)|^2}{\nu^0}\Big|_{s=0}, \qquad k = 1,2.$$

Evaluating the left side gets us to

$$\begin{aligned}&\frac{i(c_k^0(\lambda) + (c_k^0)'(\lambda) \cdot \lambda)\nu^0 - ic_k^0(\lambda) \cdot \lambda \cdot (\nu^0)'}{(\nu^0)^2} \\ =&\frac{ic_k^0(\lambda) + i(c_k^0)'(\lambda) \cdot \lambda}{\nu^0} - \frac{ic_k^0(\lambda) \cdot \lambda \cdot (\lambda^{3/2})'}{(\nu^0)^2} \\ =&\frac{ic_k^0(\lambda) + i(c_k^0)'(\lambda) \cdot \lambda}{\nu^0} - \frac{ic_k^0(\lambda) \cdot \lambda \cdot 3/2\lambda^{1/2}}{\lambda^3} \\ =&\frac{ic_k^0(\lambda) + i(c_k^0)'(\lambda) \cdot \lambda}{\nu^0} - \frac{ic_k^0(\lambda) \cdot 3/2\lambda^{3/2}}{\lambda^3} \\ =&\frac{ic_k^0(\lambda) + i(c_k^0)'(\lambda) \cdot \lambda}{\nu^0} - \frac{3ic_k^0(\lambda)}{2\nu^0}, \qquad k = 1,2.\end{aligned}$$

Now we will evaluate the right side to get to

$$\begin{aligned}
&\frac{(\dot{b}_k^0(\lambda)\cdot|b_k^+(0)|^2 + b_k^0(\lambda)\cdot|\dot{b_k^+(0)}|^2)\nu^0 - b_k^0(\lambda)\cdot|b^+(0)|^2\dot{\nu^0}}{(\nu^0)^2}\\
=&\frac{\dot{b}_k^0(\lambda)\cdot|b_k^+(0)|^2 + b_k^0(\lambda)\cdot|\dot{b_k^+(0)}|^2}{\nu^0} - \underbrace{\frac{b_k^0(\lambda)\cdot|b_k^+(0)|^2\cdot(\dot{\lambda^{3/2}})}{(\nu_0)^2}}_{=0}\\
=&\frac{\dot{b}_k^0(\lambda)\cdot|b_k^+(0)|^2 + b_k^0(\lambda)\cdot|\dot{b_k^+(0)}|^2}{\nu^0}, \qquad k=1,2.
\end{aligned}$$

Now equaling both sides and multiplying with $2\nu^0$ gets us to the equation

$$i(2c_k^0 + 2\lambda(c_k^0)' - 3c_k^0) = 2(\dot{b}_k^0(\lambda)\cdot|b^+(0)|^2 + b_k^0(\lambda)\cdot|\dot{b_k^+(0)}|^2), \quad k=1,2.$$

Simplifying gives rise to the two equations

$$i(2\lambda(c_1^0)' - c_1^0) = 2(\dot{b}_1^0(\lambda)\cdot|b_1^+(0)|^2 + b_1^0(\lambda)\cdot|\dot{b_1^+(0)}|^2) \tag{13}$$

and

$$i(2\lambda(c_2^0)' - c_2^0) = 2(\dot{b}_2^0(\lambda)\cdot|b_2^+(0)|^2 + b_2^0(\lambda)\cdot|\dot{b_2^+(0)}|^2). \tag{14}$$

Again we need to consider $\lim_{t\to 0}\tau_a$ in our parameter. We get from Lemma 4.16

$$\begin{aligned}
\lim_{t\to 0}\frac{b_1(0)}{b_2(0)} &= \lim_{t\to 0}\frac{t\cdot b_1(0)}{t\cdot b_2(0)}\\
&= \frac{b_1^0(0)\cdot|b_1^+(0)|^2}{b_2^0(0)\cdot|b_2^+(0)|^2}.
\end{aligned}$$

So we need to consider the equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\frac{b_1^0(0)\cdot|b_1^+(0)|^2}{b_2^0(0)\cdot|b_2^+(0)|^2}\bigg|_{s=0} = 0.$$

By quotient rule we get

$$\begin{aligned}
&\frac{\dot{b}_1^0(0)\cdot|b_1^+(0)|^2\cdot b_2^0(0)\cdot|b_2^+(0)|^2 + b_1^0(0)\cdot|\dot{b_1^+(0)}|^2\cdot b_2^0(0)\cdot|b_2^+(0)|^2}{(b_2^0(\lambda)\cdot|b_2^+(0)|^2)^2}\\
-&\frac{b_1^+(0)\cdot|b_1^+(0)|^2\cdot\dot{b}_2^0(0)\cdot|b_2^+(0)|^2 + b_1^0(0)\cdot|b_1^+(0)|^2\cdot b_2^+(0)\cdot|\dot{b_2^+(0)}|^2}{(b_2^0(\lambda)\cdot|b_2^+(0)|^2)^2}\\
=&\ 0.
\end{aligned}$$

That gets us to

$$\begin{aligned} &\dot{b}_1^0(0) \cdot |b_1^+(0)|^2 \cdot b_2^0(0) \cdot |b_2^+(0)|^2 + b_1^0(0) \cdot |\dot{b_1^+(0)}|^2 \cdot b_2^0(0) \cdot |b_2^+(0)|^2 \\ -\ & b_1^0(0) \cdot |b_1^+(0)|^2 \cdot \dot{b}_2^0(0) \cdot |b_2^+(0)|^2 - b_1^0(0) \cdot |b_1^+(0)|^2 \cdot b_2^0(0) \cdot |\dot{b_2^+(0)}|^2 \\ =\ & 0. \end{aligned} \tag{15}$$

We will now again consider the polynomials used in these calculations

$$\begin{aligned} b_1^0(\lambda) &= \mathfrak{b}_{1,1}^0\lambda + \mathfrak{b}_{1,2}^0 \\ b_2^0(\lambda) &= \mathfrak{b}_{2,1}^0\lambda + \mathfrak{b}_{2,2}^0 \\ \dot{b}_1^0(\lambda) &= \dot{\mathfrak{b}}_{1,1}^0\lambda + \dot{\mathfrak{b}}_{2,1}^0 \\ \dot{b}_2^0(\lambda) &= \dot{\mathfrak{b}}_{1,2}^0\lambda + \dot{\mathfrak{b}}_{2,2}^0 \\ c_1^0(\lambda) &= c_{1,1}^0\lambda + c_{2,1}^0 \\ c_2^0(\lambda) &= c_{1,2}^0\lambda + c_{2,2}^0\ . \end{aligned}$$

Now our first step will be to calculate $(c_2^0) \cdot$ (13) - $(c_1^0) \cdot$ (14) to get to

$$\begin{aligned} &i(2\lambda((c_1^0)'c_2^0 - c_1^0(c_2^0)') - c_1^0c_2^0 + c_2^0c_1^0) \\ =&2c_2^0(\dot{b}_1^0(\lambda) \cdot |b_1^+(0)|^2 + b_1^0(\lambda) \cdot |\dot{b_1^+(0)}|^2) \\ -&2c_1^0(\dot{b}_2^0(\lambda) \cdot |b_2^+(0)|^2 + b_2^0(\lambda) \cdot |\dot{b_2^+(0)}|^2). \end{aligned}$$

Simplifying yields us

$$\begin{aligned} &i\lambda((c_1^0)'c_2^0 - c_1^0(c_2^0)') \\ =&c_2^0(\dot{b}_1^0(\lambda) \cdot |b_1^+(0)|^2 + b_1^0(\lambda) \cdot |\dot{b_1^+(0)}|^2) \\ -&c_1^0(\dot{b}_2^0(\lambda) \cdot |b_2^+(0)|^2 + b_2^0(\lambda) \cdot |\dot{b_2^+(0)}|^2). \end{aligned} \tag{16}$$

Now evaluating at $\lambda = 0$ yields us

$$c_{2,2}^0(\dot{\mathfrak{b}}_{2,1}^0 \cdot |\mathfrak{b}_{2,1}^+|^2 + \mathfrak{b}_{2,1}^0 \cdot |\dot{\mathfrak{b}}_{2,1}^+|^2) = c_{2,1}^0(\dot{\mathfrak{b}}_{2,2}^0 \cdot |\mathfrak{b}_{2,2}^+|^2 + \mathfrak{b}_{2,2}^0 \cdot |\dot{\mathfrak{b}}_{2,2}^+|^2)$$

From our previous considerations, we know, that $|b_k^+(0)| = \mathfrak{b}_{2,k}^+\overline{\mathfrak{b}_{2,k}^+}$ so the s derivative is already known because of our considerations of the λ^+ parameter. Next we will evaluate Equation (13) and Equation (14) at $\lambda = 0$. That results in

$$ic_{2,1}^0 = 2(\dot{\mathfrak{b}}_{2,1}^0 \cdot |\mathfrak{b}_{2,1}^+|^2 + \mathfrak{b}_{2,1}^0 \cdot |\dot{\mathfrak{b}}_{2,1}^+|^2)$$

as well as

$$ic_{2,2}^0 = 2(\dot{\mathfrak{b}}_{2,2}^0 \cdot |\mathfrak{b}_{2,2}^+|^2 + \mathfrak{b}_{2,2}^0 \cdot |\dot{\mathfrak{b}}_{2,2}^+|^2).$$

In the next step we will use our knowledge of these polynomials and evaluate the equations at the roots λ_k of b_k^0 to reach

$$i(2\lambda_1 c_{1,1}^0 - c_1^0(\lambda_1)) = 2\dot{b}_1^0(\lambda_1) \cdot |\mathfrak{b}_{2,1}^+|^2$$

as well as

$$i(2\lambda_2 c_{1,2}^0 - c_2^0(\lambda_1)) = 2\dot{b}_2^0(\lambda_2) \cdot |\mathfrak{b}_{2,2}^+|^2.$$

Using our result before we get to

$$i(2\lambda_1 c_{1,1}^0 - c_{1,1}^0\lambda_1) = 2\dot{b}_1^0(\lambda_1) \cdot |\mathfrak{b}_{2,1}^+|^2 - 2i(\dot{\mathfrak{b}}_{2,1}^0 \cdot |\mathfrak{b}_{2,1}^+|^2 + \mathfrak{b}_{2,1}^0 \cdot |\dot{\mathfrak{b}}_{2,1}^+|^2)$$

as well as

$$i(2\lambda_2 c_{2,1}^0 - c_{2,1}^0\lambda_2) = 2\dot{b}_2^0(\lambda_2) \cdot |\mathfrak{b}_{2,2}^+|^2 - 2i(\dot{\mathfrak{b}}_{2,2}^0 \cdot |\mathfrak{b}_{2,2}^+|^2 + \mathfrak{b}_{2,2}^0 \cdot |\dot{\mathfrak{b}}_{2,2}^+|^2).$$

We can finally solve these equations and arrive at

$$c_{1,1}^0 = \frac{2}{\lambda_1}(i\dot{b}_1^0(\lambda_1) \cdot |\mathfrak{b}_{2,1}^+|^2 + (\dot{\mathfrak{b}}_{2,1}^0 \cdot |\mathfrak{b}_{2,1}^+|^2 + \mathfrak{b}_{2,1}^0 \cdot |\dot{\mathfrak{b}}_{2,1}^+|^2))$$

and

$$c_{1,2}^0 = \frac{2}{\lambda_2}(i\dot{b}_2^0(\lambda_2) \cdot |\mathfrak{b}_{2,2}^+|^2 + (\dot{\mathfrak{b}}_{2,2}^0 \cdot |\mathfrak{b}_{2,2}^+|^2 + \mathfrak{b}_{2,2}^0 \cdot |\dot{\mathfrak{b}}_{2,2}^+|^2)).$$

Now we will just as in the λ^+ limit express λ_k in terms of the coefficients of b_k^0. It is easy to see that again

$$\lambda_k^+ = -\frac{\mathfrak{b}_{2,k}^0}{\mathfrak{b}_{1,k}^0}, \qquad k = 1, 2,$$

will hold. Using this we can now simplify our solution to be

$$c_{1,k}^0 = -\frac{2\mathfrak{b}_{1,k}^0}{\mathfrak{b}_{2,k}^0}\Big(i((\dot{\mathfrak{b}}_{1,k}^0\Big(-\frac{\mathfrak{b}_{2,k}^0}{\mathfrak{b}_{1,k}^0}\Big) + \dot{\mathfrak{b}}_{2,k}^0)|\mathfrak{b}_{2,k}^+|^2 + (\dot{\mathfrak{b}}_{2,2}^0 \cdot |\mathfrak{b}_{2,2}^+|^2 + \mathfrak{b}_{2,2}^0 \cdot |\dot{\mathfrak{b}}_{2,2}^+|^2)\Big), \quad k = 1, 2.$$

We will later return to this rather large looking expression in order to simplify it. That means we have completely determined c_1^0 as well as c_2^0 but we also all the information we can get from Equation (13) and Equation (14). That means we need to turn to the other equations for further information. Therefore, we will plug our knowledge into Equation 16 which gets us

$$\begin{aligned} &-2i(\dot{\mathfrak{b}}_{2,1}^0 \cdot |\mathfrak{b}_{2,1}^+|^2 + \mathfrak{b}_{2,1}^0 \cdot |\dot{\mathfrak{b}}_{2,1}^+|^2)(\dot{\mathfrak{b}}_{2,1}^0 \cdot |\mathfrak{b}_{2,1}^+|^2 + \mathfrak{b}_{2,1}^0 \cdot |\dot{\mathfrak{b}}_{2,1}^+|^2) \\ = &-2i(\dot{\mathfrak{b}}_{2,2}^0 \cdot |\mathfrak{b}_{2,2}^+|^2 + \mathfrak{b}_{2,2}^0 \cdot |\dot{\mathfrak{b}}_{2,2}^+|^2)(\dot{\mathfrak{b}}_{2,2}^0 \cdot |\mathfrak{b}_{2,2}^+|^2 + \mathfrak{b}_{2,2}^0 \cdot |\dot{\mathfrak{b}}_{2,2}^+|^2). \end{aligned}$$

We simplify this and get

$$\dot{\mathfrak{b}}^0_{2,1} \cdot |\mathfrak{b}^+_{2,1}|^2 + \mathfrak{b}^0_{2,1} \cdot |\dot{\mathfrak{b}}^+_{2,1}|^2 = \dot{\mathfrak{b}}^0_{2,2} \cdot |\mathfrak{b}^+_{2,2}|^2 + \mathfrak{b}^0_{2,2} \cdot |\dot{\mathfrak{b}}^+_{2,2}|^2$$

which we can solve for $\dot{\mathfrak{b}}^0_{2,1}$ and arrive at

$$\dot{\mathfrak{b}}^0_{2,1} = \frac{\dot{\mathfrak{b}}^0_{2,2} \cdot |\mathfrak{b}^+_{2,2}|^2 + \mathfrak{b}^0_{2,2} \cdot |\dot{\mathfrak{b}}^+_{2,2}|^2 - \mathfrak{b}^0_{2,1} \cdot |\dot{\mathfrak{b}}^+_{2,1}|^2}{|\mathfrak{b}^+_{2,1}|^2}.$$

We will plug this equation into Equation (15) and get

$$\begin{aligned}
&\frac{\dot{\mathfrak{b}}^0_{2,2} \cdot |\mathfrak{b}^+_{2,2}|^2 + \mathfrak{b}^0_{2,2} \cdot |\dot{\mathfrak{b}}^+_{2,2}|^2 - \mathfrak{b}^0_{2,1} \cdot |\dot{\mathfrak{b}}^+_{2,1}|^2}{|\mathfrak{b}^+_{2,1}|^2} \cdot |\mathfrak{b}^+_{2,1}|^2 \cdot \mathfrak{b}^0_{2,2} \cdot |\mathfrak{b}^+_{2,2}|^2 + \mathfrak{b}^0_{2,1} \cdot |\dot{\mathfrak{b}}^+_{2,1}|^2 \cdot \mathfrak{b}^0_{2,2} \cdot |\mathfrak{b}^+_{2,2}|^2 \\
&- \mathfrak{b}^0_{2,1} \cdot |\mathfrak{b}^+_{2,1}|^2 \cdot \dot{\mathfrak{b}}^0_{2,2} \cdot |\mathfrak{b}^+_{2,2}|^2 - \mathfrak{b}^0_{2,1} \cdot |\mathfrak{b}^+_{2,1}|^2 \cdot \mathfrak{b}^0_{2,2} \cdot |\dot{\mathfrak{b}}^+_{2,2}|^2 \\
&= \dot{\mathfrak{b}}^0_{2,2} \cdot |\mathfrak{b}^+_{2,2}|^2 \cdot \mathfrak{b}^0_{2,2} \cdot |\mathfrak{b}^+_{2,2}|^2 + \mathfrak{b}^0_{2,2} \cdot |\dot{\mathfrak{b}}^+_{2,2}|^2 \cdot \mathfrak{b}^0_{2,2} \cdot |\mathfrak{b}^+_{2,2}|^2 \\
&- \mathfrak{b}^0_{2,1} \cdot |\dot{\mathfrak{b}}^+_{2,1}|^2 \cdot \mathfrak{b}^0_{2,2} \cdot |\mathfrak{b}^+_{2,2}|^2 + \mathfrak{b}^0_{2,1} \cdot |\dot{\mathfrak{b}}^+_{2,1}|^2 \cdot \mathfrak{b}^0_{2,2} \cdot |\mathfrak{b}^+_{2,2}|^2 \\
&- \mathfrak{b}^0_{2,1} \cdot |\mathfrak{b}^+_{2,1}|^2 \cdot \dot{\mathfrak{b}}^0_{2,2} \cdot |\mathfrak{b}^+_{2,2}|^2 - \mathfrak{b}^0_{2,1} \cdot |\mathfrak{b}^+_{2,1}|^2 \cdot \mathfrak{b}^0_{2,2} \cdot |\dot{\mathfrak{b}}^+_{2,2}|^2 \\
&= 0.
\end{aligned}$$

We will align the equation in the following way

$$\begin{aligned}
&\dot{\mathfrak{b}}^0_{2,2} \cdot |\mathfrak{b}^+_{2,2}|^2 \cdot \mathfrak{b}^0_{2,2} \cdot |\mathfrak{b}^+_{2,2}|^2 - \mathfrak{b}^0_{2,1} \cdot |\mathfrak{b}^+_{2,1}|^2 \cdot \dot{\mathfrak{b}}^0_{2,2} \cdot |\mathfrak{b}^+_{2,2}|^2 \\
&+ \mathfrak{b}^0_{2,2} \cdot |\dot{\mathfrak{b}}^+_{2,2}|^2 \cdot \mathfrak{b}^0_{2,2} \cdot |\mathfrak{b}^+_{2,2}|^2 + \mathfrak{b}^0_{2,1} \cdot |\dot{\mathfrak{b}}^+_{2,1}|^2 \cdot \mathfrak{b}^0_{2,2} \cdot |\mathfrak{b}^+_{2,2}|^2 \\
&- \mathfrak{b}^0_{2,1} \cdot |\dot{\mathfrak{b}}^+_{2,1}|^2 \cdot \mathfrak{b}^0_{2,2} \cdot |\mathfrak{b}^+_{2,2}|^2 - \mathfrak{b}^0_{2,1} \cdot |\mathfrak{b}^+_{2,1}|^2 \cdot \mathfrak{b}^0_{2,2} \cdot |\dot{\mathfrak{b}}^+_{2,2}|^2 \\
&= 0
\end{aligned}$$

We see that two of the terms cancel and will solve for $\dot{\mathfrak{b}}^0_{2,2}$ to get to

$$\begin{aligned}
&\dot{\mathfrak{b}}^0_{2,2} \cdot |\mathfrak{b}^+_{2,2}|^2 \cdot (\mathfrak{b}^0_{2,2} \cdot |\mathfrak{b}^+_{2,2}|^2 - \mathfrak{b}^0_{2,1} \cdot |\mathfrak{b}^+_{2,1}|^2) \\
&= \mathfrak{b}^0_{2,1} \cdot |\mathfrak{b}^+_{2,1}|^2 \cdot \mathfrak{b}^0_{2,2} \cdot |\dot{\mathfrak{b}}^+_{2,2}|^2 - \mathfrak{b}^0_{2,2} \cdot |\dot{\mathfrak{b}}^+_{2,2}|^2 \cdot \mathfrak{b}^0_{2,2} \cdot |\mathfrak{b}^+_{2,2}|^2
\end{aligned}$$

and finally

$$\begin{aligned}
\dot{\mathfrak{b}}^0_{2,2} &= \frac{\mathfrak{b}^0_{2,1} \cdot |\mathfrak{b}^+_{2,1}|^2 \cdot \mathfrak{b}^0_{2,2} \cdot |\dot{\mathfrak{b}}^+_{2,2}|^2 - \mathfrak{b}^0_{2,2} \cdot |\dot{\mathfrak{b}}^+_{2,2}|^2 \cdot \mathfrak{b}^0_{2,2} \cdot |\mathfrak{b}^+_{2,2}|^2}{|\mathfrak{b}^+_{2,2}|^2 \cdot (\mathfrak{b}^0_{2,2} \cdot |\mathfrak{b}^+_{2,2}|^2 - \mathfrak{b}^0_{2,1} \cdot |\mathfrak{b}^+_{2,1}|^2)} \\
&= \frac{\mathfrak{b}^0_{2,2} \cdot |\dot{\mathfrak{b}}^+_{2,2}|^2}{|\mathfrak{b}^+_{2,2}|^2} \frac{\mathfrak{b}^0_{2,1} \cdot |\mathfrak{b}^+_{2,1}|^2 - \mathfrak{b}^0_{2,2} \cdot |\mathfrak{b}^+_{2,2}|}{\mathfrak{b}^0_{2,2} \cdot |\mathfrak{b}^+_{2,2}|^2 - \mathfrak{b}^0_{2,1} \cdot |\mathfrak{b}^+_{2,1}|^2} \\
&= -\frac{\mathfrak{b}^0_{2,2} \cdot |\dot{\mathfrak{b}}^+_{2,2}|^2}{|\mathfrak{b}^+_{2,2}|^2}
\end{aligned}$$

Here we can see that this solution depends linearly on $|\dot{\mathfrak{b}}^+_{2,2}|$ of which we already know from the first limit that they depend linearly on μ_1. We see that this holds true because of the following

$$
\begin{aligned}
|\dot{\mathfrak{b}}^+_{2,2}|^2 &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\mathfrak{b}^+_{2,2}\overline{\mathfrak{b}^+_{2,2}}\right)\Bigg|_{s=0} \\
&= \dot{\mathfrak{b}}^+_{2,2}\overline{\mathfrak{b}^+_{2,2}} + \mathfrak{b}^+_{2,2}\overline{\dot{\mathfrak{b}}^+_{2,2}}.
\end{aligned}
$$

We will use our knowledge of $\dot{\mathfrak{b}}^0_{2,2}$ to fully determine $\dot{\mathfrak{b}}^0_{2,1}$

$$
\begin{aligned}
\dot{\mathfrak{b}}^0_{2,1} &= \frac{-\frac{\mathfrak{b}^0_{2,2}\cdot|\dot{\mathfrak{b}}^+_{2,2}|}{|\mathfrak{b}^+_{2,2}|^2}\cdot|\mathfrak{b}^+_{2,2}|^2 + \mathfrak{b}^0_{2,2}\cdot|\dot{\mathfrak{b}}^+_{2,2}|^2 - \mathfrak{b}^0_{2,1}\cdot|\dot{\mathfrak{b}}^+_{2,1}|^2}{|\mathfrak{b}^+_{2,1}|^2} \\
&= -\frac{\mathfrak{b}^0_{2,1}\cdot|\dot{\mathfrak{b}}^+_{2,1}|^2}{|\mathfrak{b}^+_{2,1}|^2}.
\end{aligned}
$$

We see that now the only factor on which this term depends is the term $|\dot{\mathfrak{b}}^+_{2,1}|^2$. Just as $|\dot{\mathfrak{b}}^+_{2,2}|^2$ before this term is linearly depedent on μ_1 so we are able to conclude the same as before. It remains to solve for the coefficients $\dot{\mathfrak{b}}^0_{1,k}$ but we have no conditions from our equations left. Although Lemma 4.8 (i) says that $\mathfrak{b}^0_{1,k} = \overline{\mathfrak{b}^0_{2,k}}$ and since both are on $\mathbb{S}^1$ that means $\mathfrak{b}^0_{2,k} = (\mathfrak{b}^0_{1,k})^{-1},\ k = 1,2$. So then for the derivative we get by chain rule that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\mathfrak{b}^0_{1,k} &= \frac{\mathrm{d}}{\mathrm{d}s}\left(\mathfrak{b}^0_{2,k}\right)^{-1} = -(\mathfrak{b}^0_{2,k})^{-2}\cdot\dot{\mathfrak{b}}^0_{2,k} \\
&= \frac{\mathfrak{b}^0_{2,k}|\dot{\mathfrak{b}}^+_{2,k}|^2}{(\mathfrak{b}^0_{2,k})^2|\mathfrak{b}^+_{2,k}|^2} = \frac{|\dot{\mathfrak{b}}^+_{2,k}|^2}{\mathfrak{b}^0_{2,k}|\mathfrak{b}^+_{2,k}|^2} = \frac{\mathfrak{b}^0_{1,k}|\dot{\mathfrak{b}}^+_{2,k}|^2}{|\mathfrak{b}^+_{2,k}|^2}, \qquad k = 1,2.
\end{aligned}
$$

Therefore, we see that if $\dot{\mathfrak{b}}^0_{2,k}$ depends linearly on μ_k it then follows that $\dot{\mathfrak{b}}^0_{1,k}$ depends linearly on μ_k^{-1} as well since they only differ by coefficients of b^0_k as well as by sign.
Now we know from Lemma 4.8 (ii) that if we would consider the limit in λ^- coordinates we would get the complex conjugates of the results established in the λ^+ parameter discussion. That also means that we can write

$$
\begin{aligned}
\dot{\beta}_k &= \frac{\mathrm{d}}{\mathrm{d}s}\left(b^0_k(0)\cdot b^-_k(0)\right)\Bigg|_{s=0} = \dot{b}^0_k(0)\cdot b^-_k(0) + b^0_k(0)\cdot\dot{b}^-_k(0) \\
&= \dot{\mathfrak{b}}^0_{2,k}\cdot\overline{\mathfrak{b}^+_{2,k}} + \mathfrak{b}^0_{2,k}\cdot\overline{\dot{\mathfrak{b}}^+_{2,k}}, \qquad k = 1,2.
\end{aligned}
$$

Now it is easy to see that both factors either depend on μ_1 or $\overline{\mu_1}$. In the last step we will consider the limit of Θ_{b_t} in the λ^-_t coordinate. In order to do so, we will first consider the following equations $\lambda = t^{-1}\lambda_t^-$ and $\lambda_t^+ = t^2(\lambda_t^-)^{-1}$ as well as Lemma 4.15 and Lemma 4.16. That gets us to

$$\begin{aligned}\Theta_{b_t} &= t \cdot \frac{b(t^{-1}\lambda_t^-)}{t^{-1}\lambda_t \cdot \sqrt{t^{-1}\lambda_t^- \cdot a(t^{-1}\lambda_t^-)}} \mathrm{d}(\lambda_t^-) \\ &= t^{-1/2} \cdot \frac{b_t(t^{-1}\lambda_t^-)}{\lambda_t^- \cdot \sqrt{\lambda_t^- \cdot a(t^{-1}\lambda_t^-)}} \mathrm{d}(\lambda_t) \\ &= t^{-3/2} \cdot \frac{t^3 \cdot b(t^{-1}\lambda_t^-)}{\lambda_t^- \cdot \sqrt{\lambda_t^- \cdot t^4 \cdot a(t^{-1}\lambda_t^-)}} \mathrm{d}(\lambda_t^-).\end{aligned}$$

Now in a similar manner as before in the λ^+ limit we can argue that the coefficients of b^+ and b^- need to admit a factor $t^{3/4}$ in order for the differential to converge. So we can calculate

$$\begin{aligned}\Theta_{b_t} &= t^{-3/2} \cdot \frac{t^3 \cdot b(t^{-1}\lambda_t^-)}{\lambda_t^- \cdot \sqrt{\lambda_t^- \cdot t^4 \cdot a(t^{-1}\lambda_t^-)}} \mathrm{d}(\lambda_t^-) \\ &= t^{-3/2} \cdot \frac{t^3 \cdot t^{3/4} \cdot \hat{b}^+ \cdot b^0 \cdot t^{3/4} \cdot \hat{b}^-}{\lambda_t^- \cdot \sqrt{\lambda_t^- \cdot t^4 \cdot a(t^{-1}\lambda_t^-)}} \mathrm{d}(\lambda_t^-) \\ &= \frac{t^3 \cdot \hat{b}(t^{-1}\lambda_t^-)}{\lambda_t^- \cdot \sqrt{\lambda_t^- \cdot t^4 \cdot a(t^{-1}\lambda_t^-)}} \mathrm{d}(\lambda_t^-) \\ &\to \frac{(\lambda^-)^2 \cdot b^-(\lambda^-) \cdot b^+(0) \cdot b^0(0)}{\lambda^- \cdot \sqrt{\lambda^- \cdot (\lambda^-)^2 \cdot a^-(\lambda^-)}} \mathrm{d}\lambda^- \\ &= (\lambda^-)^{1/2} \frac{b^-(\lambda^-) \cdot b^-(0) \cdot b^0(0)}{\sqrt{a^-(\lambda^-)}} \mathrm{d}\lambda^- att \to 0.\end{aligned}$$

It does not surprise that this is the exact analogon to the limit we calculated in the λ^+ coordinate. Therefore, we can formulate the same conditions as before and get the same equations as before, but instead of $a^+, b_1^+, b_2^+, c_1^+, c_2^+, Q^+$ and their s-derivatives we consider $a^-, b_1^-, b_2^-, c_1^-, c_2^-, Q^-$ and their s derivatives. That in turn means we get the same conditions as before for the λ^+ coordinate. Plugging this in gets us especially the four equations

$$\begin{aligned}&i(2\lambda^+(c_1^-)'a^- + c_1^- a^- - \lambda^- c_1^-(a^-)') \\ =&2\dot{b}_1^- \tilde{\beta}_1 a^- + 2b_1^- \dot{\tilde{\beta}}_1 a^- - b_1^- \beta_1 \dot{a}^-\end{aligned} \tag{17}$$

as well as

$$\begin{aligned}&i(2\lambda^-(c_2^-)'a^- + c_2^- a^- - \lambda^- c_2^-(a^-)') \\ =&2\dot{b}_2^- \tilde{\beta}_2 a^- + 2b_2^- \dot{\tilde{\beta}}_2 a^- - b_2^- \tilde{\beta}_2 \dot{a}^-\end{aligned} \tag{18}$$

as well as

$$Q^- a^- = b_2^- \tilde{\beta}_2 c_1^- - b_1^- \tilde{\beta}_1 c_2^- \tag{19}$$

and finally

$$\begin{aligned} &\dot{b}_1^-(0) \cdot \tilde{\beta}_1 \cdot b_2^-(0) \cdot \tilde{\beta}_2 + b_1^-(0) \cdot \dot{\tilde{\beta}}_1 \cdot b_2^-(0) \cdot \tilde{\beta}_2 \\ &- b_1^-(0) \cdot \tilde{\beta}_1 \cdot \dot{b}_2^-(0) \cdot \tilde{\beta}_2 - b_1^-(0) \cdot \tilde{\beta}_1 \cdot b_2^-(0) \cdot \dot{\tilde{\beta}}_2 = 0. \end{aligned} \tag{20}$$

where we defined $\tilde{\beta}_k = b_k^+(0) \cdot b_k^0(0), \quad k = 1, 2$. It is easy to see that these equations yield the same conditions and solutions as we got in the λ^+ parameter. Therefore, it holds that $Q^- = 0$ as well and we get see that for the roots λ_k^- of b_k^-

$$c_k^-(\lambda_k^-) = 0, \ k = 1, 2$$

holds true as well. So that in turn gets us to the fact that

$$c_k^-(\lambda^-) = \mu_k^- \cdot b_k^-(\lambda^-), \ k = 1, 2$$

with $\mu_k^- \in \mathbb{C}$ holds here as well. Now we know from Lemma 4.8 (ii) that $b^-(\lambda^-) = \overline{b^+(\lambda^+)}$ holds. If we plug this in our equation we get that

$$(\mu_k^-)^{-1} \cdot c_k^-(\lambda^-) = \overline{b_k^+(\lambda^+)} = \overline{(\mu_k)^{-1}} \cdot \overline{c_k^+(\lambda^+)}, \ k = 1, 2.$$

Now by our assumptions on the polynomials $b_k^\pm$ where $b_k^+ = \overline{b_k^-}$ we see that the equation $c_k^+ = \overline{c_k^-}$ needs to hold as well. We can plug that in the equation above and reach

$$(\mu_k^-)^{-1} \cdot c_k^-(\lambda^-) = \overline{(\mu_k)^{-1}} \cdot c_k^-(\lambda^-)$$

which then yields

$$\mu_k^- = \overline{\mu_k}.$$

So now if we can prove that $\mu_k = \mu_k^-$ it automatically follows that $\mu_k = \overline{\mu_k}$ holds as well. In order to reach such a result we will need extra conditions since we used up every equation from all three parameters. One condition we have not used up so far stems from Lemma 4.8 (i). We know that $\mathfrak{b}_{k,j}^0 \in \mathbb{S}^1, \ k, j = 1, 2$ as well as $\mathfrak{b}_{1,k}^0 = \overline{\mathfrak{b}_{2,k}^0}, \ k = 1, 2$. That means that $|\mathfrak{b}_{2,k}^0| = 1$ which in turn gives us $\mathfrak{b}_{2,k}^0 \overline{\mathfrak{b}_{2,k}^0} = 1$ and therefore,

$$\dot{\mathfrak{b}}_{2,k}^0 \overline{\mathfrak{b}_{2,k}^0} + \mathfrak{b}_{2,k}^0 \dot{\overline{\mathfrak{b}_{2,k}^0}} = 0, \ k = 1, 2.$$

That means that

$$\dot{\mathfrak{b}}^0_{2,k}\overline{\mathfrak{b}^0_{2,k}} = -\mathfrak{b}^0_{2,k}\dot{\overline{\mathfrak{b}^0_{2,k}}}, \quad k = 1, 2.$$

This equation yields the relationship

$$\dot{\mathfrak{b}}^0_{2,k} = i\lambda\mathfrak{b}^0_{2,k}, \quad \lambda \in \mathbb{R}, k = 1, 2.$$

From our calculations above we know that

$$\dot{\mathfrak{b}}^0_{2,k} = -\frac{\mathfrak{b}^0_{2,k} \cdot |\dot{\mathfrak{b}}^+_{2,1}|^2}{|b^+_{2,k}|^2}, \quad k = 1, 2.$$

Now we want to use our knowledge to fully determine the values of $\dot{\mathfrak{b}}^+_{2,k}$ because this is the value in turn determining $\dot{\mathfrak{b}}^0_{2,k}$. In order to do so we will need to calculate $\dot{\beta}_k$ first. We get to

$$\begin{aligned}
\dot{\beta}_k &= \dot{\mathfrak{b}}^0_{2,k} \cdot \overline{\mathfrak{b}^+_{2,k}} + \mathfrak{b}^0_{2,k} \cdot \dot{\overline{\mathfrak{b}^+_{2,k}}} \\
&= -\mathfrak{b}^0_{2,k}\overline{\mathfrak{b}^+_{2,k}}\frac{\dot{\mathfrak{b}}^+_{2,k}\overline{\mathfrak{b}^+_{2,k}} + \mathfrak{b}^+_{2,k}\dot{\overline{\mathfrak{b}^+_{2,k}}}}{|\mathfrak{b}^+_{2,k}|^2} + \frac{\mathfrak{b}^0_{2,k}\dot{\overline{\mathfrak{b}^+_{2,k}}}\mathfrak{b}^+_{2,k}}{\mathfrak{b}^+_{2,k}} \\
&= -\frac{\mathfrak{b}^0_{2,k}}{\mathfrak{b}^+_{2,k}}\left(\dot{\mathfrak{b}}^+_{k,2}\overline{\mathfrak{b}^+_{2,k}} + \mathfrak{b}^+_{2,k}\dot{\overline{\mathfrak{b}^+_{2,k}}} - \dot{\overline{\mathfrak{b}^+_{2,k}}}\mathfrak{b}^+_{2,k}\right) \\
&= -\frac{\mathfrak{b}^0_{2,k}}{\mathfrak{b}^+_{2,k}}\dot{\mathfrak{b}}^+_{2,k}\overline{\mathfrak{b}^+_{2,k}}, \qquad k = 1, 2.
\end{aligned}$$

Now we will plug this into our solution for $\dot{\mathfrak{b}}^+_{2,k}$ to get to

$$\begin{aligned}
\dot{\mathfrak{b}}^+_{2,k} &= \mathfrak{b}^+_{2,k}\frac{i\mu_k + 2\dot{\beta}_k}{2\mathfrak{b}^0_{2,k}\overline{\mathfrak{b}^+_{2,k}}} \\
&= \frac{\mathfrak{b}^+_{2,k}i\mu_k}{2\mathfrak{b}^0_{2,k}\overline{\mathfrak{b}^+_{2,k}}} - \frac{2\mathfrak{b}^+_{2,k}}{2\mathfrak{b}^0_{2,k}\overline{\mathfrak{b}^+_{2,k}}}\frac{\mathfrak{b}^0_{2,k}}{\mathfrak{b}^+_{2,k}}\dot{\mathfrak{b}}^+_{k,2}\overline{\mathfrak{b}^+_{2,k}} \\
&= \frac{\mathfrak{b}^+_{2,k}i\mu_k}{2\mathfrak{b}^0_{2,k}\overline{\mathfrak{b}^+_{2,k}}} - \frac{\dot{\mathfrak{b}}^+_{k,2}\overline{\mathfrak{b}^+_{2,k}}}{\overline{\mathfrak{b}^+_{2,k}}} \\
&= \frac{\mathfrak{b}^+_{2,k}i\mu_k}{2\mathfrak{b}^0_{2,k}\overline{\mathfrak{b}^+_{2,k}}} - \dot{\mathfrak{b}}^+_{2,k}, \qquad k = 1, 2.
\end{aligned}$$

Therefore, we can now add $\dot{\mathfrak{b}}^+_{2,k}$ to both sides and by dividing both sides by two we reach

$$\dot{\mathfrak{b}}^+_{2,k} = \frac{i\mu_k\mathfrak{b}^+_{2,k}}{4\mathfrak{b}^0_{2,k}\overline{\mathfrak{b}^+_{2,k}}} = \frac{i\mu_k}{4\mathfrak{b}^0_{2,k}}\frac{(\mathfrak{b}^+_{2,k})^2}{|\mathfrak{b}^+_{2,k}|^2}, \qquad k = 1, 2.$$

Now we want to plug our final solution for $\dot{\mathfrak{b}}^+_{2,k}$ into the solution for $\dot{\mathfrak{b}}^0_{2,k}$. In order to do so, let us recap the solution we have calculated so far

$$\dot{\mathfrak{b}}^0_{2,k} = -\mathfrak{b}^0_{2,k}\frac{\dot{\mathfrak{b}}^+_{2,k}\overline{\mathfrak{b}^+_{2,k}} + \mathfrak{b}^+_{2,k}\dot{\overline{\mathfrak{b}^+_{2,k}}}}{|\mathfrak{b}^+_{2,k}|^2}, \quad k = 1, 2.$$

We will now use our knowledge of $\dot{\mathfrak{b}}^+_{2,k}$ in order to fully determine $\dot{\mathfrak{b}}^0_{2,k}$

$$\begin{aligned}
\dot{\mathfrak{b}}^0_{2,k} &= -\mathfrak{b}^0_{2,k}\frac{\dot{\mathfrak{b}}^+_{2,k}\overline{\mathfrak{b}^+_{2,k}} + \mathfrak{b}^+_{2,k}\dot{\overline{\mathfrak{b}^+_{2,k}}}}{|\mathfrak{b}^+_{2,k}|^2} \\
&= -\frac{\mathfrak{b}^0_{2,k}}{|\mathfrak{b}^+_{2,k}|^2}\Big(\frac{i\mu_k\mathfrak{b}^+_{2,k}}{4\mathfrak{b}^0_{2,k}} + \frac{-i\overline{\mu_k}\overline{\mathfrak{b}^+_{2,k}}}{4\overline{\mathfrak{b}^0_{2,k}}}\Big) \\
&= -i\mathfrak{b}^0_{2,k}\Big(\frac{\mu_k}{4\mathfrak{b}^0_{2,k}\overline{\mathfrak{b}^+_{2,k}}} - \frac{\overline{\mu_k}}{4\overline{\mathfrak{b}^0_{2,k}}\mathfrak{b}^+_{2,k}}\Big), \qquad k = 1, 2,
\end{aligned}$$

We have now fully calculated most of our solutions and it can be seen that a lot of them depend on β_k or some other combination of $\mathfrak{b}^+_{2,k}$ as well as $\mathfrak{b}^0_{2,k}$. We want to establish an relation between these values in order to further simplify our results. Now from the unpublished paper from Carberry et al. (2020), the proof of Lemma 2.10 establishes the following relationship. Let $(t_n)_{n\in\mathbb{N}}$ be a real sequence of numbers s.t. $\lim_{n\to\infty} t_n = 0$. Then for $\delta = \lim_{n\to\infty} b^0_{t_n}(0)$ it holds that

$$\delta = \frac{\overline{b^-(0)}}{|b^-(0)|} = \frac{b^+(0)}{|b^+(0)|}.$$

Now since this relation was proven for an arbitrary sequence and arbitrary polynomials $b \in P^3_{\mathbb{R}}$ it follows that

$$b^0_k(0) = \mathfrak{b}^0_{2,k} = \frac{\mathfrak{b}^+_{2,k}}{|\mathfrak{b}^+_{2,k}|}, \qquad k = 1, 2.$$

This yields a new result for β_k, namely

$$\begin{aligned}
\beta_k = \mathfrak{b}^0_{2,k}\overline{\mathfrak{b}^+_{2,k}} &= \frac{\mathfrak{b}^+_{2,k}\overline{\mathfrak{b}^+_{2,k}}}{|\mathfrak{b}^+_{2,k}|} \\
&= |\mathfrak{b}^+_{2,k}|, \qquad k = 1, 2.
\end{aligned}$$

The new results for β_k as well as $\dot{\beta}_k$ will now be used to completely determine all derivatives in the λ^+ as well as the λ limit. However we also need to prove that the variable μ_k on which all solutions depend linearly is a one-dimensional coordinate. Therefore, we will now try to evaluate the solution for $\dot{\mathfrak{b}}_{1,k}^+$. By Lemma 4.8 (iii) proves that $\mathfrak{b}_{1,k}^+ \in \mathbb{R}$ needs to hold since $\mathfrak{b}_{1,k}^- \in \mathbb{R}_+$ and so $\mathfrak{b}_{1,k}^+ = \overline{\mathfrak{b}_{1,k}^-} = \mathfrak{b}_{1,k}^- \in \mathbb{R}$. Therefore, the same needs to hold true for the derivative. We know that

$$\dot{b}_k^+(\lambda_k^+) = -\frac{i\mu_k \mathfrak{b}_{2,k}^+}{\beta_k}, \qquad k = 1, 2,$$

holds which we can use to examine said term. In order to so, we will expand the formula in the following way

$$\dot{\mathfrak{b}}_{1,k}^+\lambda_k^+ + \dot{\mathfrak{b}}_{2,k}^+ = \dot{\mathfrak{b}}_{1,k}^+\Big(-\frac{\mathfrak{b}_{2,k}^+}{\mathfrak{b}_{1,k}^+}\Big) + \dot{\mathfrak{b}}_{2,k}^+ = -\frac{i\mu_k \mathfrak{b}_{2,k}^+}{\mathfrak{b}_{2,k}^0 \overline{\mathfrak{b}_{2,k}^+}}, \qquad k = 1, 2.$$

We will now use our solution for $\dot{\mathfrak{b}}_{2,k}^+$ in order to do further evaluations.

$$\begin{aligned}\dot{\mathfrak{b}}_{1,k}^+ &= \frac{i\mu_k \mathfrak{b}_{1,k}^+}{\mathfrak{b}_{2,k}^0 \overline{\mathfrak{b}_{2,k}^+}} + \frac{\mathfrak{b}_{1,k}^+}{\mathfrak{b}_{2,k}^+}\frac{i\mu_k \mathfrak{b}_{2,k}^+}{4\mathfrak{b}_{2,k}^0 \overline{\mathfrak{b}_{2,k}^+}} \\ &= \frac{5i\mu_k \mathfrak{b}_{1,k}^+}{4\mathfrak{b}_{2,k}^0 \overline{\mathfrak{b}_{2,k}^+}}, \qquad k = 1, 2.\end{aligned}$$

Next we will plug again our knowledge of $\mathfrak{b}_{2,k}^0$ into the equation and reach

$$\begin{aligned}\dot{\mathfrak{b}}_{1,k}^+ &= \frac{5i\mu_k \mathfrak{b}_{1,k}^+}{4\overline{\mathfrak{b}_{2,k}^+}}\frac{|\mathfrak{b}_{2,k}^+|}{\mathfrak{b}_{2,k}^+} \\ &= \frac{5i\mu_k \mathfrak{b}_{1,k}^+}{4|\mathfrak{b}_{2,k}^+|}, \qquad k = 1, 2.\end{aligned}$$

That means

$$\frac{5i\mu_k \mathfrak{b}_{1,k}^+}{4|\mathfrak{b}_{2,k}^+|} \in \mathbb{R}, \qquad k = 1, 2$$

needs to hold. So now we see that $|\mathfrak{b}_{2,k}^+| \in \mathbb{R}$ holds and by assumption as before $\mathfrak{b}_{1,k}^+ \in \mathbb{R}$ holds as well. So that means our condition becomes

$$i\mu_k \in \mathbb{R}, \qquad k = 1, 2.$$

So therefore, we see that μ_1 and μ_2 are one-dimensional parameters. Recalling the relation between μ_1 and μ_2 as well as our knowledge of β_k gets us

$$\mu_1 = \frac{\beta_1 \mu_2}{\beta_2} = \mu_2 \frac{|\mathfrak{b}_{2,1}^+|}{|\mathfrak{b}_{2,2}^+|}.$$

We now recall the solutions for all the derivatives we calculated and plug in the knowledge we have gained on β_k. First we consider the solutions we calculated for the λ^+ limit

$$\begin{aligned}
\dot{\mathfrak{a}}^+ &= \frac{i\mu_1\sqrt{\mathfrak{a}^2-4}}{|\mathfrak{b}^+_{2,1}|},\\
\dot{\mathfrak{b}}^+_{1,k} &= \frac{5i\mu_k\mathfrak{b}^+_{1,k}}{4|\mathfrak{b}^+_{2,k}|}, k=1,2,\\
\dot{\mathfrak{b}}^+_{2,k} &= \frac{i\mu_k\mathfrak{b}^+_{2,k}}{4|\mathfrak{b}^+_{2,k}|}, k=1,2,\\
c^+_{k,1} &= \mu_1\mathfrak{b}^+_{k,1}, k=1,2,\\
c^+_{k,2} &= \mu_1\mathfrak{b}^+_{k,2}\frac{|\mathfrak{b}^+_{2,2}|}{|\mathfrak{b}^+_{2,1}|}, k=1,2.
\end{aligned}$$

Here it is easy to see that in fact every solution is linearly dependent on μ_1 which we have proven to be one-dimensional. Since the solutions for the λ^- are just the complex conjugates of these solutions and $\overline{\mu_1}=-\mu_1$ holds as well these solutions also all depend linearly on μ_k. Now it remains to consider the solutions for the λ coordinate. Here we will use everything we know so far to simplify our results. First we will calculate $|\dot{\mathfrak{b}^+_{2,k}}|^2$ since this will simplify the following calculations a lot. We get

$$\begin{aligned}
|\dot{\mathfrak{b}^+_{2,k}}|^2 &= \dot{\mathfrak{b}}^+_{2,k}\overline{\mathfrak{b}^+_{2,k}} + \overline{\dot{\mathfrak{b}}^+_{2,k}\overline{\mathfrak{b}^+_{2,k}}}\\
&= \frac{i\mu_k\mathfrak{b}^+_{2,k}\overline{\mathfrak{b}^+_{2,k}}}{4|\mathfrak{b}^+_{2,k}|} + \frac{-i\overline{\mu_k}\overline{\mathfrak{b}^+_{2,k}}\mathfrak{b}^+_{2,k}}{4|\mathfrak{b}^+_{2,k}|}\\
&= \frac{1}{4}\Big(i\mu_k|\mathfrak{b}^+_{2,k}| + i\mu_k|\mathfrak{b}^+_{2,k}|\Big)\\
&= \frac{i\mu_k|\mathfrak{b}^+_{2,k}|}{2}, \qquad k=1,2.
\end{aligned}$$

Next we will use our knowledge to furhter determine $\dot{\mathfrak{b}}^0_{2,k}$ which gives

$$\begin{aligned}
\dot{\mathfrak{b}}^0_{2,k} &= -i\mathfrak{b}^0_{2,k}\Big(\frac{\mu_k}{4\mathfrak{b}^0_{2,k}\overline{\mathfrak{b}^+_{2,k}}} - \frac{\overline{\mu_k}}{4\overline{\mathfrak{b}^0_{2,k}}\mathfrak{b}^+_{2,k}}\Big)\\
&= \frac{-i\mathfrak{b}^0_{2,k}(\mu_k+\mu_k)}{4|\mathfrak{b}^+_{2,k}|}\\
&= \frac{-i\mu_k\mathfrak{b}^0_{2,k}}{2|\mathfrak{b}^+_{2,k}|}, \qquad k=1,2.
\end{aligned}$$

Now we can plug this into our result for $\dot{\mathfrak{b}}^0_{1,k}$ and get to

$$\begin{aligned}\dot{\mathfrak{b}}^0_{1,k} &= \frac{\mathfrak{b}^0_{1,k}|\dot{\mathfrak{b}^+_{2,k}}|^2}{|\mathfrak{b}^+_{2,k}|^2}\\ &= \frac{i\mu_k\mathfrak{b}^0_{1,k}}{2|\mathfrak{b}^+_{2,k}|}, \qquad k=1,2.\end{aligned}$$

Now it remains to completely determine the values of $c^0_{j,k}, j,k=1,2$. We will do so making use of our knowledge of the other derivatives. First we insert everything in the solution for $c^0_{2,k}$ to get

$$\begin{aligned}c^0_{2,k} &= -2i(\dot{\mathfrak{b}}^0_{2,k}\cdot|\mathfrak{b}^+_{2,k}|^2 + \mathfrak{b}^0_{2,k}\cdot|\dot{\mathfrak{b}^+_{2,k}}|^2)\\ &= -2i\Big(-\frac{i\mu_k\mathfrak{b}^0_{2,k}}{2|\mathfrak{b}^+_{2,k}|}\cdot|\mathfrak{b}^+_{2,k}|^2 + \mathfrak{b}^0_{2,k}\cdot\frac{i\mu_k|\mathfrak{b}^+_{2,k}|}{2}\Big)\\ &= -2i\Big(-\frac{i\mu_k\mathfrak{b}^0_{2,k}|\mathfrak{b}^+_{2,k}|}{2} + \frac{i\mu_k\mathfrak{b}^0_{2,k}|\mathfrak{b}^+_{2,k}|}{2}\Big)\\ &= 0, \qquad k=1,2.\end{aligned}$$

Now we need to calculate the leading coefficient of c^0_k where we can use our latest result to see that the last term vanishes

$$\begin{aligned}c^0_{1,k} &= -\frac{2\mathfrak{b}^0_{1,k}}{\mathfrak{b}^0_{2,k}}\Big(i((\dot{\mathfrak{b}}^0_{1,k}\Big(-\frac{\mathfrak{b}^0_{2,k}}{\mathfrak{b}^0_{1,k}}\Big) + \dot{\mathfrak{b}}^0_{2,k})|\mathfrak{b}^+_{2,k}|^2 + (\dot{\mathfrak{b}}^0_{2,2}\cdot|\mathfrak{b}^+_{2,2}|^2 + \mathfrak{b}^0_{2,2}\cdot|\dot{\mathfrak{b}^+_{2,2}}|^2)\Big)\\ &= -\frac{2\mathfrak{b}^0_{1,k}}{\mathfrak{b}^0_{2,k}}\Big(i\Big(-\frac{\mathfrak{b}^0_{2,k}}{\mathfrak{b}^0_{1,k}}\frac{i\mu_k\mathfrak{b}^0_{1,k}}{2|\mathfrak{b}^+_{2,k}|} - \frac{i\mu_k\mathfrak{b}^0_{2,k}}{2|\mathfrak{b}^+_{2,k}|}\Big)|\mathfrak{b}^+_{2,k}|^2\Big)\\ &= -\frac{2i\mathfrak{b}^0_{1,k}}{\mathfrak{b}^0_{2,k}}\Big(-\frac{i\mu_k\mathfrak{b}^0_{2,k}}{2|\mathfrak{b}^+_{2,k}|} - \frac{i\mu_k\mathfrak{b}^0_{2,k}}{2|\mathfrak{b}^+_{2,k}|}\Big)|\mathfrak{b}^+_{2,k}|^2\Big)\\ &= -\frac{2i\mathfrak{b}^0_{1,k}}{\mathfrak{b}^0_{2,k}}\Big(-\frac{i\mu_k\mathfrak{b}^0_{2,k}}{|\mathfrak{b}^+_{2,k}|}\Big)|\mathfrak{b}^+_{2,k}|^2\Big)\\ &= -2\mu_k\mathfrak{b}^0_{1,k}|\mathfrak{b}^+_{2,k}|, \qquad k=1,2.\end{aligned}$$

So all in all we get here

$$\begin{aligned}\dot{\mathfrak{b}}^0_{1,k} &= \frac{i\mu_k\mathfrak{b}^0_{1,k}}{2|\mathfrak{b}^+_{2,k}|}, \qquad k=1,2,\\ \dot{\mathfrak{b}}^0_{2,k} &= -\frac{i\mu_k\mathfrak{b}^0_{2,k}}{2|\mathfrak{b}^+_{2,k}|}, \qquad k=1,2,\\ c^0_{1,k} &= -2\mu_k\mathfrak{b}^0_{1,k}|\mathfrak{b}^+_{2,k}|, \qquad k=1,2,\\ c^0_{2,k} &= 0, \qquad k=1,2.\end{aligned}$$

Therefore, we have now also fully proven that every polynomial in the λ limit is linearly dependent on μ_1 as well. We have also established that $(\dot{a}^-, \dot{b}_1^-, \dot{b}_2^-, c_1^-, c_2^-)$ but from our considerations before $\mu_1 \in i\mathbb{R}$ holds which means $\overline{\mu_k} = -\mu_k$ and therefore, this tuple depends linearly on μ_k as well. Therefore, all three tuples depend linearly on $\overline{\mu_1}$ so by the implicit function theorem $(\dot{a}^+, \dot{b}_1^+, \dot{b}_2^+, c_1^+, c_2^+, \dot{b}_1^0, \dot{b}_2^0, c_1^0, c_2^0, \dot{a}^-, \dot{b}_1^-, \dot{b}_2^-, c_1^-, c_2^-)$ make up a one-dimensional manifold just as in the case of bounded coefficients established in the Master's thesis by B.Schmidt (2020) one could assume if the blow-up was chosen correctly. However one can observe that all of the derivatives are not depending on t. So this poses the problem that the solutions in this point don't represent the solutions outside of the point $t = 0$ so we are not able to use the implicit function theorem here. A potential solution to this problem is that we have before first calculated the limits $t \to 0$ and then calculated the s-derivatives which explains why none of the solutions depend on t and why $\dot{t}$ doesn't occur. In order to do so we will now do the opposite of what we have been doing before. We start with the λ^+ coordinate again becausse we can use our calculations for the λ^--coordinate as well. We start with $a(\lambda)$ and get

$$\begin{aligned}
&\lim_{t\to 0} \frac{\mathrm{d}}{\mathrm{d}s} (\lambda^+)^{-2} a^+(\lambda^+) a^-(t^2/(\lambda^+)) \\
&= \lim_{t\to 0} (((\dot{\alpha}\lambda^+ a^-(t^2/(\lambda^+)) + a^+(\lambda^+) 2t(\dot{\overline{\alpha}} t + \dot{t}\overline{\alpha})) \\
&= \dot{\alpha}\lambda^+.
\end{aligned}$$

So the derivatives for $a(\lambda)$ do not change from this procedure. Next we will do the same calculation for $b(\lambda)$ in the λ^+-coordinate. We get

$$\begin{aligned}
&\lim_{t\to 0} \frac{\mathrm{d}}{\mathrm{d}s} (\lambda^+)^{-1} b^+(\lambda^+) b^0(t/(\lambda^+)) b^-(t^2/(\lambda^+)) \\
&= \lim_{t\to 0} (\lambda^+)^{-1} (\dot{b}^+(\lambda^+) \cdot b^0(t/\lambda^+) b^-(t^2/\lambda^+) + b^+(\lambda^+)(\dot{t}\mathfrak{b}_1^0 \lambda^+ + \dot{b}^0(t/\lambda^+)) b^-(t^2/\lambda^+)) \\
&\quad + t(b^+(\lambda^+)\dot{b}^0(t/\lambda^+) b^-(t^2/\lambda^+) + b^+(\lambda^+) b^0(t/\lambda^+) 2t(\dot{t} b^-(t^2/\lambda^+) + \dot{b}^-(t^2/(\lambda^+))) \\
&= (\lambda^+)^{-1} (\dot{b}^+(\lambda^+) \cdot \beta + (\dot{t}\mathfrak{b}_1^0 \lambda^+ + \dot{b}^0(0))(b^+(\lambda^+) b^-(0))).
\end{aligned}$$

Here we see the first change to our previous approach which is that $\dot{t}$ occurs. Now we only need to calculate the limit of the s-derivative of $b(\lambda)$ in the limit $t \to 0$. We calculate

$$
\begin{aligned}
&\lim_{t\to 0} \frac{\mathrm{d}}{\mathrm{d}s} t \cdot (t/\lambda)^{-1} b^+(t/\lambda) b^0(\lambda) b^-(t\lambda) \\
&= \lim_{t\to 0} \frac{\mathrm{d}}{\mathrm{d}s} (t\mathfrak{b}_1^+ + \mathfrak{b}_2^+ \lambda) b^0(\lambda) b^-(t\lambda) \\
&= \lim_{t\to 0} \frac{\mathrm{d}}{\mathrm{d}s} ((\dot{t}\mathfrak{b}_1^+ + t\dot{\mathfrak{b}}_1^+ + \dot{\mathfrak{b}}_2^+ \lambda) b^0(\lambda) b^-(t\lambda) \\
&\quad + \lambda b^+(t/\lambda) \dot{b}^0(\lambda) b^-(t\lambda) + \lambda b^+(t/\lambda) b^0(\lambda)(\dot{b}^-(t\lambda) + \dot{t}\mathfrak{b}_1^- \lambda) \\
&= (\dot{t} b^+(0) + \dot{\mathfrak{b}}_2^+ \lambda) b^0(\lambda) b^-(0) + \lambda |b^+(0)|^2 \dot{b}^0(\lambda) + \lambda b^+(0) b^0(\lambda)(\dot{b}^-(0) + \dot{t}\mathfrak{b}_1^- \lambda) \\
&= \lambda |b^+(0)|^2 \dot{b}^0(\lambda) + b^0(\lambda)(\dot{t}(\mathfrak{b}_1^+ + \overline{\mathfrak{b}_1^+}\lambda) + \dot{\mathfrak{b}}_2^+ \lambda + \dot{\overline{\mathfrak{b}_2^+}}).
\end{aligned}
$$

Again this changes the previous approach since new derivatives occur in this calculation, especially $\dot{t}$ again. Therefore, the next step here is to recalculate everything using the now, correct approach and solve for the derivatives, including $\dot{t}$. But since this mistake was only discovered in the last weeks of this thesis, it was not possible to solve the right equations in time. This is a topic for future research.

5 Intersections with $\mathcal{S}^2$

In this chapter we will use the results of chapter 6 from B.Schmidt (2020) with the goal of examining the intersections of the one-dimensional manifolds we considered in Chapter 4 with $\mathcal{S}^2$. From Definition 3.28 we see hat if we want to consider the case of an intersection with $\mathcal{S}^2$ that gives us the case that b_1 and b_2 have a common root, namely on $\mathbb{S}^1$ because of the reality condition. Theorem 3.2 from Carberry and Schmidt (2016) proves that because we consider the case of genus $g = 2$ then $\gcd(b_1, b_2) \leq 1$ needs to hold so in this chapter we will consider the case where $\gcd(b_1, b_2) = 1$ holds. In chapter three we have defined functions $\mu_\omega \colon \Sigma^* \to \mathbb{C}^\times$ which are defined by the action of the monodromies $M_\omega = F(\omega)$. We also defined a relation between our polynomials b_k and such function μ_k as follows The equations

$$\Theta_{b_k} = \mathrm{d}q_k = \mathrm{d}\log\mu_k, \qquad k = 1, 2.$$

Now we can write this in terms of μ_k differently where we get

$$\Theta_k = \frac{\mathrm{d}\mu_k}{\mu_k}, \qquad k = 1, 2.$$

So we know that $q_k = \log\mu_k$ where we obviously need to consider the complex logarithm. So then it follows that this is not uniquely defined because we can only consider this relation on branches of the complex logarithm. That means in consequence that $\Im(\log\mu_k)$ is only defined up to $2\pi i\mathbb{Z}$. Now we will define the following

Definition 5.1. *Let $\Sigma|_{\mathbb{S}^1}$ be the following set*

$$\Sigma|_{\mathbb{S}^1} = \{(\lambda, \nu) \in \mathbb{S}^1 \times \mathbb{C} \mid (\lambda, \nu) \in \Sigma\}$$

which is the restriction of our Riemann surface to the unit circle. Geometrically it looks like a circle where each value is assumed twice because Σ is a two-sheeted covering of $\mathbb{C}\mathbf{P}^1$ it covers $\mathbb{S}^1$ as well.

We will now note that Corollary 4.13 from B.Schmidt (2020) specifically treats the case where the polynomials b_1 and b_2 have a common root on $\mathbb{S}^1$. Therefore, we can follow that $T^{-1}(\tau_a)$ is still a one-dimensional manifold if the set intersects $\mathcal{S}^2$.
Now we will define an algebraic curve in the following way

Definition 5.2. *Let $a \in \mathcal{M}_2^1$ and $b_1, b_2 \in \mathcal{B}_a$ uniquely defined as usual. Now we will restrict ourselves to one branch of the complex logarithm and define*

$$V(q_1, q_2) = \{(\Im(q_1(\lambda)), \Im(q_2(\lambda))) \in \mathbb{R}^2 \mid \lambda \in \mathbb{S}^1\}$$

which is a curve in the real plane.

Now from our results in chapter 4 as well the results from B. Schmidt (2020) we can consider $T^{-1}(\tau_a)$ as a one dimensional manifold $(a, b_1, b_2)(s)$ where s is a real parameter and therefore, also consider the set of curves $V(q_1, q_2)(s)$ defined by said manifolds. The goal in this section is to show that each set of curves $V(q_1, q_2)(s)$ pertaining to a connected component of $T^{-1}(\tau_a)$ intersects with $\mathcal{S}^2$ at most once.

Proposition 5.3. *The curve $V(q_1, q_2) \subset \mathbb{R}^2$ is a closed curve.*

Proof: The Riemann surface Σ is constructed by connecting 0 with ∞ and connecting the roots α_i each with $\overline{\alpha_i}^{-1}$ as in Figure 2.2 which then gives a two-sheeted covering of $\mathbb{C}\mathbf{P}^1$ where we change the sheet three times if pass these points. That gets us that the preimage of $\mathbb{S}^1$ in this manifold is exactly a manifold that is diffeomorphic to $\mathbb{S}^1$ which gets assumed twice because we consider a two-sheeted covering. Therefore, we change the sheet every time we pass one of these connected points. That also means that we need to travel twice in the plane to reach the point we originated from in Σ. Now we know that the map σ interchanges the sheets of Σ and we also know that it acts on q as $\sigma^* q = -q$ and therefore, $\sigma^*\Theta = -\Theta$. Consequently we get that for every curve that goes around twice in the plane and so especially for $\Sigma|_{\mathbb{S}^1}$ it follows that

$$\begin{aligned}\int_{\Sigma|_{\mathbb{S}^1}} \Theta &= \int_{\gamma_1} \Theta + \int_{\sigma^*\gamma_1} \Theta \\ &= \int_{\gamma_1} \Theta + \int_{\gamma_1} \sigma^*\Theta \\ &= \int_{\gamma_1} \Theta - \int_{\gamma_1} \Theta \\ &= 0\end{aligned}$$

where $\Sigma|_{\mathbb{S}^1} = \gamma_1 + \sigma^*\gamma_1$. However since the relation $\Theta = \mathrm{d}q$ holds it follows that the integral of the derivative of q - and so especially of the imaginary part - vanishes making the curve closed. **q.e.d.**

Proposition 5.4. *The winding number $n(\tilde{f})$ corresponding to a triple $(a, b_1, b_2) \in T^{-1}(\tau_a)$ changes when the level set $T^{-1}(\tau_a)$ intersects with $\mathcal{S}^2$*

Proof: From the paper by Carberry and Schmidt (2016) we know that $n(\tilde{f}) = \pm 1$ holds. **q.e.d.**

Our next goal is to establish a connection between the Willmore functional defined in Definition 3.20 and $V(q_1, q_2)$.

Proposition 5.5. *The Willmore functional is proportional to* $\mathrm{Vol}(V(q_1, q_2))$*.*

Proof: In the paper from Knopf et al. (2018) Theorem 5.7 establishes the following formula for the Willmore functional

$$W(a) = 4i \operatorname{Res}_{\lambda=0} \log(\mu_2) \mathrm{d} \log(\mu_1).$$

Since we know that $q_k = \log(\mu_k)$ so we can write this as

$$W(a) = 4i \operatorname{Res}_{\lambda=0} q_2 \, \mathrm{d}q_1.$$

Now we will try to further calculate the right hand side to get to

$$\begin{aligned} 4i \operatorname{Res}_{\lambda=0} q_2 \, \mathrm{d}q_1 &= \frac{4i}{2\pi i} \int_{\mathbb{S}^1} q_2 \, \mathrm{d}q_1 \\ &= \frac{2}{\pi} \int_{\mathbb{S}^1} q_2 \, \mathrm{d}q_1 \\ &= \frac{2}{\pi} \int_{\mathbb{S}^1} \Im(q_2) \, \mathrm{d}\Im(q_1) \\ &= \frac{2}{\pi} \int_0^{2\pi} y(s) \mathrm{d}(x(s)) \\ &= \frac{2}{\pi} \int_0^{2\pi} y(s) x'(s) \mathrm{d}(x(s)). \end{aligned}$$

Here we have used that the functions are purely imaginary on the unit circle, the definition of the residuum and have written q_1 and q_2 as the coordinates x and y of $V(q_1, q_2)$. On the other hand since the volume can be determined as

$$\mathrm{Vol}(V(q_1, q_2)) = \frac{1}{2} \int_0^{2\pi} y(s) x'(s) \mathrm{d}(x(s))$$

we see that $W(a)$ is proportional to $\mathrm{Vol}(V(q_1, q_2))$. **q.e.d.**

Proposition 5.6. $\mathrm{Vol}(V(q_1, q_2))$ *is either monotonously increasing or decreasing along each connected component of* $T^{-1}(\tau_a)$ *not intersecting* $\mathcal{S}^2$*. The monotonicity only changes at intersections with* $\mathcal{S}^2$*.*

Proof: We will now use several calculations concerning $\dot{W}(a)$ in order to prove our claims. They stem from the 6th chapter of B.Schmidt (2020).

$$\begin{aligned} \dot{W}(a) &= 4i \frac{c_2 b_1 - c_1 b_2}{\nu^2} \frac{\mathrm{d}\lambda}{\lambda} \\ &= 4i Q_{11} \frac{\mathrm{d}\lambda}{\lambda}. \end{aligned}$$

Now looking at the second formula we also recall that in B.Schmidt (2020) it is proven that $Q_{11} \neq 0$ holds for all $a \in \mathcal{M}_2^1 \setminus \mathcal{S}^2$. Since $\frac{d\lambda}{\lambda}$ has no roots on $\mathbb{S}^1$ that in turn means that $\dot{W}(a) \neq 0$ holds for all $a \in \mathcal{M}_2^1 \setminus \mathcal{S}^2$. Therefore, by Proposition 5.4 Vol$(V(q_1, q_2))$ is either monotonously decreasing or increasing. However if $a \in \mathcal{M}_2^1 \cap \mathcal{S}^2$ it follows that $Q_{11} = 0$ needs to hold. Therefore, we get that $\dot{W}(a)$ vanishes in these instances. So the sign of $W(a)$ changes for $a \in \mathcal{M}_2^1 \cap \mathcal{S}^2$. By way of Proposition 5.4 we get the same result for Vol$(V(q_1, q_2))$ which completes the proof. **q.e.d.**

Our new goal will be to continue the calculations done in chapter 6 by B.Schmidt (2020). We will assume that every polynomial we consider in the Whitham equations can be written as a formal power series and try to calculate the coefficients using the results already established in the chapters 4 and 6 from B.Schmidt (2020). So we will use another ansatz where we calculate the Whitham equations in the Cayley transform, as in chapter 6 of the Bachelor thesis by B.Schmidt (2017). To recall this we define the Cayley transform

Definition 5.7. *The Cayley transform is the map*

$$\kappa : \begin{cases} \mathbb{CP}^1 & \to \mathbb{CP}^1 \\ \lambda & \mapsto \kappa(\lambda) = \frac{\lambda - i}{\lambda + i} \end{cases}$$

We then get that $\nu^2 = (\kappa^2 + 1)a(\kappa)$ where $a(\kappa) = \kappa^4 + a_1\kappa^3 + a_2\kappa^2 + a_3\kappa + a_4$ with $a_i \in \mathbb{R}$ holds. Further we recall that we are still in the case of intersecting with $\mathcal{S}^2$. Here we can now assume that $\lambda_0 \in \mathbb{S}^1$ which denotes the common root of b_1 and b_2 gets mapped to 0 under the Cayley transform so $b_1(0) = 0 = b_2(0)$ now holds. Further the polynomials b_k also have real coefficients. So we can write the polynomials b_k in the following form $b_k(\kappa) = b_{1,k}\kappa^3 + b_{2,k}\kappa^2 + b_{3,k}\kappa$ with $b_{i,k} \in \mathbb{R}$ for $i = 1, 2$ and $k = 1, 2, 3$. Next we take a look at Θ_k and see that

$$\Theta_k = \frac{b_k(\kappa)}{\nu} \frac{d\kappa}{\kappa^2 + 1}, \qquad k = 1, 2$$

holds. By Schwarz lemma we get

$$\frac{\partial^2 q_k}{\partial t \partial \kappa} = \frac{\partial^2 q_k}{\partial \kappa \partial t}, \qquad k = 1, 2.$$

Here we consider the same q_k we defined in the beginning of this chapter in the κ-coordinates. However we know that

$$\frac{\partial q_k}{\partial t} = \frac{i c_k(\kappa)}{\nu}, \qquad k = 1, 2$$

needs to hold as well where $c_k(\kappa) \in P^3_{\mathbb{R}}$. Therefore, we get the following equation if we plug this in the equation above and cancel the $\mathrm{d}\kappa$ on the left hand side

$$\frac{\partial}{\partial t}\frac{b_k(\kappa)}{\nu}\frac{1}{\kappa^2+1} = \frac{\partial}{\partial \kappa}\frac{ic_k(\kappa)}{\nu}, \qquad k = 1, 2.$$

In a next step we will calculate both sides. Starting with the left hand side gets us

$$\begin{aligned}\frac{\partial}{\partial t}\frac{b_k(\kappa)}{\nu}\frac{1}{\kappa^2+1} &= \frac{i\dot{b}_k(\kappa)\nu(\kappa^2+1) - ib_k(\kappa)\dot{\nu}(\kappa^2+1)}{\nu^2(\kappa^2+1)^2} \\ &= \frac{i\dot{b}(\kappa)}{\nu(\kappa^2+1)} - \frac{ib_k\frac{\dot{a}(\kappa)}{2\nu}}{\nu^2} \\ &= \frac{2i\dot{b}_k(\kappa)a(\kappa) - ib_k(\kappa)\dot{a}(\kappa)}{2(\kappa^2+1)a(\kappa)\nu}, \qquad k = 1, 2.\end{aligned}$$

We now further evaluate the right hand side as well to get to

$$\begin{aligned}\frac{\partial}{\partial \kappa}\frac{ic_k(\kappa)}{\nu} &= \frac{ic_k'(\kappa)\nu - ic_k(\kappa)\nu'}{\nu^2} \\ &= \frac{ic_k'(\kappa)}{\nu} - \frac{ic_k(\kappa)\frac{2\kappa a(\kappa)+(\kappa^2+1)a'(\kappa)}{2\nu}}{\nu^2} \\ &= \frac{2ic_k(\kappa)a(\kappa)(\kappa^2+1) - 2ic_k(\kappa)a(\kappa)\kappa - ic_k(\kappa)a'(\kappa)(\kappa^2+1)}{2(\kappa^2+1)a(\kappa)\nu}, \qquad k = 1, 2.\end{aligned}$$

Now equaling both sides we see that the denominator cancels as well as i. So our equations become

$$\begin{aligned}&2c_1'(\kappa)a(\kappa)(\kappa^2+1) - 2c_1(\kappa)a(\kappa)\kappa - c_1(\kappa)a'(\kappa)(\kappa^2+1) \\ &= 2\dot{b}_1(\kappa)a(\kappa) - b_1(\kappa)\dot{a}(\kappa)\end{aligned} \tag{21}$$

as well as

$$\begin{aligned}&2c_2'(\kappa)a(\kappa)(\kappa^2+1) - 2c_2(\kappa)a(\kappa)\kappa - c_2(\kappa)a'(\kappa)(\kappa^2+1) \\ &= 2\dot{b}_2(\kappa)a(\kappa) - b_2(\kappa)\dot{a}(\kappa).\end{aligned} \tag{22}$$

We note here that the degrees of both sides do not seem to match up because the left hand side appears to have degree 8 but the right hand side only has degree 7 at most. But if we calculate the leading coefficient on the left hand side and use that the highest coefficient of $a(\kappa)$ is one we get that the leading coefficient appears to be

$$6c_{1,k} - 2c_{1,k} - 4c_{1,k} = 0, \qquad k = 1, 2,$$

so since this vanishes we get that the degree of the left hand side is at most seven as well and therefore, the sides of the equation match up. Now we will also evaluate the equation

$$\dot{q}_1 \mathrm{d}q_2 - \dot{q}_2 \mathrm{d}q_1 = Q \frac{\mathrm{d}\kappa}{\kappa^2 + 1}$$

from chapter 6.1 of the master's thesis B.Schmidt (2020) in our new κ-variable. Evaluating this with our knowledge of before we get that this equation is equivalent to

$$\frac{ic_1(\kappa)}{\nu} \frac{ib_2(\kappa)}{\nu(\kappa^2+1)} \mathrm{d}\kappa - \frac{ic_2(\kappa)}{\nu} \frac{ib_1(\kappa)}{\nu(\kappa^2+1)} \mathrm{d}\kappa = \frac{Q(\kappa)}{\kappa^2+1} \mathrm{d}\kappa.$$

Canceling the 1-form $\mathrm{d}\kappa$ from both sides then gets us to

$$\frac{c_2 b_1 - c_1 b_2}{a(\kappa)(\kappa^2+1)} = \frac{Q(\kappa)}{\kappa^2+1}.$$

Now multiplying both sides by $a(\kappa)$ and canceling the $\kappa^2 + 1$ terms gives us the third Whitham equation

$$c_2(\kappa) b_1(\kappa) - c_1(\kappa) b_2(\kappa) = Q(\kappa) a(\kappa). \tag{23}$$

Now we will use these equations in the same way as in B.Schmidt (2020) chapter 4 in the case that b_1 and b_2 have a common root. We have already established that $\kappa = 0$ is the new common root of these polynomials. Our goal will now be to prove an analogon of Corollary 4.11 from B.Schmidt (2020). First we need to consider τ_a in our κ-variable. Since we define κ by the Cayley transformation we will need to define τ_a in the following way

$$\tau_a = \frac{b_1(\kappa = i)}{b_2(\kappa = i)}.$$

However again since we look at $T^{-1}(\tau_a)$ we know that $\tau_a = const.$ and therefore, we can calculate

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \frac{b_1(\kappa = i)}{b_2(\kappa = i)} = 0.$$

Calculating this gets us to

$$\frac{\dot{b}_1(i) b_2(i) - b_1(i) \dot{b}_2(i)}{b_2(i)^2} = 0$$

and then

$$\dot{b}_1(i)b_2(i) - b_1(i)\dot{b}_2(i) = 0.$$

Looking at Equation (23) we see that if we plug in $\kappa = i$ we get

$$c_2(i)b_1(i) - c_1(i)b_2(i) = Q(i)a(i).$$

However if we look at Equation (21) as well as Equation (22) at $\kappa = i$ that yields

$$-2c_1(i)a(i)i = 2\dot{b}_1(i)a(i) - b_1(i)\dot{a}(i)$$

and

$$-2c_2(i)a(i)i = 2\dot{b}_2(i)a(i) - b_2(i)\dot{a}(i)$$

That gets us

$$c_1(i) = i(\dot{b}_1(i) - b_1(i)\frac{\dot{a}(i)}{2a(i)})$$

and also

$$c_2(i) = i(\dot{b}_2(i) - b_2(i)\frac{\dot{a}(i)}{2a(i)}).$$

Now inserting this into our equation above yields

$$\begin{aligned}
Q(i)a(i) &= i(\dot{b}_2(i) - b_2(i)\frac{\dot{a}(i)}{2a(i)})b_1(i) - i(\dot{b}_1(i) - b_1(i)\frac{\dot{a}(i)}{2a(i)})b_2(i) \\
&= i(\dot{b}_2(i)b_1(i) - \dot{b}_1(i)b_2(i)) + \frac{\dot{a}(i)}{2a(i)}\Big(b_1(i)b_2(i) - b_1(i)b_2(i)\Big) \\
&= i\dot{\tau}_a + 0 \\
&= 0.
\end{aligned}$$

Although since $a(\kappa = i)$ is equal to $a(\lambda = 0)$ we see that this can't vanish and therefore, $Q(i) = 0$ needs to hold. By the reality condition that all coefficients of Q need to be real it holds that the complex conjugate of i is a root as well so $Q(-i)$ needs to vanish as well and therefore, $Q(\kappa)$ needs to be proportional to $\kappa^2 + 1$. Because of the fact that Q is a polynomial of degree two we then get $Q(\kappa) = \mu(\kappa^2 + 1)$ needs to hold. However if we now recall equation that gets us

$$c_2(\kappa)b_1(\kappa) - c_1(\kappa)b_2(\kappa) = a(\kappa)\mu(\kappa^2 + 1).$$

Because of our assumptions we see that the left hand side needs to vanish at $\kappa = 0$ but $a(\lambda)$ doesn't vanish at $\kappa = 0$ because of our assumptions. So that means that Q needs to vanish at $\kappa = 0$. So since Q is a multiple of $\kappa^2 + 1$ that means that $\mu = 0$ needs to hold. That in turn implies $Q = 0$. Therefore, we get that equation (23) can be transformed into

$$c_2(\kappa)b_1(\kappa) - c_1(\kappa)b_2(\kappa) = 0.$$

Since $\gcd(b_1, b_2) = 1$ holds we can now evaluate the equation at the roots of b_1 unequal to zero because we know that b_2 doesn't vanish there. We also see that since c_k fulfilled the reality condition as well in the λ-parameter before the new coefficients need to be real concerning the κ-parameter as well. Therefore, we can write $c_k(\kappa) = c_{1,k}\kappa^3 + c_{2,k}\kappa^2 + c_{3,k}\kappa + c_{4,k}$ where $c_{i,k} \in \mathbb{R}, i = 1, \dots, 4,\ k = 1, 2$. Now we can define $\frac{b_k(\kappa)}{\kappa} = \tilde{b}_k(\kappa)$. Because of our previous considerations it needs to hold true that $c_k(\kappa) = \gamma_k(\kappa - \kappa_{0,k})\tilde{b}_k(\kappa)$ since we can write $\tilde{b}_k(\kappa) = b_{1,k}(\kappa - \kappa_{1,k})(\kappa - \kappa_{2,k})$ where $\kappa_{1,k}$ and $\kappa_{2,k}$ are the roots of b_k not zero (in the case of a root of higher order at zero they can obviously still be equal to zero as well). Now we can write out our polynomial c_k in the following way

$$\begin{aligned}
c_k(\kappa) &= \gamma_k(\kappa - \kappa_{0,k})\tilde{b}_k(\kappa) \\
&= \gamma_k(\kappa - \kappa_{0,k})b_{1,k}(\kappa - \kappa_{1,k})(\kappa - \kappa_{2,k}) \\
&= \gamma_k(\kappa - \kappa_{0,k})\tilde{b}_k(\kappa) \\
&= \gamma_k\kappa\tilde{b}_k(\kappa) - \gamma_k\kappa_{0,k}\tilde{b}_k(\kappa) \\
&= \gamma_k b_k(\kappa) - \gamma_k\kappa_{0,k}\tilde{b}_k(\kappa), \qquad k = 1, 2.
\end{aligned}$$

Because of the assumption that all the coefficients of c_k and b_k are real we know that γ_k needs to be real as well since $c_{1,k} = \gamma_k b_{1,k}$. Therefore, we have two unknowns out of which one is real and one is complex. Therefore, we have reduced the dimension of c_k from four to three. Now if we recall that the Cayley transform transforms polynomials that fulfill the reality condition into polynomials whose coefficients are real it holds that c_k needs to have real coefficients as well and therefore, $c_{1,k} \in \mathbb{R}$ needs to hold. However we have just calculated that $c_{1,k} = \gamma_k b_{1,k}$ holds and $b_{1,k} \in \mathbb{R}$ holds as seen before. Then in turn it needs to be true that $\kappa_{0,k} \in \mathbb{R}$ holds as well. This reduces the dimension of our polynomials from three to two. Now we will insert our solutions into equation (23) to get to

$$\begin{aligned}
&\gamma_2(\kappa - \kappa_{0,2})\tilde{b}_2(\kappa)\kappa\tilde{b}_1(\kappa) - \gamma_1(\kappa - \kappa_{0,1})\tilde{b}_1(\kappa)\kappa\tilde{b}_2(\kappa) \\
&= (\gamma_2(\kappa - \kappa_{0,2}) - \gamma_1(\kappa - \kappa_{0,1}))\kappa\tilde{b}_1\kappa\tilde{b}_2(\kappa) \\
&= 0.
\end{aligned}$$

Since the right factor doesn't identically vanish we can divide both sides by $\kappa\tilde{b}_1\kappa\tilde{b}_2(\kappa)$ which leaves us with

$$\gamma_2(\kappa - \kappa_{0,2}) - \gamma_1(\kappa - \kappa_{0,1}) = 0.$$

Comparing coefficients we quickly see that $\gamma_1 = \gamma_2$ as well as $\kappa_{0,1} = \kappa_{0,2}$. In the following we will denote these parameters with γ and $\hat{\kappa}$. So now our solution space appears to be two-dimensional. We can write our polynomials c_k now in the following way

$$c_1(\kappa) = \gamma(\kappa - \hat{\kappa})\tilde{b}_1(\kappa)$$
$$c_2(\kappa) = \gamma(\kappa - \hat{\kappa})\tilde{b}_2(\kappa).$$

Our goal now is to determine $\dot{a}(\kappa)$. In order to do so we will now consider the four roots of $a(\lambda)$ which we will denote $\alpha_1, \ldots, \alpha_4$. Then we get the following formula for the derivative

$$\dot{a}(\kappa) = -\sum_{i=1}^{4} \dot{\alpha}_i \frac{a(\kappa)}{\kappa - \alpha_i}.$$

It is easy to see that if we now plug in a root α_i in $\dot{a}(\kappa)$ the only term that doesn't vanish is the term with $\dot{\kappa}_i$. Now if we calculate the κ-derivative of $a(\lambda)$ as well we see that this has the following form

$$a'(\kappa) = \sum_{i=1}^{4} \frac{a(\kappa)}{\kappa - \alpha_i}.$$

So that means

$$\dot{a}(\alpha_i) = -\dot{\alpha}_i a'(\alpha_i), \qquad i = 1, \ldots, 4.$$

Therefore, we can use the four roots of $a(\lambda)$ to fully determine $\dot{a}$. Now if we recall the equations Equation (21) as well as Equation (22) and insert the roots α_i of $a(\lambda)$ we get the following equations

$$c_k(\alpha_i)a'(\alpha_i)(\alpha_i^2 + 1) = b_k(\alpha_i)\dot{a}(\alpha_i), \qquad k = 1, 2, i = 1, \ldots, 4.$$

Now if we use our solution of $\dot{a}(\alpha_i)$ we get

$$c_k(\alpha_i)a'(\alpha_i)(\alpha_i^2 + 1) = -b_k(\alpha_i)\dot{\alpha}_i a'(\alpha_i), \qquad k = 1, 2, i = 1, \ldots, 4.$$

Now the $a'(\alpha_i)$ cancel out and we can now solve for $\dot{\kappa}_i$

$$\dot{\alpha}_i = -\frac{c_k(\alpha_i)(\alpha_i^2 + 1)}{b_k(\alpha_i)}, \qquad k = 1, 2, i = 1, \ldots, 4.$$

Now if we use our results for $c_k(\kappa)$ we see that

$$\begin{aligned}\frac{c_k(\alpha_i)}{b_k(\alpha_i)} &= \frac{\gamma(\alpha_i - \hat{\kappa})\tilde{b}_k(\alpha_i)}{\alpha_i \tilde{b}_k(\alpha_i)} \\ &= \frac{\gamma(\alpha_i - \hat{\kappa})}{\alpha_i}, \qquad k = 1, 2, i = 1, \dots, 4.\end{aligned}$$

So that means for our solution that it is independent of k. We get

$$\dot{\alpha}_i = -\frac{\gamma(\alpha_i - \hat{\kappa})(\alpha_i^2 + 1)}{\alpha_i}, \qquad i = 1, \dots, 4.$$

From our previous considerations we see that this fully determines $\dot{a}(\kappa)$. Therefore, we can insert our solutions of $\dot{\alpha}_i$ into the formula we have calculated in order to fully determine $\dot{a}(\kappa)$

$$\begin{aligned}\dot{a}(\kappa) &= -\sum_{i=1}^{4} \dot{\alpha}_i \frac{a(\kappa)}{\kappa - \alpha_i} \\ &= \sum_{i=1}^{4} \frac{\gamma(\alpha_i - \hat{\kappa})(\alpha_i^2 + 1)}{\alpha_i} \frac{a(\kappa)}{\kappa - \alpha_i}\end{aligned}$$

Now it remains to solve Equation (21) and Equation (22) for $\dot{b}_k$. In order to do so we will add $b_k(\kappa)\dot{a}(\kappa)$ on both sides of the respective equations which gets us

$$\begin{aligned}2\dot{b}_k(\kappa)a(\kappa) =& 2c_k'(\kappa)a(\kappa)(\kappa^2 + 1) - 2c_k(\kappa)a(\kappa)\kappa \\ & - c_k(\kappa)a'(\kappa)(\kappa^2 + 1) + b_k(\kappa)\dot{a}(\kappa), \quad k = 1, 2.\end{aligned}$$

Now we will prove that both sides of these equations are divisible by $a(\kappa)$. In order to do so we will show that both sides vanish at all roots α_i of $a(\kappa)$. We easily see that the first summands of the right hand side depend on $a(\kappa)$. It remains to show that the last two terms also vanish there. Obviously the terms directly depending on $a(\kappa)$ vanish which leaves us with the last two terms. Here we will need to look into the way we have defined $\dot{a}(\kappa)$ because it needs to be aligned in such a way that these two terms agree at the roots of $a(\kappa)$ down to the sign and in consequence vanish. But we have already established that $\dot{a}(\alpha_i) = \dot{\alpha}_i a'(\alpha_i)$ holds at the roots of $a(\kappa)$. Now if we evaluate the two terms not depending on $a(\kappa)$ directly we get

$$\begin{aligned}& b_k(\alpha_i)\dot{\alpha}_i a'(\alpha_i) - c_k(\alpha_i)a'(\alpha_i)(\alpha_i^2 + 1) \\ &= b_k(\alpha_i)\frac{\gamma(\alpha_i - \hat{\kappa})a'(\alpha_i)(\alpha_i^2 + 1)}{\alpha_i} - \gamma(\alpha_i - \hat{\kappa})\tilde{b}_k(\alpha_i)a'(\alpha_i)(\alpha_i^2 + 1) \\ &= \tilde{b}_k(\alpha_i)\gamma(\alpha_i - \hat{\kappa})a'(\alpha_i)(\alpha_i^2 + 1) - \gamma(\alpha_i - \hat{\kappa})\tilde{b}_k(\alpha_i)a'(\alpha_i)(\alpha_i^2 + 1) \\ &= 0.\end{aligned}$$

That proves that the last part of the equations above vanishes at all roots of $a(\kappa)$ so both sides of these equations are divisible by $a(\kappa)$. Therefore, we can now write these two terms as a product of $a(\kappa)$ with another polynomial which we call $d_k(\kappa)$ so we get

$$b_k(\kappa)\dot{a}(\kappa) - c_k(\kappa)a'(\kappa)(\kappa^2+1) = a(\kappa)d_k(\kappa), \qquad k = 1, 2.$$

So now we can solve Equation (21) and Equation (22) for $\dot{b}_1$ and $\dot{b}_2$. We see that the following needs to hold then for $\dot{b}_k$.

$$\dot{b}_k = c_k'(\kappa)(\kappa^2+1) - c_k(\kappa)\kappa + d_k(\kappa), \qquad k = 1, 2.$$

We are now able to fully determine $\dot{b}_k$ by comparing coefficients. So now we see that every solution depends on a linear combination of γ and $\hat{\kappa}$. Since $c_k = \gamma(\kappa - \hat{\kappa})\tilde{b}_k$ holds we see that our solution space is two-dimensional. Our next goal is to determine how these solutions are related. In order to do so we calculate a certain Möbius transformation. We know that

$$\lambda = \frac{\kappa - i}{\kappa + i}$$

holds and therefore,

$$\kappa = i\,\frac{\lambda+1}{\lambda-1}.$$

Our goal is now to construct a Möbius transformation that is a rotation of λ. Therefore, we will replace λ with $e^{it}\lambda$ in our equation. That means

$$\begin{aligned}
\kappa &= i\,\frac{e^{it}\lambda+1}{e^{it}\lambda-1}\\
&= i\,\frac{e^{it}\frac{\kappa-i}{\kappa+i}+1}{e^{it}\frac{\kappa-i}{\kappa+i}-1}\\
&= i\,\frac{e^{it}(\kappa-i)+\kappa+i}{e^{it}(\kappa-i)-(\kappa+i)}\\
&= \frac{\kappa(e^{it/2}+e^{-it/2})+i(e^{-it/2}-e^{it/2})}{\kappa(ie^{it/2}-ie^{it/2})+i(ie^{it/2}+ie^{-it/2})}\\
&= \frac{\cos(t/2)\kappa-\sin(t/2)}{\sin(t/2)\kappa+\cos(t/2)}.
\end{aligned}$$

needs to hold. Now it is possible to dilate $t/2$ to t without loss of generality.

We then calculate the derivative of κ at $t = 0$. That gets us

$$\begin{aligned}\dot{\kappa} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\cos(t)\kappa - \sin(t)}{\sin(t)\kappa + \cos(t)}\bigg|_{t=0} \\ &= \frac{(-\sin(t)\kappa - \cos(\kappa))(\sin(t)\kappa + \cos(t)) - (\cos(t)\kappa - \sin(t))(\cos(t)\kappa - \sin(t))}{(\sin(t)\kappa + \cos(t))^2}\bigg|_{t=0} \\ &= \frac{-1-\kappa^2}{1} = -(\kappa^2+1).\end{aligned}$$

Now if we use this result on $\dot{q}_k$ we will see

$$\begin{aligned}\frac{c_k}{\nu} &= \dot{q}_k \\ &= -(1+\kappa^2)\frac{\mathrm{d}}{\mathrm{d}\kappa}q_k \\ &= -(1+\kappa^2)\frac{b_k}{\nu}\frac{1}{1+\kappa^2} \\ &= -\frac{b_k}{\nu}.\end{aligned}$$

Therefore, the solution $c_k = -b_k$ corresponds to this infinitesimal Möbius transformation proving in turn that our solutions are one-dimensional except for this. In a next step we can look at what happens if we add μb_k to our solutions of c_k. Therefore, we recall Equation (23) and get

$$\begin{aligned}&(c_1 + \mu b_1)b_2 - (c_2 + \mu b_2)b_1 \\ =&c_1b_2 + \mu b_1 b_2 - c_2 b_1 - \mu b_1 b_2 \\ =&c_1 b_2 - c_2 b_1 \\ =&Qa.\end{aligned}$$

Therefore, this doesn't change our solution as well. In order to reduce the dimension of our solutions to one we need to reduce the dimension of the space we consider by one as well. In order to do so we try the ansatz b_1 has only a simple root at $\kappa = 0$ and $\dot{b}_{1,4} = 0$. Now we will use Equation (21) at $\kappa = 0$ in order to get a condition for $\hat{\kappa}$ and γ. We will use the solutions for c_1 and $\dot{a}$ to do so. Because of our assumptions we see that $b_1(0) = 0$ but $b_1'(0) \neq 0$ as well as $\tilde{b}_1(0) \neq 0$ so we get

$$\begin{aligned}&2c_1'(0)a(0) - c_1(0)a'(0) + b_1(0)\dot{a}(0) \\ =& 2(\gamma b_1'(0) - \gamma\hat{\kappa}\tilde{b}_1'(0))a(0) + \gamma\hat{\kappa}\tilde{b}_1(\kappa)a'(0) \\ =& 2(\gamma b_{3,1} - \gamma\hat{\kappa}b_{2,1})a_4 + \gamma\hat{\kappa}b_{3,1}a_3 \\ =& 0.\end{aligned}$$

So now we can align this in the following way

$$\gamma(2b_{3,1}a_4 + (b_{3,1}a_3 - 2b_{2,1}a_4)\hat{\kappa}) = 0.$$

This gets us two possible solutions

$$\gamma = 0$$

or

$$\hat{\kappa} = -\frac{2b_{3,1}a_4}{b_{3,1}a_3 - 2b_{2,1}a_4}.$$

However since c_1 and c_2 depend linearly on γ the first solution would lead to these polynomials vanishing which means that the second equation is the solution we need making our space of solutions one-dimensional. We can now use our results to simplify the solutions for c_k as well as $\dot{b}_k$. First we will plug the solution for $\hat{\kappa}$ into the solution for c_k. That yields

$$\begin{aligned} c_k(\kappa) &= \gamma(\kappa - \hat{\kappa})\tilde{b}_k(\kappa) \\ &= \gamma\Big(\kappa + \frac{2b_{3,1}a_4}{b_{3,1}a_3 - 2b_{2,1}a_4}\Big)\tilde{b}_k(\kappa). \end{aligned}$$

Now we can also calculate $\dot{b}_k$ using this and get

$$\begin{aligned} \dot{b}_k(\kappa) &= \Big(\gamma b_k'(\kappa) + \frac{2b_{3,1}a_4\gamma}{b_{3,1}a_3 - 2b_{2,1}a_4}\tilde{b}_k'(\kappa)\Big)(\kappa^2+1) \\ &\quad - \gamma\Big(\kappa + \frac{2b_{3,1}a_4}{b_{3,1}a_3 - 2b_{2,1}a_4}\Big)\tilde{b}_k(\kappa)\Big(\kappa + \frac{a'(\kappa)}{2a(\kappa)}(\kappa^2+1)\Big) \\ &\quad + \frac{a(\kappa)}{2a(\kappa)}\sum_{i=1}^{4}\frac{\gamma(\alpha_i - \hat{\kappa})(\alpha_i^2+1)}{\alpha_i}\frac{b_k(\kappa)}{\kappa - \alpha_i} \\ &= \gamma\Big((b_k'(\kappa) + \frac{2b_{3,1}a_4}{b_{3,1}a_3 - 2b_{2,1}a_4}\tilde{b}_k'(\kappa))(\kappa^2+1) \\ &\quad - \Big(\kappa + \frac{2b_{3,1}}{b_{3,1}a_3 - 2b_{2,1}a_4}\Big)\Big(\sum_{i=1}^{4}\frac{(\kappa+1)^2}{2(\kappa - \alpha_i)} + \kappa\Big)\tilde{b}_k(\kappa) \\ &\quad + \sum_{i=1}^{4}\frac{\Big(\alpha_i + \frac{2b_{3,1}a_4}{b_{3,1}a_3 - 2b_{2,1}a_4}\Big)(\alpha_i^2+1)}{2\alpha_i(\kappa - \alpha_i)}b_k(\kappa).\Big), \qquad k = 1, 2. \end{aligned}$$

The next step is now as in chapter 6 of B.Schmidt (2020) to consider the case

$t \neq 0$ where we write the functions a, b_1, b_2, c_1, c_2, Q as Taylor series. That means we can write them in the following way

$$Q(\kappa) = \sum_{i=0}^{\infty} \frac{Q_i}{i!} t^i.$$

So we already know that Q vanishes at $t = 0$ as well as that it needs to be a multiple of $\kappa^2 + 1$. That means that we can consider $Q_1 t = (\kappa^2 + 1)t$ as the lowest coefficient of Q that doesn't vanish. That means we will plug this as well as the Taylor series expansions for the other functions into the equation Equation (23). In order to do so we will define likewise

$$a(\kappa) = \sum_{i=0}^{\infty} \frac{A_i(\kappa)}{i!} t^i$$

where $A_i(\kappa)$ are polynomials of degree four as well as

$$b_k(\kappa) = \sum_{i=0}^{\infty} \frac{B_{i,k}(\kappa)}{i!} t^i, \qquad k = 1, 2,$$

where $B_{i,k}$ are polynomials of degree three and

$$c_k(\kappa) = \sum_{i=0}^{\infty} \frac{C_{i,k}(\kappa)}{i!} t^i, \qquad k = 1, 2$$

where $C_{i,k}$ are polynomials of degree as well. Using these formulas and comparing the linear coefficients then yields

$$A_0(\kappa^2 + 1)t = (C_{0,2}B_{1,1} + C_{1,2}B_{0,1} - C_{0,1}B_{1,2} - C_{1,1}B_{0,2})t.$$

This means we can now insert our solutions for the linear coefficients of all polynomials in this solution because we already calculated them in the previous considerations. We get the following equation

$$a(\kappa)(\kappa^2 + 1) = c_2(\kappa)\dot{b}_1(\kappa) - c_1(\kappa)\dot{b}_2(\kappa) + C_{2,1}(\kappa)b_1(\kappa) - C_{1,1}(\kappa)b_2(\kappa)$$

Our goal is now to fully determine our solutions of $\dot{b}_k$ which still depend on γ. In order to do so we will evaluate the equation at $\kappa = 0$ because we know that $b_1(\kappa)$ as well as a $b_2(\kappa)$ vanish at $\kappa = 0$ and so the linear coefficients $C_{k,1}$ vanish as well. In addition to that we recall that $\dot{b}_1(0) = 0$ holds as well. So now we can look at the equation for $\kappa = 0$ and get

$$a(0) = c_1(0)\dot{b}_2(0)$$

into which we plug in our solutions and see that

$$\begin{aligned}a_4 &= \gamma\Big(\frac{2b_{3,1}a_4}{b_{3,1}a_3 - 2b_{2,1}a_4}\Big)\tilde{b}_1(0)\gamma\Big(b_2'(0) + \frac{2b_{3,1}a_4}{b_{3,1}a_3 - 2b_{2,1}a_4}\tilde{b}_1'(0) - \frac{2b_{3,1}}{b_{3,1}a_3 - 2b_{2,1}a_4}\sum_{i=1}^{4}\frac{\tilde{b}_2(0)}{-2\alpha_i}\Big)\\ &= \gamma^2\Big(\frac{2b_{3,1}^2a_4}{b_{3,1}a_3 - 2b_{2,1}a_4}\Big)\Big(\frac{2b_{3,1}a_4}{b_{3,1}a_3 - 2b_{2,1}a_4}(b_{2,1} + \sum_{i=1}^{4}\frac{b_{3,2}}{2\alpha_i}) + b_{3,2}\Big)\end{aligned}$$

holds. Now we can solve this for γ and get

$$\begin{aligned}\gamma &= \pm\sqrt{\frac{a_4}{\Big(\frac{2b_{3,1}^2a_4}{b_{3,1}a_3-2b_{2,1}a_4}\Big)\Big(\frac{2b_{3,1}a_4}{b_{3,1}a_3-2b_{2,1}a_4}(b_{2,1} + \sum_{i=1}^{4}\frac{b_{3,2}}{2\alpha_i}) + b_{3,2}\Big)}}\\ &= \pm\sqrt{\Big(\Big(\frac{2b_{3,1}^2}{b_{3,1}a_3 - 2b_{2,1}a_4}\Big)\Big(\frac{2b_{3,1}a_4}{b_{3,1}a_3 - 2b_{2,1}a_4}(b_{2,1} - \sum_{i=1}^{4}\frac{b_{3,2}}{2\alpha_i}) + b_{3,2}\Big)\Big)^{-1}}.\end{aligned}$$

That means we have two solutions for γ so the solutions is unique up to sign. So we have uniquely determined all the solutions right now. Our next goal is to determine $C_{1,2}$ and $C_{1,1}$ as well. In order to do so we isolate them as the only terms not determined right now. So then we get the equation

$$a(\kappa)(\kappa^2 + 1) + c_1(\kappa)\dot{b}_2(\kappa) - c_2(\kappa)\dot{b}_1(\kappa) = C_{2,1}(\kappa)b_1(\kappa) - C_{1,1}(\kappa)b_2(\kappa).$$

Here we see again that if have solutions for this equation which we will call $\dot{c}_1$ and $\dot{c}_2$ because this is a linear equation we can add μb_1 and μ_2 to both sides without changing the solution which we see here

$$\begin{aligned}&a(\kappa)(\kappa^2 + 1) + c_1(\kappa)\dot{b}_2(\kappa) - c_2(\kappa)\dot{b}_1(\kappa)\\ &= (\dot{c}_1(\kappa) + \mu b_2(\kappa))b_1(\kappa) - (\dot{c}_2(\kappa) + \mu b_1(\kappa))b_2(\kappa)\\ &= \dot{c}_1(\kappa)b_2(\kappa) - \dot{c}_2(\kappa)b_1(\kappa).\end{aligned}$$

Therefore we get in the same way as before with the constant coefficients that the solution space is our general solution we will now determine plus the linear term we now introduced allowing us to reduce the dimension of the solution space from two to one using another condition that eliminates the Möbius transforms. Now we see that the left hand side of our equation is divisible by κ because both $b_1(\kappa)$ and $b_2(\kappa)$ vanish at $\kappa = 0$. That reduces the degree of the equation by one and we get

$$\kappa^{-1}(a(\kappa)(\kappa^2 + 1) + c_1(\kappa)\dot{b}_2(\kappa) - c_2(\kappa)\dot{b}_1(\kappa)) = C_{2,1}(\kappa)\tilde{b}_1(\kappa) - C_{1,1}(\kappa)\tilde{b}_2(\kappa).$$

Now in a next step the goal is to reduce the degree of the equation by doing polynomial division with $\tilde{b}_1$ of the left hand side of our equation. So we need to calculate

$$\frac{a(\kappa)(\kappa^2+1)+c_1(\kappa)\dot{b}_2(\kappa)-c_2\dot{b}_1(\kappa)}{\kappa} : \tilde{b}_1$$

In order to do so we will plug in our solutions for the derivatives $\dot{b}_k$ as well as c_k and calculate how the term we want to do polynomial division with turns out. So we get

$$\begin{aligned}
&\kappa^{-1}(a(\kappa)(\kappa^2+1)+\gamma(\kappa-\hat{\kappa})\tilde{b}_1(\kappa)\gamma((b_2'(\kappa)-\hat{\kappa}\tilde{b}_2'(\kappa))(\kappa^2+1)\\
&-(\kappa-\hat{\kappa})\tilde{b}_2\Big(\sum_{i=1}^{4}\frac{\kappa^2+1}{2(\kappa-\alpha_i)}+\kappa\Big)+\dot{a}(\kappa)b_2(\kappa))-\gamma(\kappa-\hat{\kappa})\tilde{b}_2(\kappa)\gamma((b_1'(\kappa)\\
&-\hat{\kappa}\tilde{b}_1'(\kappa))(\kappa^2+1)-(\kappa-\hat{\kappa})\tilde{b}_1\Big(\sum_{i=1}^{4}\frac{\kappa^2+1}{2(\kappa-\alpha_i)}+\kappa\Big)+\dot{a}(\kappa)b_1(\kappa)))\\
=&\kappa^{-1}(a(\kappa)(\kappa^2+1)+\gamma^2(\kappa-\hat{\kappa})(\tilde{b}_1(\kappa)(b_2'(\kappa)-\hat{\kappa}\tilde{b}_2'(\kappa))(\kappa^2+1)\\
&-\tilde{b}_2(\kappa)(b_1'(\kappa)-\hat{\kappa}\tilde{b}_1'(\kappa))(\kappa^2+1)))\\
=&\kappa^{-1}(\kappa^2+1)(a(\kappa)+\gamma^2(\kappa-\hat{\kappa})(\tilde{b}_1(\kappa)(b_2'(\kappa)-\hat{\kappa}\tilde{b}_2'(\kappa))-\tilde{b}_2(\kappa)(b_1'(\kappa)-\hat{\kappa}\tilde{b}_1'(\kappa))).
\end{aligned}$$

Evaluating the right bracket at $\kappa=0$ gets us

$$\begin{aligned}
&a_4+\gamma^2(-\hat{\kappa})(b_{3,1}(b_{3,2}-\hat{\kappa}b_{2,2})-b_{3,2}(b_{3,1}-\hat{\kappa}b_{2,1})))\\
=&\,a_4+\gamma^2\hat{\kappa}^2(b_{3,1}b_{2,2}-b_{3,2}b_{2,1})\\
=&\,a_4+\frac{\hat{\kappa}^2(b_{3,1}b_{2,2}-b_{3,2}b_{2,1})}{\hat{\kappa}b_{3,1}(\hat{\kappa}(b_{2,1}-\sum_{i=1}^{4}\frac{b_{3,2}}{2\alpha_i})+b_{3,2})}.
\end{aligned}$$

Since the other side of our equation vanishes at $\kappa=0$ we know that the right bracket we just evaluated needs to vanish at $\kappa=0$. Now we are able to proceed with our polynomial division to get

$$\frac{(\kappa^2+1)(a(\kappa)+\gamma^2(\kappa-\hat{\kappa})(\tilde{b}_1(\kappa)(b_2'(\kappa)-\hat{\kappa}\tilde{b}_2'(\kappa))-\tilde{b}_2(\kappa)(b_1'(\kappa)-\hat{\kappa}\tilde{b}_1'(\kappa))}{\kappa} : \tilde{b}_1$$

We now calculate the leading coefficient of the expression we want to divide in order to start so we get that this has the form

$$\gamma^2(3b_{1,1}b_{1,2}-3b_{1,2}b_{1,1})=0$$

which doesn't come as a surprise since this is the same result we established for Equation (21) and Equation (22) in order for the solutions to match up by degree. So we need to actually calculate the second highest coefficient which is

$$\begin{aligned}
&1+\gamma^2(2b_{1,1}b_{2,2}-2\hat{\kappa}b_{1,1}b_{1,2}-3\hat{\kappa}b_{1,1}b_{1,2}-2b_{1,2}b_{2,1}+2\hat{\kappa}b_{1,2}b_{1,1}+3\hat{\kappa}b_{1,2}b_{1,1})\\
=&\,1+\gamma^2(2b_{1,1}b_{2,2}-2b_{1,2}b_{2,1}).
\end{aligned}$$

So that means that the first coefficient we get from polynomial division is

$$\frac{a(\kappa)(\kappa^2+1)+c_1(\kappa)\dot{b}_2(\kappa)-c_2(\kappa)\dot{b}_1(\kappa)}{\kappa}:\tilde{b}_1=\Big(\frac{1}{b_{1,1}}+2\gamma^2\Big(b_{2,2}-\frac{b_{1,2}b_{2,1}}{b_{1,1}}\Big)\Big)\kappa^3.$$

In order to calculate further coefficients we need to determine the next coefficients of the term we are dividing. The second highest coefficient that doesn't vanish is

$$\begin{aligned}
&a_1+\gamma^2(b_{1,1}b_{3,2}+2b_{2,1}b_{2,2}+3b_{3,1}b_{1,2}-\hat{\kappa}(2b_{1,1}b_{2,2}+3b_{2,1}b_{1,2}+2b_{1,1}b_{1,2})\\
&+2\hat{\kappa}^2b_{1,2}b_{1,1}-(b_{1,2}b_{3,1}+2b_{2,2}b_{2,1}+3b_{3,2}b_{1,1})\\
&+\hat{\kappa}(2b_{1,2}b_{2,1}+3b_{1,2}b_{2,1}+2b_{2,1}b_{1,1}))-2\hat{\kappa}^2b_{1,2}b_{1,1}\\
&=a_1+\gamma^2(b_{1,1}b_{3,2}-b_{1,2}b_{3,1}+3b_{3,1}b_{1,2}-3b_{3,2}b_{1,1}))\\
&=a_1+2\gamma^2(b_{3,1}b_{1,2}-b_{1,1}b_{3,2}).
\end{aligned}$$

Next we calculate the third highest coefficient to get

$$\begin{aligned}
&a_2+\gamma^2(b_{2,1}b_{3,2}+2b_{3,1}b_{2,2}-\hat{\kappa}(b_{1,1}b_{3,2}+2b_{2,1}b_{2,2}+3b_{3,1}b_{1,2}+b_{2,1}b_{2,2}+2b_{3,1}b_{1,2})\\
&+\hat{\kappa}^2(b_{1,1}b_{2,2}+2b_{2,1}b_{2,2})-(b_{2,2}b_{3,1}+2b_{3,1}b_{2,1})\\
&+\hat{\kappa}(b_{1,2}b_{3,1}+2b_{2,2}b_{2,1}+3b_{3,2}b_{1,1}+b_{2,2}b_{2,1}+2b_{3,2}b_{1,1})\\
&-\hat{\kappa}^2(b_{2,1}b_{1,2}+2b_{2,2}b_{2,1})+1+\gamma^2(2b_{1,1}b_{2,2}-2b_{1,2}b_{2,1})\\
&=1+a_2+\gamma^2(b_{3,1}b_{2,2}-b_{2,1}b_{3,2}-\hat{\kappa}(4b_{3,1}b_{1,2}-4b_{3,2}b_{1,1})\\
&+\hat{\kappa}^2(b_{1,1}b_{2,2}-b_{2,1}b_{1,2})+2b_{1,1}b_{2,2}-2b_{1,2}b_{2,1})).
\end{aligned}$$

Now we calculate the next coefficient which is the one belonging to κ^3/κ

$$\begin{aligned}
&a_3+a_1+2\gamma^2(b_{3,1}b_{1,2}-b_{1,1}b_{3,2})+\gamma^2(b_{3,1}b_{3,2}-\hat{\kappa}(b_{2,1}b_{3,2}+2b_{3,1}b_{2,2}+2b_{3,1}b_{3,2})\\
&+\hat{\kappa}^2(b_{2,1}b_{2,2}+2b_{3,1}b_{1,2})-b_{3,1}b_{3,2}+\hat{\kappa}(b_{3,1}b_{2,2}+2b_{2,1}b_{3,2}+2b_{3,1}b_{3,2})\\
&-\hat{\kappa}^2(b_{2,1}b_{2,2}+2b_{1,1}b_{3,2}))\\
&=a_3+a_1+\gamma^2(2b_{3,1}b_{1,2}-2b_{1,1}b_{3,2}-\hat{\kappa}(b_{3,1}b_{2,2}-b_{2,1}b_{3,2})+\hat{\kappa}^2(2b_{3,1}b_{1,2}-2b_{1,1}b_{3,2})).
\end{aligned}$$

Second to last, we calculate the coefficient belonging to the linear term which is

$$\begin{aligned}
&a_2+\gamma^2(b_{2,1}b_{3,2}-b_{2,1}b_{2,2}-\hat{\kappa}(3b_{3,1}b_{1,2}+2b_{2,1}b_{2,2}+b_{1,1}b_{3,2}+b_{2,1}b_{2,2}+2b_{3,1}b_{1,2})\\
&+\hat{\kappa}^2(b_{1,1}b_{2,2}+2b_{2,1}b_{2,2})-(b_{3,1}b_{2,2}-b_{2,1}b_{2,2})\\
&+\hat{\kappa}(3b_{1,1}b_{3,2}+2b_{2,1}b_{2,2}+b_{3,1}b_{1,2}+b_{2,1}b_{2,2}+2b_{1,1}b_{3,2})-\hat{\kappa}^2(b_{2,1}b_{1,2}+2b_{2,1}b_{2,2}))\\
&=a_2+\gamma^2(b_{2,1}b_{3,2}-b_{3,1}b_{2,2}-\hat{\kappa}(4b_{3,1}b_{1,2}-4b_{1,1}b_{3,2})+\hat{\kappa}^2(b_{1,1}b_{2,2}-b_{2,1}b_{1,2})).
\end{aligned}$$

We don't need to consider the constant coefficients times κ^2 because we already know that this vanishes by recent argumentation. And lastly we get the lowest coefficient which is

$$
\begin{aligned}
& a_3 + \gamma^2(b_{3,1}b_{3,2} - \hat{\kappa}(b_{2,1}b_{3,2} + 2b_{3,1}b_{2,2} + 2b_{3,1}b_{3,2}) + \hat{\kappa}^2(b_{2,1}b_{2,2} + 2b_{3,1}b_{1,2}) \\
& - b_{3,1}b_{3,2} + \hat{\kappa}(b_{3,1}b_{2,2} + 2b_{2,1}b_{3,2} + 2b_{3,1}b_{3,2}) - \hat{\kappa}^2(b_{2,1}b_{2,2} + 2b_{1,1}b_{3,2})) \\
& = a_3 + \gamma^2(\hat{\kappa}(b_{2,1}b_{3,2} - b_{3,1}b_{2,2}) + 2\hat{\kappa}^2(b_{3,1}b_{1,2} - b_{1,1}b_{3,2})).
\end{aligned}
$$

That means we can now finish our polynomial division.

$$
\begin{aligned}
& \Big(1 + \gamma^2(2b_{1,1}b_{2,2} - 2b_{1,2}b_{2,1})\Big)\kappa^5 + \Big(a_1 + 2\gamma^2(b_{3,1}b_{1,2} - b_{1,1}b_{3,2})\Big)\kappa^4 \\
& + \Big(1 + a_2 + \gamma^2(b_{3,1}b_{2,2} - b_{2,1}b_{3,2} - \hat{\kappa}(4b_{3,1}b_{1,2} - 4b_{3,2}b_{1,1}) \\
& + \hat{\kappa}^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}) + 2b_{1,1}b_{2,2} - 2b_{1,2}b_{2,1}))\Big)\kappa^3 \\
& + \Big(a_3 + a_1 + \gamma^2(2b_{3,1}b_{1,2} - 2b_{1,1}b_{3,2} - \hat{\kappa}(b_{3,1}b_{2,2} - b_{2,1}b_{3,2}) + \hat{\kappa}^2(2b_{3,1}b_{1,2} - 2b_{1,1}b_{3,2}))\Big)\kappa^2 \\
& + \Big(a_2 + \gamma^2(b_{2,1}b_{3,2} - b_{3,1}b_{2,2} - \hat{\kappa}(4b_{3,1}b_{1,2} - 4b_{1,1}b_{3,2}) + \hat{\kappa}^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}))\Big)\kappa \\
& + a_3 + \gamma^2(\hat{\kappa}(b_{2,1}b_{3,2} - b_{3,1}b_{2,2}) + 2\hat{\kappa}^2(b_{3,1}b_{1,2} - b_{1,1}b_{3,2})) : b_{1,1}\kappa^2 + b_{2,1}\kappa + b_{3,1} \\
& = \frac{1 + 2\gamma^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2})}{b_{1,1}}\kappa^3 + \mathcal{O}(\kappa^2).
\end{aligned}
$$

Now we need to calculate the new coefficients to proceed which yields

$$
\begin{aligned}
& \Big(a_1 + 2\gamma^2(b_{3,1}b_{1,2} - b_{1,1}b_{3,2}) - \frac{b_{2,1}(1 + 2\gamma^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}))}{b_{1,1}}\Big)\kappa^4 \\
& + \Big(1 + a_2 + \gamma^2(b_{3,1}b_{2,2} - b_{2,1}b_{3,2} - \hat{\kappa}(4b_{3,1}b_{1,2} - 4b_{3,2}b_{1,1}) \\
& + \hat{\kappa}^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}) + 2b_{1,1}b_{2,2} - 2b_{1,2}b_{2,1})) - \frac{b_{3,1}(1 + 2\gamma^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}))}{b_{1,1}}\Big)\kappa^3 \\
& + \mathcal{O}(\kappa^2) : b_{1,1}\kappa^2 + b_{2,1}\kappa + b_{3,1} \\
& = \Big(\frac{a_1 + 2\gamma^2(b_{3,1}b_{1,2} - b_{1,1}b_{3,2})}{b_{1,1}} - \frac{b_{2,1}(1 + 2\gamma^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}))}{b_{1,1}^2}\Big)\kappa^2 + \mathcal{O}(\kappa).
\end{aligned}
$$

Iterating this another step gets us

$$
\begin{aligned}
&\Big(1 + a_2 + \gamma^2(b_{3,1}b_{2,2} - b_{2,1}b_{3,2} - \hat{\kappa}(4b_{3,1}b_{1,2} - 4b_{3,2}b_{1,1})\\
&+ \hat{\kappa}^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}) + 2b_{1,1}b_{2,2} - 2b_{1,2}b_{2,1})) - \frac{b_{3,1}(1 + 2\gamma^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}))}{b_{1,1}}\\
&- \frac{b_{2,1}(a_1 + 2\gamma^2(b_{3,1}b_{1,2} - b_{1,1}b_{3,2}))}{b_{1,1}} + \frac{b_{2,1}^2(1 + 2\gamma^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}))}{b_{1,1}^2}\Big)\kappa^3\\
&+ \Big(a_3 + a_1 + \gamma^2(2b_{3,1}b_{1,2} - 2b_{1,1}b_{3,2} - \hat{\kappa}(b_{3,1}b_{2,2} - b_{2,1}b_{3,2}) + \hat{\kappa}^2(2b_{3,1}b_{1,2} - 2b_{1,1}b_{3,2}))\\
&- \frac{b_{3,1}(a_1 + 2\gamma^2(b_{3,1}b_{1,2} - b_{1,1}b_{3,2}))}{b_{1,1}} + \frac{b_{2,1}b_{3,1}(1 + 2\gamma^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}))}{b_{1,1}^2}\Big)\kappa^2\\
&+ \mathcal{O}(\kappa) : b_{1,1}\kappa^2 + b_{2,1}\kappa + b_{3,1}\\
&= \Big(b_{1,1}^{-1}(1 + a_2 + \gamma^2(b_{3,1}b_{2,2} - b_{2,1}b_{3,2} - \hat{\kappa}(4b_{3,1}b_{1,2} - 4b_{3,2}b_{1,1})\\
&+ \hat{\kappa}^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}) + 2b_{1,1}b_{2,2} - 2b_{1,2}b_{2,1}))) - \frac{b_{3,1}(1 + 2\gamma^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}))}{b_{1,1}^2}\\
&- \frac{b_{2,1}(a_1 + 2\gamma^2(b_{3,1}b_{1,2} - b_{1,1}b_{3,2}))}{b_{1,1}^2} + \frac{b_{2,1}^2(1 + 2\gamma^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}))}{b_{1,1}^3}\Big)\kappa + C.
\end{aligned}
$$

Now the last step to finish this procedure is

$$
\begin{aligned}
&\Big(a_3 + a_1 + \gamma^2(2b_{3,1}b_{1,2} - 2b_{1,1}b_{3,2} - \hat{\kappa}(b_{3,1}b_{2,2} - b_{2,1}b_{3,2}) + \hat{\kappa}^2(2b_{3,1}b_{1,2} - 2b_{1,1}b_{3,2}))\\
&- \frac{b_{3,1}(a_1 + 2\gamma^2(b_{3,1}b_{1,2} - b_{1,1}b_{3,2}))}{b_{1,1}} + \frac{b_{2,1}b_{3,1}(1 + 2\gamma^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}))}{b_{1,1}^2}\\
&-\frac{b_{2,1}}{b_{1,1}}(1 + a_2 + \gamma^2(b_{3,1}b_{2,2} - b_{2,1}b_{3,2} - \hat{\kappa}(4b_{3,1}b_{1,2} - 4b_{3,2}b_{1,1})\\
&+ \hat{\kappa}^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}) + 2b_{1,1}b_{2,2} - 2b_{1,2}b_{2,1}))) + \frac{b_{3,1}b_{2,1}(1 + 2\gamma^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}))}{b_{1,1}^2}\\
&+ \frac{b_{2,1}^2(a_1 + 2\gamma^2(b_{3,1}b_{1,2} - b_{1,1}b_{3,2}))}{b_{1,1}^2} - \frac{b_{2,1}^3(1 + 2\gamma^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}))}{b_{1,1}^3}\Big)\kappa^2\\
&+ \Big(a_2 + \gamma^2(b_{2,1}b_{3,2} - b_{3,1}b_{2,2} - \hat{\kappa}(4b_{3,1}b_{1,2} - 4b_{1,1}b_{3,2}) + \hat{\kappa}^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}))\\
&- \frac{b_{3,1}}{b_{1,1}}\Big((1 + a_2 + \gamma^2(b_{3,1}b_{2,2} - b_{2,1}b_{3,2} - \hat{\kappa}(4b_{3,1}b_{1,2} - 4b_{3,2}b_{1,1})\\
&+ \hat{\kappa}^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}) + 2b_{1,1}b_{2,2} - 2b_{1,2}b_{2,1}))) - \frac{b_{3,1}(1 + 2\gamma^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}))}{b_{1,1}^2}
\end{aligned}
$$

$$- \frac{b_{2,1}(a_1 + 2\gamma^2(b_{3,1}b_{1,2} - b_{1,1}b_{3,2}))}{b_{1,1}^2} + \frac{b_{2,1}^2(1 + 2\gamma^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}))}{b_{1,1}^3}\Big)\Big)\kappa$$

$$+ a_3 + \gamma^2(\hat{\kappa}(b_{2,1}b_{3,2} - b_{3,1}b_{2,2}) + 2\hat{\kappa}^2(b_{3,1}b_{1,2} - b_{1,1}b_{3,2})) : b_{1,1}\kappa^2 + b_{2,1}\kappa + b_{3,1}$$

$$= \Big(a_3 + a_1 + \gamma^2(2b_{3,1}b_{1,2} - 2b_{1,1}b_{3,2} - \hat{\kappa}(b_{3,1}b_{2,2} - b_{2,1}b_{3,2}) + \hat{\kappa}^2(2b_{3,1}b_{1,2} - 2b_{1,1}b_{3,2}))$$

$$- \frac{b_{3,1}(a_1 + 2\gamma^2(b_{3,1}b_{1,2} - b_{1,1}b_{3,2}))}{b_{1,1}} + \frac{b_{2,1}b_{3,1}(1 + 2\gamma^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}))}{b_{1,1}^2}$$

$$- \frac{b_{2,1}}{b_{1,1}}(1 + a_2 + \gamma^2(b_{3,1}b_{2,2} - b_{2,1}b_{3,2} - \hat{\kappa}(4b_{3,1}b_{1,2} - 4b_{3,2}b_{1,1})$$

$$+ \hat{\kappa}^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}) + 2b_{1,1}b_{2,2} - 2b_{1,2}b_{2,1}))) + \frac{b_{3,1}b_{2,1}(1 + 2\gamma^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}))}{b_{1,1}^2}$$

$$+ \frac{b_{2,1}^2(a_1 + 2\gamma^2(b_{3,1}b_{1,2} - b_{1,1}b_{3,2}))}{b_{1,1}^2} - \frac{b_{2,1}^3(1 + 2\gamma^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}))}{b_{1,1}^3}\Big)\frac{1}{b_{1,1}}.$$

We see that polynomial division yields that the left hand side of the equation in question can be written as

$$\mathcal{O}(\kappa^3)\tilde{b}_1(\kappa) + \mathcal{O}(\kappa).$$

It is now our goal to write down the $\mathcal{O}(\kappa)$ term which yields

$$\Big(a_2 + \gamma^2(b_{2,1}b_{3,2} - b_{3,1}b_{2,2} - \hat{\kappa}(4b_{3,1}b_{1,2} - 4b_{1,1}b_{3,2}) + \hat{\kappa}^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}))$$

$$- \frac{b_{3,1}}{b_{1,1}}\Big((1 + a_2 + \gamma^2(b_{3,1}b_{2,2} - b_{2,1}b_{3,2} - \hat{\kappa}(4b_{3,1}b_{1,2} - 4b_{3,2}b_{1,1})$$

$$+ \hat{\kappa}^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}) + 2b_{1,1}b_{2,2} - 2b_{1,2}b_{2,1}))) - \frac{b_{3,1}(1 + 2\gamma^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}))}{b_{1,1}^2}$$

$$- \frac{b_{2,1}(a_1 + 2\gamma^2(b_{3,1}b_{1,2} - b_{1,1}b_{3,2}))}{b_{1,1}^2} + \frac{b_{2,1}^2(1 + 2\gamma^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}))}{b_{1,1}^3}\Big)$$

$$- \frac{b_{2,1}}{b_{1,1}}\Big(a_3 + a_1 + \gamma^2(2b_{3,1}b_{1,2} - 2b_{1,1}b_{3,2} - \hat{\kappa}(b_{3,1}b_{2,2} - b_{2,1}b_{3,2}) + \hat{\kappa}^2(2b_{3,1}b_{1,2} - 2b_{1,1}b_{3,2}))$$

$$- \frac{b_{3,1}(a_1 + 2\gamma^2(b_{3,1}b_{1,2} - b_{1,1}b_{3,2}))}{b_{1,1}} + \frac{b_{2,1}b_{3,1}(1 + 2\gamma^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}))}{b_{1,1}^2}$$

$$
\begin{aligned}
&-\frac{b_{2,1}}{b_{1,1}}(1 + a_2 + \gamma^2(b_{3,1}b_{2,2} - b_{2,1}b_{3,2} - \hat{\kappa}(4b_{3,1}b_{1,2} - 4b_{3,2}b_{1,1}) \\
&+ \hat{\kappa}^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}) + 2b_{1,1}b_{2,2} - 2b_{1,2}b_{2,1}))) + \frac{b_{3,1}b_{2,1}(1 + 2\gamma^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}))}{b_{1,1}^2} \\
&+ \frac{b_{2,1}^2(a_1 + 2\gamma^2(b_{3,1}b_{1,2} - b_{1,1}b_{3,2}))}{b_{1,1}^2} - \frac{b_{2,1}^3(1 + 2\gamma^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}))}{b_{1,1}^3}\Big)\Big)\kappa \\
&+ a_3 + \gamma^2(\hat{\kappa}(b_{2,1}b_{3,2} - b_{3,1}b_{2,2}) + 2\hat{\kappa}^2(b_{3,1}b_{1,2} - b_{1,1}b_{3,2})) \\
&- \frac{b_{3,1}}{b_{1,1}}\Big(a_3 + a_1 + \gamma^2(2b_{3,1}b_{1,2} - 2b_{1,1}b_{3,2} - \hat{\kappa}(b_{3,1}b_{2,2} - b_{2,1}b_{3,2}) + \hat{\kappa}^2(2b_{3,1}b_{1,2} - 2b_{1,1}b_{3,2})) \\
&- \frac{b_{3,1}(a_1 + 2\gamma^2(b_{3,1}b_{1,2} - b_{1,1}b_{3,2}))}{b_{1,1}} + \frac{b_{2,1}b_{3,1}(1 + 2\gamma^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}))}{b_{1,1}^2} \\
&-\frac{b_{2,1}}{b_{1,1}}(1 + a_2 + \gamma^2(b_{3,1}b_{2,2} - b_{2,1}b_{3,2} - \hat{\kappa}(4b_{3,1}b_{1,2} - 4b_{3,2}b_{1,1}) \\
&+ \hat{\kappa}^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}) + 2b_{1,1}b_{2,2} - 2b_{1,2}b_{2,1}))) + \frac{b_{3,1}b_{2,1}(1 + 2\gamma^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}))}{b_{1,1}^2} \\
&+ \frac{b_{2,1}^2(a_1 + 2\gamma^2(b_{3,1}b_{1,2} - b_{1,1}b_{3,2}))}{b_{1,1}^2} - \frac{b_{2,1}^3(1 + 2\gamma^2(b_{1,1}b_{2,2} - b_{2,1}b_{1,2}))}{b_{1,1}^3}\Big).
\end{aligned}
$$

This rather large polynomial now will be defined as $\varepsilon_1\kappa + \varepsilon_2$ in order to make the following calculations remotely understandable. So we can write our equation now in the form

$$\mathcal{O}(\kappa^3)\tilde{b}_1(\kappa) + \varepsilon_1\kappa + \varepsilon_2 - \dot{c}_2(\kappa)\tilde{b}_1(\kappa) = -\dot{c}_1(\kappa)\tilde{b}_2(\kappa).$$

Therefore, we have isolated the term $-\dot{c}_1(\kappa)b_2(\kappa)$ on the left hand side. Evaluating the equation at the two roots of $\tilde{b}_1$ we get a solution of degree one for $\dot{c}_2(\kappa)$. We can also fully determine $C_{1,2}(\kappa)$ using the result from our polynomial division, so we can fully solve for the coefficients of our polynomial. In order to determine $C_{1,2}(\kappa)$ we will need to consider the part of the left side of the equation that is a multiple of $\tilde{b}_1(\kappa)$ because then we know that this polynomial is the solution for $C_{1,2}(\kappa)$. That means the solution of our equation is now two-dimensional. We denote the roots of $\tilde{b}_1$ with $\beta_{1,1}$ and $\beta_{2,1}$ and get that

$$\dot{c}_1(\beta_{k,1}) = -\frac{\varepsilon_1\beta_{k,1} + \varepsilon_2}{\tilde{b}_2(\beta_{k,1})}, \qquad k = 1, 2.$$

To further determine our solution we need to consider the equivalent of Equation (21) for the linear term t. This means we need to calculate the

t-derivative on both sides yielding

$$\begin{aligned}&2(\kappa^2+1)(C'_{1,1}(\kappa)A_0(\kappa)+C'_{0,1}(\kappa)A_1(\kappa))-2\kappa(C_{1,1}(\kappa)A_0(\kappa)\\&+C_{0,1}(\kappa)A_1(\kappa))-(\kappa^2+1)(C_{1,1}(\kappa)A'_0(\kappa)+C_{0,1}(\kappa)A'_1(\kappa))\\&=2(B_{1,1}(\kappa)A_1(\kappa)+B_{2,1}(\kappa)A_0(\kappa))-(B_{0,1}(\kappa)A_2(\kappa)+B_{1,1}(\kappa)A_1(\kappa)).\end{aligned}\tag{24}$$

As before we have argued again that since we can add multiples of b_k to our solutions of $C_{1,k}$ we see that the solution space can only consist of one solution plus a certain Möbius transform. Therefore, our solution space is a priori a two-dimensional space but as in the case for the constant coefficient we can fix the Möbius transform and make the solution space one-dimensional by assuming that $B_{2,1}(0)=0$ holds as well. Using this condition we can evaluate Equation (24) at $\kappa=0$ where the right side vanishes and therefore, we get an extra condition to determine $C_{1,1}(\kappa)$ so the solution space is now only a one-dimensional space as in the case before. So we have now proven that the solution space for the linear coefficients are a one-dimensional solution space just like we have for the constant coefficients. That means that we can fully determine the solutions again by using the third of the Whitham equations for the quadratic coefficient and evaluating it at $\kappa=0$. That means we have solved the coefficient equations for the constant and linear coefficients. Using the algorithm we established in solving for the linear coefficients permits one to solve these equations inducitvely for all coefficients. That means we can now fully determine the coefficients of the power series expansions we have defined before.
The next step here is to evaluate the Taylor series of q_k at $\kappa=0$ using the results we just established. Here we still know that b_1 and b_2 have a root at $\kappa=0$. Using a change of base we can easily assume that b_2 has a simple root at $\kappa=0$ and b_1 has a root of order two at $\kappa=0$. Recallig

$$\mathrm{d}q_k=\frac{b_k(\kappa)}{\nu}\mathrm{d}\kappa,\qquad k=1,2,$$

we see that q_1 has a simple root at $\kappa=0$ as well and q_2 has a root of order two at $\kappa=0$ since $\nu|_{\kappa=0}\neq 0$ holds. So now if we consider q_k to have a Taylor series expansion at $\kappa=0$ in the following way

$$q_k(\kappa)=\sum_{i=0}^{\infty}\frac{q_{i,k}(\kappa,t)}{i!}\kappa^i,\qquad k=1,2,$$

we can easily see that the constant coefficients need to vanish because both q_k have a root at $\kappa=0$. We even see that the linear coefficient of q_2 needs to vanish as well. That means we can now look at the Taylor series expansion of $\mathrm{d}q_k$ because we already know the other coefficients. Using the formula for $b_k(\kappa)$ as well as the Taylor series expansion for ν^{-1} we will have the following ansatz

$$\mathrm{d}q_k = b_k(\kappa)\Big(\sum_{i=0}^{\infty}\frac{\nu_i}{i!}\kappa^i\Big) = 1, 2.$$

Here we will now calculate both factors of this product. By definition we know that

$$b_k(\kappa) = b_0(t) + b_1(t)\kappa + \mathcal{O}(\kappa^2)$$

needs to hold. Now using the power series expansion for $\sqrt{1+\kappa}$ while bracketing gets us

$$\begin{aligned}\nu^{-1} &= \sqrt{(\kappa^2+1)a(\kappa)}^{-1} = \sqrt{(\kappa^2+1)(a_4 + a_3\kappa + a_2\kappa^2 + a_1\kappa^3 + \kappa^4)}^{-1}\\ &= \sqrt{a_4}^{-1}\sqrt{1+\mathcal{O}(\kappa)}\\ &= \sqrt{a_4}^{-1}\Big(1 - \frac{a'(0)}{2a_4}\kappa + \mathcal{O}(\kappa)^2\Big).\end{aligned}$$

That means we can now calculate the Taylor series expansion of $\mathrm{d}q_k$ to get

$$\begin{aligned}q_k &= \Big(b_0(t) + b_1(t)\kappa + b_1(t) + \mathcal{O}(\kappa^3)\Big)\Big(\sqrt{a_4}^{-1}\Big(1 - \frac{a'(0)}{2a_4}\kappa + \mathcal{O}(\kappa)^2\Big)\Big)\\ &= \sqrt{a_4}^{-1}\Big(b_0(t) + \Big(b_1(t) - \frac{a'(0)b_0(t)}{2a_4}\Big)\kappa - \frac{b_1(t)a'(0)}{2a_4}\kappa^2 + \mathcal{O}(\kappa^3)\Big).\end{aligned}$$

The next step is to insert the definition of $b_1(\kappa)$ and $b_2(\kappa)$ into these polynomials and then define $q_1 = y$ and $q_2 = x$. Then the goal is to find a polynomial in x and y such that $f(x,y) = 0$ holds. However since no time was left at the end of this thesis, this was not done. Now we will use a sequence of plots to explain how a family of curves for one connected component of $T^{-1}(\tau_a)$ looks like. Here we fix that the curves are all contracting up until we intersect with $\mathcal{S}^2$.

Figure 5.1: Family of curves $V(q_1, q_2)$ before intersecting with $\mathcal{S}^2$

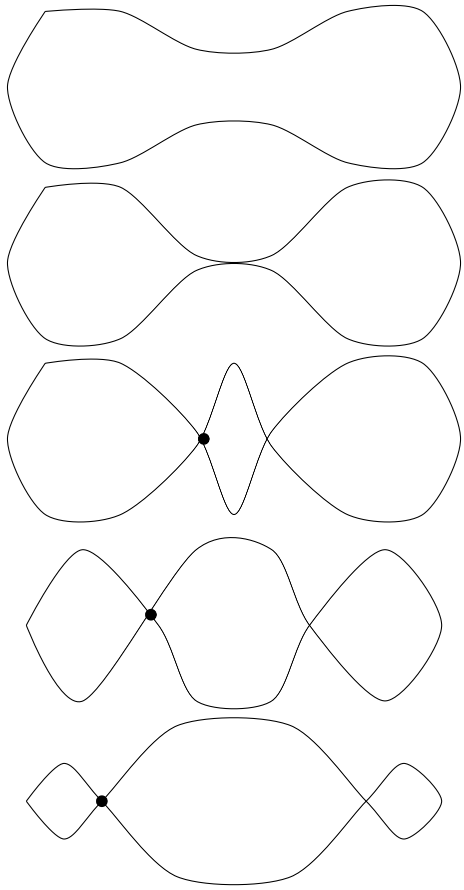


Figure 5.2: Intersection with $\mathcal{S}^2$

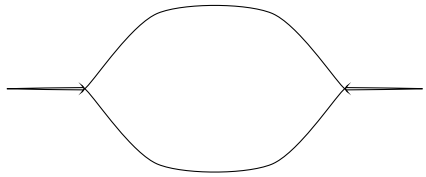
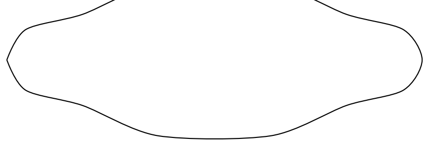


Figure 5.3: $V(q_1, q_2)$ after intersecting with $\mathcal{S}^2$



The dot in the graphics describes the double point of the curve where it intersects with itself. The second to last graphic symbolizes how a curve looks if $T^{-1}(\tau_a)$ intersects with $\mathcal{S}^2$. It would be a goal of future research to prove that the familiy of curves looks similar to the one pictured above where the type of singularity the curves have changes after intersecting with $\mathcal{S}^2$. In order to prove this one needs to consider $f(x, y) = 0$ as before. Then it should be possible to write $f(x, y) = 0$ in the following way

$$g(x, y) = p(x)$$

where $p(x)$ is a polynomial of degree three. Then $p(x)$ should always be a polynomial that has one double root and one simple root. The conjecture is that this changes when intersecting with $\mathcal{S}^2$ from one to the other. This would prove that $V(q_1, q_2)$ needs to have a cusp when intersecting with $\mathcal{S}^2$.

6 Conclusion

In chapter 4 of this thesis we have used the results from B.Schmidt (2020) and considered the of each connected component of the one-dimensional manifold $T^{-1}(\tau_a)$.

First we considered the case $a \in \mathcal{M}_2^2 \cup \mathcal{M}_2^3$ where the polynomials $a(\lambda)$ have a double root on $\mathbb{S}^1$. In this case we examined the fraction $\frac{a(\lambda)}{\lambda^2}$ and deduced that since it's derivative by s doesn't vanish at the double roots the sign of the fraction needs to change if we go further along the manifold which in turn means that $a \in \mathcal{M}_2^2 \cup \mathcal{M}_2^3$ is a true boundary value.

In a next step we constructed a blow-up to consider the case where coefficients of $a(\lambda)$ go to infinity. Here we were able to prove that the limit is continuous and that we can continuously extend the definition of τ_a. We also proved that the connected components of this manifold are biholomorphic to the unique hyperelliptic curve defined by the limit of b_1 and b_2. We also tried to prove that this boundary now also defines a one-dimensional manifold using the Whitham equations. Here we made the mistake of calculating the limits first and then calculating the derivatives, so we didn't succeed here. A point of further research would be to use the correct derivatives and then try to solve the Whitham equations on $T^{-1}(\tau_a)$ proving that this is a one-dimensional manifold.

In chapter 5 we have constructed a curve $V(q_1, q_2)$ in the real plane. We have established certain properties that concern this curve using the results in chapter 6 of B.Schmidt (2020). We have also proven that the Whitham equations are solvable inductively for every coefficient of the Taylor series expansions. A point of further research would be trying to prove that each connected component of $T^{-1}(\tau_a)$ intersects with $\mathcal{S}^2$ at most once and proving that the family of curves has a cusp while intersecting $\mathcal{S}^2$.

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8 List of Figures