University of Mannheim
School of Business Informatics and Mathematics

Master's Thesis

A New Parametrization of the Solutions of the sinh-Gordon Equation of Spectral Genus Two

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Abstract

The study of constant mean curvature (CMC) tori inevitably leads to studying the sinh-Gordon equation. This is an elliptic partial differential equation. Examining this equation is a very challenging task and requires several tools. Among those are the so-called polynomial Killing fields. Solutions of the sinh-Gordon equation form a space of potentials that is a space of matrix valued polynomials. The determinants of these potentials yield polynomials and the degree of these polynomials is connected to the so-called spectral genus. We are going to look at the case of spectral genus two. This means we will have polynomials of degree four. Studying the properties of these polynomials is central in the current research on the sinh-Gordon equation. In this thesis we will construct a mapping that connects each of these polynomials to their corresponding period lattice. Moreover, the roots of these polynomials can be used to construct a homology basis that in return can be used to derive additional polynomials. All these polynomials can be used to utilize the so-called Whitham equations. With the help of these Whitham equations we will see that the level sets of the mapping to the period lattice form one-dimensional submanifolds. In the end we will begin to connect our results to the Willmore energy and point to possible future research.

Contents

1 Introduction

In the twentieth century geometry developed rapidly. Among the aspiring geometers was Heinz Hopf, who simultaneously to developing algebraic topology theory also conjectured the following:

If Σ is an immersion of an oriented, closed hypersurface of constant mean curvature $H \neq 0$ in $\mathbb{R}^n$ then Σ is the $(n-1)$-sphere.

This conjecture proves to be true with some additional constraints. This short survey is based on a survey Bobenko did in [Bob91]. Alexandrov [Ale62] proved that it holds if Σ is an embedded surface not only an immersion. Hopf himself showed a proof for a simple connected surface [Hop03]. However, in 1986 Wente published a counterexample in the general case stated above [Wen86]. He introduced the so-called Wente torus 1.1

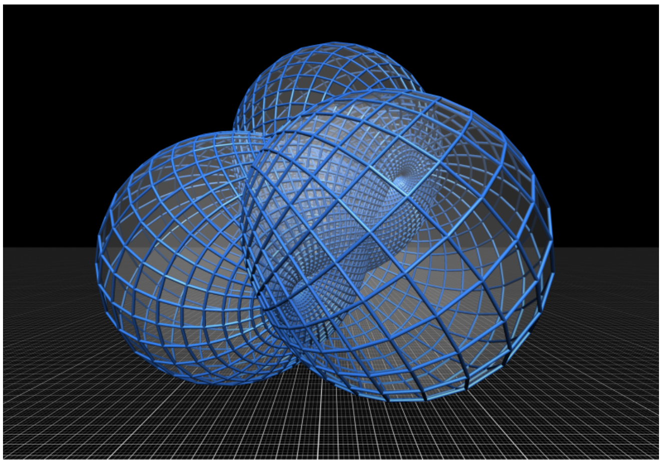


Figure 1.1: Wente torus visualization from TU Berlin:http://page.math.tu-berlin.de/ knoeppel/cmctorivr.html

This torus gave rise to more general tori of constant mean curvature, which have then been extensively studied but still yield further research potential. The family of these tori has been classified by Pinkall and Sterling in [PS89]. All these tori have the following Gauß-Codazzi equation

$$\Delta u + 2\sinh(2u) = 0. \tag{1.1}$$

Equation (1.1) is known as the sinh-Gordon equation. It can be examined within the theory of integrable systems. The study of solutions touches several areas within complex analysis and algebraic geometry. We will look at solutions of said equation parametrized through coefficients of related polynomials. A side goal of this thesis is also to give future students of Prof. Schmidt in Mannheim an introduction to the field. Therefore, it touches several introductory subjects. In order to do this, it is structured as follows.

In *chapter two* we will see the geometric origin of the sinh-Gordon equation as well as the origin of the frames that give rise to methods currently used. Those frames initially use 3×3 matrices. But with the so-called loop group method it is possible to transform them into 2×2 matrices which will be used directly in this thesis.

Chapter three gives an overview about the relevant areas of Riemann surfaces such as hyperelliptic Riemann surfaces or the canonical homology basis and connects it to algebraic curves. Furthermore, we visit some standard submanifold theory which is useful since our main goal is to prove that certain sets are submanifolds.

With *chapter four* we will start explicitly working with the spectral curves of the sinh-Gordon equation. In the first part of this chapter we see what the period lattice are and introduces several important spaces of polynomials. Those polynomials are mainly characterized by their roots. We will exclusively treat the case in which the polynomials have degree four. In the second part we will prove a first result. That is that the level sets of a mapping that connects the mentioned polynomials to their period lattices are one-dimensional submanifolds if the polynomials have four distinct roots.

The *fifth chapter* deals with a problem related to special polynomials that provide us with a singularity. This singularity occurs when the polynomials have a double root. In this chapter we tried two equivalent approaches because the first approach turned out to be very complex. The second approach is somewhat similar to the approach used in chapter four. Through several steps we were able to deduct new conditions and therefore, also prove that the level sets also form one-dimensional submanifolds in the case in which the polynomials have at least one double root. In other words we were able to get rid of the singularity.

In *chapter six* we see what happens if we intersect with another important set of polynomials. Due to a time constraint the step could not fully be finished. However, we were able to obtain a new result that can be connected to the Willmore energy.

Chapter seven summarizes the work done in this thesis and draws a conclusion.

2 Differential geometric origin of the sinh-Gordon equation

My interest in differential geometry was sparked by my bachelor's thesis, which introduced me to constant mean curvature tori. I had not thought about curvature before and was fascinated with the idea. Even though, the topics of this thesis will be further away from the differential geometric origin, I felt the need to go through the derivations once. This will motivate the sinh-Gordon equation as well as the Lax-pair, which is heavily used in the theory of integrable systems due to the inverse scattering transform. Vania Neugebauer investigated this differential geometric origin in her diploma thesis [Neu08] in a similar way. Therefore, this chapter partly draws from her thesis and partly draws from [Bob91]. However, both used the following form

$$\alpha = Udz + Vd\bar{z} \tag{2.1}$$

but we need

$$\tilde{\alpha} = Udx + Vdy. \tag{2.2}$$

Therefore, the calculations become a little longer.

2.1 The sinh-Gordon equation

Let $S \subset \mathbb{R}^3$ be a smooth surface and $\bigcup U_i$ a suitable covering of S such that $f_i : U_i \mapsto \mathbb{R}^2$ forms a chart on S and holds a smooth structure. Hence, S is a 2-manifold. The Euclidean 3-space surrounding S induces the Euclidean metric $< \cdot, \cdot >$ on S. The identification

$$\mathbb{R}^2 \to \mathbb{C}$$
$$(x, y) \mapsto x + iy$$

gives rise to an important link to Riemann surfaces. This link enables us to choose coordinates in a way that there exists another family of charts such that they generate a complex structure to which g is conformal. Now let f denote the corresponding immersion of S, i.e.

$$f : S \to \mathbb{R}^3.$$

With this we can now calculate the fundamental forms. For the first fundamental form we have

$$g_{ij} =< f_i, f_j > .$$

Since g is conformal f_x and f_y are orthogonal. This leads to

$$g_{xy} = 0 = g_{yx}.$$

Furthermore, we set $g_{xx} =< f_x, f_x >= 4e^{2u}$. Thus, we arrive at

$$g = 4 \begin{pmatrix} e^{2u} & 0 \\ 0 & e^{2u} \end{pmatrix}$$

for the first fundamental form. We will use the following common partial derivative operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

and

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

This gives us the following expressions

$$\begin{aligned} < f_z, f_{\overline{z}} >= & < \frac{1}{2}(f_x - if_y), \frac{1}{2}(f_x + if_y) > \\ = & < \frac{1}{2}f_x, \frac{1}{2}f_x > + < -\frac{1}{2}if_y, \frac{1}{2}f_x > + < \frac{1}{2}f_x, i\frac{1}{2}f_y > + < -i\frac{1}{2}f_y, i\frac{1}{2}f_y > \\ = & \frac{1}{4} < f_x, f_x > + \frac{1}{4} < f_y, f_y > \\ = & 2e^{2u}, \end{aligned}$$

$$\begin{aligned} < f_z, f_z >= & < \frac{1}{2}(f_x - if_y), \frac{1}{2}(f_x - if_y) > \\ = & < \frac{1}{2}f_x, \frac{1}{2}f_x > + < -\frac{1}{2}if_y, \frac{1}{2}f_x > + < \frac{1}{2}f_x, -i\frac{1}{2}f_y > + < -i\frac{1}{2}f_y, -i\frac{1}{2}f_y > \\ = & 0 \end{aligned}$$

and

$$\begin{aligned} < f_{\overline{z}}, f_{\overline{z}} >= & < \frac{1}{2}(f_x + if_y), \frac{1}{2}(f_x + if_y) > \\ = & < \frac{1}{2}f_x, \frac{1}{2}f_x > + < \frac{1}{2}if_y, \frac{1}{2}f_x > + < \frac{1}{2}f_x, i\frac{1}{2}f_y > + < i\frac{1}{2}f_y, i\frac{1}{2}f_y > \\ = & 0. \end{aligned}$$

Let N now denote the unit normal to f_x and f_y. Then $< f_x, N >= 0 =< f_y, N >$ and $< N, N >= 1$ holds. This gives the first fundamental form

$$I = 4e^{2u} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now we want to calculate the second fundamental form. It is given through

$$II = \begin{pmatrix} < N, f_{xx} > & < N, f_{xy} > \\ < N, f_{yx} > & < N, f_{yy} > \end{pmatrix}.$$

We set $< f_{zz}, N >= Q$ and $< f_{z\bar{z}}, N >= 2\tilde{H}e^{2u}$. It holds

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \Leftrightarrow \frac{\partial}{\partial x} = 2\frac{\partial}{\partial z} + i\frac{\partial}{\partial y}$$

and

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \Leftrightarrow \frac{\partial}{\partial x} = 2\frac{\partial}{\partial \bar{z}} - i\frac{\partial}{\partial y}.$$

This yields

$$\frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)$$

and therefore

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}.$$

Hence, the second partial derivatives become

$$\begin{aligned}
\frac{\partial^2}{\partial z^2} &= \frac{\partial}{\partial z}\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x}\frac{\partial}{\partial z} - i\frac{\partial}{\partial y}\frac{\partial}{\partial z}\right) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x}\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) - i\frac{\partial}{\partial y}\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\right) \\
&= \frac{1}{4}\frac{\partial^2}{\partial x^2} - \frac{1}{2}i\frac{\partial^2}{\partial x \partial y} - \frac{1}{4}\frac{\partial^2}{\partial y^2},
\end{aligned}$$

likewise one gets

$$\frac{\partial^2}{\partial \bar{z}^2} = \frac{1}{4}\frac{\partial^2}{\partial x^2} + \frac{1}{2}i\frac{\partial^2}{\partial x \partial y} - \frac{1}{4}\frac{\partial^2}{\partial y^2}$$

and

$$\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4}\frac{\partial^2}{\partial x^2} + \frac{1}{4}\frac{\partial^2}{\partial y^2}.$$

Solving this system of equations yields

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial z^2} + 2\frac{\partial^2}{\partial z \partial \overline{z}} + \frac{\partial^2}{\partial \overline{z}^2},$$
$$\frac{\partial^2}{\partial y^2} = -\frac{\partial^2}{\partial z^2} + 2\frac{\partial^2}{\partial z \partial \overline{z}} - \frac{\partial^2}{\partial \overline{z}^2}$$

and

$$\frac{\partial^2}{\partial x \partial y} = i\left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \overline{z}^2}\right).$$

Thus, we calculate

$$< f_{xx}, N > = < f_{zz}, N > +2 < f_{z\bar{z}}, N > + < f_{\bar{z}\bar{z}}, N > = Q + 4\tilde{H}e^{2u} + \overline{Q},$$
$$< f_{yy}, N > = - < f_{zz}, N > +2 < f_{z\bar{z}}, N > - < f_{\bar{z}\bar{z}}, N > = -Q + 4\tilde{H}e^{2u} - \overline{Q}$$

and

$$< f_{xy}, N > = i < f_{zz}, N > -i < f_{\bar{z}\bar{z}}, N > = iQ - i\overline{Q}.$$

Hence, we obtain

$$II = \begin{pmatrix} Q + 4\tilde{H}e^{2u} + \overline{Q} & iQ - i\overline{Q} \\ iQ - i\overline{Q} & -Q + 4\tilde{H}e^{2u} - \overline{Q} \end{pmatrix}.$$

Now we can compute the mean curvature H.

$$H = \frac{1}{2}tr(I^{-1}II) = \frac{1}{8e^{2u}}tr\begin{pmatrix} Q + 4\tilde{H}e^{2u} + \overline{Q} & iQ - i\overline{Q} \\ iQ - i\overline{Q} & -Q + 4\tilde{H}e^{2u} - \overline{Q} \end{pmatrix} = \frac{8\tilde{H}e^{2u}}{8e^{2u}} = \tilde{H}.$$

Thus, the mean curvature H is equal to the term $\tilde{H}$ used in the second fundamental form. This fact will be of great importance in the next calculations. We can now proceed and construct the Lax pair U and V and the frame F, which is the fundamental solution to the following ODE for the immersion f.

$$F_z = UF,\ F_{\overline{z}} = VF,\ \text{with } F = (f_z, f_{\overline{z}}, N)^T,\ \text{where}$$
$$U = \begin{pmatrix} 2u_z & 0 & Q \\ 0 & 0 & 2He^{2u} \\ -H & -\frac{1}{2}e^{-2u}Q & 0 \end{pmatrix},\ V = \begin{pmatrix} 0 & 0 & 2He^{2u} \\ 0 & 2u_{\overline{z}} & \overline{Q} \\ -\frac{1}{2}e^{-2u}\overline{Q} & -H & 0 \end{pmatrix}.$$

The fundamental theorem of surface theory states that such an immersion f with the corresponding first and second fundamental form exists exactly when the Gauß-Mainardi-Codazzi equations are satisfied. Due to Bonnet's theorem the first and second fundamental form determine a surface in $\mathbb{R}^3$ uniquely up to a rigid motion. The Gauß-Mainardi-Codazzi equations become

$$4u_{z\overline{z}} + 4H^2e^{2u} - Q\overline{Q}e^{-2u} = 0$$
$$Q_{\overline{z}} = 2H_ze^{2u}$$
$$\overline{Q}_z = 2H_{\overline{z}}e^{2u}.$$

Now we can finally come back to constant mean curvature tori in which we were initially interested. We can without loss of generality set $H = \frac{1}{2}$. Since H is now constant Q must be holomorphic. We choose $Q = e^{i\varphi}$ where $\varphi \in \mathbb{R}$ is constant. Inserting these assumptions into the Gauß-Mainardi-Codazzi equations we obtain

$$\begin{aligned} & 4u_{z\bar{z}} + e^{2u} - e^{-2u} = 0 \\ \Leftrightarrow\ & 4u_{z\bar{z}} + 2\sinh(2u) = 0 \\ \Leftrightarrow\ & 4\left(\frac{1}{4}u_{xx} + \frac{1}{4}u_{yy}\right) + 2\sinh(2u) = 0 \\ \Leftrightarrow\ & u_{xx} + u_{yy} + 2\sinh(2u) = 0 \\ \Leftrightarrow\ & \Delta u + 2\sinh(2u) = 0 \end{aligned}$$

Therefore, the sinh-Gordon equation is simply a transformed version of the Gauß-Mainardi-Codazzi equation.

2.2 Loop group methods for 2×2 matrices

We will now look again at the Lax pair. It is possible to transform the 3×3 frame into a 2×2 frame, which is heavily used in the theory developed around CMC tori and especially in this thesis. This Lax pair of 3×3 matrices can be transformed into 2×2 matrices. This procedure is described in [FKR06]. First, we need to define the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

All fulfill the equality $\sigma_k^2 = \mathbb{1}$. Then,

$$\{\mathbb{1}, -i\sigma_1, -i\sigma_2, -i\sigma_3\}$$

forms a basis. It is now possible to identify $\mathbb{R}^3$ with SU_2 through these matrices with

$$-x_1\frac{i}{2}\sigma_1 + x_2\frac{i}{2}\sigma_2 + x_3\frac{i}{2}\sigma_3 = \frac{-i}{2}\begin{pmatrix} -x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_3 \end{pmatrix}.$$

We now set $F = F(x, y, \lambda) \in SU_2$. That is the matrix such that

$$Fe_k = F\frac{-i\sigma_k}{2} \ \forall\ k \in \{1, 2, 3\}.$$

In other words we obtain

$$e_1 = F\frac{-i\sigma_1}{2}F^{-1}, \ e_2 = F\frac{-i\sigma_2}{2}F^{-1} \text{ and } e_3 = F\frac{-i\sigma_3}{2}F^{-1}.$$

We denote $e_3 = N$. Now looking at e_1 and e_2 we obtain

$$e_1 = \frac{f_x}{|f_x|} = \frac{f_x}{2e^u} = \frac{-i}{2} F \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} F^{-1} \tag{2.3}$$

and

$$e_2 = \frac{f_y}{|f_y|} = \frac{f_y}{2e^u} = \frac{-i}{2} F \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} F^{-1} = \frac{1}{2} F \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.4}$$

(2.3) and (2.4) then yield

$$f_x = -ie^u F \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} F^{-1} \tag{2.5}$$

and

$$f_y = e^u F \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} F^{-1}. \tag{2.6}$$

We define

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} := F^{-1} F_x$$

and

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} := F^{-1} F_y.$$

Therefore, we get $FU = F_x$ and $U^{-1}F^{-1} = F_x^{-1}$ as well as $FV = F_y$ and $V^{-1}F^{-1} = F_y^{-1}$. Now we calculate

$$\begin{aligned}
f_{xy} &= u_y f_x - ie^u \left(F_y \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} F^{-1} + F \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} F_y^{-1} \right) \\
&= u_y f_x - ie^u \left(FV \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} F^{-1} + F \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} V^{-1} F^{-1} \right) \\
&= u_y f_x - ie^u \left(F \begin{pmatrix} V_{12} & V_{11} \\ V_{22} & V_{21} \end{pmatrix} F^{-1} + F \begin{pmatrix} -V_{21} & V_{11} \\ V_{22} & -V_{12} \end{pmatrix} F^{-1} \right) \\
&= u_y f_x - ie^u \left(F \begin{pmatrix} V_{12} - V_{21} & 2V_{11} \\ 2V_{22} & V_{21} - V_{12} \end{pmatrix} F^{-1} \right)
\end{aligned}$$

and

$$
\begin{aligned}
f_{yx} &= u_y f_x + e^u\left(F_x\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}F^{-1} + F\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}F_x^{-1}\right)\\
&= u_y f_x + e^u\left(FU\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}F^{-1} + F\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}U^{-1}F^{-1}\right)\\
&= u_x f_y + e^u\left(F\begin{pmatrix}U_{12} & -U_{11}\\ U_{22} & -U_{21}\end{pmatrix}F^{-1} + F\begin{pmatrix}U_{21} & -U_{11}\\ U_{22} & -U_{12}\end{pmatrix}F^{-1}\right)\\
&= u_x f_y + e^u\left(F\begin{pmatrix}U_{12}+U_{21} & -2U_{11}\\ 2U_{22} & -U_{21}-U_{12}\end{pmatrix}F^{-1}\right)
\end{aligned}
$$

Due to Schwarz's theorem $f_{xy} = f_{yx}$ holds since f is smooth. Using the corresponding expressions calculated above we obtain

$$
u_y f_x - u_x f_y = e^u\left(F\begin{pmatrix}U_{12}+U_{21}+i(V_{12}-V_{21}) & -2U_{11}+2iV_{11}\\ 2U_{22}+2iV_{22} & -U_{21}-U_{12}+i(V_{21}-V_{12})\end{pmatrix}F^{-1}\right). \tag{2.7}
$$

We can further compute the left-hand side of (2.7) with

$$
u_y f_x = -ie^u F\begin{pmatrix}0 & u_y\\ u_y & 0\end{pmatrix}F^{-1}
$$

and

$$
u_x f_y = e^u F\begin{pmatrix}0 & -u_x\\ u_x & 0\end{pmatrix}F^{-1}.
$$

Thus, (2.7) yields the following four conditions on the coefficients of U and V:

1. $U_{12} + U_{21} + i(V_{12} - V_{21}) = 0$
2. $-2U_{11} + 2iV_{11} = u_x - iu_y$
3. $2U_{22} + 2iV_{22} = -u_x - iu_y$
4. $-U_{21} - U_{12} + i(V_{21} - V_{12}) = 0$

Looking at f_{xx} will yield further conditions. We already know that

$$
\begin{aligned}
f_{xx} &= f_{\bar{z}\bar{z}} + 2f_{z\bar{z}} + f_{zz} \\
&= \bar{Q}N + 2u_{\bar{z}}f_{\bar{z}} + 4He^{2u}N + QN + 2u_z f_z \\
&= \bar{Q}N + \frac{1}{2}(u_x + iu_y)(f_x + if_y) + 4He^{2u}N + QN + \frac{1}{2}(u_x - iu_y)(f_x - if_y) \\
&= (\bar{Q} + 4He^{2u} + Q)N + \frac{1}{2}u_x f_x + \frac{1}{2}iu_x f_y + \frac{1}{2}iu_y f_x - \frac{1}{2}u_y f_y \\
&\quad + \frac{1}{2}u_x f_x - \frac{1}{2}iu_x f_y - \frac{1}{2}iu_y f_x - \frac{1}{2}u_y f_y \\
&= (\bar{Q} + 4He^{2u} + Q)N + u_x f_x - u_y f_y.
\end{aligned}
$$

At the same time we can also compute f_{xx} from f_x. This yields

$$
\begin{aligned}
f_{xx} &= u_x f_x + (ie^u)\left(F_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} F^{-1} + F \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} F_x^{-1}\right) \\
&= u_x f_x + (ie^u)\left(FU \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} F^{-1} + F \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U^{-1}F^{-1}\right) \\
&= u_x f_x - ie^u\left(F \begin{pmatrix} U_{12} - U_{21} & 2U_{11} \\ 2U_{22} & U_{21} - U_{12} \end{pmatrix} F^{-1}\right).
\end{aligned}
$$

Equating both expressions for f_{xx} therefore yields

$$
F \begin{pmatrix} U_{12} - U_{21} & 2U_{11} \\ 2U_{22} & U_{21} - U_{12} \end{pmatrix} F^{-1} = F \begin{pmatrix} -\bar{Q}e^{-u} - Qe^{-u} - 4He^{u} & iu_y \\ -iu_y & \bar{Q}e^{-u} + Qe^{-u} + 4He^{u} \end{pmatrix} F^{-1}
$$

This gives us

$$
U_{11} = \frac{iu_y}{2},\ U_{22} = \frac{-iu_y}{2} \text{ and } U_{12} - U_{21} = -\bar{Q}e^{-u} - Qe^{-u} - 4He^{u}.
$$

The same reasoning for f_{yy} yields

$$
\begin{aligned}
f_{yy} &= -f_{\bar{z}\bar{z}} + 2f_{z\bar{z}} - f_{zz} \\
&= -\bar{Q}N - 2u_{\bar{z}}f_{\bar{z}} + 4He^{2u}N - QN - 2u_z f_z \\
&= -\bar{Q}N - \frac{1}{2}(u_x + iu_y)(f_x + if_y) + 4He^{2u}N - QN - \frac{1}{2}(u_x - iu_y)(f_x - if_y) \\
&= (-\bar{Q} + 4He^{2u} - Q)N - \frac{1}{2}u_x f_x - \frac{1}{2}iu_x f_y - \frac{1}{2}iu_y f_x + \frac{1}{2}u_y f_y \\
&\quad - \frac{1}{2}u_x f_x + \frac{1}{2}iu_x f_y + \frac{1}{2}iu_y f_x + \frac{1}{2}u_y f_y \\
&= (-\bar{Q} + 4He^{2u} - Q)N - u_x f_x + u_y f_y.
\end{aligned}
$$

Computing f_{yy} from f_y yields

$$\begin{aligned} f_{yy} &= u_y f_y + e^u \left(F_y \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} F^{-1} + F \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} F_y^{-1} \right) \\ &= u_y f_y + e^u \left(FV \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} F^{-1} + F \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} V^{-1} F^{-1} \right) \\ &= u_y f_y + e^u \left(F \begin{pmatrix} V_{12} + V_{21} & -2V_{11} \\ 2V_{22} & -V_{21} - V_{12} \end{pmatrix} F^{-1} \right). \end{aligned}$$

Equating both expressions gives us

$$F \begin{pmatrix} V_{12} + V_{21} & -2V_{11} \\ 2V_{22} & -V_{21} - V_{12} \end{pmatrix} F^{-1} = F \begin{pmatrix} (-\bar{Q}e^{-u} - Qe^{-u} + 4He^u)i & iu_x \\ iu_x & (\bar{Q}e^{-u} + Qe^{-u} - 4He^u)i \end{pmatrix} F^{-1}.$$

This gives us

$$V_{11} = \frac{-iu_x}{2},\ V_{22} = \frac{iu_x}{2} \text{ and } V_{12} + V_{21} = (-\bar{Q}e^{-u} - Qe^{-u} + 4He^u)i.$$

Now, we can further calculate f_{yx} with $U_{11} = \frac{iu_y}{2}$ and $U_{22} = \frac{-iu_y}{2}$

$$f_{yx} = e^u \left(F \begin{pmatrix} U_{12} + U_{21} & -iu_y - u_x \\ -iu_y + u_x & -U_{21} - U_{12} \end{pmatrix} F^{-1} \right).$$

We also know

$$\begin{aligned} f_{yx} &= if_{zz} - if_{\bar{z}\bar{z}} \\ &= iQN + 2iu_u f_z - i\bar{Q}N - 2iu_{\bar{z}} f_{\bar{z}} \\ &= (iQ - i\bar{Q})N + u_x f_y + u_y f_x \\ &= F \begin{pmatrix} -i(iQ - i\bar{Q}) & -u_x e^u - iu_y e^u \\ u_x e^u - iu_y e^u & i(iQ - i\bar{Q}) \end{pmatrix} F^{-1}. \end{aligned}$$

Therefore, we obtain $e^u(U_{12} + U_{21}) = -i(iQ - i\bar{Q})$. This yields

$$U_{12} = (Q - \bar{Q})e^{-u} - U_{21}.$$

Together with $U_{12} - U_{21} = -\bar{Q}e^{-u} - Qe^{-u} - 4He^u$ we obtain

$$U_{21} = Qe^{-u} + 2He^u$$

and

$$U_{12} = -\bar{Q}e^{-u} - 2He^u.$$

Now, recall

$$U_{12} + U_{21} + i(V_{12} - V_{21}) = 0, \tag{2.8}$$

and

$$V_{12} + V_{21} = (-\bar{Q}e^{-u} - Qe^{-u} + 4He^{u})i \Leftrightarrow V_{12} = (-\bar{Q}e^{-u} - Qe^{-u} + 4He^{u})i - V_{21}.$$

Those yield

$$V_{21} = -iQe^{-u} + 2iHe^{u}$$

and

$$V_{12} = -i\bar{Q}e^{-u} + 2iHe^{u}.$$

Thus, we obtain

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} = \begin{pmatrix} \frac{iu_y}{2} & -\bar{Q}e^{-u} - 2He^{u} \\ Qe^{-u} + 2He^{u} & \frac{-iu_y}{2} \end{pmatrix}$$

and

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} = \begin{pmatrix} \frac{-iu_x}{2} & -i\bar{Q}e^{-u} + 2iHe^{u} \\ -iQe^{-u} + 2iHe^{u} & \frac{iu_x}{2} \end{pmatrix}.$$

For CMC tori we choose again $H = \frac{1}{2}$ and also $Q = \lambda \in \mathbb{S}^1$. We also define $\gamma = e^u$. Furthermore, we define $u_z = -\alpha$ and $u_{\bar{z}} = -\bar{\alpha}$. This gives us

$$\frac{\alpha - \bar{\alpha}}{2} = \frac{-u_z + u_{\bar{z}}}{2} = \frac{-u_x + iu_y + u_x + iu_y}{2} = \frac{iu_y}{2}, \tag{2.9}$$

$$\frac{\bar{\alpha} - \alpha}{2} = \frac{-u_{\bar{z}} + u_z}{2} = \frac{-u_x - iu_y + u_x - iu_y}{2} = \frac{-iu_y}{2}, \tag{2.10}$$

$$i\frac{\alpha + \bar{\alpha}}{2} = i\frac{-u_z - u_{\bar{z}}}{2} = i\frac{-u_x + iu_y - u_x - iu_y}{2} = \frac{-iu_x}{2} \tag{2.11}$$

and

$$-i\frac{\alpha + \bar{\alpha}}{2} = -i\frac{-u_z - u_{\bar{z}}}{2} = -i\frac{-u_x + iu_y - u_x - iu_y}{2} = \frac{iu_x}{2}. \tag{2.12}$$

Therefore, we obtain

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} = \begin{pmatrix} \frac{\alpha-\bar{\alpha}}{2} & -\lambda^{-1}\gamma^{-1} - \gamma \\ \lambda\gamma^{-1} + \gamma & \frac{\bar{\alpha}-\alpha}{2} \end{pmatrix}$$

and

$$V = i\begin{pmatrix} \frac{\alpha+\bar{\alpha}}{2} & -\lambda^{-1}\gamma^{-1} + \gamma \\ -\lambda\gamma^{-1} + \gamma & -\frac{\alpha+\bar{\alpha}}{2} \end{pmatrix}.$$

These matrices will be essential in the remainder of this thesis since they form an existential part of the Lax equations:

$$\tfrac{\partial\zeta}{\partial x} = [\zeta, U(\zeta)] \qquad \tfrac{\partial\zeta}{\partial y} = [\zeta, V(\zeta)],$$

where $[\cdot,\cdot]$ is the Lie bracket. We will now look at some basic theory before we continue our work on CMC tori in chapter 4.

3 Theoretic background for Riemann surfaces and integrable systems

In order to use the appropriate tools to examine the sinh-Gordon equation we need to introduce some theory regarding Riemann surfaces and their connection to integrable systems. Riemann surfaces are complex one-dimensional manifolds. We assume basic knowledge of real and complex manifold theory and will not go into detail about charts, complex structures etc., we will rather focus on the precise definitions, methods and concepts relevant for the study of CMC tori. The reader can be referred to the lecture series in complex analysis of Sebastian Klein [Kle20] which thoroughly introduces Riemann surfaces and then builds the theory around them or to the standard literature [For12] or [BF09]. A nice overview of the homology theory of Riemann surfaces can yet again be found in a survey done by Bobenko [Bob11].

3.1 Algebraic curves and Riemann surfaces

As it turns out we are going to work with so-called hyperelliptic Riemann surfaces.

Definition 3.1 (Hyperelliptic Riemann surface)**.** *A hyperelliptic Riemann surface is a Riemann surface X on which a meromorphic function f with exactly two poles exists (counted by multiplicity).*

Of special interest are compact Riemann surfaces because they bear a very nice connection to algebraic curves.

Theorem 3.2. *Any compact Riemann surface can be described as an algebraic curve.*

The proof of theorem 3.2 is non trivial. It can be shown that any Riemann surface can be embedded in a suitable complex projective space. CMC tori will lead to Riemann surfaces that can be derived from algebraic curves. Therefore, it is useful to shortly visit some facts of algebraic curves.

Definition 3.3 (Algebraic curve). *An algebraic curve C is a so-called one-dimensional algebraic variety, i.e. it is defined as*

$$C = \{(\lambda,\mu) \in \mathbb{C}^2 | P(\lambda,\mu) = 0\},$$

where P is an irreducible polynomial. An algebraic curve C is called non-singular if the gradient of the polynomial does not vanish.

The study of CMC tori leads to hyperelliptic curves.

Definition 3.4 (Hyperelliptic curve). *A curve that arises from the following equation*

$$\mu^2 = \prod_{j=1}^{N} (\lambda - \lambda_j)$$

for $N \geq 3$ is called hyperelliptic curve.

A hyperelliptic curve is therefore exactly non-singular if all the λ_j are unique (otherwise the gradient of $\prod_{j=1}^{N}(\lambda-\lambda_j)-\mu^2$ would vanish at the (at least) double root). Depending on the genus of the surface g it either has one puncture or two, i.e. for $N = 2g+1$ there is one puncture $P \to \infty \Leftrightarrow \lambda \to \infty$ and for $N = 2g+2$ there are two punctures $\infty^{\pm}$ described through $P \to \infty^{\pm} \Leftrightarrow \frac{\mu}{\lambda^{g+1}} \to \pm 1,\ \lambda \to \infty$. Furthermore, it is advantageous to work with compact Riemann surfaces. Therefore, it is common to construct so-called compactifications denoted by $\hat{C}$. In the cases above we obtain $\hat{C} = C \cup \{\infty\}$ for odd N and $\hat{C} = C \cup \{\infty^{\pm}\}$ for even N. Algebraic curves can be understood as coverings of $\hat{\mathbb{C}}$ with the standard projection

$$\hat{C} \to \hat{\mathbb{C}}$$
$$(\lambda,\mu) \mapsto \lambda.$$

CMC tori will lead to a situation where we have an equation of the form $\nu^2 = \lambda a(\lambda)$, where a is a special polynomial. As the title of this thesis already suggests we are going to treat the case of spectral genus two. This means that a is of degree four or in other words $2g$. Thus, we can define a Riemann surface of genus 2

$$\Sigma^* = \{(\lambda,\nu) \in \mathbb{C} \setminus \{0\} \times \mathbb{C} | \nu^2 = \lambda a(\lambda)\}.$$

In the case that a has four distinct roots we obtain a smooth Riemann surface. Otherwise there are singularities at these roots. Σ^* is not compact. Therefore, those Riemann surfaces will depend on the roots of a and be central to the observations in the following chapters. We can compactify them through adding 0 and ∞. Then we obtain a compact Riemann surface $\bar{\Sigma}$.

3.2 Homology of Riemann surfaces

Now, it is also useful to revisit some facts about the topology of Riemann surfaces because we want to consider homotopy and homology on Riemann surfaces.

Theorem 3.5. *Any compact Riemann surface X is homeomorphic to a sphere with handles. The number g of handles (i.e. holes of the surface as a real 2-dimensional manifold) is called genus of X.*

Therefore, two compact Riemann surfaces with different genus cannot be homeomorphic. The fundamental group $\pi_1(X)$ of a Riemann surface is generated by cycles γ_i going around the holes of the surface.
To introduce the first homology group of a Riemann surface. We need to establish the common homology theory, i.e. the chain groups C_0, C_1 and C_2, the boundary operator ∂_k that acts as connecting homeomorphism (i.e. $\partial_k : C_k \to C_{k-1}$ and the two subgroups $Z_k = \ker \partial_k$ and $B_k = im\partial_{k+1}$. Hereby, C_0 consists of a sum of points, C_1 consists of a sum of oriented curves and C_2 consists of a sum of oriented domains. The first homology group H_1 is then defined as $H_1 = Z_1/B_1$. We are interested in the first homology of a Riemann surface since we want to define a basis of this group later on. In fact, the fundamental group taken modulo the commutator group gives the first homology group. Thus, both are strongly related. In order to define a canonical basis of the first homology group we need to introduce intersection numbers.

Definition 3.6 (Intersection numbers)**.** *Let γ_1 and γ_2 be two smooth cycles (i.e. elements of Z_1) that transversely intersect in finitely many points $P_1, \dots, P_n$. Then we define $(\gamma_1 \circ \gamma_2)_{P_k} = \pm 1$. The sign depends on the orientation of γ_1 and γ_2. The total intersection number of both cycles is then defined as*

$$\sum_{P \in \gamma_1 \cap \gamma_2} (\gamma_1 \circ \gamma_2)_P.$$

Now we can define the canonical homology basis.

Definition 3.7 (Canonical homology basis)**.** *Let $A_1, B_1, \dots A_g, B_g$ be a homology basis of a compact Riemann surface of genus g with intersection number*

$$A_k \circ B_l = \delta_{kl} \text{ and } A_k \circ A_l = B_k \circ B_l = 0.$$

Then the basis is called a canonical homology basis.

A canonical basis for a hyperelliptic Riemann surface of genus 2 looks like 3.1.

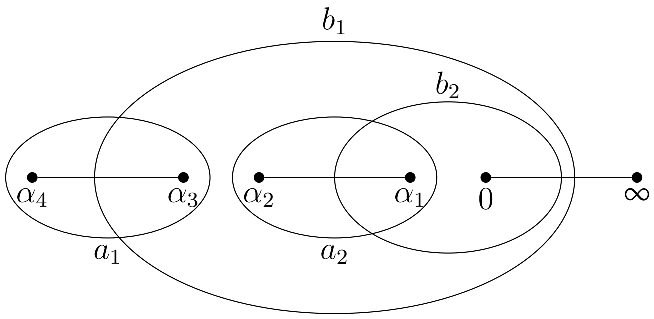


Figure 3.1: Canoncial basis of a Riemann surface of genus 2

Once an homology basis is obtained it is possible to determine the period of differentials. This means the following.

Definition 3.8 (Period of differentials)**.** *Let X be a Riemann surface and $(\gamma_i)_{i\in I}$ a homology basis. Then the period of a closed differential ω is defined as*

$$\Lambda_i = \int_{\gamma_i} \omega.$$

3.3 Line bundles

Another important concept in the study of geometry are line bundles. They can be used to connect integrable systems and Riemann surfaces. We will introduce the concept through the following example.

Example 3.9. *Let the base space be $X = S^1$ (See Figure 3.2).*

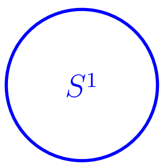


Figure 3.2: Base space

If we attach an orthogonal line F_{x_0} to S^1 at x_0 (see Figure 3.3) we get:

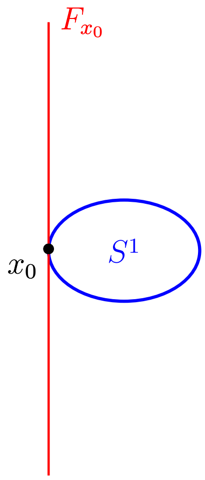


Figure 3.3: Base space with single fiber

The intersection with the line F_{x_0} and S^1 is a single point x_0. All the points on F_{x_0} can be projected to x_0. We will denote the projection by π (see Figure 3.4).

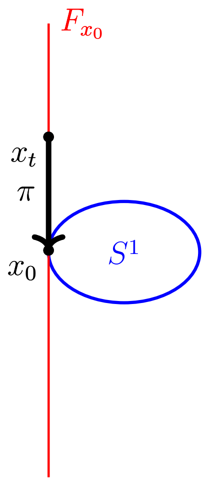


Figure 3.4: Illustration of projection

This sketch motivates $\pi^{-1}(x_0) = F_{x_0}$, which is called a fiber at x_0. Since we can construct this for all points x on S^1 $E = S^1 \times F$, we get a cylinder (see Figure 3.5).

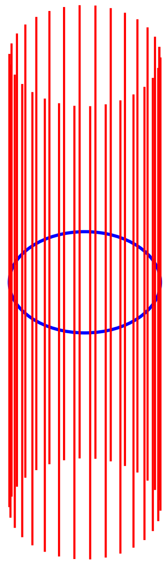


Figure 3.5: Visualization of fiber bundle

Thus, we have seen an example for a trivial fibration (E, S^1, π).
Since all the fibers are linear spaces they carry a vector space structure. Hence, the bundle is a so-called vector bundle. The vector spaces have dimension one. Therefore, the vector bundle is called line bundle.

We can extend this example to the complex space to motivate the definition of complex line bundles.

Definition 3.10 (Complex line bundle)**.** *Let* X *be a Riemann surface,* E *be a topological space and* $\pi : X \to E$ *a continuous mapping. Furthermore, for any* $x \in X$ *the fiber* $F = E_x = \pi^{-1}(\{x\})$ *is a complex one-dimensional vector space. Then* $\pi : E \to X$ *(sometimes only denoted as* E*) is called a (complex) line bundle over* X *if for any* $x \in X$ *there is an open neighborhood* U *and an homeomorphism* $h : E_U := \pi^{-1}(U) \to U \times \mathbb{C}$ *such that the following diagram commutes*

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\quad h \quad} & U \times \mathbb{C} \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle pr_U} & \\ & U_{x_0} & \end{array}$$

and for any $x \in U$ *is the mapping* $pr_{\mathbb{C}} \circ h|_{E_x} : E_x \to \mathbb{C}$ *a vector space isomorphism.*
E *is called the total space of the fibration.*
X *is called the base space of the fibration.*
$F = E_x = \pi^{-1}(\{x\})$ *is called the fiber over* $x \in X$.
The sets $\{(U_{x_i}, h_{U_{x_i}})\}$ *are called local trivialization of the bundle.*

Definition 3.11 (Equivalent fibrations). *Let E and E^* be two fibrations over X with projections π and π^*. Both fibrations are called equivalent if there exists a homomorphism $f : E \to E^*$ with the commutative diagram:*

$$\begin{array}{ccc} E & \xrightarrow{\quad f \quad} & E^* \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi^*} & \\ & X & \end{array}$$

Definition 3.12 (Trivial fibration). *A locally trivial fibration $(E, B, \pi; F)$ is called trivial fibration if it is equivalent to $(B \times F, B, \pi; F)$.*

Thus, we have seen a trivial fibration in the example above. Now, we are ready to show a link to spectral theory of integrable systems as presented in Sebastian Klein's lecture [Kle20]. Let $X = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and let

$$M(\lambda) = \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) \\ \gamma(\lambda) & \delta(\lambda) \end{pmatrix} : X \to \mathrm{SL}(2, \mathbb{C})$$

be a holomorphic function. The eigenvalues of $M(\lambda)$ can be obtained through

$$\det(M(\lambda) - \mu\mathbb{1}) = 0 \;\Leftrightarrow\; (\alpha(\lambda) - \mu)(\delta(\lambda) - \mu) - \gamma(\lambda)\beta(\lambda) = 0.$$

This is equivalent to

$$\mu^2 - \underbrace{(\alpha(\lambda) + \delta(\lambda))}_{tr(M(\lambda)}\mu + 1 = 0 \;\Leftrightarrow\; \mu^2 - tr(M(\lambda)\mu = -1.$$

We define $\nu := \mu - \frac{1}{2}tr(M(\lambda))$ and obtain

$$\nu^2 = \underbrace{\mu^2 - tr(M(\lambda))}_{-1} + \frac{1}{4}tr(M(\lambda))^2 = \frac{1}{4}tr(M(\lambda))^2 - 1.$$

Now we define $a(\lambda) := \frac{1}{4}tr(M(\lambda))^2 - 1$. Therefore, we obtain an algebraic curve that gives us the following complex variety

$$\Sigma^\circ = \{(\lambda, \nu) \in \mathbb{C}^* \times \mathbb{C} | \nu^2 = a(\lambda)\}. \tag{3.1}$$

If we add points for $\lambda = 0$ and $\lambda = \infty$ we obtain a compact hyperelliptic Riemann surface. If $a(\lambda)$ only has distinct roots we obtain the so-called spectral curve corresponding to $M(\lambda)$. Now,

$$\pi : E \to \Sigma,\ ((\lambda, \nu), v) \mapsto (\lambda, \nu) \text{ mit } E = \bigcup_{(\lambda,\nu)} \{(\lambda, \nu)\} \times \ker(M(\Lambda) - (\nu + \frac{1}{2}tr(M(\lambda)))$$

forms a holomorphic line bundle on Σ and is called the eigenbundle corresponding to $M(\lambda)$.

3.4 Submanifold theory

Now, we will review some standard theory concerning submanifolds. It is not really necessary to present the following theorems here, but since I used them in my Bachelor's and my Master's thesis it might be handy for future students to have a short survey at hand when working on their thesis.

Definition 3.13 (Rank)**.** *Let $F : X \to Y$ be a smooth map between a manifold X of dimension n and a manifold Y of dimension m. Then consider $\tilde{F} = g \circ F \circ f^{-1}$, which is a smooth map $\tilde{F} : \mathbb{R}^n \to \mathbb{R}^m$. Then we define the rank of F at x as the rank of the derivative*

$$D\tilde{F}\Big|_{f(x)} : \mathbb{R}^n \to \mathbb{R}^m$$

of $\tilde{F}$ at $f(x)$.

Definition 3.14 (Regular point and value)**.** *Let $F : X \to Y$ be a smooth function between two manifolds X of dimension n and Y of dimension m. We say that a point $x \in X$ is a regular point of F if the rank of F at x is equal to m. If x is not a regular point it is called critical point. We say that a point $y \in Y$ is a regular value if every point $x \in F^{-1}(\{y\})$ is a regular point. If y is not a regular value then it is called a critical value.*

Proposition 3.15. *Let $F : X \to Y$ be a smooth function between two manifolds X of dimension n and Y of dimension m. Let $y \in Y$ be a regular value of F. Then the level set*

$$Z_y = F^{-1}(y) \in X$$

is a submanifold of X of codimension m.

Proof. Let $x \in Z_y$ and (U, φ) be a chart containing x. Now, let (V, ψ) be a chart containing the image $F(U)$. Naturally consider the composition

$$\tilde{F} = \psi \circ F \circ \varphi^{-1} : \underbrace{\tilde{U}}_{\subset \mathbb{R}^n} \to \underbrace{\tilde{V}}_{\subset \mathbb{R}^m} .$$

This gives us

$$\tilde{F}^{-1}(\psi(y)) = \varphi(F^{-1}(y) \cap U) = \varphi(Z_y \cap U).$$

We know that y is a regular value of F. Hence, x is a regular point of F. This gives us that $\varphi(x)$ is a regular point of $\tilde{F}$. Since $\tilde{F}$ is a smooth map between $\mathbb{R}^n$ and $\mathbb{R}^m$ we can apply the implicit function theorem and get that there is a chart (W, ξ) on $\mathbb{R}^n$ such that $\varphi(x) \in W \subset \tilde{U}$ and $\xi(\tilde{F}^{-1}(\psi(y))) = \mathbb{R}^{n-m} \cap \tilde{W}$. Thus, the chart $(\varphi^{-1}(W), \xi \circ \varphi)$ on X can be used on Z_y to show that Z_y satisfies the submanifold condition at x. Since we chose an arbitrary $x \in Z_y$ we get that Z_y is a submanifold of codimension m. □

Definition 3.16 (Immersion). *A smooth function $F : X \to Y$ is called an immersion if the rank of F at any point $x \in X$ is equal to the dimension of X (i.e. dF is injective).*

We can apply the theory above also for $\mathbb{C}$ instead of $\mathbb{R}$ either if we see $\mathbb{C}$ as isomorphic to $\mathbb{R}^2$ or if we are interested in the complex structure carried by a complex manifold through using holomorphic functions and charts.

4 Period lattices of CMC tori

4.1 Spectral curves of CMC tori

Solutions of the sinh-Gordon equation $u : \mathbb{R}^2 \to \mathbb{R}$ are parametrized by flow like actions on the space of so-called Potentials, which in this case are matrix-valued polynomials. Their degree gives rise to the so-called spectral genus. This thesis focuses exclusively on the spectral genus two case. The corresponding space of Potentials is defined as below.

Definition 4.1 (Set of Potentials of spectral genus two)**.**

$$\mathcal{P}_2 := \left\{ \zeta = \begin{pmatrix} \alpha\lambda - \bar{\alpha}\lambda^2 & -\gamma^{-1} + \beta\lambda - \gamma\lambda^2 \\ \gamma\lambda - \bar{\beta}\lambda^2 + \gamma^{-1}\lambda^3 & -\alpha\lambda + \bar{\alpha}\lambda^2 \end{pmatrix} \middle| \alpha, \beta \in \mathbb{C}, \gamma \in \mathbb{R}^+ \right\}$$

The orbit of the aforementioned flow like actions are formed by so-called Polynomial Killing fields. Those are solutions of the Lax equations which we have already seen in chapter 2.2 along with the origin of the matrices U and V.

Definition 4.2 (Polynomial Killing fields)**.** *Polynomial Killing fields are maps* $(x, y) \mapsto \zeta(x, y)$ *that solve the Lax equations*

$$\tfrac{\partial \zeta}{\partial x} = [\zeta, U(\zeta)] \qquad \tfrac{\partial \zeta}{\partial y} = [\zeta, V(\zeta)],$$

where $[\cdot, \cdot]$ *is the Lie bracket,*

$$U(\zeta) := \begin{pmatrix} \frac{\alpha - \bar{\alpha}}{2} & -\gamma^{-1}\lambda^{-1} - \gamma \\ \gamma + \gamma^{-1}\lambda & \frac{\bar{\alpha} - \alpha}{2} \end{pmatrix}$$

and

$$V(\zeta) := \begin{pmatrix} \frac{\alpha + \bar{\alpha}}{2} & -\gamma^{-1}\lambda^{-1} + \gamma \\ \gamma - \gamma^{-1}\lambda & -\frac{\bar{\alpha} + \alpha}{2} \end{pmatrix}.$$

The solution of

$$\Delta u + 2\sinh(2u) = 0$$

is then given through $u(x, y) = \ln(\gamma(x, y))$. The determinant of the ζ leads to so-called spectral curves.

Definition 4.3 (Spectral curves)**.** *The smooth algebraic curve X that is described through*

$$\nu^2 = \det(\zeta) = \lambda a(\lambda)$$

is called spectral curve. The degree of the polynomial a divided by two gives the spectral genus (in this case two).

Since a is a polynomial of degree $2g = 4$ the spectral curve has degree $2g + 1 = 5$ and therefore, forms a hyperelliptic Riemann surface. The definition above leads to complex polynomials of degree four such that $a \in \mathbb{C}^4[\lambda]$.

$$\begin{aligned}
\det(\zeta) =& \det\begin{pmatrix} \alpha\lambda - \alpha\bar{\lambda}^2 & -\gamma^{-1} + \beta\lambda - \gamma\lambda^2 \\ \gamma\lambda - \bar{\beta}\lambda^2 + \gamma^{-1}\lambda^3 & -\alpha\lambda + \bar{\alpha}\lambda^2 \end{pmatrix} \\
=& (\alpha\lambda - \bar{\alpha}\lambda^2)(-\alpha\lambda + \bar{\alpha}\lambda^2) - (\gamma\lambda - \bar{\beta}\lambda^2 + \gamma^{-1}\lambda^3)(-\gamma^{-1} + \beta\lambda - \gamma\lambda^2) \\
=& -\alpha^2\lambda^2 + 2\alpha\bar{\alpha}\lambda^3 - \bar{\alpha}^2\lambda^4 + \gamma\gamma^{-1}\lambda - \gamma\beta\lambda^2 + \gamma^2\lambda^3 - \bar{\beta}\gamma^{-1}\lambda^2 + \beta\bar{\beta}\lambda^3 - \bar{\beta}\gamma\lambda^4 + \gamma^{-2}\lambda^3 \\
& - \gamma^{-1}\beta\lambda^4 + \lambda^5 \\
=& \lambda(1 + \underbrace{(-\bar{\alpha}^2 - \bar{\beta}\gamma - \gamma^{-1}\beta)}_{=:\, a_1}\lambda^3 + \underbrace{(2\bar{\alpha}\alpha + \beta\bar{\beta} + \gamma^{-2} + \gamma^2)}_{=:\, a_2}\lambda^2 + \underbrace{(-\alpha^2 - \gamma\beta - \bar{\beta}\gamma^{-1})}_{=:\, \bar{a}_1} + 1) \\
=& \lambda\underbrace{(\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + \bar{a}_1\lambda + 1)}_{=:\, a(\lambda)} \\
=& \lambda a(\lambda)
\end{aligned}$$

These polynomials fulfill a so-called reality condition i.e. $a(\lambda) = \lambda^4\overline{a(\bar{\lambda}^{-1})}$. We will often encounter polynomials of different degree satisfying the corresponding reality condition. Therefore, it is useful to define the space of polynomials of degree n satisfying the reality condition.

Definition 4.4. *The space of polynomials f of degree n satisfying the reality condition $f(\lambda) = \lambda^n\overline{f(\bar{\lambda}^{-1})}$ is denoted by $P^n_{\mathbb{R}}$.*

Since a is a complex polynomial of degree four it has exactly four roots in $\mathbb{C}$. The space of these polynomials can be defined as

$$\mathcal{M}_2 := \{a \in \mathbb{C}^4[\lambda] |\ \lambda a(\lambda) = det(\zeta) \text{ for a } \zeta \in \mathcal{P}_2\}.$$

Since only polynomials a with four distinct roots yield smooth curves it is necessary to structure the solutions mentioned above. We will divide $\mathcal{M}_2$ into the following subspaces

$$\mathcal{M}_2^1 := \{a \in \mathcal{M}_2 |\ a \text{ has four pairwise distinct simple roots absent } \mathbb{S}^1\},$$

$\mathcal{M}_2^2 := \{a \in \mathcal{M}_2 | \, a$ has one double root on $\mathbb{S}^1$ and two simple roots absent $\mathbb{S}^1\}$,

$\mathcal{M}_2^3 := \{a \in \mathcal{M}_2 | \, a$ has two distinct double roots on $\mathbb{S}^1\}$,

$\mathcal{M}_2^4 := \{a \in \mathcal{M}_2 | \, a$ has a fourth-order root on $\mathbb{S}^1\}$

and

$\mathcal{M}_2^5 := \{a \in \mathcal{M}_2 | \, a$ as two distinct double roots absent $\mathbb{S}^1\}$.

Then,

$$\mathcal{M}_2 = \mathcal{M}_2^1 \uplus \mathcal{M}_2^2 \uplus \mathcal{M}_2^3 \uplus \mathcal{M}_2^4 \uplus \mathcal{M}_2^5$$

holds where $\uplus$ denotes the disjoint union.

In [KHS17] related isospectral sets and lattices are examined. Both will be used in this thesis. Therefore, it is necessary to closely examine the theory developed.

Definition 4.5 (Isospectral set)**.** *The level sets of these polynomials a*

$$I(a) := \{\zeta \in \mathcal{P}_2 | \det \zeta = \lambda a(\lambda)\}$$

are going to be called isospectral sets.

The Lax equations induce actions of $\mathbb{R}^2$ on the set of potentials $\mathcal{P}_2$.

$$\begin{aligned} \mathcal{P}_2 &\to \mathcal{P}_2 \\ \zeta &\mapsto \phi(x,y)\zeta, \end{aligned}$$

with $\phi : (x,y) \in \mathbb{R}^2 \mapsto \phi(x,y)$.

Definition 4.6 (Isomorphic lattices)**.** *Two lattices $\Gamma, \Gamma' \subset \mathbb{C}$ are called isomorphic if they originate from one another through a rotation-dilation.*

It is proven in [BF09], that each lattice Γ in $\mathbb{C}$ is isomorphic to $\Gamma_\tau := \mathbb{Z} + \mathbb{Z}\tau$ up to a rotation-dilation with

$$\tau \in \left\{ \tau \in \mathbb{C} \middle| \mathfrak{I}(\tau) > 0, \; |\mathfrak{R}(\tau)| \leq \frac{1}{2}, ||\tau|| \geq 1 \right\}.$$

Definition 4.7. *Let $\mathcal{F}$ denote the space of such τ with the quotient topology of the subset of $\mathbb{C}$ divided by the relation $\sim$.*

This gives the following picture of $\mathcal{F}$

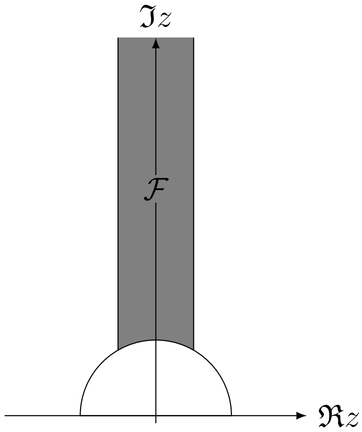


Figure 4.1: Initial set of τ

Let $a \in \mathcal{M}_2^1$. Then $\Gamma_a = \{x + iy \in \mathbb{C} | \forall \zeta \in I(a) : \phi(x,y)(\zeta) = \zeta\}$ is a lattice and hence there are complex numbers ω_1 and ω_2 such that $\Gamma_a \cong \omega_1\mathbb{C} + \omega_2\mathbb{C}$ and ω_1 and ω_2 have minimal length. Then Γ_a is isomorphic to $\Gamma_{\tau_a} = \mathbb{Z} + \tau_a\mathbb{Z}$ where τ_a is a complex number in $\mathcal{F}$. Thus, we can construct the following mapping

$$T : \mathcal{M}_2^1 \to \mathcal{F}$$
$$a \mapsto \tau_a,$$

such that Γ_a is isomorphic to Γ_{τ_a}. In order to understand how this mapping works we need to closely examine the frame F and the corresponding monodromy M_ω. The frame is the fundamental solution of the ODE system

$$\tfrac{\partial F}{\partial x} = FU(\zeta), \qquad \tfrac{\partial F}{\partial y} = FV(\zeta), \quad F(0,0) = Id,$$

where ζ is a polynomial Killing field with initial potential $\zeta_0 \in \mathcal{P}_2$. The existence of this fundamental solution F is granted by the Picard-Lindelöf theorem. Now we can define the monodromies

$$M_\omega = F(\omega).$$

in [Hoe15] it is shown that the monodromy commutes with ζ_0 and maps eigenspaces of ζ_0 into themselves. For $\zeta_0 \in I(a)$ and $a \in \mathcal{M}_2^1$ the one-dimensional eigenspaces of ζ_0 can be parametrized as the smooth Riemann surface

$$\Sigma^* := \{(\lambda, \nu) \in (\mathbb{C}/\{0\} \times \mathbb{C}) | \det(\nu\mathbb{1} - \zeta_o) = \nu^2 + \lambda a(\lambda) = 0\}.$$

The reality condition of a gives an involution ρ through

$$\rho : (\lambda, \nu) \longmapsto (\bar{\lambda}^{-1}, -\bar{\lambda}^{-3}\bar{\nu}).$$

Another involution σ is given by

$$\sigma : (\lambda, \nu) \longmapsto (\lambda, -\nu).$$

In [KHS17] it is shown that the monodromies M_ω act on the one-dimensional eigenspaces of ζ_0 like multiplication with its eigenvalues μ_ω, which maps Σ^* to $\mathbb{C} \setminus \{0\}$. Taking the logarithmic derivative of μ_ω gives us a meromorphic differential of second kind with poles at $\lambda = 0$ and $\lambda = \infty$. [KHS17] showed that it has the following form

$$\Theta_{b_\omega} := d\ln(\mu_\omega) = \frac{b_\omega(\lambda)}{\nu} d\ln\lambda = \frac{b_\omega(\lambda)}{\lambda\nu} d\lambda \text{ with } b \in P^3_{\mathbb{R}}. \tag{4.1}$$

Since $a \in \mathcal{M}^1_2$ there are four distinct roots α_i for $i = 1, \dots, 4$ of a. Due to the reality condition of a the roots are as follows α_1, $\alpha_2 = \bar{\alpha}_1^{-1}$,$\alpha_3$, $\alpha_4 = \bar{\alpha}_3^{-1}$. They are visualized in the following.

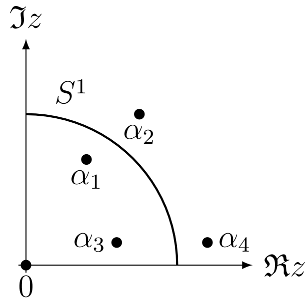


Figure 4.2: Roots of $a \in \mathcal{M}^1_2$

We are interested in when such μ_ω exist, i.e. what conditions for b arise. We obtain these conditions by looking at the periods of Θ_b. Hence, we are interested in the integrals along the cycles that form a homology basis of the Riemann surface. Let $\bar{\Sigma}$ denote the compact Riemann surface of genus two that is a two-sheeted cover of the complex projective line $\mathbb{CP}^1$ branched at the four roots α_i of a, at $\lambda = 0$ and $\lambda = \infty$. A suitable homology basis can be obtained in the following way. First of all we encircle α_1, α_2 and α_3, α_4 since they can be obtained from another through the involution ρ. The corresponding cycles will be called A_1 and A_2. Additionally, we will use the cycle B_1 surrounding α_1 and 0 and B_2 surrounding the roots α_1, α_2 and α_3 and 0. This gives the following image:

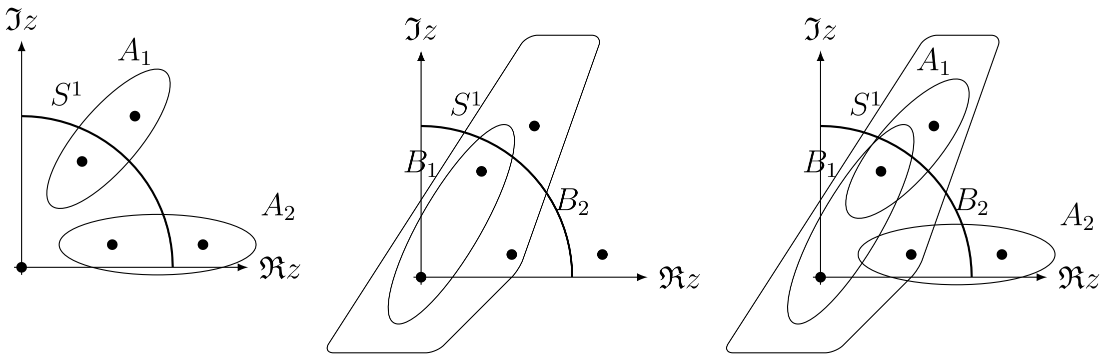


Figure 4.3: Canonical homology basis of $\overline{\Sigma}$

Therefore, the chosen homology basis is the canonical basis of a hyperelliptic Riemann surface of genus 2 described in 3.1. In [KHS17] the following existence lemma is shown.

Corollary 4.8. *For all $a \in \mathcal{M}_2^1 \cup \mathcal{M}_2^2 \cup \mathcal{M}_2^3$ and $\omega \in \mathbb{C}$ there exists a unique $b_\omega \in P_{\mathbb{R}}^3$ with*

1. $b_\omega(0) = \omega$ *and*
2. $\int_{A_1} \theta_{b_\omega} = 0 = \int_{A_2} \theta_{b_\omega}$.

It is easy to see that the involution ρ preserves the B-cycles (up to an addition of A-cycles) since it only reverses the orientation and intersection numbers. This also gives that $\int_{B_k} \theta_{b_\omega}$ is purely imaginary for $k = 1, 2$. Furthermore, if we impose $\mu_\omega = \pm 1$ at the roots of a we immediately get the following corollary from $d\ln(\mu_\omega) = \Theta_{b_\omega}$.

Corollary 4.9. *It holds $\int_{B_1} \theta_{b_\omega} = 2\pi i\mathbb{Z}$ and $\int_{B_2} \theta_{b_\omega} = 2i\pi\mathbb{Z}$.*

This shows that if Θ_{b_ω} has period $2\pi i\mathbb{Z}$ such a μ_ω exists. Therefore, we need to restrict us to polynomials b_k that have a period of the form $2\pi i\mathbb{Z}$.

4.2 Level sets of $\mathcal{M}_2^1$

Our goal is to show that level sets of T for any $a \in \mathcal{M}_2^1$ are one-dimensional submanifolds. In order to do this we first want to extend the map T to a map $\hat{T}$ that maps a triple (a, b_1, b_2), where b_1, b_2 form a basis of

$$\mathcal{B}_a = \left\{ b \in P_{\mathbb{R}}^3 | \ \Theta_b := \frac{b(\lambda)d\lambda}{\lambda\nu} \text{ has purely imaginary periods} \right\}$$

as defined in [CS16]. In [KHS17] the following corollary is proved.

Corollary 4.10. *For $a \in \mathcal{M}_2^1$ the elements of Γ_a are the values of $\omega = b_\omega(0)$ of the b_ω from* (4.1) *whose one-forms $d\ln\mu$ are the logarithmic derivative of a holomorphic function μ_ω on Σ^* with*

$$\Sigma^* = \{(\lambda,\nu) \in (\mathbb{C} \setminus \{0\}) \times \mathbb{C} |\ \det(\nu 1 - \zeta_0) = y^2 + \lambda a(\lambda) = 0\}$$

Thus, we have to restrict $\mathcal{B}_a$ to polynomials b that have period $2\pi i\mathbb{Z}$ and vanish along the A cycles in order to get a lattice generated by $b_1(0) = \omega_1$ and $b_2(0) = \omega_2$. Therefore, we have gathered the necessary conditions to implicitly describe how $\hat{T}$ works. That is, once we obtained b_1 and b_2 from a that build a basis of $\mathcal{B}_a$ we simply calculate $\tau_a = \frac{b_1(0)}{b_2(0)}$. Unfortunately, this implicit description is the only way to do this since the b_i are transcendent functions only described through the conditions. Therefore, it is impossible to calculate an explicit expression for T. Thus, we can alter the mapping T to a mapping

$$\begin{aligned} \mathcal{M}_2^1 &\to \mathcal{B}_a \times \mathcal{B}_a \to \mathcal{F} \\ a &\mapsto (b_1, b_2) \mapsto \tau_a. \end{aligned}$$

We may write it as

$$\begin{aligned} \hat{T} : \mathcal{M}_2^1 \times \mathcal{B}_a \times \mathcal{B}_a &\to \mathcal{F} \\ (a, b_1, b_2) &\mapsto \frac{b_1(0)}{b_2(0)} = \tau_a. \end{aligned}$$

Now we want to calculate the rank of this map. Before we do this we want to closer examine $\mathcal{M}_2^1$. Since an element looks like $a(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + \bar{a}_1\lambda + 1$, where $a_1 \in \mathbb{C}$ and $a_2 \in \mathbb{R}$. We conclude that $\mathcal{M}_2^1$ is three-dimensional. The rank of a map $F : U \to V$ at a point $x \in U$ is defined as the rank of its derivative dF at x. Therefore, we need to calculate the rank of $dT : \mathbb{R}^3 \to \mathbb{R}^2$. Due to the rank-nullity theorem

$$rank(dT) = \dim(\mathbb{R}^3) - \dim\ker(dT) = 3 - \dim\ker(dT)$$

holds. The same holds true for $\hat{T}$ and we know that due to [CS16] the tangent space is described through the triples $(\dot{a}, \dot{b}_1, \dot{b}_2)$. Now we want to apply the Whitham deformations to the derivative of this mapping to describe its kernel as subspace of the tangent space of (a, b_1, b_2). Let $(\dot{a}, \dot{b}_1, \dot{b}_2)$ denote the tangent vector at $t = 0$ that preserves the periods of Θ_{b_1} and Θ_{b_2}, i.e. $(\dot{a}, \dot{b}_1, \dot{b}_2) \in T_{(a,b_1,b_2)}F^2$, where F^2 is the frame bundle of $\mathcal{B}$. Since the meromorphic differential forms $\frac{d}{dt}\big|_{t=0}\Theta_{b_1}$ and $\frac{d}{dt}\big|_{t=0}\Theta_{b_2}$ have vanishing periods and no residues there exist meromorphic functions $\dot{q}_1$ and $\dot{q}_2$ on the Riemann surface X_a that satisfy

$$d\dot{q}_k = \frac{d}{dt}\bigg|_{t=0} \Theta_{b_k}$$

for $k = 1, 2$. This gives

$$\dot{q}_k = \frac{ic_k(\lambda)}{y}$$

with $c_k \in P_{\mathbb{R}}^3$ and $\nu = \sqrt{\lambda a(\lambda)}$. Together with the equation above we get the Whitham equation

$$\frac{\partial}{\partial\lambda}\frac{ic_k(\lambda)}{\nu} = \frac{\partial}{\partial t}\frac{b_k(\lambda)}{\nu\lambda}\bigg|_{t=0}.$$

Using product and chain rule we get the following expressions

$$(2\lambda a c_1' - a c_1 - \lambda a' c_1)i = 2a\dot{b}_1 - \dot{a}b_1 \tag{4.2}$$

and

$$(2\lambda a c_2' - a c_2 - \lambda a' c_2)i = 2a\dot{b}_2 - \dot{a}b_2, \tag{4.3}$$

where a dot (e.g. $\dot{a}$) denotes the derivative with respect to t, evaluated at $t = 0$ and a prime (e.g. a') denotes the derivative with respect to λ. Now $c_2\cdot$ (4.2) $-c_1\cdot$ (4.3) yields

$$2a(ic_1'c_2\lambda - ic_2'c_1\lambda + c_1\dot{b}_2 - c_2\dot{b}_1) = \dot{a}(c_1b_2 - c_2b_1).$$

An argumentation with the roots of a and $\dot{a}$ (see [Sch17]) in return yields

$$c_1b_2 - c_2b_1 = Qa, \tag{4.4}$$

where $Q \in P_{\mathbb{R}}^2$. The kernel of $d\hat{T}$ consists exactly of the triples $(\dot{a}, \dot{b}_1, \dot{b}_2)$ that leave τ_a unchanged. Since $\tau_a = \frac{b_1(0)}{b_2(0)}$ holds the condition for the triple to belong to $\ker(d\hat{T})$ is

$$d\left(\frac{b_1(0)}{b_2(0)}\right)\bigg|_{t=0} = 0.$$

Due to product and chain rule we arrive at

$$\frac{\dot{b}_1(0)b_2(0) - b_1(0)\dot{b}_2(0)}{b_2(0)^2} = 0.$$

Simplified we obtain the following condition

$$\dot{b}_1(0)b_2(0) - b_1(0)\dot{b}_2(0) = 0. \tag{4.5}$$

Before we make use of this new found condition in combination with the Whitham equations, it is useful to look at the form of the involved polynomials

$$\begin{aligned}
a(\lambda) &= \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + \bar{a}_1\lambda + 1 \\
b_1(\lambda) &= b_{31}\lambda^3 + b_{21}\lambda^2 + \bar{b}_{21}\lambda + \bar{b}_{31} \\
\dot{b}_1(\lambda) &= \dot{b}_{31}\lambda^3 + \dot{b}_{21}\lambda^2 + \bar{\dot{b}}_{21}\lambda + \bar{\dot{b}}_{31} \\
b_2(\lambda) &= b_{32}\lambda^3 + b_{22}\lambda^2 + \bar{b}_{22}\lambda + \bar{b}_{32} \\
\dot{b}_2(\lambda) &= \dot{b}_{32}\lambda^3 + \dot{b}_{22}\lambda^2 + \bar{\dot{b}}_{22}\lambda + \bar{\dot{b}}_{32} \\
c_1(\lambda) &= c_{31}\lambda^3 + c_{21}\lambda^2 + \bar{c}_{21}\lambda + \bar{c}_{31} \\
c_2(\lambda) &= c_{32}\lambda^3 + c_{22}\lambda^2 + \bar{c}_{22}\lambda + \bar{c}_{32} \\
Q(\lambda) &= Q_2\lambda^2 + Q_1\lambda + \bar{Q}_2.
\end{aligned}$$

Therefore, (4.5) becomes

$$\bar{\dot{b}}_{31}\bar{b}_{32} - \bar{b}_{31}\bar{\dot{b}}_{32} = 0. \tag{4.6}$$

From (4.4) we obtain for $\lambda = 0$

$$\bar{c}_{31}\bar{b}_{32} - \bar{c}_{32}\bar{b}_{31} = \bar{Q}_2. \tag{4.7}$$

Setting $\lambda = 0$ in the Whitham equations then give us equations for $\bar{c}_{31}$ and $\bar{c}_{32}$. From (4.2) we obtain

$$\bar{c}_{31}i = 2\bar{\dot{b}}_{31} - \bar{b}_{31} \tag{4.8}$$

and from (4.3) we obtain

$$\bar{c}_{32}i = 2\bar{\dot{b}}_{32} - \bar{b}_{32}. \tag{4.9}$$

Substituting (4.8) and (4.9) in (4.7) gives

$$\begin{aligned}
& -i(2\bar{\dot{b}}_{31} - \bar{b}_{31})\bar{b}_{32} + i(2\bar{\dot{b}}_{32} - \bar{b}_{32})\bar{b}_{31} &= \bar{Q}_2 \\
\Leftrightarrow\quad & -2i\bar{\dot{b}}_{31}\bar{b}_{32} + i\bar{b}_{31}\bar{b}_{32} + 2i\bar{\dot{b}}_{32}\bar{b}_{31} - i\bar{b}_{32}\bar{b}_{31} &= \bar{Q}_2 \\
\Leftrightarrow\quad & -2i\underbrace{\left(\bar{\dot{b}}_{31}\bar{b}_{32} - \bar{\dot{b}}_{32}\bar{b}_{31}\right)}_{=0 \text{ due to } (4.6)} &= \bar{Q}_2 \\
\Leftrightarrow\quad & 0 &= \bar{Q}_2.
\end{aligned}$$

This yields $Q_2 = 0$ as well. Therefore, Q only consists of the middle term $Q(\lambda) = Q_1\lambda$.

Corollary 4.11. *If b_1 and b_2 have a common root that is no root of a we can conclude that $Q_1 = 0$ has to hold.*

Proof. Let λ_0 be the common root of b_1 and b_2. It cannot be a root of the polynomial a as well because then we would have $\lambda_0 \in \mathbb{S}^1$. Thus, the structure of a immediately yields that a would have a double root at λ_0. But this is a contradiction to $a \in \mathcal{M}_2^1$. Hence, the left-hand side of (4.4) will vanish at λ_0 and a will not. Therefore, the right-hand side only vanishes if Q vanishes at this root. Since $Q = Q_1\lambda$ only vanishes when $Q_1 = 0$ we obtain $Q = 0$. □

Now it remains to show that the solutions obtained from (4.2), (4.3) and (4.4) form a one-dimensional set of solutions. There are two possible situations

1. b_1 and b_2 have no common root
2. b_1 and b_2 have a common root $\lambda_0 \in \mathbb{S}^1$.

Using $Q = Q_1\lambda$ we obtain

$$c_1 b_2 - c_2 b_1 = Q_1 \lambda a.$$

If b_1 and b_2 do not have common roots we can use (4.4) to get enough conditions for c_1 and c_2. Let λ_{11}, λ_{21} and λ_{31} be the roots of b_1 and let λ_{12}, λ_{22} and λ_{32} be the roots of b_2. Evaluating (4.4) at these roots yields the desired conditions. For the roots of b_1 we obtain the following system of equations

1. $c_1(\lambda_{11})b_2(\lambda_{11}) = Q_1\lambda_{11}a(\lambda_{11}) \Leftrightarrow c_1(\lambda_{11}) = \frac{Q_1\lambda_{11}a(\lambda_{11})}{b_2(\lambda_{11})}$
2. $c_1(\lambda_{21})b_2(\lambda_{21}) = Q_1\lambda_{21}a(\lambda_{21}) \Leftrightarrow c_1(\lambda_{21}) = \frac{Q_1\lambda_{21}a(\lambda_{21})}{b_2(\lambda_{21})}$
3. $c_1(\lambda_{31})b_2(\lambda_{31}) = Q_1\lambda_{31}a(\lambda_{31}) \Leftrightarrow c_1(\lambda_{31}) = \frac{Q_1\lambda_{31}a(\lambda_{31})}{b_2(\lambda_{31})}$

and for the roots of b_2 we obtain

1. $c_2(\lambda_{12})b_1(\lambda_{12}) = Q_1\lambda_{12}a(\lambda_{12}) \Leftrightarrow c_2(\lambda_{12}) = \frac{Q_1\lambda_{12}a(\lambda_{12})}{b_1(\lambda_{12})}$
2. $c_2(\lambda_{22})b_1(\lambda_{22}) = Q_1\lambda_{22}a(\lambda_{22}) \Leftrightarrow c_2(\lambda_{22}) = \frac{Q_1\lambda_{22}a(\lambda_{22})}{b_1(\lambda_{22})}$
3. $c_2(\lambda_{32})b_1(\lambda_{32}) = Q_1\lambda_{32}a(\lambda_{32}) \Leftrightarrow c_2(\lambda_{32}) = \frac{Q_1\lambda_{32}a(\lambda_{32})}{b_1(\lambda_{32})}$

Therefore, we have enough expressions to evaluate (4.2) and (4.3) at λ_{lk}. They have the form

$$(2\lambda_{lk}a(\lambda_{lk})c_k'(\lambda_{lk}) - a(\lambda_{lk})c_k(\lambda_{lk}) - \lambda_{lk}a'(\lambda_{lk})c_k)i = 2a(\lambda_{lk})\dot{b}_k(\lambda_{lk}). \tag{4.10}$$

Thus, we can calculate $\dot{b}_1$ and $\dot{b}_2$. Since the roots of a are distinct and a has no common roots with b_1 and b_2 we also obtain enough conditions to calculate $\dot{a}$. Let λ_k for $k = 1, \ldots 4$ denote the four distinct roots of a. Then at λ_k (4.2) and (4.3) become

$$-\lambda a'(\lambda_k)c_l(\lambda_k)i = -\dot{a}(\lambda_k)b_l(\lambda_k). \tag{4.11}$$

for $l = 1, 2$. All expressions for $(c_1, c_2, \dot{b}_1, \dot{b}_2, \dot{a})$ depend on $Q_1 \in \mathbb{R}$. Therefore, we have a one-dimensional solution set. If any of the b_k have a multiple root the conditions alter. Instead of inserting all roots into (4.4) we can differentiate it once or twice to obtain one or two additional conditions. The first derivative reads

$$c_1' b_2 + c_1 b_2' - c_2' b_1 - c_2 b_1' = Q_1 a + Q_1 \lambda a'. \tag{4.12}$$

Assuming λ_{1k} is a double root of b_k we now obtain

$$\pm c_k'(\lambda_{1k}) b_{3-k}(\lambda_{1k}) \pm c_k(\lambda_{1k}) b_{3-k}'(\lambda_{1k}) = Q_1 a(\lambda_{1k}) + Q_1 \lambda_{1k} a'(\lambda_{1k}), \tag{4.13}$$

where we have $+$ for $k = 1$ and $-$ for $k = 2$. Thus, in case of a double root we again have three equations. In case of a triple root we can differentiate (5.60) once more to obtain another condition.
If b_1 and b_2 have a common root we obtain $Q = 0$ as explained in corollary 4.11. Therefore, we get

$$c_1 b_2 - c_2 b_1 = 0, \tag{4.14}$$

where b_1 and b_2 have a common root $\lambda_0 \in \mathbb{S}^1$. Therefore they have the following form

$$b_k(\lambda) = \kappa_k (\lambda - \lambda_0) \tilde{b}_k(\lambda),$$

where $\tilde{b}_k(\lambda)$ is a polynomial of degree two that fulfills the reality condition and therefore is of the form

$$\tilde{b}_k(\lambda) = \tilde{b}_{2k} \lambda^2 + \tilde{b}_{1k} \lambda + \bar{\tilde{b}}_{2k}.$$

We can calculate the factor κ_k explicitly with $b_k(\lambda) = \lambda^3 \overline{b_k(\bar{\lambda}^{-1})}$

$$\kappa_k (\lambda - \lambda_0) \tilde{b}_k(\lambda) = \bar{\kappa}_k \lambda \overline{(\bar{\lambda}^{-1} - \lambda_0)} \lambda^2 \overline{\tilde{b}_k(\bar{\lambda}^{-1})}.$$

With $\tilde{b}_k(\lambda) = \lambda^2 \overline{\tilde{b}_k(\bar{\lambda}^{-1})}$ the following remains

$$\begin{aligned} \kappa_k (\lambda - \lambda_0) &= \bar{\kappa}_k \lambda \overline{(\bar{\lambda}^{-1} - \lambda_0)} \\ &= -\bar{\kappa}_k \lambda_0^{-1} (\lambda - \lambda_0). \end{aligned}$$

This gives us

$$\kappa_k = \sqrt{-\frac{|\kappa_k|^2}{\lambda_0}} = i \frac{|\kappa_k|}{\sqrt{\lambda_0}}. \tag{4.15}$$

Since we also have a scaling factor included in the coefficients of $\tilde{b}_k$ we can without loss of generality set $|\kappa_k| = 1$. If we plug in the remaining two roots of b_1 (i.e. the two roots of $\tilde{b}_1$) in (4.14) we know that b_2 does not vanish at these λ. Hence, c_1 has to have the same roots as b_1. Therefore, c_1 is a linear combination of b_1 and of $\frac{b_1}{\lambda-\lambda_0}$. Likewise, looking at the roots of b_2 gives us that c_2 is a linear combination of b_2 and of $\frac{b_2}{\lambda-\lambda_0}$. Therefore, the polynomials c_k have the following form

$$c_k(\lambda) = w_1 b_k(\lambda) + w_2 \frac{b_k(\lambda)}{\lambda - \lambda_0}, \text{ for } k \in \{1,2\} \text{ with } w_1, w_2 \in \mathbb{C}.$$

Thus, the solution set a priori appears to be four-dimensional. But the reality condition and the form the polynomial a has eliminates dimensions through (4.2) and (4.3).

We require c_k to fulfill the reality condition. That is

$$c_k(\lambda) = \lambda^3 \overline{c_k(\bar{\lambda}^{-1})}.$$

Hence, we can calculate the w_n further.

The second term of the linear combination becomes

$$\frac{b_k(\lambda)}{\lambda_0 - \lambda} = \kappa_k(\tilde{b}_{2k}\lambda^2 + \tilde{b}_{1k}\lambda + \bar{\tilde{b}}_{2k}).$$

Therefore, $c_k(\lambda) = \lambda^3 \overline{c_k(\bar{\lambda}^{-1})}$ becomes

$$\begin{aligned}
&w_1 \tfrac{i}{\sqrt{\lambda_0}}(\tilde{b}_{2k}\lambda^3 + \lambda^2(\tilde{b}_{1k} - \tilde{b}_{2k}\lambda_0) + \lambda(\bar{\tilde{b}}_{2k} - \tilde{b}_{1k}\lambda_0) - \bar{\tilde{b}}_{2k}\lambda_0) + w_2 \tfrac{i}{\sqrt{\lambda_0}}(\tilde{b}_{2k}\lambda^2 + \tilde{b}_{1k}\lambda + \bar{\tilde{b}}_{2k}) \\
&= \bar{w}_1 \tfrac{-i}{\sqrt{\bar{\lambda}_0}}(\bar{\tilde{b}}_{2k} + \lambda(\tilde{b}_{1k} - \bar{\tilde{b}}_{2k}\bar{\lambda}_0) + \lambda^2(\tilde{b}_{2k} - \tilde{b}_{1k}\bar{\lambda}_0) - \tilde{b}_{2k}\bar{\lambda}_0\lambda^3) + \bar{w}_2 \tfrac{-i}{\sqrt{\bar{\lambda}_0}}(\bar{\tilde{b}}_{2k}\lambda + \tilde{b}_{1k}\lambda^2 + \tilde{b}_{2k}\lambda^3)
\end{aligned}$$

This yields the following system of equations

1. $w_1 \frac{i}{\sqrt{\lambda_0}} \tilde{b}_{2k} = -\bar{w}_1 \frac{-i}{\sqrt{\bar{\lambda}_0}} \tilde{b}_{2k}\bar{\lambda}_0 + \bar{w}_2 \frac{-i}{\sqrt{\bar{\lambda}_0}} \tilde{b}_{2k}$

2. $w_1 \frac{i}{\sqrt{\lambda_0}}(\tilde{b}_{1k} - \tilde{b}_{2k}\lambda_0) + w_2 \frac{i}{\sqrt{\lambda_0}} \tilde{b}_{2k} = \bar{w}_1 \frac{-i}{\sqrt{\bar{\lambda}_0}}(\tilde{b}_{2k} - \tilde{b}_{1k}\bar{\lambda}_0) + \bar{w}_2 \frac{-i}{\sqrt{\bar{\lambda}_0}} \tilde{b}_{1k}$

3. $w_1 \frac{i}{\sqrt{\lambda_0}}(\bar{\tilde{b}}_{2k} - \tilde{b}_{1k}\lambda_0) + w_2 \frac{i}{\sqrt{\lambda_0}} \tilde{b}_{1k} = \bar{w}_1 \frac{-i}{\sqrt{\bar{\lambda}_0}}(\tilde{b}_{1k} - \bar{\tilde{b}}_{2k}\bar{\lambda}_0) + \bar{w}_2 \frac{-i}{\sqrt{\bar{\lambda}_0}} \bar{\tilde{b}}_{2k}$

4. $-w_1 \frac{i}{\sqrt{\lambda_0}} \bar{\tilde{b}}_{2k}\lambda_0 + w_2 \frac{i}{\sqrt{\lambda_0}} \bar{\tilde{b}}_{2k} = \bar{w}_1 \frac{-i}{\sqrt{\bar{\lambda}_0}} \bar{\tilde{b}}_{2k}.$

This gives us

$$w_{2k} = -\lambda_0(\bar{w}_{1k} - w_{1k})$$

and

$$\bar{w}_{2k} = -\frac{w_{1k} - \bar{w}_{1k}}{\lambda_0} = -\bar{\lambda}_0(w_{1k} - \bar{w}_{1k}).$$

The second and third equation do not yield another condition since both sides vanish. The second equation becomes

$$\begin{aligned} & w_1 \tfrac{i}{\sqrt{\lambda_0}}(\tilde{b}_{1k} - \tilde{b}_{2k}\lambda_0) + (-\lambda_0(\bar{w}_{1k} - w_{1k}))\tfrac{i}{\sqrt{\lambda_0}}\tilde{b}_{2k} \\ & = \bar{w}_1 \tfrac{-i}{\sqrt{\bar{\lambda}_0}}(\tilde{b}_{2k} - \tilde{b}_{1k}\bar{\lambda}_0) - \tfrac{w_{1k}-\bar{w}_{1k}}{\lambda_0}\tfrac{-i}{\sqrt{\bar{\lambda}_0}}\tilde{b}_{1k} \\ \Leftrightarrow \quad & w_1(\tilde{b}_1 - \tilde{b}_2\lambda_0) + \tfrac{w_1-\tilde{w}_1}{\bar{\lambda}_0}\tilde{b}_2 = -\tilde{w}_1 \tfrac{\sqrt{\lambda_0}}{\sqrt{\bar{\lambda}_0}}(\tilde{b}_2 - \tilde{b}_1\bar{\lambda}_0) - \tfrac{\bar{w}_1-w_1}{\lambda_0}\tfrac{\sqrt{\lambda_0}}{\bar{\lambda}_0}\tilde{b}_1 \\ \Leftrightarrow \quad & w_1 \underbrace{(\tilde{b}_1 - \underbrace{\frac{\sqrt{\lambda_0}}{\lambda_0\sqrt{\bar{\lambda}_0}}}_{=\frac{\sqrt{\lambda_0}}{\sqrt{\lambda_0}}=1}\tilde{b}_1)}_{=0} = -\bar{w}_1 \underbrace{(\underbrace{\frac{\sqrt{\lambda_0}}{\sqrt{\bar{\lambda}_0}}}_{=\lambda_0}\tilde{b}_2 - \underbrace{\frac{1}{\bar{\lambda}_0}}_{=\lambda_0})}_{=0} \end{aligned}$$

and the third equation becomes

$$\begin{aligned} & w_1 \tfrac{i}{\sqrt{\lambda_0}}(\bar{\tilde{b}}_{2k} - \tilde{b}_{1k}\lambda_0) - \lambda_0(\bar{w}_{1k} - w_{1k})\tfrac{i}{\sqrt{\lambda_0}}\tilde{b}_{1k} \\ & = \bar{w}_1 \tfrac{-i}{\sqrt{\bar{\lambda}_0}}(\tilde{b}_{1k} - \bar{\tilde{b}}_{2k}\bar{\lambda}_0) - \tfrac{w_{1k}-\bar{w}_{1k}}{\lambda_0}\tfrac{-i}{\sqrt{\bar{\lambda}_0}}\bar{\tilde{b}}_{2k} \\ \Leftrightarrow \quad & w_1(\bar{\tilde{b}}_{2k} - \tilde{b}_{1k}\lambda_0) - \lambda_0(\bar{w}_{1k} - w_{1k})\tilde{b}_{1k} = -\bar{w}_1 \tfrac{\sqrt{\lambda_0}}{\sqrt{\bar{\lambda}_0}}(\tilde{b}_{1k} - \bar{\tilde{b}}_{2k}\bar{\lambda}_0) + \tfrac{w_{1k}-\bar{w}_{1k}}{\lambda_0}\tfrac{\sqrt{\lambda_0}}{\sqrt{\bar{\lambda}_0}}\bar{\tilde{b}}_{2k} \\ \Leftrightarrow \quad & w_{1k}\bar{\tilde{b}}_{2k} - w_{1k}\bar{\tilde{b}}_{2k} = -\bar{w}_{1k}\tfrac{\sqrt{\lambda_0}}{\sqrt{\bar{\lambda}_0}}\tilde{b}_{1k} + \bar{w}_{1k}\lambda_0\tilde{b}_{1k} \\ \Leftrightarrow \quad & 0 = 0. \end{aligned}$$

Thus, a priori the dimension of solutions seems to be two dimensional. Therefore, we need to use the other Whitham equations to fully determine the dimension of solutions. Since we have a linear space of solutions it will suffice to show that there are combinations that are no solutions as well as combinations that form a solution. First, we will look at $w_{2k} = 0$ (i.e. $w_1 \in \mathbb{R}$). Therefore, we choose $c_1 = b_1$ and $c_2 = b_2$ which is one of the possibilities due to the recent argumentation. Thus, (4.2) and (4.3) now have the form

$$(2\lambda a b_1' - a b_1 - \lambda a' b_1)i = 2a\dot{b}_1 - \dot{a}b_1 \tag{4.16}$$

and

$$(2\lambda a b_2' - a b_2 - \lambda a' b_2)i = 2a\dot{b}_2 - \dot{a}b_2. \tag{4.17}$$

Now (4.16) unambiguously defines $\dot{b}_1$ in the following way

$$\dot{b}_1(\lambda) = \lambda b_1'(\lambda)i + m_1 b_1(\lambda) \tag{4.18}$$

and (4.17) unambiguously defines $\dot{b}_2$ as

$$\dot{b}_2(\lambda) = \lambda b_2'(\lambda)i + m_2 b_2(\lambda). \tag{4.19}$$

Both give an unambiguous solution for $\dot{a}$ as follows

$$\begin{aligned}\dot{a}(\lambda) &= \lambda a'(\lambda)i + na(\lambda) &(4.20)\\ &= 4i\lambda^4 + 3ia_1\lambda^3 + 2ia_2\lambda^2 + i\bar{a}_1\lambda + n(\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + \bar{a}_1\lambda + 1). &(4.21)\end{aligned}$$

Therefore, there is no $\dot{a}$ where the highest $(4i+n)$ and lowest (n) coefficient is zero. Hence, there is no solution for $w_2 = 0$ (i.e. $\Im(w_1) \neq 0$. Let λ_1 be one of the four distinct roots of a. Let λ_l for $l \in \{1,2,3,4\}$ be the four distinct roots of a. Then (4.2) and (4.3) at λ_l become

$$-\lambda_l a'(\lambda_l)c_k(\lambda_l)i = -\dot{a}(\lambda_l)b_k(\lambda_l). \tag{4.22}$$

Inserting $c_k(\lambda) = w_1 b_k(\lambda) - 2\lambda_0 \frac{\Im(w_1)}{\sqrt{\lambda_0}}\tilde{b}_k(\lambda)$ gives

$$\begin{aligned}\dot{a}(\lambda_l) &= \lambda_l a'(\lambda_l)\frac{c_k(\lambda_l)i}{b_k(\lambda_l)} &(4.23)\\ &= \lambda_l a'(\lambda_l)\frac{iw_1 b_k(\lambda_l) - 2i\lambda_0\frac{-2i\Im(w_1)}{\sqrt{\lambda_0}}\tilde{b}_k(\lambda_l)}{b_k(\lambda_l)} &(4.24)\\ &= \lambda_l a'(\lambda_l)\left(iw_1 - 2\lambda_0\frac{\Im(w_1)}{(\lambda_0-\lambda_l)}\right). &(4.25)\end{aligned}$$

Now, we can conclude that

$$\begin{aligned}\dot{a}(\lambda) &= (\lambda_2-\lambda)(\lambda_3-\lambda)(\lambda_4-\lambda)\lambda_1 a'(\lambda_1)\left(iw_1 - 2\lambda_0\frac{\Im(w_1)}{(\lambda_0-\lambda_1)}\right)\\ &\quad+(\lambda_1-\lambda)(\lambda_3-\lambda)(\lambda_4-\lambda)\lambda_2 a'(\lambda_2)\left(iw_1 - 2\lambda_0\frac{\Im(w_1)}{(\lambda_0-\lambda_2)}\right)\\ &\quad+(\lambda_1-\lambda)(\lambda_2-\lambda)(\lambda_4-\lambda)\lambda_3 a'(\lambda_3)\left(iw_1 - 2\lambda_0\frac{\Im(w_1)}{(\lambda_0-\lambda_3)}\right)\\ &\quad+(\lambda_1-\lambda)(\lambda_2-\lambda)(\lambda_3-\lambda)\lambda_4 a'(\lambda_4)\left(iw_1 - 2\lambda_0\frac{\Im(w_1)}{(\lambda_0-\lambda_4)}\right)\\ &\quad+na(\lambda).\end{aligned}$$

We also know that we need $\dot{a}(0) = 0$, $\dot{a}$ has degree three and that $\lambda_1\lambda_2\lambda_3\lambda_4 = 1$ holds. Therefore, we can further calculate w_1 and n. Since $na(\lambda)$ is the only summand that contains λ^4 we can conclude that $n = 0$ has to hold. With $\dot{a}(0) = 1$ we obtain

$$\begin{aligned}0 &= a'(\lambda_1)\left(iw_1 - 2\lambda_0\frac{\Im(w_1)}{(\lambda_0-\lambda_1)}\right) + a'(\lambda_2)\left(iw_1 - 2\lambda_0\frac{\Im(w_1)}{(\lambda_0-\lambda_2)}\right)\\ &\quad+a'(\lambda_3)\left(iw_1 - 2\lambda_0\frac{\Im(w_1)}{(\lambda_0-\lambda_3)}\right) + a'(\lambda_4)\left(iw_1 - 2\lambda_0\frac{\Im(w_1)}{(\lambda_0-\lambda_4)}\right)\\ &= \sum_{l=1}^{4} a'(\lambda_k)\left(i\Re(w_1) - \Im(w_1) - 2\lambda_0\frac{\Im(w_1)}{\lambda_0-\lambda_k}\right).\end{aligned}$$

This gives

$$\Re(w_1) = \frac{-i\sum_{l=1}^{4} a'(\lambda_k)\left(\Im(w_1) + 2\lambda_0\frac{\Im(w_1)}{\lambda_0-\lambda_k}\right)}{\sum_{l=1}^{4} a'(\lambda_k)}.$$

Since all solutions only depend on $\Im(w_1)$ and we have a linear space of solutions we also obtain a one-dimensional set of solutions in the second case. Thus, in all possible cases the set of solutions is one-dimensional. Therefore, $\dim\ker(d\hat{T}) = 1$. Due to the rank-nullity theorem we obtain $rank(d\hat{T}) = 3-1 = 2$. Thus, $rank(d\hat{T}) = \dim\mathcal{F} = \dim\mathbb{C} = 2$ and therefore, all $a \in \mathcal{M}_2^1$ are regular points of $\hat{T}$. Furthermore, any $\tau_a \in \hat{T}(\mathcal{M}_2^1 \times \mathcal{B}'_a \times \mathcal{B}'_a)$ is a regular value (since all $(a, b_1, b_2) \in \mathcal{M}_2^1 \times \mathcal{B}'_a \times \mathcal{B}'_a$ are regular points), where $\mathcal{B}'_a$ describes the $b \in \mathcal{B}'_a$ that have period $2\pi i$. Together with the implicit function theorem this yields the following theorem.

Theorem 4.12. *The level sets $\hat{T}^{-1}(\tau_a)$ of $\hat{T}$ are submanifolds of dimension one for any $(a, b_1, b_2) \in \mathcal{M}_2^1 \times \mathcal{B}'_a \times \mathcal{B}'_a$.*

Now, it remains to show that $(a, b_1, b_2) \mapsto a$ is an immersion. This will generalize theorem (4.12) to the mapping T and therefore, proof corollary (4.13). We are only interested in tangent vectors $(\dot{a}, \dot{b}_1.\dot{b}_2)$ that infinitesimally preserve the periods of Θ_{b_1} and Θ_{b_2}. Those are exactly the solutions of the Whitham equations (4.2), (4.3) and (4.4). We need to argue that the solution $\dot{a} = 0$ of (4.2), (4.3) and (4.4) implies $\dot{b}_1 = 0 = \dot{b}_2$. Since a and b_1, b_2 have no common roots looking at (4.4) at the roots of a yields that c_1 and c_2 have to vanish at these roots as well. But since a has degree four and c_1 and c_2 have degree three and a has no double root since it is in $\mathcal{M}_2^1$ it follows that $c_1 = 0 = c_2$ has to hold. This implies that the right-hand side of (4.4) has to vanish as well. Hence, $Q = 0$ holds. Assuming now that also $\dot{a} = 0$ holds (4.2) implies that $\dot{b}_1 = 0$ and (4.3) implies that $\dot{b}_2 = 0$.

Corollary 4.13. *The level sets $T^{-1}(\tau_a)$ of T are submanifolds of dimension one for any $a \in \mathcal{M}_2^1$.*

5 Singularities for $\mathcal{M}_2^2 \cup \mathcal{M}_2^3$

Unfortunately, polynomials a outside of $\mathcal{M}_2^1$ yield singularities and therefore need a more advanced treatment than in the previous chapter. We will focus on $a \in \mathcal{M}_2^2 \cup \mathcal{M}_2^3$. Therefore, a has now at least one double root. In both cases [KHS17] showed that b_1 and b_2 then also have at least one root at the corresponding double roots at $\mathbb{S}^1$. For $a \in \mathcal{M}_2^2$ let $\alpha_1 \in \mathbb{S}^1$ denote the double root and let α_2 and α_3 be the other distinct roots. Then b_1 and b_2 also have at least one root at $\lambda = \alpha_1$. Thus, the differential forms Θ_{b_k} change as follows for $k = 1, 2$

$$\begin{aligned} \Theta_{b_k} = \frac{b_k(\lambda)}{\lambda\nu} d\lambda &= \frac{b_k(\lambda)}{\lambda\sqrt{\lambda a(\lambda)}} d\lambda \\ &= \frac{(\lambda-\alpha_1)\tilde{b}_k(\lambda)}{\lambda\sqrt{\lambda(\lambda-\alpha_1)^2(\lambda-\alpha_2)(\lambda-\alpha_3)}} d\lambda \\ &= \frac{\tilde{b}_k(\lambda)}{\lambda\sqrt{\lambda(\lambda-\alpha_2)(\lambda-\alpha_3)}} d\lambda, \end{aligned}$$

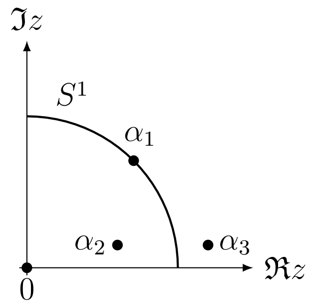
where $\tilde{b}_k(\lambda)$ describes the reminder of b_k such that $b_k(\lambda) = (\lambda - \alpha_1)\tilde{b}_k(\lambda)$. Similarly, for $a \in \mathcal{M}_2^3$ let $\alpha_1 \in \mathbb{S}^1$ and $\alpha_2 = \alpha_3 \in \mathbb{S}^1$ denote the double roots of a

$$\begin{aligned} \Theta_{b_k} &= \frac{b_k(\lambda)}{\lambda\sqrt{\lambda a(\lambda)}} d\lambda \\ &= \frac{(\lambda-\alpha_1)(\lambda-\alpha_2)\hat{b}_k(\lambda)}{\lambda\sqrt{\lambda(\lambda-\alpha_1)^2(\lambda-\alpha_2)^2}} d\lambda \\ &= \frac{\hat{b}_k(\lambda)}{\lambda\sqrt{\lambda}} d\lambda, \end{aligned}$$

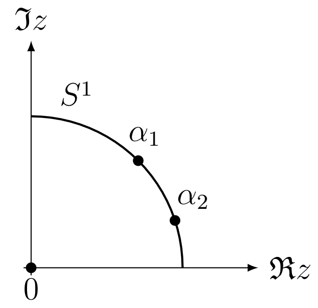
where $\hat{b}_k(\lambda)$ describes the reminder of b_k such that $b_k(\lambda) = (\lambda - \alpha_1)(\lambda - \alpha_2)\hat{b}_k(\lambda)$. This shows that in both cases $\frac{b_k(\lambda)}{\lambda\nu} d\lambda$ is independent of the choice of the double roots. Therefore, they can be moved arbitrarily. This in return leads to a singularity such that the level sets of T as subsets of the form (a, b_1, b_2) are no longer smooth manifolds.

5.1 Local integrals

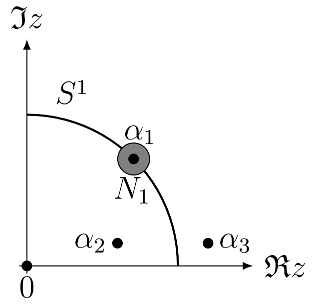
A first approach was to look at local connected neighborhoods N_k of the double roots.



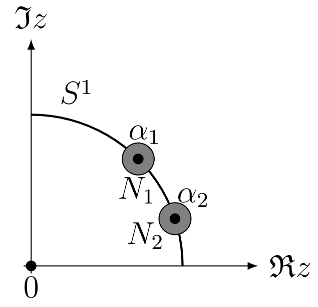
(a) For $a \in \mathcal{M}_2^2$



(b) For $a \in \mathcal{M}_2^3$



(c) For $a \in \mathcal{M}_2^2$ with neighborhood N_1



(d) For $a \in \mathcal{M}_2^3$ with neighborhoods N_1 and N_2

Figure 5.1: Local neighborhoods

In order to get rid of the singularities we will from now on look at the local integrals $q_1 = \int_{N_k} \frac{b_1(\lambda)}{\lambda\nu} d\lambda$ and $q_2 = \int_{N_k} \frac{b_2(\lambda)}{\lambda\nu} d\lambda$ for the local connected neighborhoods N_k of the double roots α_k. q_1 and q_2 can be chosen in a way that they are anti-symmetric with regard to the hyperelliptic involution

$$\sigma : (\lambda, \nu) \mapsto (\lambda, -\nu).$$

Then q_1^2 and $\frac{q_2}{q_1}$ are symmetric with respect to σ. Furthermore, it holds

$$\begin{aligned}\dot{\left(\frac{q_2}{q_1}\right)} d(q_1^2) - (\dot{q_1^2}) d\left(\frac{q_2}{q_1}\right) =& \left(\frac{\dot{q}_2 q_1 - q_2 \dot{q}_1}{q_1^2}\right) 2q_1 dq_1 - 2q_1\dot{q}_1 \left(\frac{dq_2 q_1 - q_2 dq_1}{q_1^2}\right) \\ =& 2\dot{q}_2 dq_1 - 2\frac{q_2 \dot{q}_1 dq_1}{q_1} - 2\dot{q}_1 dq_2 + 2\frac{\dot{q}_1 q_2 dq_1}{q_1} \\ =& 2(\dot{q}_2 dq_1 - \dot{q}_1 dq_2).\end{aligned} \tag{5.1}$$

Furthermore, we know

$$\dot{q}_k = \frac{ic_k}{\nu}$$

and

$$dq_k = \frac{b_k}{\lambda\nu} d\lambda$$

for $k = 1, 2$ with $\nu = \sqrt{\lambda a(\lambda)}$. This, together with (4.4) yields

$$\begin{aligned} c_1 b_2 - c_2 b_1 = && Qa \\ \Leftrightarrow \quad \left(\tfrac{ic_1}{\nu} \tfrac{b_2}{\lambda\nu} d\lambda - \tfrac{ic_2}{\nu} \tfrac{b_1}{\lambda\nu} d\lambda \right)(-i)\nu^2 \tfrac{\lambda}{d\lambda} = && Qa \\ \Leftrightarrow \quad (\dot{q}_1 dq_2 - \dot{q}_2 dq_1)(-i)\nu^2 \tfrac{\lambda}{d\lambda} = && Qa \\ \Leftrightarrow \quad (\dot{q}_1 dq_2 - \dot{q}_2 dq_1)(-i)\lambda a \tfrac{\lambda}{d\lambda} = && Qa \\ \Leftrightarrow \quad (\dot{q}_1 dq_2 - \dot{q}_2 dq_1)(-i) = && \frac{Q}{\lambda^2} d\lambda \end{aligned}$$

Therefore, with (5.1) we get

$$\dot{\left(\frac{q_2}{q_1} \right)} d(q_1^2) - (\dot{q_1^2}) d\left(\frac{q_2}{q_1} \right) = \frac{iQ}{2\lambda^2} d\lambda. \tag{5.2}$$

No we choose local parameters z around the double root such that

$$q_1^2 = P_1(z),$$

where P_1 is a polynomial which highest coefficient is equal to one and which second highest coefficient is equal to zero. Furthermore, we obtain

$$\frac{q_2}{q_1} = P_2(z) f(z),$$

where P_2 is a polynomial which highest coefficient is equal to one and where $f(z)$ is a locally holomorphic function that is invertible. Thus, we can transform (5.2) into

$$\dot{\left(P_2(z) f(z) \right)} dP_1(z) - \dot{P}_1(z) d\left(P_2(z) f(z) \right) = \frac{iQ}{2\lambda^2} d\lambda.$$

The product rule then yields

$$\left(\dot{P}_2(z) f(z) + P_2(z) \dot{f}(z) \right) dP_1(z) - \dot{P}_1(z) \left(dP_2(z) f(z) + P_2(z) df(z) \right) = \frac{iQ}{2\lambda^2} d\lambda. \tag{5.3}$$

We choose

$$P_1(z) = z^2 + k_1 \tag{5.4}$$

and

$$P_2(Z) = z - k_2. \tag{5.5}$$

Therefore, we obtain

$$\left(\dot{k}_2 f(z) + (z - k_2) \dot{f}(z) \right) 2z dz - \dot{k}_1 \left(f(z) + (z - k_2) f'(z) \right) dz = \tilde{Q}(z) dz, \tag{5.6}$$

where $\tilde{Q}(z)$ describes the right-hand side in (5.3) as a series in z, which in turn can be expressed as a convergent power series in λ. Thus, it can be written as

$$\tilde{Q}(z) = \tilde{Q}_0 + \tilde{Q}_1 z + \frac{\tilde{Q}_2}{2} z^2 + \dots .$$

Together we get

$$\Big(\dot{k}_2 f(z) + (z - k_2) \dot{f}(z) \Big) 2z dz - \dot{k}_1 \Big(f(z) + (z - k_2) f'(z) \Big) dz = (\tilde{Q}_0 + \tilde{Q}_1 z + \frac{\tilde{Q}_2}{2} z^2 + \dots) dz. \tag{5.7}$$

Looking at $z = 0$ we get

$$-\dot{k}_1 \Big(f(0) - k_2 f'(0) \Big) dz = \tilde{Q}_0 dz. \tag{5.8}$$

This gives

$$-\dot{k}_1 \Big(f(0) - k_2 f'(0) \Big) = \tilde{Q}_0 \tag{5.9}$$

from which we can deduce a differential equation for k_1

$$\dot{k}_1 = \frac{-\tilde{Q}_0}{f(0) - k_2 f'(0)}. \tag{5.10}$$

In general, we know

$$\Big(\dot{k}_2 f(z) + (z - k_2) \dot{f}(z) \Big) 2z - \dot{k}_1 \Big(f(z) + (z - k_2) f'(z) \Big) = (\tilde{Q}_0 + \tilde{Q}_1 z + \frac{\tilde{Q}_2}{2} z^2 + \dots). \tag{5.11}$$

Setting $z = k_2$ in equation (5.11) we obtain

$$\Big(\dot{k}_2 f(k_2) \Big) 2k_2 - \dot{k}_1 \Big(f(k_2) \Big) = (\tilde{Q}_0 + \tilde{Q}_1 k_2 + \frac{\tilde{Q}_2}{2} k_2^2 + \dots). \tag{5.12}$$

This yields

$$\dot{k}_2 = \frac{(\tilde{Q}_0 + \tilde{Q}_1 k_2 + \frac{\tilde{Q}_2}{2} k_2^2 + \dots) + \dot{k}_1 f(k_2)}{f(k_2) 2k_2}. \tag{5.13}$$

Now, we want to use (5.10) and (5.13) to calculate $(\dot{a}, \dot{b}_1, \dot{b}_2)$. In order to do this, we will first look at the local parameter z. We know that $P_1'(z) = 2z$. Thus, $z = 0$ is a root of P_1' and therefore a root of $d(q_1^2) = 2q_1 dq_1$. Thus, $z = 0$ is a root of b_1 (since $dq_1 = b_1$). In other words we obtain

$$q_1^2 \Big|_{\underbrace{z=0}_{\text{root of } b_1}} = k_1. \tag{5.14}$$

Therefore, P_1 smoothly depends on b_1. Now, we will look at $P_2 = z - k_2$. This has a root at $z = k_2$. Since we defined $\frac{q_2(z)}{q_1(z)} = P_2(z) f(z)$ we see that $z = k_2$ also is a root of q_2. At the same time, q_1^2 evaluated at $z = k_2$ gives us

$$q_1^2\Big|_{z=k_2} = k_2^2 + k_1. \tag{5.15}$$

In other words we obtain

$$q_1\Big|_{z=k_2} = \sqrt{k_2^2 + k_1}. \tag{5.16}$$

We wanted to define the parameter in a way such that in a neighborhood of $z = 0$ we can map the real axis to $\mathbb{S}^1$, i.e. we have

$$\lambda \mapsto \bar{\lambda}^{-1} \; z \mapsto \bar{z}^{-1}.$$

Thus, we now have to choose a sign of z to make the root in (5.16) unambiguous. Thus, we choose k_2 positive. Now, we want to show that we can obtain a unique $\dot{a}$ for $\dot{k}_1$ and $\dot{k}_2$. $z = 0$ is the (former) double root of a and a single root of b_1. Thus, $q_1^2 = 0$ at the (former) double root of a. Therefore, we obtain

$$a(z) = P_1(z)\tilde{a}(z), \tag{5.17}$$

where $\tilde{a}(z)$ describes the remaining two roots of a. This gives us

$$\begin{aligned} \dot{a}(z) &= \dot{P}_1(z)\tilde{a}(z) + P_1(z)\dot{\tilde{a}}(z) && (5.18)\\ &= \dot{k}_1\tilde{a}(z) + P_1(z)\dot{\tilde{a}}(z). && (5.19) \end{aligned}$$

We also know

$$b_1(z) = z\tilde{b}_1(z),$$

where $\tilde{b}_1(z)$ describes the other roots of b_1. Thus, we get

$$\dot{b}_1(z) = \dot{z}\tilde{b}_1(z) + z\dot{\tilde{b}}_1(z). \tag{5.20}$$

For b_2 it is more difficult. Trying to obtain the structure of b_2 in the local parameter z gives a new problem. We would need to look at the derivative of the function f. Since this is not a polynomial the effort would have been very large and instead to further pursue this approach we decided to try an alternate approach. Prof. Schmidt is currently working on a paper that might be applied to this case. Therefore, the observations obtained in section 5.1 might be interesting in the future.

5.2 Additional conditions

This section will apply an alternative (yet equivalent) approach to the one seen in 5.1. We will further follow the reaonsing done in [KHS17] to obtain more formulas connected to the Whitham equations (4.2), (4.3) and (4.4). Together with these new conditions we will be able to remove the singularity or in other words obtain a smooth manifold again. The new conditions arise from the following equation

$$\mu = f(\lambda) + g(\lambda)\nu, \tag{5.21}$$

where f and g are holomorphic functions mapping $\mathbb{C} \setminus \{0\} \to \mathbb{C}$. Since this approach is equivalent to taking the exponential of the approach in section 5.1 and since taking the exponential is an immersion both approaches are equivalent. Now, recall that $\nu = \sqrt{\lambda a(\lambda)}$. Differentiating (5.21) by t gives us

$$\dot{\mu} = \dot{f} + \dot{g}\nu + g\dot{\nu}. \tag{5.22}$$

We also recall $\dot{\ln\mu} = i\frac{c}{\nu}$. We will have μ_1 and μ_2 since we have c_1 and c_2 as well as b_1 and b_2. Thus, we have the following reasoning for $k = 1, 2$. From the equations above we get

$$i\frac{c_k}{\nu} = \dot{\ln\mu_k} = \frac{\dot{\mu}_k}{\mu_k} \Leftrightarrow \dot{\mu}_k = \mu_k i\frac{c_k}{\nu}. \tag{5.23}$$

Equating (5.22) and (5.23) gives

$$i\mu_k\frac{c_k}{\nu} = \dot{f}_k + \dot{g}_k\nu + g_k\dot{\nu}. \tag{5.24}$$

Multiplying (5.21) by $i\frac{c_k}{\nu}$ gives us

$$i\mu_k\frac{c_k}{\nu} = if_k\frac{c_k}{\nu} + ig_kc_k. \tag{5.25}$$

Equating (5.24) and (5.25) yields

$$if_k\frac{c_k}{\nu} + ig_kc_k = \dot{f}_k + \dot{g}_k\nu + g_k\frac{\lambda\dot{a}}{2\nu}. \tag{5.26}$$

Pairing the terms with ν and without gives us the following two equations

$$\dot{f}_k = ig_kc_k \tag{5.27}$$

and

$$\dot{g}_k\lambda a = if_kc_k - g_k\frac{\lambda\dot{a}}{2}. \tag{5.28}$$

Now, we impose that $\mu_k = \pm 1$ at the roots of a. This translates into the following condition

$$(f_k + g_k\nu)(f_k - g_k\nu) = 1.$$

In other words, we obtain

$$f_k^2 - g_k^2\lambda a = 1. \tag{5.29}$$

(5.29) shows that at λ_0 we have $f_k(\lambda_0) = \pm 1$. First, we are going to look at the case where a has no double root. In the case of a single root we simply obtain the following at λ_0 looking at (5.28) at λ_0

$$f_k(\lambda_0)i2c_k(\lambda_0) = g_k(\lambda_0)\lambda_0\dot{a}(\lambda_0). \tag{5.30}$$

We can now calculate $g_k(\lambda_0)$ to show that we obtain the same condition as we obtain from (4.2) or (4.3) at the single roots (e.g. λ_0). We start from the relation $d\ln\mu_k = \Theta_{b_k}$ introduced in 4.1. This gives us

$$d\ln(\mu_k) = \frac{b_k}{\lambda\nu}d\lambda \Leftrightarrow d\mu_k = \mu_k\frac{b_k}{\lambda\nu}d\lambda. \tag{5.31}$$

Taking the derivative with respect to λ from (5.21) gives

$$d\mu_k = (f_k' + g_k'\nu + g_k\frac{a + \lambda a'}{2\nu})d\lambda. \tag{5.32}$$

Equating (5.31) and (5.32) establishes the following equation

$$\mu_k\frac{b_k}{\lambda\nu}d\lambda = (f_k' + g_k'\nu + g_k\frac{a + \lambda a'}{2\nu})d\lambda \Leftrightarrow \mu_k\frac{b_k}{\lambda} = f_k'\nu + g_k'\nu^2 + g_k\frac{a + \lambda a'}{2}. \tag{5.33}$$

(5.33) can be written as

$$(f_k + g_k\nu)\frac{b_k}{\lambda} = f_k'\nu + g_k'\lambda a + g_k\frac{a + \lambda a'}{2}. \tag{5.34}$$

Again, we compare the terms with ν and without to get

$$g_k\frac{b_k}{\lambda} = f_k' \tag{5.35}$$

and

$$f_k\frac{b_k}{\lambda} = g_k'\lambda a + g_k\frac{a + \lambda a'}{2}. \tag{5.36}$$

Evaluating (5.36) at λ_0 gives us

$$f_k(\lambda_0)\frac{b_k(\lambda_0)}{\lambda_0} = g_k(\lambda_0)\frac{\lambda_0 a'(\lambda_0)}{2}. \tag{5.37}$$

Now we can obtain an expression for $g_k(\lambda_0)$.

$$g_k(\lambda_0) = f(\lambda_0)\frac{2b_k(\lambda_0)}{\lambda_0^2 a'(\lambda_0)} \tag{5.38}$$

Inserting (5.38) in (5.30) gives us

$$f_k(\lambda_0)i2c_k(\lambda_0) = f_k(\lambda_0)\frac{2b_k(\lambda_0)}{\lambda_0^2 a'(\lambda_0)}\lambda_0\dot{a}(\lambda_0) \Leftrightarrow \lambda_0 a'(\lambda_0)c_k(\lambda_0)i = \dot{a}(\lambda_0)b_k(\lambda_0). \tag{5.39}$$

Hence, we have obtained the same equation as we obtain from (4.2) and (4.3) at λ_0

$$-\lambda_0 a'(\lambda_0)c_k(\lambda_0)i = -\dot{a}(\lambda_0)b_k(\lambda_0) \Leftrightarrow \lambda_0 a'(\lambda_0)c_k(\lambda_0)i = \dot{a}(\lambda_0)b_k(\lambda_0).$$

Thus, in this case the additional conditions impose no new conditions. Now, we are going to treat the interesting case of a double root of a at λ_0. First of all, we note that we can assume that at least one $b_k(\lambda_0) \neq 0$ and $a''(\lambda_0) \neq 0$ (otherwise we can achieve this through a change of basis).(5.28) shows that at λ_0 also $f_k c_k - g_k \frac{\lambda \dot{a}}{2}$ has to have a double root. Therefore, the derivative with respect to λ of (5.28) has to vanish at λ_0 as well. It is

$$\dot{g}_k'\lambda a + \dot{g}_k a + \dot{g}_k \lambda a' = i f_k' c_k + i f_k c_k' - g_k' \frac{\lambda \dot{a}}{2} - g_k \frac{\dot{a} + \lambda \dot{a}'}{2}. \tag{5.40}$$

The left-hand side of (5.40) vanishes at λ_0. Therefore, we obtain

$$0 = i f_k'(\lambda_0)c_k(\lambda_0) + i f_k(\lambda_0)c_k'(\lambda_0) - g_k'(\lambda_0)\frac{\lambda_0\dot{a}(\lambda_0)}{2} - g_k(\lambda_0)\frac{\dot{a}(\lambda_0) + \lambda_0\dot{a}'(\lambda_0)}{2}. \tag{5.41}$$

Now, let λ_0 be the double root. Then we already know that b_k also has a root at λ_0. Furthermore, we know that only one b_k can have a root of higher order than one. Thus, (5.39) and (5.38) with the rule of L'Hospital become

$$\dot{a}(\lambda_0) = \frac{\lambda_0 a''(\lambda_0)c_k(\lambda_0)i}{\lambda_0 b_k'(\lambda_0)} \tag{5.42}$$

and

$$g_k(\lambda_0) = f_k(\lambda_0)\frac{2b_k'(\lambda_0)}{\lambda_0^2 a''(\lambda_0)} \tag{5.43}$$

for at least one polynomial b_k with $b_k'(\lambda_0) \neq 0$. Thus, there is a $k \in \{1, 2\}$ such that $g_k(\lambda_0) \neq 0$. Furthermore (5.35) at λ_0 provides us with

$$f_k'(\lambda_0) = g(\lambda_0)\frac{b_k(\lambda_0)}{\lambda_0} = 0.$$

Therefore, equation (5.41) becomes

$$0 = f_k(\lambda_0)ic_k'(\lambda_0) - g_k'(\lambda_0)\frac{\lambda_0\dot{a}(\lambda_0)}{2} - f_k(\lambda_0)\frac{2b_k'(\lambda_0)}{\lambda_0^2 a''(\lambda_0)}\frac{\dot{a}(\lambda_0) + \lambda_0\dot{a}'(\lambda_0)}{2}. \tag{5.44}$$

We need to find $g_k'(\lambda_0)$ to further utilize equation (5.44). In order to find $g_k'(\lambda_0)$ we will differentiate (5.36) twice. The first derivative is

$$f_k'\frac{b_k}{\lambda} + f_k\frac{b_k'\lambda - b_k}{\lambda^2} = g_k''\lambda a + g_k' a + g_k'\lambda a' + g_k'\frac{a + \lambda a'}{2} + g_k\frac{a' + a' + \lambda a''}{2}. \tag{5.45}$$

This again gives

$$g_k(\lambda_0) = f_k(\lambda_0)\frac{2b_k'(\lambda_0)}{\lambda_0^2 a''(\lambda_0)}. \tag{5.46}$$

The second derivative is

$$\begin{aligned} & f_k''\frac{b_k}{\lambda} + f_k'\frac{b_k'\lambda - b_k}{\lambda^2} + f_k'\frac{b_k'\lambda - b_k}{\lambda^2} + f_k\frac{(b_k''\lambda + b_k' - b_k')\lambda^2 - 2\lambda(b_k'\lambda - b_k)}{\lambda^4} \\ = \; & g_k'''\lambda a + g_k'' a + g_k''\lambda a' + g_k'' a + g_k' a' + g_k''\lambda a' + g_k' a' + g_k'\lambda a'' \\ & + g_k''\frac{a + \lambda a'}{2} + g_k'\frac{a' + \lambda a'' + a'}{2} + g_k'\frac{a' + a' + \lambda a''}{2} + g_k\frac{a'' + a'' + a'' + \lambda a'''}{2}. \end{aligned}$$

Evaluating the second derivative at λ_0 gives

$$\begin{aligned} & f_k(\lambda_0)\frac{b_k''(\lambda_0)\lambda_0^3 - 2\lambda_0 b_k'(\lambda_0)}{\lambda_0^4} = g_k'(\lambda_0)\lambda_0 a''(\lambda_0) + g_k'(\lambda_0)\frac{\lambda_0 a''(\lambda_0)}{2} \\ & + g_k'(\lambda_0)\frac{\lambda_0 a''(\lambda_0)}{2} + g_k(\lambda_0)\frac{3a''(\lambda_0) + \lambda_0 a'''(\lambda_0)}{2}. \end{aligned}$$

This gives

$$f_k(\lambda_0)\frac{b_k''(\lambda_0)\lambda_0^3 - 2\lambda_0 b_k'(\lambda_0)}{\lambda_0^4} - g_k(\lambda_0)\frac{3a''(\lambda_0) + \lambda_0 a'''(\lambda_0)}{2} = 2g_k'(\lambda_0)\lambda_0 a''(\lambda_0). \tag{5.47}$$

Therefore, we obtain

$$\begin{aligned} g_k'(\lambda_0) &= f_k(\lambda_0)\frac{b_k''(\lambda_0)\lambda_0^3 - 2\lambda_0 b_k'(\lambda_0)}{2\lambda_0^4\lambda_0 a''(\lambda_0)} - g_k(\lambda_0)\frac{3a''(\lambda_0) + \lambda_0 a'''(\lambda_0)}{4\lambda_0 a''(\lambda_0)} &(5.48)\\ &= f_k(\lambda_0)\frac{b_k''(\lambda_0)\lambda_0^3 - 2\lambda_0 b_k'(\lambda_0)}{2\lambda_0^4\lambda_0 a''(\lambda_0)} - f_k(\lambda_0)\frac{2b_k'(\lambda_0)}{\lambda_0^2 a''(\lambda_0)}\frac{3a''(\lambda_0) + \lambda_0 a'''(\lambda_0)}{4\lambda_0 a''(\lambda_0)} &(5.49) \end{aligned}$$

Now, we can look at (5.44) again and insert (5.49).

$$\begin{aligned} 0 = f_k(\lambda_0) i c_k'(\lambda_0) - \Bigg(& f_k(\lambda_0)\frac{b_k''(\lambda_0)\lambda_0^3 - 2\lambda_0 b_k'(\lambda_0)}{2\lambda_0^4\lambda_0 a''(\lambda_0)} \\ & - f_k(\lambda_0)\frac{2b_k'(\lambda_0)}{\lambda_0^2 a''(\lambda_0)}\frac{3a''(\lambda_0) + \lambda_0 a'''(\lambda_0)}{4\lambda_0 a''(\lambda_0)}\Bigg)\frac{\lambda_0\dot{a}(\lambda_0)}{2} \\ & - f_k(\lambda_0)\frac{2b_k'(\lambda_0)}{\lambda_0^2 a''(\lambda_0)}\frac{\dot{a}(\lambda_0) + \lambda_0\dot{a}'(\lambda_0)}{2}. \end{aligned} \tag{5.50}$$

Using the formula for $\dot{a}(\lambda_0)$ we obtain

$$
\begin{aligned}
0 =& f_k(\lambda_0) i c_k'(\lambda_0) - \Bigg(f_k(\lambda_0) \frac{b_k''(\lambda_0)\lambda_0^3 - 2\lambda_0 b_k'(\lambda_0)}{2\lambda_0^4 \lambda_0 a''(\lambda_0)} \\
&- f_k(\lambda_0) \frac{2b_k'(\lambda_0)}{\lambda_0^2 a''(\lambda_0)} \frac{3a''(\lambda_0) + \lambda_0 a'''(\lambda_0)}{4\lambda_0 a''(\lambda_0)} \Bigg) \frac{\lambda_0}{2} \frac{\lambda_0 a''(\lambda_0) c_k(\lambda_0) i}{\lambda_0 b_k'(\lambda_0)} \\
&- f_k(\lambda_0) \frac{2b_k'(\lambda_0)}{\lambda_0^2 a''(\lambda_0)} \frac{\lambda_0 a''(\lambda_0) c_k(\lambda_0) i}{\lambda_0 b_k'(\lambda_0)} \frac{1}{2} - f_k(\lambda_0) \frac{2b_k'(\lambda_0)}{\lambda_0^2 a''(\lambda_0)} \frac{\lambda_0 \dot{a}'(\lambda_0)}{2}.
\end{aligned}
\tag{5.51}
$$

Now, we can deduce a formula for $\dot{a}'(\lambda_0)$

$$
\begin{aligned}
\dot{a}'(\lambda_0) =& \frac{a''(\lambda_0)\lambda_0 i c_k'(\lambda_0)}{b_k'(\lambda_0)} - \frac{a''(\lambda_0) b_k''(\lambda_0) c_k(\lambda_0) i}{4 b_k'(\lambda_0)^2} + \frac{a''(\lambda_0) c_k(\lambda_0) i}{2\lambda_0^2 b_k'(\lambda_0)} \\
&+ \frac{a''(\lambda_0) 3 c_k(\lambda_0) i}{4\lambda_0 b_k'(\lambda_0)} + \frac{a'''(\lambda_0) c_k(\lambda_0) i}{4 b_k'(\lambda_0)} - \frac{a''(\lambda_0) c_k(\lambda_0) i}{\lambda_0 b_k'(\lambda_0)}.
\end{aligned}
\tag{5.52}
$$

Since a has a double root at λ_0 we also get

$$
c_1(\lambda_0) b_2'(\lambda_0) - c_2(\lambda_0) b_1'(\lambda_0) = 0 \;\Leftrightarrow\; \frac{c_1(\lambda_0)}{b_1'(\lambda_0)} = \frac{c_2(\lambda_0)}{b_2'(\lambda_0)}. \tag{5.53}
$$

form (4.4) with L'Hospital. Since (5.52) has to hold for $k = 1, 2$ and since we have (5.53) we obtain a new condition

$$
\frac{\lambda_0 c_1'(\lambda_0)}{b_1'(\lambda_0)} - \frac{b_1''(\lambda_0) c_1(\lambda_0)}{4 b_1'(\lambda_0)^2} = \frac{\lambda_0 c_2'(\lambda_0)}{b_2'(\lambda_0)} - \frac{b_2''(\lambda_0) c_2(\lambda_0)}{4 b_2'(\lambda_0)^2}. \tag{5.54}
$$

We can rearrange (5.54) to get

$$
\frac{\lambda_0 c_1'(\lambda_0)}{b_1'(\lambda_0)} - \frac{\lambda_0 c_2'(\lambda_0)}{b_2'(\lambda_0)} = \frac{b_1''(\lambda_0) c_1(\lambda_0)}{4 b_1'(\lambda_0)^2} - \frac{b_2''(\lambda_0) c_2(\lambda_0)}{4 b_2'(\lambda_0)^2} \tag{5.55}
$$

or

$$
c_1'(\lambda_0) b_2'(\lambda_0) - c_2'(\lambda_0) b_1'(\lambda_0) = \frac{b_2'(\lambda_0) b_1''(\lambda_0) c_1(\lambda_0)}{4\lambda_0 b_1'(\lambda_0)} - \frac{b_1'(\lambda_0) b_2''(\lambda_0) c_2(\lambda_0)}{4\lambda_0 b_2'(\lambda_0)}. \tag{5.56}
$$

Together with (5.53) we get

$$
c_1'(\lambda_0) b_2'(\lambda_0) - c_2'(\lambda_0) b_1'(\lambda_0) = \frac{b_2'(\lambda_0) b_1''(\lambda_0) c_2(\lambda_0)}{4\lambda_0 b_2'(\lambda_0)} - \frac{b_1'(\lambda_0) b_2''(\lambda_0) c_2(\lambda_0)}{4\lambda_0 b_2'(\lambda_0)}. \tag{5.57}
$$

Now, we return to the initial Whitham equations (4.2), (4.3) and (4.4) to deduce the dimension of the solution set. Looking at (4.4) we can divide through $(\lambda - \lambda_0)$ on both sides and get

$$
c_1 \tilde{b}_2 - c_2 \tilde{b}_1 = Q_1 \lambda (\lambda - \lambda_0)(\lambda - \lambda_2)(\lambda - \lambda_3). \tag{5.58}
$$

Thus, equation (5.58) at λ_0 gives us

$$\frac{c_1(\lambda_0)}{\tilde{b}_1(\lambda_0)} = \frac{c_2(\lambda_0)}{\tilde{b}_2(\lambda_0)}. \tag{5.59}$$

We can use the roots of the polynomials b_k that are distinct from the roots of a. This gives first conditions for c_k. But since the left- and the right-hand side both vanish at λ_0 we need to differentiate (4.4) and get

$$c_1'b_2 + c_1b_2' - c_2'b_1 - c_2b_2' = Q_1a + Q_1\lambda a'. \tag{5.60}$$

Differentiating (5.60) again yields

$$c_1''b_2 + 2c_1'b_2' + c_1b_2'' - c_2''b_1 - 2c_2'b_1' - c_2b_1'' = 2Q_1a' + Q_1\lambda a''. \tag{5.61}$$

Inserting λ_0 gives us

$$2c_1'(\lambda_0)b_2'(\lambda_0) + c_1(\lambda_0)b_2''(\lambda_0) - 2c_2'(\lambda_0)b_1'(\lambda_0) - c_2(\lambda_0)b_1''(\lambda_0) = Q_1\lambda_0a''(\lambda_0). \tag{5.62}$$

Now, we can utilize (5.57) and get

$$\begin{aligned}&\frac{b_2'(\lambda_0)b_1''(\lambda_0)c_2(\lambda_0)}{2\lambda_0b_2'(\lambda_0)} - \frac{b_1'(\lambda_0)b_2''(\lambda_0)c_2(\lambda_0)}{2\lambda_0b_2'(\lambda_0)} + c_1(\lambda_0)b_2''(\lambda_0) - c_2(\lambda_0)b_1''(\lambda_0)\\ &= Q_1\lambda_0a''(\lambda_0)\end{aligned} \tag{5.63}$$

using $c_1(\lambda_0) = \frac{c_2(\lambda_0)b_1'(\lambda_0)}{b_2'(\lambda_0)}$ gives us

$$\begin{aligned}&\frac{b_2'(\lambda_0)b_1''(\lambda_0)c_2(\lambda_0)}{2\lambda_0b_2'(\lambda_0)} - \frac{b_1'(\lambda_0)b_2''(\lambda_0)c_2(\lambda_0)}{2\lambda_0b_2'(\lambda_0)} + \frac{c_2(\lambda_0)b_1'(\lambda_0)}{b_2'(\lambda_0)}b_2''(\lambda_0) - c_2(\lambda_0)b_1''(\lambda_0)\\ &= Q_1\lambda_0a''(\lambda_0).\end{aligned} \tag{5.64}$$

Therefore, we can find a new condition on c_2 (and for c_1 in the same way)

$$c_2(\lambda_0) = \frac{Q_1\lambda_0a''(\lambda_0)2\lambda_0b_2'(\lambda_0)}{b_2'(\lambda_0)b_1''(\lambda_0) - b_1'(\lambda_0)b_2''(\lambda_0) + 2\lambda_0b_1'(\lambda_0)b_2''(\lambda_0) - 2\lambda_0b_2'(\lambda_0)b_1''(\lambda_0)}. \tag{5.65}$$

This condition holds exactly if

$$b_2'(\lambda_0)b_1''(\lambda_0) - b_1'(\lambda_0)b_2''(\lambda_0) + 2\lambda_0b_1'(\lambda_0)b_2''(\lambda_0) - 2\lambda_0b_2'(\lambda_0)b_1''(\lambda_0) \neq 0. \tag{5.66}$$

In other words the new found condition holds only if there is no linear combination of b_1 and b_2 with a root of degree three at λ_0 (otherwise the second derivatives would vanish). Since this case cannot happen the new condition holds. This means we found overall three conditions for each c_k. The other two conditions arise from inserting the other roots of the polynomials b_k in (4.4). They read

$$(-1)^{3-k} c_k(\lambda_{2k}) b_{3-k}(\lambda_{2k}) = Q_1 \lambda_{2k} a(\lambda_{2k}) \tag{5.67}$$

and

$$(-1)^{3-k} c_k(\lambda_{3k}) b_{3-k}(\lambda_{3k}) = Q_1 \lambda_{3k} a(\lambda_{3k}) \tag{5.68}$$

for $k = 1, 2$. In the case that b_k has an higher order root we can use the differentiated equation (5.60) to get three conditions. Thus, the c_k only depend on $Q_1 \in \mathbb{R}$.

Solving (4.2) and (4.3) for $\dot{b}_1$ and $\dot{b}_2$ is now possible through inserting the roots of b_k (if b_k has a higher order root we can obtain new conditions by taking the derivative of the equation as seen in 4.2). A special treatment is necessary for λ_0 since it is also a (double) root of a now. At the other roots of b_k we have

$$(2\lambda_{2k} a(\lambda_{2k}) c_k'(\lambda_{2k}) - a(\lambda_{2k}) c_k(\lambda_{2k}) - \lambda_{2k} a'(\lambda_{2k}) c_k(\lambda_{2k}))i = 2a(\lambda_{2k}) \dot{b}_k(\lambda_{2k}) \tag{5.69}$$

and

$$(2\lambda_{3k} a(\lambda_{3k}) c_k'(\lambda_{3k}) - a(\lambda_{3k}) c_k(\lambda_{3k}) - \lambda_{3k} a'(\lambda_{3k}) c_k(\lambda_{3k}))i = 2a(\lambda_{3k}) \dot{b}_k(\lambda_{3k}). \tag{5.70}$$

Therefore, we have two conditions for $\dot{b}_k$. If b_k has a double root we differentiate the Whitham equation. Differentiating (4.2) and (4.3) gives us

$$\begin{aligned}&(2ac_k' + 2\lambda a' c_k' + 2\lambda a c_k'' - a' c_k - a c_k' - a' c_k - \lambda a'' c_k - \lambda a' c_k')i \\ &= 2a' \dot{b}_k + 2a \dot{b}_k' - \dot{a}' b_k - \dot{a} b_k'.\end{aligned} \tag{5.71}$$

Assuming b_k has a double root we can obtain a condition on $\dot{b}_k'$ at the double root. Therefore, we already have two conditions for $\dot{b}_k$ only depending on c_k. At λ_0 which also is the double root of a both equations vanish.

Inserting the double root λ_0 in (5.71) gives us

$$\lambda_0 a''(\lambda_0) c_k(\lambda_0) i = \dot{a}(\lambda_0) b_k'(\lambda_0) \Leftrightarrow \frac{\lambda_0 a''(\lambda_0) c_k(\lambda_0) i}{b_k'(\lambda_0)} = \dot{a}(\lambda_0). \tag{5.72}$$

That is the same equation as we obtained from the L'Hospital rule and also gives us

$$\frac{c_1(\lambda_0)}{b_1'(\lambda_0)} = \frac{c_2(\lambda_0)}{b_2'(\lambda_0)} \tag{5.73}$$

Differentiating the Whitham equations (4.2) and (4.3) a second time yields

$$\begin{aligned}&(2a' c_k' + 2a c_k'' + 2a' c_k' + 2\lambda a'' c_k' + 2\lambda a' c_k'' + 2a c_k'' + 2\lambda a' c_k'' + 2\lambda a c_k''' \\ &- a'' c_k - a' c_k' - a' c_k' - a c_k'' - a'' c_k - a' c_k' - a'' c_k - \lambda a''' c_k - \lambda a'' c_k' - a' c_k' - \lambda a'' c_k' - \lambda a' c_k'')i \\ &= 2a'' \dot{b}_k + 2a' \dot{b}_k' + 2a' \dot{b}_k' + 2a \dot{b}_k'' - \dot{a}'' b_k - \dot{a}' b_k' - \dot{a}' b_k' - \dot{a} b_k''.\end{aligned} \tag{5.74}$$

Inserting λ_0 gives us

$$\begin{aligned}&(2\lambda_0 a''(\lambda_0)c_k'(\lambda_0) - a''(\lambda_0)c_k(\lambda_0) - a''(\lambda_0)c_k(\lambda_0) - a''(\lambda_0)c_k(\lambda_0)\\&- \lambda_0 a'''(\lambda_0)c_k(\lambda_0) - \lambda_0 a''(\lambda_0)c_k'(\lambda_0) - \lambda_0 a''(\lambda_0)c_k'(\lambda_0))i\\&= 2a''(\lambda_0)\dot{b}_k(\lambda_0) - \dot{a}'(\lambda_0)b_k'(\lambda_0) - \dot{a}'(\lambda_0)b_k'(\lambda_0) - \dot{a}(\lambda_0)b_k''(\lambda_0).\end{aligned} \tag{5.75}$$

This gives us a new condition for $\dot{b}_k$

$$\begin{aligned}&\frac{1}{2a''(\lambda_0)}((2\lambda a''(\lambda_0)c_k'(\lambda_0) - a''(\lambda_0)c_k(\lambda_0) - a''(\lambda_0)c_k(\lambda_0) - a''(\lambda_0)c_k(\lambda_0)\\&- \lambda_0 a'''(\lambda_0)c_k(\lambda_0) - \lambda_0 a''(\lambda_0)c_k'(\lambda_0) - \lambda_0 a''(\lambda_0)c_k'(\lambda_0))i\\&+ \dot{a}'(\lambda_0)b_k'(\lambda_0) + \dot{a}'(\lambda_0)b_k'(\lambda_0) + \dot{a}(\lambda_0)b_k''(\lambda_0))\\&= \dot{b}_k(\lambda_0),\end{aligned} \tag{5.76}$$

where we can replace $\dot{a}(\lambda_0)$ and $\dot{a}'(\lambda_0)$ with the formulas obtained above in (5.42) and (5.52). Therefore, $\dot{b}_1$ and $\dot{b}_2$ only depend on c_1 and c_2. It remains to solve for $\dot{a}$. Using the two distinct roots λ_2 and λ_3 of a we obtain

$$\lambda a'(\lambda_l)c_l(\lambda_l)i = \dot{a}(\lambda_l)b_k(\lambda_l). \tag{5.77}$$

Earlier we collected conditions on the double root for $\dot{a}(\lambda_0)$ in (5.42) and $\dot{a}'(\lambda_0)$ in (5.52). Together with $\dot{a} = 0$ and (5.57) $\dot{a}$ only depends on c_k. Therefore, we obtain a one-dimensional solution set.

Since $\dot{b}_1$, $\dot{b}_2$ and $\dot{a}$ only depend on c_1 and c_2 that in return only depend on $Q_1 \in \mathbb{R}$ we also have a one-dimensional set of solutions for polynomials $a \in \mathcal{M}_2^2$. Due to time problems we cannot explicitly visit the case of $a \in \mathcal{M}_2^3$. But this not necessary since it is merely a limiting case of $a \in \mathcal{M}_2^2$ and the argumentation will be very much the same. Therefore, we can conclude that in the case of double roots the level sets of T also are one-dimensional submanifolds.

6 Intersection of the level sets of T and the set $\mathcal{S}_2$

The goal of this chapter is to examine the intersection of the level sets of T and the set $\mathcal{S}_2$.

Definition 6.1. *The set $\mathcal{S}_2$ is defined as*

$$\mathcal{S}_2 := \bigcup_{\lambda_0 \in \mathbb{S}^1} \{a \in \mathcal{H}^2 | b_1(\lambda_0) = 0 = b_2(\lambda_0)\},$$

where $\mathcal{H}^2$ denotes the space of spectral curves of CMC tori of finite type (i.e. the ones that can be described through spectral data see [CS16] or [Sch17] for more).

Therefore, we are now interested in the case in which b_1 and b_2 have a common root. Until now we were only interested in the case in which $t = 0$. Now, we want to deviate from this case. In order to do this, we will describe Q in a Taylor series in t. From 4.2 we know that $Q(\lambda) = Q_1 \lambda$. Thus, we obtain

$$Q(\lambda) = \sum_{j=0}^{\infty} \frac{Q_{j1}}{j!} t^j = \lambda(Q_{01} + Q_{11} t + \dots).$$

For the derivative we obtain

$$\frac{dQ(\lambda)}{dt} = \sum_{j=1}^{\infty} \frac{Q_{j1}(\lambda)}{(j-1)!} t^{j-1} = \lambda(Q_{11} + \dots).$$

Likewise, we proceed for the other polynomials and obtain

$$a(\lambda) = \sum_{j=0}^{\infty} \frac{A_j(\lambda)}{j!} t^j = A_0(\lambda) + A_1(\lambda) t + \dots,$$

where A_j are polynomials in λ of degree four.

$$b_1(\lambda) = \sum_{j=0}^{\infty} \frac{B_{j1}(\lambda)}{j!} t^j = B_{01}(\lambda) + B_{11}(\lambda) t + \dots,$$

where B_{j1} are polynomials in λ of degree three.

$$b_2(\lambda) = \sum_{j=0}^{\infty} \frac{B_{j2}(\lambda)}{j!} t^j = B_{02}(\lambda) + B_{12}(\lambda)t + \dots,$$

where B_{j2} are polynomials in λ of degree three.

$$c_1(\lambda) = \sum_{j=0}^{\infty} \frac{C_{j1}(\lambda)}{j!} t^j = C_{01}(\lambda) + C_{11}(\lambda)t + \dots,$$

where C_{j1} are polynomials in λ of degree three.

$$c_2(\lambda) = \sum_{j=0}^{\infty} \frac{C_{j2}(\lambda)}{j!} t^j = C_{02}(\lambda) + C_{12}(\lambda)t + \dots,$$

where C_{j2} are polynomials in λ of degree three. With these representations (4.4) becomes

$$\begin{aligned} &(C_{01}(\lambda) + C_{11}(\lambda)t + \dots)(B_{02}(\lambda) + B_{12}(\lambda)t + \dots) \\ &- (C_{02}(\lambda) + C_{12}(\lambda)t + \dots)(B_{01}(\lambda) + B_{11}(\lambda)t + \dots) \\ &= \lambda(Q_{01} + Q_{11}t + \dots)(A_0(\lambda) + A_1(\lambda)t + \dots). \end{aligned} \tag{6.1}$$

For $t = 0$ we get

$$C_{01}(\lambda)B_{02}(\lambda) - C_{02}(\lambda)B_{01}(\lambda) = Q_{01}\lambda A_0(\lambda). \tag{6.2}$$

In the case that B_{02} and B_{01} have a common root λ_0 that is no root of A_0 we obtain the following equation at $\lambda = \lambda_0$

$$0 = Q_{01}\lambda_0 \underbrace{A_0(\lambda)}_{\neq 0}.$$

Thus, we obtain $Q_{01} = 0$. This now gives us that the right-hand side of (6.2) is 0. Hence, we obtain

$$C_{01}(\lambda)B_{02}(\lambda) - C_{02}(\lambda)B_{01}(\lambda) = 0.$$

Hence, we are almost in the same situation as in 4.2 in the case $Q = 0$. Therefore, we obtain that C_{0k} is a linear combination of B_{0k} and $\frac{B_{0k}}{\lambda_0 - \lambda}$ where λ_0 is the common root of B_{01} and B_{02}. We will denote them again as

$$C_{0k}(\lambda) = w_1 B_{0k} + w_2 \frac{B_{0k}}{\lambda_0 - \lambda} \tag{6.3}$$

for $k = 1, 2$. Now we differentiate (4.4) in this representation and obtain

$$\dot{c}_1 b_2 + c_1 \dot{b}_2 - \dot{c}_2 b_1 - c_2 \dot{b}_1 = \dot{Q} a + Q \dot{a}.$$

For $t = 0$ this becomes

$$C_{11}(\lambda)B_{02}(\lambda) + C_{01}(\lambda)B_{12}(\lambda) - C_{12}(\lambda)B_{01}(\lambda) - C_{02}(\lambda)B_{11}(\lambda) = \lambda Q_{11}A_0(\lambda) \qquad (6.4)$$

since $Q_{01} = 0$. To deduce properties of B_{11} and B_{12} we look at (4.2) and (4.3) in the representation with Taylor coefficients. For $t = 0$ this gives

$$(2\lambda A_0(\lambda)C'_{01}(\lambda) - A_0(\lambda)C_{01}(\lambda) - \lambda A'_0(\lambda)C_{01}(\lambda))i = 2A_0(\lambda)B_{11}(\lambda) - A_1(\lambda)B_{01}(\lambda) \quad (6.5)$$

and

$$(2\lambda A_0(\lambda)C'_{02}(\lambda) - A_0(\lambda)C_{02}(\lambda) - \lambda A'_0(\lambda)C_{02}(\lambda))i = 2A_0(\lambda)B_{12}(\lambda) - A_1(\lambda)B_{02}(\lambda). \quad (6.6)$$

Using $C_{01}(\lambda) = w_1 B_{01} + w_2\frac{B_{01}(\lambda)}{\lambda_0-\lambda}$ and $C_{02}(\lambda) = w_1 B_{02} + w_2\frac{B_{02}(\lambda)}{\lambda_0-\lambda}$ therefore gives

$$\begin{aligned}&\Bigg(2\lambda A_0(\lambda)\frac{d}{d\lambda}\left(w_1 B_{01}(\lambda) + w_2\frac{B_{01}(\lambda)}{\lambda_0-\lambda}\right) - A_0(\lambda)\left(w_1 B_{01}(\lambda) + w_2\frac{B_{01}(\lambda)}{\lambda_0-\lambda}\right)\\ &- \lambda A'_0(\lambda)\left(w_1 B_{01}(\lambda) + w_2\frac{B_{01}(\lambda)}{\lambda_0-\lambda}\right)\Bigg)i = 2A_0(\lambda)B_{11}(\lambda) - A_1(\lambda)B_{01}(\lambda)\end{aligned} \qquad (6.7)$$

and

$$\begin{aligned}&\Bigg(2\lambda A_0(\lambda)\frac{d}{d\lambda}\left(w_1 B_{02}(\lambda) + w_2\frac{B_{02}(\lambda)}{\lambda_0-\lambda}\right) - A_0(\lambda)\left(w_1 B_{02}(\lambda) + w_2\frac{B_{02}(\lambda)}{\lambda_0-\lambda}\right)\\ &- \lambda A'_0(\lambda)\left(w_1 B_{02}(\lambda) + w_2\frac{B_{02}(\lambda)}{\lambda_0-\lambda}\right)\Bigg)i = 2A_0(\lambda)B_{12}(\lambda) - A_1(\lambda)B_{02}(\lambda).\end{aligned} \qquad (6.8)$$

Note that with $B_{0k} = \frac{i}{\sqrt{\lambda_0}}(\lambda_0 - \lambda)\tilde{B}_{0k}(\lambda)$ where $\tilde{B}_{0k} = \tilde{B}_{2k}\lambda^2 + \tilde{B}_{1k}\lambda + \tilde{\tilde{B}}_{2k}$ we get

$$\frac{d}{d\lambda}\left(w_1 B_{0k} + w_2\frac{B_{0k}(\lambda)}{\lambda_0-\lambda}\right) = w_1 B'_{0k}(\lambda) + w_2\frac{i}{\sqrt{\lambda_0}}\tilde{B}'_{0k}(\lambda)$$

with

$$\tilde{B}'_{0k}(\lambda) = 2\tilde{B}_{2k}\lambda + \tilde{B}_{1k}$$

where $\tilde{B}'_{0k}(\lambda_0) \neq 0$ for at least one $k = 1, 2$. Thus, at $\lambda = \lambda_0$ we get

$$\frac{d}{d\lambda}\left(w_1 B_{0k} + w_2\frac{B_{0k}(\lambda)}{\lambda_0-\lambda}\right)\Bigg|_{\lambda=\lambda_0} = w_1 B'_{0k}(\lambda_0) + w_2\frac{i}{\sqrt{\lambda_0}}\tilde{B}'_{0k}(\lambda_0). \qquad (6.9)$$

Since the other roots are distinct from the common root λ_0 the derivative (6.9) of the C_{0k} is not zero at $\lambda = \lambda_0$. Thus (6.7) and (6.8) evaluated at the common root λ_0 yield

$$
\begin{aligned}
&\Bigg(2\lambda_0 A_0(\lambda_0)\Bigg(w_1 B'_{01}(\lambda_0) + w_2 \frac{i}{\sqrt{\lambda_0}} \tilde{B}'_{01}(\lambda_0))\Bigg) \\
&- A_0(\lambda_0)\Bigg(w_1 \underbrace{B_{01}(\lambda_0)}_{=0} + w_2 \frac{i}{\sqrt{\lambda_0}} \tilde{B}_{01}(\lambda_0))\Bigg) \\
&- \lambda_0 A'_0(\lambda_0)\Bigg(w_1 \underbrace{B_{01}(\lambda_0)}_{=0} + w_2 \frac{i}{\sqrt{\lambda_0}} \tilde{B}_{01}(\lambda_0)\Bigg)\Bigg) i \\
&= 2A_0(\lambda_0) B_{11}(\lambda_0) - A_1(\lambda_0) \underbrace{B_{01}(\lambda_0)}_{=0}
\end{aligned}
$$

and

$$
\begin{aligned}
&\Bigg(2\lambda_0 A_0(\lambda_0)\Bigg(w_1 B'_{0k}(\lambda_0) + w_2 \frac{i}{\sqrt{\lambda_0}} \tilde{B}'_{0k}(\lambda_0))\Bigg) \\
&- A_0(\lambda_0)\Bigg(w_1 \underbrace{B_{02}(\lambda_0)}_{=0} + w_2 \frac{i}{\sqrt{\lambda_0}} \tilde{B}_{02}(\lambda_0)\Bigg) \\
&- \lambda_0 A'_0(\lambda_0)\Bigg(w_1 \underbrace{B_{02}(\lambda_0)}_{=0} + w_2 \frac{i}{\sqrt{\lambda_0}} \tilde{B}_{02}(\lambda_0))\Bigg)\Bigg) i \\
&= 2A_0(\lambda_0) B_{12}(\lambda_0) - A_1(\lambda_0) \underbrace{B_{02}(\lambda_0)}_{=0}.
\end{aligned}
$$

This gives us

$$
\begin{aligned}
B_{11}(\lambda_0) = &\Bigg(2\lambda_0 A_0(\lambda_0)\Bigg(w_1 B'_{01}(\lambda_0) + w_2 \frac{i}{\sqrt{\lambda_0}} \tilde{B}'_{01}(\lambda_0))\Bigg) \\
&- A_0(\lambda_0) w_2 \frac{i}{\sqrt{\lambda_0}} \tilde{B}_{01}(\lambda_0)) - \lambda_0 A'_0(\lambda_0) w_2 \frac{i}{\sqrt{\lambda_0}} \tilde{B}_{01}(\lambda_0))\Bigg) i (2A_0(\lambda_0))^{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
B_{12}(\lambda_0) = &\Bigg(2\lambda_0 A_0(\lambda_0)\Bigg(w_1 B'_{0k}(\lambda_0) + w_2 \frac{i}{\sqrt{\lambda_0}} \tilde{B}'_{0k}(\lambda_0))\Bigg) \\
&- A_0(\lambda_0) w_2 \frac{i}{\sqrt{\lambda_0}} \tilde{B}_{02}(\lambda_0) - \lambda_0 A'_0(\lambda_0) w_2 \frac{i}{\sqrt{\lambda_0}} \tilde{B}_{02}(\lambda_0))\Bigg) i (2A_0(\lambda_0))^{-1}
\end{aligned}
$$

Therefore, $B_{11}(\lambda_0) \neq 0$ and $B_{12}(\lambda_0) \neq 0$. Furthermore, we also have $B_{11}(\lambda_0) \neq B_{12}(\lambda_0)$. At $\lambda = \lambda_0$ (6.4) becomes

$$
C_{01}(\lambda_0) B_{12}(\lambda_0) - C_{02}(\lambda_0) B_{11}(\lambda_0) = \lambda_0 Q_{11} A_0(\lambda_0) \tag{6.10}
$$

Since the left-hand side does not vanish we obtain $Q_{11} \neq 0$. For simplicity we choose $Q_{11} = 1$.

Thus, Q now simply has the form $Q(\lambda) = t\lambda$. The reasoning above shows that the level sets of T intersect transversely with $\mathcal{S}_2$. Looking again at (6.1) now gives us

$$\begin{aligned}&(C_{01}(\lambda)+C_{11}(\lambda)t+\dots)(B_{02}(\lambda)+B_{12}(\lambda)t+\dots)\\&-(C_{02}(\lambda)+C_{12}(\lambda)t+\dots)(B_{01}(\lambda)+B_{11}(\lambda)t+\dots)\\&=\lambda(A_0(\lambda)t+A_1(\lambda)t^2+\dots).\end{aligned}\tag{6.11}$$

Equating coefficients gives us

$$C_{01}B_{02}-C_{02}B_{01}=0$$

fort t^0 and

$$C_{01}B_{12}+C_{11}B_{02}-C_{02}B_{11}-C_{12}B_{01}=\lambda A_1$$

for t^1. In general, we obtain

$$\sum_{k=0}^{n}\frac{C_{k1}}{k!}\frac{B_{(n-k)2}}{(n-k)!}-\sum_{l=0}^{n}\frac{C_{k2}}{k!}\frac{B_{(n-k)1}}{(n-k)!}=\frac{\lambda A_n}{n!}$$

for t^n. Using the solution for C_{01} and C_{02} we obtain

$$(w_{11}B_{01}+w_{21}\frac{B_{01}}{\lambda_0-\lambda})B_{02}-(w_{12}B_{02}+w_{22}\frac{B_{02}}{\lambda_0-\lambda})B_{01}=0$$

fort t^0.

Looking at (4.2) and (4.3) we obtain

$$\begin{aligned}&(2\lambda\sum_{j=0}^{\infty}\frac{A_j(\lambda)}{j!}t^j\sum_{j=0}^{\infty}\frac{C'_{j1}(\lambda)}{j!}t^j-\sum_{j=0}^{\infty}\frac{A_j(\lambda)}{j!}t^j\sum_{j=0}^{\infty}\frac{C_{j1}(\lambda)}{j!}t^j-\lambda\sum_{j=0}^{\infty}\frac{A'_j(\lambda)}{j!}t^j\sum_{j=0}^{\infty}\frac{C_{j1}(\lambda)}{j!}t^j\\&=2\sum_{j=0}^{\infty}\frac{A_j(\lambda)}{j!}t^j\sum_{j=1}^{\infty}\frac{B_{j1}(\lambda)}{(j-1)!}t^j-\sum_{j=1}^{\infty}\frac{A_j(\lambda)}{(j-1)!}t^{j-1}\sum_{j=0}^{\infty}\frac{B_{j1}(\lambda)}{j!}t^j\end{aligned}\tag{6.12}$$

and

$$\begin{aligned}&(2\lambda\sum_{j=0}^{\infty}\frac{A_j(\lambda)}{j!}t^j\sum_{j=0}^{\infty}\frac{C'_{j2}(\lambda)}{j!}t^j-\sum_{j=0}^{\infty}\frac{A_j(\lambda)}{j!}t^j\sum_{j=0}^{\infty}\frac{C_{j2}(\lambda)}{j!}t^j-\lambda\sum_{j=0}^{\infty}\frac{A'_j(\lambda)}{j!}t^j\sum_{j=0}^{\infty}\frac{C_{j2}(\lambda)}{j!}t^j\\&=2\sum_{j=0}^{\infty}\frac{A_j(\lambda)}{j!}t^j\sum_{j=1}^{\infty}\frac{B_{j2}(\lambda)}{(j-1)!}t^j-\sum_{j=1}^{\infty}\frac{A_j(\lambda)}{(j-1)!}t^{j-1}\sum_{j=0}^{\infty}\frac{B_{j2}(\lambda)}{j!}t^j.\end{aligned}\tag{6.13}$$

Equating coefficients we obtain the following for t^0

$$\begin{aligned}&(2\lambda A_0(\lambda)C'_{0k}(\lambda)-A_0C_{0k}(\lambda)-\lambda A'_0(\lambda)C_{0k}(\lambda))i\\&=2A_0(\lambda)B_{1k}(\lambda)-A_1(\lambda)B_{0k}(\lambda)\end{aligned}\tag{6.14}$$

and for t^1

$$\begin{aligned}&(2\lambda(A_1(\lambda)C'_{0k}(\lambda) + A_0(\lambda)C'_{1k}(\lambda)) - (A_1(\lambda)C_{0k}(\lambda) + A_0(\lambda)C_{1k}(\lambda))\\&- \lambda(A'_1(\lambda)C_{0k}(\lambda) + A'_0(\lambda)C_{1k}(\lambda)))i\\&= 2(A_1(\lambda)B_{1k}(\lambda) + A_0(\lambda)B_{2k}(\lambda)) - (A_2(\lambda)B_{0k}(\lambda) + A_1(\lambda)B_{1k}(\lambda)).\end{aligned}$$

Therefore, we get

$$\begin{aligned}&\frac{1}{2A_0(\lambda)}(((2\lambda(A_1(\lambda)C'_{0k}(\lambda) + A_0(\lambda)C'_{1k}(\lambda)) - (A_1(\lambda)C_{0k}(\lambda) + A_0(\lambda)C_{1k}(\lambda))\\&- \lambda(A'_1(\lambda)C_{0k}(\lambda) + A'_0(\lambda)C_{1k}(\lambda)))i - 2A_1(\lambda)B_{1k}(\lambda) + (A_2(\lambda)B_{0k}(\lambda) + A_1(\lambda)B_{1k}(\lambda)))\\&= B_{2k}(\lambda).\end{aligned}$$

The solution set for t^0 consists of a quintuple $(C_{01}, C_{02}, B_{11}, B_{12}, A_1)$ and is proportional to a solution $(c_1, c_2, \dot{b}_1, \dot{b}_2, \dot{a})$ of the equations in (4.2) in the case where b_1 and b_2 have a common root. Both solutions only differ by a real multiple. This multiple can be denoted as θ. Therefore, equation (6.10) becomes

$$\theta^2(c_1(\lambda_0)\dot{b}_2(\lambda_0) - c_2(\lambda_0)\dot{b}_1(\lambda_0)) = \lambda_0 Q_{11} A_0(\lambda_0). \tag{6.15}$$

With this equation and the solutions from (4.2) it is possible to solve for θ. This then gives us a solution $(C_{01}, C_{02}, B_{11}, B_{12}, A_1)$. It would now be possible to construct a inductive formula for all the other components $(C_{k1}, C_{k2}, B_{k+1,1}, B_{k+1,2}, A_{k+1})$. However, this calculation is omitted due to a time constraint. Nevertheless, the result obtained so far in this chapter ($Q_{11} \neq 0$) can be used to get a new result about the Willmore energy.

6.1 Willmore energy

It is possible to connect the result we just obtained with the Willmore energy. The Willmore energy is a common measure in differential geometry. It measures the curving energy of an embedded surface. It is defined as follows.

Definition 6.2 (Willmore energy)**.** *Let Σ be a smooth, embedded, compact and oriented surface in $\mathbb{R}^3$. Let H denote its mean curvature. Then the Willmore energy W is defined as*

$$W(\Sigma) = \int_\sigma H^2 dA,$$

where dA is the induced volume form.

In [KHS17] the relation between a polynomial $a \in \mathcal{M}_2^1 \cup \mathcal{M}_2^2 \cup \mathcal{M}_2^3$ and a conformal immersion $f_a : \mathbb{C}/\tilde{\Gamma}_a \to \mathbb{H}$ is explained. Which makes it possible to look at the Willmore energy in dependence of the polynomial a. They found the following theorem.

Theorem 6.3. *For all $a \in \mathcal{M}_2^1 \cup \mathcal{M}_2^2 \cup \mathcal{M}_2^3$ the Willmore energy of f_a is equal to*

$$W(a) = \int_{\mathbb{C}/\hat{\Gamma}} 4\gamma^2 dx \wedge dy = \int_{\mathbb{C}/\tilde{\Gamma}} 8\gamma^2 dx \wedge dy = 4iRes_{\lambda=0} \log(\mu_2) d\log(\mu_1). \tag{6.16}$$

We know that $Q_1 = \lambda t$ which gives us that the sign of Q_1 changes around zero. Differentiating the expression in (6.16) with respect to t now gives us

$$\begin{aligned} \dot{W}(a) &= 4i\text{Res}_{\lambda=0}\dot{\log(\mu_2)} d\log(\mu_1) - \dot{\log(\mu_1)} d\log(\mu_2) && (6.17)\\ &= 4i\text{Res}_{\lambda=0}\frac{c_2 b_1 - c_1 b_2}{\nu^2}\frac{d\lambda}{\lambda} && (6.18)\\ &= 4i\text{Res}_{\lambda=0}\frac{Q_{11}\lambda a}{\lambda a}\frac{d\lambda}{\lambda} && (6.19)\\ &= 4i\text{Res}_{\lambda=0} t\frac{d\lambda}{\lambda}. && (6.20) \end{aligned}$$

Therefore, Q_1 is proportional to the derivative of the Willmore functional. Thus, we get that the monotony of the Willmore functional changes around zero. For the function $f = \frac{b_2}{b_1}$ it is possible to see that the index changes. However, this step is not included in this thesis due to time limitations.

7 Conclusion

In this final chapter we summarize the findings of this work and point to possible future research.

At the beginning we revisited some differential geometric theory to deduce the sinh-Gordon equation and the 2×2 matrices existential for the current research on CMC tori. Later on we altered the map $T : \mathcal{M}_2^1 \to \mathcal{F}$ from [KHS17] in a way that we got a mapping

$$\hat{T} : (a, b_1, b_2) \mapsto \tau_a.$$

Since $\hat{T}$ uses on the tripel (a, b_1, b_2) it was now possible to use the Whitham equations from [CS16].

In a first step we determined that the level sets of T for $a \in \mathcal{M}_2^1$ are one-dimensional submanifolds, i.e. the Whitham equations yield a one-dimensional set of solutions.

In a second step we examined the case of $a \in \mathcal{M}_2^2 \cup \mathcal{M}_2^3$. In this case a singularity appears. At first we tried to remove the singularity with local integrals. This approach led to even more complex equations. Therefore, we tried an equivalent approach through introducing additional equations to the Whitham equations. With these additional equations we were able to show that the set of solutions is again one-dimensional.

In a third step we examined a special case in which the polynomials b_k have a common root. This case leads to the derivative of the Willmore functional changing its sign.

Further research can be done on the third step. It is possible to use the imaginary part of q_1 and q_2 to construct a closed curve on $\lambda \in \mathbb{S}^1$ in the real plane. It might then be possible to show that between two points of a connected component of the level sets of T the number of points intersection points of the curve changes. Furthermore, it might be possible to determine existence and number of such points on the levels sets through investigating the double points along the level set.

Bibliography

[Ale62] Aleksandr D Aleksandrov. "Uniqueness theorems for surfaces in the large". In: *Amer. Math. Soc. Transl.(2)* 21 (1962), pp. 341–354.

[BF09] Rolf Busam and Eberhard Freitag. *Complex analysis*. Springer, 2009.

[Bob11] Alexander I Bobenko. "Introduction to compact Riemann surfaces". In: *Computational Approach to Riemann Surfaces*. Springer, 2011, pp. 3–64.

[Bob91] Alexander Ivanovich Bobenko. "Constant mean curvature surfaces and integrable equations". In: *Russian Mathematical Surveys* 46.4 (1991), pp. 1–45.

[CS16] Emma Carberry and Martin Ulrich Schmidt. "The prevalence of tori amongst constant mean curvature planes in R3". In: *Journal of Geometry and Physics* 106 (2016), pp. 352–366.

[FKR06] Shoichi Fujimori, Shimpei Kobayashi, and Wayne Rossman. "Loop group methods for constant mean curvature surfaces". In: *arXiv preprint math/0602570* (2006).

[For12] Otto Forster. *Lectures on Riemann surfaces*. Vol. 81. Springer Science & Business Media, 2012.

[Hoe15] Ricardo Pena Hoepner. *Solutions of the Sinh-Gordon Equation of Spectral Genus Two*. 2015. URL: https://www.wim.uni-mannheim.de/media/Lehrstuehle/wim/schmidt/Paper/pena_hoepner_masterarbeit.pdf.

[Hop03] Heinz Hopf. *Differential geometry in the large: seminar lectures New York University 1946 and Stanford University 1956*. Vol. 1000. Springer, 2003.

[KHS17] Markus Knopf, Ricardo Pena Hoepner, and Martin U. Schmidt. "Solutions of the Sinh-Gordon Equation of Spectral Genus Two and constrained Willmore Tori I". In: (2017). arXiv: 1708.00887 [math.DG].

[Kle20] Sebastian Klein. *Funktionentheorie 2*. 2020. URL: https://www.wim.uni-mannheim.de/media/Lehrstuehle/wim/schmidt/FSS2020/Funktionentheorie_II/FT2.pdf.

[Neu08] Vania Neugebauer. *Constant Mean Curvature Tori in the 3-Hyperbolic Space*. 2008. URL: https://www.wim.uni-mannheim.de/media/Lehrstuehle/wim/schmidt/Paper/neugebauer_diplom.pdf.

[PS89] Ulrich Pinkall and Ivan Sterling. "On the classification of constant mean curvature tori". In: *Annals of Mathematics* 130.2 (1989), pp. 407–451.

[Sch17] Benedikt Schmidt. *The Closure of Spectral Curves of Constant Mean Curvature Tori of Spectral Genus 2*. 2017. URL: https://www.wim.uni-mannheim.de/media/Lehrstuehle/wim/schmidt/Paper/schmidt_benedikt_arbeit.pdf.

[Wen86] Henry Wente. "Counterexample to a conjecture of H. Hopf". In: *Pacific Journal of Mathematics* 121.1 (1986), pp. 193–243.

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