Bachelor’s Thesis

The Wente Family

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Abstract

The Wente Family is the space of polynomials in $\mathcal{S}_1^2$ [1] whose coefficients are all real valued. The behaviour under the Whitham deformation can be described by a vector field. We explicitly constructed the vector field and tried to prove uniqueness of the maximal integral curves.

[1]Definition 3.4

Contents

1 Introduction

A subject in differential geometry is the construction of tori of constant mean curvature (CMC tori). These tori are described by the solutions of the sinh-Gordon equation

$$\Delta u + sinh(2u) = 0$$

where Δu means the Laplace operator of u. We only consider the solutions of finite type. These solutions can be described by the space of potentials, which is a space of matrix polynomials. The determinants of these matrices are called spectral curves and are of the following form

$$y^2 = \lambda a(\lambda) = (-1)^g \lambda \prod_{j=1}^{g} \frac{\eta_j}{|\eta_j|}(\lambda - \eta_j)(\lambda - \bar{\eta}_j^{-1})$$

Here g is the genus of the spectral curve. In the following we will consider the spectral curves of genus 2. The polynomials $a(\lambda)$ define the space $\mathcal{H}^g$ of spectral curves of genus g. The submanifold $\mathcal{S}_1^2$ has a subset of polynomials with all real coefficients, called the Wente Family. Any polynomial in the Wente Family is of the form $a(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_1\lambda + 1$. Every $a(\lambda)$ defines a two-dimensional vector space $\mathcal{B}_a$.
We will use the explicit form of $a(\lambda)$ and b_1, b_2 to solve the Whitham equations and construct a vector field that maps the coefficients (a, b_1, b_2) to $(\dot{a}, \dot{b}_1, \dot{b}_2)$.
In chapter four we will try to show uniqueness of the maximal integral curves by examining the roots of the vector field constructed before.

2 Preliminaries

Definition 2.1. Let X be a two-dimensional manifold. A complex chart on X is a homeomorphism $\phi : U \to V$ on an open set $U \subset X$ to an open set $V \subset \mathbb{C}$. Two complex charts $\phi_i : U_i \to V_i, i = 1, 2$ are called biholomorphic compatible, if the transition map $\phi_2 \circ \phi_1^{-1} : \phi_1[U_1 \cap U_2] \to \phi_2[U_1 \cap U_2]$ is biholomorphic.

Definition 2.2. A complex atlas is a system $\mathfrak{A} = \{\phi_i : U_i \to V_i, i \in I\}$ of pairwise biholomorphic compatible charts with $\bigcup_{i \in I} U_i = X$.

Definition 2.3. A complex structure on a two-dimensional manifold is an equivalence relation of biholomorphic equivalent atlases on X.

Definition 2.4. A Riemann Surface is a pair (X, Σ) where X is a connected, two-dimensional manifold and Σ is a complex structure.

Definition 2.5. We will define the real projective space. Therefore we will first define an equivalence relation on $\mathbb{R}^{n+1} \setminus \{0\}$:
$x \sim y :\Leftrightarrow \exists \lambda \in \mathbb{R} \setminus \{0\} : x = \lambda y$
We define $\mathbb{R}P^n := (\mathbb{R}^{n+1} \setminus \{0\}) / \sim$ as the real projective space.

Definition 2.6. We will now define the process of Blowing Up.
Let $\Delta \subset \mathbb{C}^n$ be a disc. We will look at the following coordinates: $z = (z_1, ...z_n) \in \Delta, l = [l_1, ..., l_n] \in \mathbb{C}P^{n-1}$. Let $\tilde{\Delta} \subset \Delta \times \mathbb{C}P^{n-1}$ be the submanifold given by

$$\tilde{\Delta} = \{(z, l) \in \Delta \times \mathbb{C}P^{n-1} | z_i l_j = z_j l_i \forall i, j\}$$

We now define a mapping

$$\begin{aligned} \pi : \tilde{\Delta} &\to \Delta \\ (z, l) &\mapsto z \end{aligned}$$

This is an isomorphism $\forall z \neq 0$ in Δ and $\pi^{-1}[0] = \{0\} \times \mathbb{C}P^{n-1}$. We call $(\tilde{\Delta}, \pi)$ the blow up of Δ at 0.

3 Spectral curves of CMC tori in $\mathbb{R}^3$

Constant mean curvature tori of genus one surfaces in $\mathbb{R}^3$ can be described by spectral data in a way that every such immersion corresponds to a quintuple $(X, \lambda, \rho, \lambda_0, L)$ where X is a spectral curve, which is a special kind of algebraic curve, λ is a degree two meromorphic function which has branches at $x_0 = \lambda^{-1}\{0\}$ and $x_\infty = \lambda^{-1}\{\infty\}$. ρ is an anti-holomorphic involution that has the set $\mathbb{S}^1$ as fixed points. λ_0 is a so called sym point in $\mathbb{S}^1$ and finally L is a quaternionic line bundle.

Definition 3.1. A polynomial of degree n is said to satisfy the reality condition if the following equation holds

$$\lambda^n \overline{f(\overline{\lambda}^{-1})} = f(\lambda) \tag{1}$$

The space of these polynomials of degree n is called $P^n_{\mathbb{R}}$

We define $\mathcal{H}^g = \{a(\lambda) \mid a \text{ is a spectral curve of a CMC immersion of finite type}\} \subset P^{2g}$. These polynomials $a(\lambda)$ satisfy

(i) the reality condition

(ii) $\frac{a(\lambda)}{\lambda^g} \geq 0$

(iii) the highest coefficient of a has absolute value 1

(iv) the roots of a are pairwise distinct, meaning X_a is smooth

X_a is described by the following equation

$$y^2 = \lambda a(\lambda) = (-1)^g \lambda \prod_{j=1}^{g} \frac{\eta_j}{|\eta_j|}(\lambda - \eta_j)(\lambda - \bar{\eta}_j^{-1}) \tag{2}$$

Definition 3.2. $\forall a \in \mathcal{H}^g$ we define $\mathcal{B}_a$ as the real two-dimensional space of polynomials of degree $g+1$ with $b \in P^{g+1}_{\mathbb{R}}$ and

$$\Theta_b := \frac{b(\lambda)\mathrm{d}\lambda}{\lambda y}$$

has purely imaginary periods

Definition 3.3. $\forall (a, \lambda_0) \in \mathcal{H}^g \times \mathbb{S}^1$ $\exists$ linearly independent $b_1, b_2 \in \mathcal{B}_a$, μ_1, μ_2 functions on X_a satisfying

(i) $\log(\mu_1)$ and $\log(\mu_2)$ are holomorphic on $X_a \setminus \{x_0, x_\infty\}$ and on $\{x_0, x_\infty\}$ they have simple poles and linearly independent residues.

(ii) $\Theta_{b_1} = \mathrm{d}\log(\mu_1)$, $\Theta_{b_2} = \mathrm{d}\log(\mu_2)$

(iii) $\mu_1(\lambda_0) = \mu_2(\lambda_0) = \pm 1$

(iv) $b_1(\lambda_0) = b_2(\lambda_0) = 0$

Definition 3.4. (i) $\mathcal{S}^g_{\lambda_0} := \{a \in \mathcal{H}^g | \forall b \in \mathcal{B}_a : b(\lambda_0) = 0 \text{ holds}\}$

(ii) $\mathcal{P}^2_{\lambda_0} := \{a \in \mathcal{H}^g | X_a \text{ is the spectral curve of a CMC torus in } \mathbb{R}^3\}$ and $\mathcal{P}^g_{\lambda_0} \subset \mathcal{S}^g_{\lambda_0}$

Definition 3.5.

$$\mathcal{P}^g = \bigcup_{\lambda_0 \in \mathbb{S}^1} \mathcal{P}^g_{\lambda_0}$$

$$\mathcal{S}^g = \bigcup_{\lambda_0 \in \mathbb{S}^1} \mathcal{S}^g_{\lambda_0}$$

We will consider the genus g $= 2$ and $\lambda_0 = 1$ in this work. As mentioned in the introduction, the solutions of the sinh-Gordon equation

$$\Delta u + sinh(2u) = 0 \tag{3}$$

describe CMC tori. For genus 2 as we consider the solution of this equation is defined on the space of potentials which we will now define.

Definition 3.6.

$$\mathcal{P}^2 = \{\zeta_\lambda = \begin{pmatrix} \alpha\lambda - \overline{\alpha}\lambda^2 & -\gamma^{-1} + \beta\lambda - \gamma\lambda^2 \\ \gamma\lambda - \overline{\beta}\lambda^2 + \gamma^{-1}\lambda^3 & -\alpha\lambda + \overline{\alpha}\lambda^2 \end{pmatrix}, \alpha, \beta \in \mathbb{C}, \gamma \in \mathbb{R}^+\}$$

is called the set of potentials

Every ζ_λ satisfies reality condition, as shown in Hoepner (2015).
The determinant of these matrices will describe our spectral curve X_a as in (2) where g is now 2. X_a is a Riemann Surface.

Definition 3.7. The Wente family is the subset of $\mathcal{S}^2_1$ where all coefficients are real and can be written as $\{a(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_1\lambda + 1 | a_1, a_2 \in \mathbb{R}\}$. For all these $a(\lambda)$, we get two linearly independent polynomials in $\mathcal{B}_a$ of the form $b_1 = (\lambda - 1)^2(\lambda + 1)$ and $ib_2 = i(\lambda - 1)(\lambda^2 + \beta\lambda + 1)$. Since $\mathcal{B}_a$ has dimension two, those form a basis.

4 Vector Field

In this chapter we will use the results of chapter 4 of B.Schmidt (2017) to construct a vector field that describes how the Wente Family is affected by Whitham deformation. This will describe the mapping $(a_1, a_2, \beta, k_1, k_2) \mapsto (\dot{a}_1, \dot{a}_2, \dot{\beta}, \dot{k}_1, \dot{k}_2)$. The latter are the tangent vectors at $t = 0$. We get 3 equations which we will solve using Mathematica to get the explicit forms of our tangent vectors.

As seen in Definition 3.7 we have $a(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_1\lambda + 1$ and $b_1 = k_1(\lambda - 1)^2(\lambda + 1)$, $ib_2 = ik_2(\lambda - 1)(\lambda^2 + \beta\lambda + 1)$. From B.Schmidt (2017) we get

$$\mathrm{d}\dot{q}_k = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \Theta_{b_k}, \quad k = 1, 2$$

where $\dot{q}_k$ are meromorphic functions on X_a.
The functions $\dot{q}_k$ have the following form

$$\dot{q}_k = \frac{ic_k(\lambda)}{y}, \quad k = 1, 2$$

where $c_k(\lambda) \in P^3_{\mathbb{R}}, \quad k = 1, 2$ and the y is from (2).
The three equations from B.Schmidt (2017) are now

$$(2\lambda a c_1' - a c_1 - \lambda a' c_1)i = 2a\dot{b}_1 - \dot{a}b_1 \tag{4}$$

$$(2\lambda a c_2' - a c_2 - \lambda a' c_2)i = 2ai\dot{b}_2 - \dot{a}ib_2 \tag{5}$$

$$c_1 b_2 - c_2 b_1 = Qa, \quad Q \in P^2_{\mathbb{R}} \tag{6}$$

First we will try to determine the coefficients of $c_1(\lambda), c_2(\lambda)$ so we can eliminate them from our equations.
Since $\dot{b}_1 = \dot{k}_1(\lambda - 1)^2(\lambda + 1)$ the right side of (4) vanishes at ± 1 and we get

$$(2a(1)c_1'(1) - a(1)c_1(1) - a'(1)c_1(1)) = 0 \tag{7}$$

$$(-2a(-1)c_1'(-1) - a(-1)c_1(-1) + a'(-1)c_1(-1)) = 0 \tag{8}$$

We will now use Mathematica to solve these equations with the following code.

```
a[x_] = x^4 + a1*x^3 + a2*x^2 + a1*x + 1;
c1[x_] = c13*(x)^3 + c12*(x)^2 + c11*(x)+ c10;
pol1[x_] = Simplify[2*x*a[x]*D[c1[x],x] - a[x]*c1[x] - x*D[a[x],x]*c1[x]];
Lsg1 = Solve[pol1[1] == 0, pol1[-1] == 0, c12, c11];
```

The polynomials $c_1(\lambda), c_2(\lambda)$ are here described as polynomials with arbitrary coefficients: $c_1(\lambda) = c_{13}\lambda^3 + c_{12}\lambda^2 + c_{11}\lambda + c_{10}, c_2(\lambda) = c_{23}\lambda^3 + c_{22}\lambda^2 + c_{21}\lambda + c_{20}$ with $c_{ij} \in \mathbb{C} \forall i, j$ Doing so gives us the result $c_{12} = 3c_{10} \wedge c_{11} = 3c_{13}$.
We will now do the same with $c_2(\lambda)$ but since $\dot{b}_2(\lambda) = i\dot{k}_2(\lambda-1)(\lambda^2+\beta\lambda+1)+ik_2(\lambda-1)\dot{\beta}\lambda$ we can see that b_2 and $\dot{b}_2$ only have one common root at -1 so we only get one equation. We will again use Mathematica to solve this.

```
c2[x_] = c23*(x)^3 + c22*(x)^2 + c21*(x)+ c20;
pol2[x_] = Simplify[2*x*a[x]*D[c2[x],x] - a[x]*c2[x] - x*D[a[x],x]*c2[x]];
Lsg2 = Solve[pol2[1] == 0, c21];
```

That yields in $c_{21} = -3c_{20} + c_{22} + 3c_{23}$.
Since we now have used all conditions on the roots of our polynomials, we have to calculate our equations (4) - (6) and hope to get some new results for our polynomials $c_1(\lambda)$ and $c_2\lambda)$.
We will again use Mathematica for these calculations:

```
b1dot[x_] = k1dot*(x-1)^2*(x+1);
adot[x_] = a1dot*x^3 + a2dot*x^2 + a1dot*x;
c12 = 3*c10;   c11 = 3*c13;
Dgl1 = (2*x*a[x]*D[c1[x],x] - a[x]*c1[x] - x*D[a[x],x]*c1[x])*I -
(2*a[x]*b1dot[x] - adot[x]*b1[x]);
ListDgl1 = Simplify[CoefficientList[Dgl1,x]];
```

We get the following equation

$$
\begin{aligned}
&- ic_{10}\lambda^7 + (-2ia_1c_{10} + ic_{11})\lambda^6 + (-3ia_2c_{10} + 3ic_{12})\lambda^5 + \\
&(-4ia_1c_{10} - ia_2c_{11} + 2ia_1c_{12} + 5ic_{13})\lambda^4 + (-5ic_{10} + ia_2c_{12} - 2a_1c_{11} + 4ia_1c_{13})\lambda^3 \\
&+ (-3ic_{11} + 3ia_2c_{13})\lambda^2 + (-ic_{12} + 2ia_1c_{13})\lambda + ic_{13} = \\
&2\dot{k}_1\lambda^7 + (2a_1\dot{k}_1 - \dot{a}_1k_1 - 2\dot{k}_1)\lambda^6 + (\dot{a}_1k_1 - \dot{a}_2k_1 - 2\dot{k}_1 - 2a_1\dot{k}_1 + 2a_2\dot{k}_1)\lambda^5 \\
&(\dot{a}_2k_1 + 2\dot{k}_1 - 2a_2\dot{k}_1)\lambda^4 + (\dot{a}_2k_1 + 2\dot{k}_1 - 2a_2\dot{k}_1)\lambda^3 \\
&+ (\dot{a}_1k_1 - \dot{a}_2k_1 - 2\dot{k}_1 - 2a_1\dot{k}_1 - 2a_1\dot{k}_1 + 2a_2\dot{k}_1)\lambda^2 \\
&+ (2a_1\dot{k}_1 - 2\dot{k}_1 - \dot{a}_1k_1)\lambda + 2\dot{k}_1
\end{aligned}
$$

By equating coefficients, we extract 8 new equations which we will use to describe $c_1(\lambda)$ further and later construct our vector field.

I $2\dot{k}_1 = -ic_{10}$

II $2a_1\dot{k}_1 - \dot{a}_1k_1 - 2\dot{k}_1 = -2ia_1c_{10} + ic_{11}$

III $\dot{a}_1k_1 - \dot{a}_2k_1 - 2\dot{k}_1 - 2a_1\dot{k}_1 + 2a_2\dot{k}_1 = -3ia_2c_{10} + 3ic_{12}$

IV $\dot{a}_2k_1 + 2\dot{k}_1 - 2a_2\dot{k}_1 = -4ia_1c_{10} - ia_2c_{11} + 2ia_1c_{12} + 5ic_{13}$

V $\dot{a}_2k_1 + 2\dot{k}_1 - 2a_2\dot{k}_1 = -5ic_{10} + ia_2c_{12} - 2a_1c_{11} + 4ia_1c_{13}$

VI $\dot{a}_1k_1 - \dot{a}_2k_1 - 2\dot{k}_1 - 2a_1\dot{k}_1 - 2a_1\dot{k}_1 + 2a_2\dot{k}_1 = -3ic_{11} + 3ia_2c_{13}$

VII $2a_1\dot{k}_1 - 2\dot{k}_1 - \dot{a}_1k_1 = -ic_{12} + 2ia_1c_{13}$

VIII $2\dot{k}_1 = ic_{13}$

Since the equations I and $VIII$ have the same term on the left side, we see that $c_{13} = -c_{10}$. Therefore, we can now describe $c_1(\lambda)$ with only one coefficient:
$c_1(\lambda) = c_{13}(\lambda^3 - 3\lambda^2 + 3\lambda - 1)$

We will now do the same with $c_2(\lambda)$, where we only have one condition as of now. We use similar coding to extract equation (5)

```
b2dot[x_] = I*(k2dot*(x-1)*(x^2 + b*x +1) + k2*(x-1)*bdot*x);
c21 = -3*c20 + c22 + 3*c23;
Dgl2 = (2*x*a[x]*D[c2[x],x] - a[x]*c2[x] - x*D[a[x],x]*c2[x])*I -
(2*a[x]*b2dot[x] - adot[x]*b2[x]);
ListDgl2 = Simplify[CoefficientList[Dgl2,x]];
```

The result is the full version of (5)

$$
\begin{aligned}
&- ic_{20}\lambda^7 - i((3 + 2a_1)c_{20} - c_{22} - 3c_{23})\lambda^6 + i(-3a_2c_{20} + 3c_{22})\lambda^5 \\
&- i(a_2(c_{22} - 3c_{20} + 3c_{23}) - 5c_{23} + 2a_1(2c_{22} - c_{22}))\lambda^4 + i((6a_1 - 5)c_{20} + a_2c_{22} - 2a_1(c_{22} + c_{23}))\lambda^3 \\
&+ i(9c_{20} - 3c_{22} - 9c_{23} + 3a_2c_{23})\lambda^2 + -i(c_{22} - 2a_1c_{23})\lambda + ic_{23} = \\
&- 2i\dot{k}_2\lambda^7 + i(k_2(\dot{a}_1 - 2\dot{\beta}) + \dot{k}_2(2 - 2a_1 - 2\beta))\lambda^6 \\
&- i(k_2(\dot{a}_1 - \dot{a}_2 - \dot{a}_1\beta - 2\dot{\beta} + 2a_1\dot{\beta}) + \dot{k}_2(2 - 2a_1 + 2a_2 - 2\beta + 2a_1\beta))\lambda^5 \\
&+ i(k_2(2\dot{a}_1 - \dot{a}_2 - \dot{a}_1\beta + \dot{a}_2\beta + 2a_1\dot{\beta} - 2a_2\dot{\beta}) + \dot{k}_2(2 + 2a_2 - 4a_1 - 2\beta a_2 + 2\beta a_1))\lambda^4 \\
&- i(k_2(2\dot{a}_1 - \dot{a}_2 - \dot{a}_1\beta + \dot{a}_2\beta - 2a_2\dot{\beta} + 2a_1\dot{\beta}) + \dot{k}_2(2 + 2a_2 - 4a_1 - 2a_2\beta + 2a_1\beta)\lambda^3 \\
&- i(k_2(-\dot{a}_1 + \dot{a}_2 + \dot{a}_1\beta - 2a_1\dot{\beta} + 2\dot{\beta}) + \dot{k}_2(-2 + 2a_1 - 2a_2 + 2\beta - 2a_1\beta))\lambda^2 \\
&+ i(k_2(-\dot{a}_1 + 2\dot{\beta}) + \dot{k}_2(-2 + 2a_1 + 2\beta))\lambda + 2i\dot{k}_2
\end{aligned}
$$

Again we equate coefficients and get

I $-ic_{20} = -2i\dot{k}_2$

II $-i((3 + 2a_1)c_{20} - c_{22} - 3c_{23}) = i(k_2(\dot{a}_1 - 2\dot{\beta}) + \dot{k}_2(2 - 2a_1 - 2\beta))$

III $i(-3a_2c_{20}+3c_{22}) = -i(k_2(\dot{a}_1-\dot{a}_2-\dot{a}_1\beta-2\dot{\beta}+2a_1\dot{\beta})+\dot{k}_2(2-2a_1+2a_2-2\beta+2a_1\beta))$

IV $-i(a_2(c_{22}-3c_{20}+3c_{23})-5c_{23}+2a_1(2c_{22}-c_{22})) = i(k_2(2\dot{a}_1-\dot{a}_2-\dot{a}_1\beta+\dot{a}_2\beta+2a_1\dot{\beta}-2a_2\dot{\beta})+\dot{k}_2(2+2a_2-4a_1-2\beta a_2+2\beta a_1))$

V $i((6a_1-5)c_{20}+a_2c_{22}-2a_1(c_{22}+c_{23})) = -i(k_2(2\dot{a}_1-\dot{a}_2-\dot{a}_1\beta+\dot{a}_2\beta-2a_2\dot{\beta}+2a_1\dot{\beta})+\dot{k}_2(2+2a_2-4a_1-2a_2\beta+2a_1\beta$

VI $i(9c_{20}-3c_{22}-9c_{23}+3a_2c_{23}) = -i(k_2(-\dot{a}_1+\dot{a}_2+\dot{a}_1\beta-2a_1\dot{\beta}+2\dot{\beta})+\dot{k}_2(-2+2a_1-2a_2+2\beta-2a_1\beta))$

VII $-i(c_{22}-2a_1c_{23}) = i(k_2(-\dot{a}_1+2\dot{\beta})+\dot{k}_2(-2+2a_1+2\beta))$

VIII $ic_{23} = 2i\dot{k}_2$

We see that that the left side of I is the left side of $-VIII$ which yields $c_{23}i = -(-c_{20}i)$, therefore it follows $c_{23} = c_{20}$
We can see that there are no other conditions we can use to describe $c_2(\lambda)$ more exact, so we will now make use of (6). The goal is to have $c_2(\lambda)$ and $c_1(\lambda)$ depend from only one variable.

In B.Schmidt (2017) we see the exact values of $Q(1), Q'(1), Q''(1)$ which we will use to calculate $Q(\lambda)$ by using the Taylor series at $\lambda = 1$. The values are

$$Q(1) = 0$$
$$Q'(1) = \frac{c_1(1)b_2'(1) - c_2(1)b_1'(1)}{a(1)}$$
$$Q''(1) = \frac{[c_1(1) - \frac{a'(1)}{a(1)}c_1(1)]b_2'(1) + c_1(1)b_2''(1) - [c_2(1) - \frac{a'(1)}{a(1)}c_2(1)]b_1'(1) - c_2(1)b_1''(1)}{a(1)}$$

We will now calculate $Q(\lambda) = Q(1) + Q'(1)(\lambda-1) + \frac{Q''(1)}{2}(\lambda-1)^2$ with Mathematica and the results we already have.

```
Q1'[x_] = (c1[x]*D[b2[x],x] - c2[x]*D[b1[x],x])/a[x];
Q1"[x_] = ((c1[x] - D[a[x],x]/a[x]*c1[x])*D[b2[x],x] +
c1[x]*D[D[b2[x],x],x] - (c2[x] - D[a[x],x]/a[x]*c2[x])*D[b1[x],x] -
c2[x]*D[D[b1[x],x],x])/a[x];
Q1"[1]
```

We see that $Q'(1) = 0$ because c_1 has a root at $\lambda = 1$ and b_1 has a double root at $\lambda = 1$ which means that b_2' also has a root there. Hence, we can calculate $Q(\lambda)$ fairly easy

```
Q[x_] = (Q1"[1]*(x-1)^2)/2;
Dgl3 = c1[x]*b2[x] - c2[x]*c1[x] - Q[x]*a[x];
ListDgl3 = Simplify[CoefficientList[Dgl3,x]];
Lsg5 = Solve[ListDgl3 == {0,0,0,0,0,0,0}, {c22, k1, k2, c23, a1, a2, b}];
```

That yields in the following results

$$
\begin{aligned}
c_{22} &= \frac{-i(2 - 2a_1 - 3a_2 - 2\beta + 2a_1\beta + a_2\beta)c_{13}k_2}{(-2 + 2a_1 - a_2)k_1} \\
c_{23} &= \frac{i(6 + 2a_1 - a_2 - 4\beta)c_{13}k_2}{(-2 + 2a_1 - a_2)k2}
\end{aligned}
$$

So now we have the following polynomials

$$
\begin{aligned}
&c_1(\lambda) = c_{13}(\lambda^3 - 3\lambda^2 + 3\lambda - 1) \\
&c_2(\lambda) = c_{13}(\frac{i(6 + 2a_1 - a_2 - 4\beta}{-2 + 2a_1 - a_2}\lambda^3 - \frac{i(2 - 2a_1 - 3a_2 - 2\beta + 2a_1\beta + a_2\beta)k_2}{(-2 + 2a_1 - a_2)k_1}\lambda^2 \\
&- \frac{i(2 - 2a_1 - 3a_2 - 2\beta + 2a_1\beta + a_2\beta)k_2}{(-2 + 2a_1 - a_2)k_1}\lambda + \frac{i(6 + 2a_1 - a_2 - 4\beta}{-2 + 2a_1 - a_2})
\end{aligned}
$$

who both only depend on $c_{13} \in \mathbb{C}$. We will now prove a theorem found in M.U.Schmidt (2018) about the integral curves of vector fields.

Theorem 4.1. *V is a finite dimensional Banach space, $X \subset V$ open.*
Let $F : X \to V$ be a locally Lipschitz continuous vector field, $f : X \to \mathbb{K}$ be a continuous function.

(i) If $\exists C_1 > C_2 > 0$ s.t. $C_1 > f > C_2 > 0$ every maximal integral curve of fF is the composition of the maximal integral curves of F with bijective maps from the respective intervals.

(ii) If f is locally Lipschitz continuous, fF is locally Lipschitz continuous.

(iii) If f is locally Lipschitz continuous, F and fF are complete vector fields and $\exists C_1 > C_2 > 0$ s.t. $C_1 > f > C_2 > 0$ the respective dynamical systems have the same trajectories.

Proof. (i) Let $t_0 \in \mathbb{R}$ be arbitrary, $q_0 \in X$ and $\gamma_1(t) : (\alpha, \beta) \to X$ a maximal integral curve of fF for the initial value $\gamma_1(t_0) = q_0$. We now search a real valued map $\phi : (\alpha, \beta) \to \mathbb{R}$. For the initial value exists a unique integral curve $\gamma_2 : (a, b) \to X$ s.t.

$\dot{\gamma}_2 = F(\gamma_2(t))$. We now search a function $\phi : (\alpha, \beta) \to \mathbb{R}$ where $\gamma_1 = \gamma_2 \circ \phi$ holds. We will now differentiate both sides to get a differential equation for ϕ.

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma_2(\phi(t)) = \dot{\gamma}_2(\phi(t))\dot{\phi(t)} = F(\gamma_2(\phi(t))\dot{\phi}(t) = F(\gamma_1(t))f(\gamma_1(t)) = \dot{\gamma}_2(t)$$

So we get the differential equation

$$\dot{\phi}(t) = f(\gamma_1(t))$$

Since the right hand side is integrable, we know that there exists such a function $\phi(t)$ that is continuous. Since we know $\dot{\phi}(t) = f(\gamma_1(t)) > C_2 > 0$ we know that $\phi(t)$ is injective. We also know that the image of a connected space under a continuous function is connected. Therefore $\phi[(\alpha, \beta)] \subset \mathbb{R}$ is connected and as every connected subspace of $\mathbb{R}$ an interval. So ϕ is a bijective continuous map that maps an interval to an interval and therefore homeomorphic. We will call the image of $\phi[(\alpha, \beta)] = (a, b)$ and show that it is the maximal existence interval for the initial value problem γ_2 solves. Since γ_1 is a maximal integral curve we know that on the boundaries, one of three things can occur:

(i) $\alpha = -\infty, \beta = \infty$

(ii) $t \to ||f(\gamma_1(t))F(\gamma_1(t))||$ is unbounded for small $\epsilon > 0$ on $(\alpha, \alpha + \epsilon)$, $(\beta - \epsilon, \beta)$

(iii) The curve $\gamma_1(t)$ can be extended continuously to $[\alpha, \beta)$, $(\alpha, \beta]$ but $\lim_{t\downarrow\alpha}(t, \gamma_1(t)) \notin \mathbb{R} \times X$, $\lim_{t\uparrow\beta}(t, \gamma_1(t)) \notin \mathbb{R} \times X$.

We will only show what happens for the lower boundaries. We start with (i). If $\alpha = -\infty$, the interval has no lower bound. We know that $(\phi^{-1})' = \frac{1}{\phi'}$ and hence $0 < \dot{\phi}^{-1} < C_2^{-1}$. Therefore, ϕ^{-1} is bijective and Lipschitz continuous because the derivative is bounded. So ϕ^{-1} is a bijective, Lipschitz continuous map from (a, b) to $(-\infty, \beta)$. Therefore (a, b) has to have no lower bound as well and we get $a = -\infty$.
(ii) We assume that $t \to ||f(\gamma_1(t))F(\gamma_1(t))||$ is unbounded for small $\epsilon > 0$ on $(\alpha, \alpha + \epsilon)$. Therefore, we know that $||f(\gamma_1(t))F(\gamma_1(t))||$ is unbounded on $(\alpha, \alpha + \epsilon)$. We also know that $||f(\gamma_1(t))F(\gamma_1(t))|| < C_1||F(\gamma_1(t)|| = C_1||F(\gamma_2(\phi(t))||$. So we know that $||F(\gamma_2(t))||$ is unbounded on $\phi[(\alpha, \alpha+\epsilon)]$. Since ϕ is monotonous, we also know that that is an interval of the form $(a, a + \tilde{\epsilon})$. So (ii) holds for γ_2 as well.
(iii) We see that the condition (iii) can equivalently be seen as that the curve $\gamma_1(t)$ can be extended continuously to $[\alpha, \beta)$, but $\lim_{t\downarrow\alpha} \gamma_1(t) \notin X$. We want to show that the same thing will happen to $\gamma_2 = \gamma_1 \circ \phi$. We know that ϕ is a bijective continuous mapping from an interval to an interval and therefore monotonous. That means that $\lim_{t\downarrow\alpha} \phi(t) = a \wedge \lim_{t\uparrow\beta} \phi(t) = b$. Therefore we know $\lim_{t\downarrow a} \gamma_2(t) = \lim_{t\downarrow a} \gamma_1(\phi(t)) = \lim_{t\downarrow\alpha} \gamma_1(t) \notin X$. Therefore the condition (iii) holds for γ_2 as well. So we see that γ_2 also has to be a maximal integral curve, completing the proof.

(ii) Let $x \in V$ arbitrary. Since V is a finite dimensional Banach space, it is also locally compact with respect to the norm topology. We choose U_f, U_F, K_x as neighborhoods of x s.t. U_f is the neighborhood where f is Lipschitz continuous, U_F is the same with F and K_x is a compact neighborhood of x. Therefore we get

$$||F(x) - F(y)|| \leq L_1 ||x - y|| \quad \forall x, y \in U_F$$
$$||f(x) - f(y)|| \leq L_2 ||x - y|| \quad \forall x, y \in U_f$$

with $L_1, L_2 \geq 0$. So now we will look at $x, y \in U_f \cap U_F \cap K_x$. We get

$$\begin{aligned}
&||f(x)F(x) - f(y)F(y)|| = ||f(x)F(x) - f(y)F(y) + f(y)F(x) - f(y)F(x)|| \leq \\
&||f(x)F(x) - f(y)F(x)|| + ||f(y)F(x) - f(y)F(y)|| \leq ||F(x)|| \cdot ||f(x) - f(y)|| + \\
&||f(y)|| \cdot ||F(x) - F(y)|| \leq ||F(x)|| \cdot L_2 ||x - y|| + ||f(y)|| \cdot L_1 ||x - y|| \leq \\
&(L_2 \cdot \sup_{x \in U_f \cap U_F \cap K_x} ||F(x)|| + L_1 \cdot \sup_{y \in U_f \cap U_F \cap K_x} ||f(y)||)||x - y||
\end{aligned}$$

We know that $U_f \cap U_F \cap K_x \subset K_x$ is obviously a subset of a compact space we can look at the supremum on K_x instead of a subset, where both f and F as continuous functions have a maximum instead of a supremum. Therefore we get

$$\begin{aligned}
&(L_2 \cdot \sup_{x \in U_f \cap U_F \cap K_x} ||F(x)|| + L_1 \cdot \sup_{y \in U_f \cap U_F \cap K_x} ||f(y)||)||x - y|| \leq (L_2 \cdot \sup_{x \in K_x} ||F(x)|| + \\
&L_1 \cdot \sup_{x \in K_x} ||f(x)||)||x - y|| = (L_2 \cdot \max_{x \in K_x} ||F(x)|| + L_1 \cdot \max_{x \in K_x} ||f(x)||)||x - y|| = \\
&(L_1 \cdot c_2 + L_2 \cdot c_1)||x - y|| \quad \forall x, y \in U_f \cap U_F \cap K_x := U_{fF}
\end{aligned}$$

We can now define $L_{fF} = L_1 \cdot c_2 + L_2 \cdot c_1$ and get

$$||f(x)F(x) - f(y)F(y)|| \leq L_{fF} ||x - y|| \quad \forall x, y \in U_{fF}$$

Since we constructed this for an arbitrary x, fF is locally Lipschitz continuous.
(iii) From (ii) we know that fF is locally Lipschitz continuous. Take $t_0 \in \mathbb{R}$ arbitrary. The images of the maximal integral curves of fF, F for t_0 are the trajectories $Orb(\Phi_{fF}, t_0)$ and $Orb(\Phi_F, t_0)$. Since both vector fields are locally Lipschitz continuous, both integral curves $\gamma_1 : I_1 \to X$ for fF and $\gamma_2 : I_2 \to X$ for F are unique. We will use the same approach as in (i) and get the equation

$$\dot{\phi}(t) = f(\gamma_1(t)) > C_2 > 0$$

Therefore, we have a monotonous function ϕ that again maps the intervals on which the integral curves are defined. In the same way as in (i) we get

$$(\phi^{-1})' = \frac{1}{\phi'}, \quad 0 < C_2^{-1} < \frac{1}{\phi'} < C_1^{-1}$$

meaning that ϕ^{-1} is Lipschitz continuous and the derivative is bounded and can't go to zero. Therefore, $\phi^{-1}[I_1]$ can only be unbounded, if I_1 is unbounded and $\phi^{-1}[I_1]$ has to be unbounded if I_1 is unbounded, because ϕ is monotonous and the derivative is bounded by a constant greater than zero. Therefore, $\phi^{-1}[\mathbb{R}]$ has to be the whole space $\mathbb{R}$ as well, meaning ϕ maps the maximal integral curves of F to the maximal integral curves of fF, allowing us to write the orbits as the images of our integral curves. We get

$$Orb(\Phi_{fF}, t_0) = \gamma_2[\mathbb{R}] = \gamma_1 \circ \phi[\mathbb{R}] = \gamma_1[\mathbb{R}] = Orb(\Phi_F, t_0)$$

Therefore, they have the same trajectories. $\square$

We already mentioned in the beginning that in the Wente family every polynomial has only real valued coefficients. We also know that β is real valued. If we look at our equations (4) and (5) we see that both left sides are complelety imaginary. $a(\lambda), \dot{a}(\lambda), b_1(\lambda), \dot{b}_1(\lambda)$ are all completely real valued, which means the left side in the brackets has to be completely imaginary. Since $c_1(\lambda)$ or $c_1'(\lambda)$ appear in every coefficient, this means $c_1(\lambda)$ must be imaginary, so we get $c_{13} \in i\mathbb{R}$. Looking at (5) we see that $b_2(\lambda)$, $\dot{b}_2(\lambda)$ are completely imaginary, making the whole right side imaginary. So both sides are completely imaginary if $c_2(\lambda)$ is real valued, which is equivalent to $c_{13} \in i\mathbb{R}$ as well. Hence, we choose $c_{13} = i$, and for every other version of our vector field we only need to multiply F with an $\lambda \in \mathbb{R}$ to get the new vector field for another coefficient $\tilde{c}_{13}$. With our completely calculated polynomials we will now go back to the equations (4) and (5) to calculate our vector field.

```
c22 = -I*((2-2*a1-3*a2-2*b+2*a1*b+a2*b)*c13*k2)/((-2+2*a1-a2)*k1);
c23 = I*((6+2*a1-a2-4*b)*c13*k2)/((-2+2*a1-a2)*k2);
c13 = I;
Dgl1 = (2*x*a[x]*D[c1[x],x] - a[x]*c1[x] - x*D[a[x],x]*c1[x])*I -
    (2*a[x]*b1dot[x] - adot[x]*b1[x]);
Dgl2 = (2*x*a[x]*D[c2[x],x] - a[x]*c2[x] - x*D[a[x],x]*c2[x])*I -
    (2*a[x]*b2dot[x] - adot[x]*b2[x]);
ListDgl1 = Simplify[CoefficientList[Dgl1,x]];
ListDgl2 = Simplify[CoefficientList[Dgl2,x]];
Lsg6 = Solve[ListDgl1 == {0,0,0,0,0,0,0,0},
    {a1dot, a2dot, k1dot, k2dot, k1, k2, a1, a2}];
Lsg7 = Solve[ListDgl2 == {0,0,0,0,0,0,0,0},
    {k2dot, bdot, a1dot, a2dot, k2, k1dot, k1, a1}];
```

Those calculations yield the following solutions:

$$\dot{k}_1 = -\frac{1}{2} \tag{9}$$

$$\dot{k}_2 = \frac{-6 - 2a_1 + a_2 + 4\beta}{2(-2 + 2a_1 - a_2)} \tag{10}$$

$$\dot{a}_1 = \frac{(4 + a_1)}{k_1} \tag{11}$$

$$\dot{a}_2 = \frac{2(a_1 + a_2 - 2)}{k_1} \tag{12}$$

$$\dot{\beta} = \frac{-8 + 6\beta + 2a_1\beta - a_2\beta - 2\beta^2}{(-2 + 2a_1 - a_2)k_1} \tag{13}$$

So our vector field F maps the quintuple $(k_1, k_2, a_1, a_2, \beta)$ to these solutions. One can easily see that the last three variables depend on k_1, k_2 the same way. We will now try to multiplicate F with a function f s.t. fF is a polynomial vector field. Therefore we will define

$$\begin{aligned} &f : \mathbb{R}^5 \to \mathbb{R} \\ &\begin{pmatrix} k_1 & k_2 & a_1 & a_2 & \beta \end{pmatrix} \mapsto k_1(-2 + 2a_1 - a_2) \end{aligned}$$

That leaves us with

$$\begin{aligned} &fF : \mathbb{R}^5 \to \mathbb{R}^5 \\ &\begin{pmatrix} k_1 \\ k_2 \\ a_1 \\ a_2 \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} \frac{k_1(-2+2a_1-a_2)}{2} \\ \frac{k_1(-6-2a_1+a_2+4\beta)}{2} \\ (4 + a_1)(-2 + 2a_1 - a_2) \\ 2(a_1 + a_2 - 2)(-2 + 2a_1 - a_2) \\ -8 + 6\beta + 2a_1\beta - a_2\beta - 2\beta^2 \end{pmatrix} \end{aligned}$$

We want to justify this approach using Theorem 4.1. But since we can't prove that fF and F are complete, we can't use Theorem 4.1 (iii). Instead we will try to use Theorem 4.1 (i). Later, we are only interested in the vector field in certain points and their neighborhoods. Therefore, we can restrict $\mathbb{R}^5$ to compact neighborhoods of these points. There we know that f will have a definite sign in these special neighborhoods, and since f is continuous we see that there are C_1, C_2 s.t. either $0 < C_1 < f < C_2$ or $0 < -C_1 < -f < -C_2$ hold and we can then use Theorem 4.1 (i). So we know that the integral curves are homeomorphic. Since the last three variables don't depend on the first two, we will now only look at the last three and try to parametrize the vector field in an easier way. We can easily see that the term $(-2 + 2a_1 - a_2)$ appears both in $\dot{a}_1$ and $\dot{a}_2$ because we multiplied them with that term. One can also see that this term is $-a(-1)$ so we will now look at $a_1, -a(-1)$ instead of a_1, a_2. We get the equations

$\tilde{a} = -2 + 2a_1 - a_2$ and similar $a_2 = -2 + 2a_1 - \tilde{a}$ which we will use to calculate $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \tilde{a}$. We know $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \tilde{a} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (-2 + 2a_1 - a_2) = 2\dot{a}_1 - \dot{a}_2$. We will again use Mathematica to calculate $\tilde{F}$

```
a1dot = (4+a1)/k1;
a2dot = 2*(-2+a1+a2)/k1;
k1dot = -1/2;
atilde = -2 + 2*a1 - a2;
atildedot = Simplify[2*a1dot - a2dot,x];
c13 = k1*atildeneu*I;
a2dot = 2*a1dot - atildedot;
a1dot = 4*atildeneu + a1*atildeneu;
```

We now see that our new vector field has a much easier form

$$
\begin{aligned}
&f\tilde{F} : \mathbb{C}^3 \to \mathbb{C}^3 \\
&\begin{pmatrix} a_1 \\ \tilde{a} \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} \tilde{a}(a_1 + 4) \\ 2\tilde{a}(8 - 2a_1 + \tilde{a}) \\ -(2\beta^2 - (8 - \tilde{a})\beta + 8) \end{pmatrix}
\end{aligned}
$$

5 Stability Analysis

In this chapter we will examine the vector field of chapter 4. We will try to show uniqueness of the maximal integral curves. Therefore, we will try to see whether the integral curves are monotonous and look at roots of our vector field and how the integral curves behave there. We will also examine the boundaries of the integral curves and look in which ways the flows escape the Wente family there.

We will examine the roots of our vector field because they are fixed points of the corresponding dynamical system. There we will examine whether the fixed points are hyperbolic, because we want to use theorem 2.40 about hyperbolic fixed points and stable and unstable manifolds found in chapter 2.6 of M.U.Schmidt (2018). In a first step we will look at the last version of our vector field $f\tilde{F}$ and calculate every root using Mathematica

```
a2 = -2 + 2*a1 - atildeneu
atildedot = Simplify[2*a1dot - a2dot,x];
c13 = k1*atildeneu*I;
a2dot = 2*a1dot - atildedot;
atildedot = 2*(8*atildeneu - 2*a1*atildeneu+atildeneu^2);
bdot = -I*(8 - 8*b - atildeneu*b + 2*b^2);
Erg = Solve[{a1dot, atildedot, bdot} == {0,0,0}, {a1, atildeneu, b}];
```

We see that the only roots of this are $(a_1, \tilde{a}, \beta) = (-4, -16, -2)$ and $(x, 0, 2), x \in \mathbb{R}$. We see that the second root is a whole linear subspace. We will now examine the first root we found and try to apply the principle of linearized stability so we will examine $\nabla F(-4, -16, -2)$.

$$\nabla f\tilde{F}(a_1, \tilde{a}, \beta) = \begin{pmatrix} \tilde{a} & 4 + a_1 & 0 \\ -4\tilde{a} & 2(8 - 2a_1 + 2\tilde{a}) & 0 \\ 0 & \beta & -(4\beta - 8 - \tilde{a}) \end{pmatrix}$$

Therefore, we get

$$\nabla f\tilde{F}(-4, -16, -2) = \begin{pmatrix} -16 & 0 & 0 \\ 64 & -32 & 0 \\ 0 & -2 & 0 \end{pmatrix}$$

One can easily see that the eigenvalues of $\nabla f\tilde{F}(-4, -16, -2)$ are -16, -32 and 0. Therefore $(-4, -16, -2)$ is not a hyperbolic fixed point and we have to try something else to examine this root. First we will again use a different parametrization for our vector field. We already substituted a_2 with $-a(-1)$, and now we will substitute a_1 with

$a(1)$. Since this is only a linear transformation, we can't expect the roots to change in their behaviour. Using Mathematica to transform our vector field, we get the following formula

$$\begin{aligned}\dot{\tilde{a}}_1 &= -2\tilde{a}_1\tilde{a}_2\\ \dot{\tilde{a}}_2 &= -\tilde{a}_2(\tilde{a}_1+\tilde{a}_2-16)\\ \dot{\beta} &= -(2\beta^2-(8-\tilde{a}_2)\beta+8)\end{aligned}$$

The new roots of our vector field are $(x,0,2), x\in\mathbb{R}$ and $(0,16,-2)$. Since the first two tangent vectors don't depend on β and we didn't change the third, we still get an eigenvalue of zero. Therefore, we will now blow up the root $(0,16,-2)$. To do so we first must parametrize our vector field so that our root is at $(0,0,0)$. So we just look at $(\tilde{a}_1, \tilde{a}_2+16, \beta-2)$ and we get

$$\begin{aligned}\dot{\tilde{a}}_1 &= -2\tilde{a}_1(\tilde{a}_2+16)\\ \dot{\tilde{a}}_2 &= -(\tilde{a}_2+16)(\tilde{a}_1+\tilde{a}_2)\\ \dot{\beta} &= -(2(\beta-2)^2-(8-(\tilde{a}_2+16))(\beta-2)+8)\end{aligned}$$

If we look at the definition of the blow up, we see that $\tilde{\Delta}$ is a manifold. The condition $z_i l_j = z_j l_i \quad \forall i,j$ means that if $l_j \neq 0$ we get $z_i = \frac{l_i}{l_j}z_j$. Therefore, if there exists a coordinate where l does not vanish we see that our coordinates are $(z,l)=(\lambda l, l)$. We will now define $U_i = \{y\in\mathbb{P}^{n-1}|y_i\neq 0\} = \{y\in\mathbb{P}^{n-1}|y_i=1\}\ \forall i=1,\dots n$. Now we will define a chart $\phi_j : U_j \to \mathbb{C}^{n-1}$. If we take $y\in U_i$ we can map $y\to\frac{1}{y_i}(y_1,\dots y_n) = (\frac{y_1}{y_i},\dots,1,\dots,\frac{y_n}{y_n})$. If we take an arbitrary $[y]\in U_i$ and do the calculation we used before, we get $(y_1,\dots,1,\dots y_n)\in[y]$ where $y_i\in\mathbb{C}\quad\forall i$. Therefore, we can identify $[y]\in\mathbb{P}^{n-1}$ with an element $(y_1,\dots,y_{i-1},y_{i+1},\dots,y_n)\in\mathbb{C}^{n-1}$. So we get a map

$$\begin{aligned}\phi_i :&U_i\to\mathbb{C}^{n-1}\\ &y\to(y_1,\dots,y_{i-1},y_{i+1},\dots,y_n)\end{aligned}$$

That is obviously a chart. We can therefore construct a chart from $\mathbb{C}^n\times U_i$ as $\mathbb{1}_{\mathbb{C}^n}\times\phi_i$. We obviously get n charts which we will consider. The next step is to look at our vector field F in these charts. For ϕ_i we get the parametrization of an $x=y_i(y_1,\dots,1\dots,y_n)$ where we set $x_i=y_i$ and $x_j=y_iy_j \quad \forall i\neq 0$. If we now look at the derivatives, we get the following using the product rule:

$$\begin{aligned}\dot{x}_i &= \dot{y}_i\\ \dot{x}_j &= \dot{y}_iy_j+y_i\dot{y}_j \quad \forall j\neq i\end{aligned}$$

We will now solve the second equation because our vector field $(\dot{x}_1,\dots\dot{x}_n)$ is mapped to $(\dot{y}_1\dots\dot{y}_n)$. We here get the equation

$$\dot{y}_j = \frac{\dot{x}_j-\dot{y}_iy_j}{y_i}$$

If we solve these equations in every chart, we get the blow up of our vector field. Since we blew up the origin, we are only looking at roots of this new vector field in the so called exceptional fibre which means the projective space we replaced the origin with. Therefore, we have to set $y_i = 0$ before we look at roots.
For U_1 we get the new parameters $y = (y_1, y_2, y_3)$

$$\begin{aligned} \tilde{a}_1 &= y_1 \\ \tilde{a}_2 &= y_1 y_2 \\ \beta &= y_1 y_3 \end{aligned}$$

That means we receive the derivatives

$$\begin{aligned} \dot{y}_1 &= \dot{\tilde{a}}_1(y) \\ \dot{y}_2 &= \frac{\dot{\tilde{a}}_2(y) - y_2 \dot{y}_1}{y_1} \\ \dot{y}_3 &= \frac{\dot{\beta}(y) - y_3 \dot{y}_1}{y_1} \end{aligned}$$

If we use Mathematica for that we have the following code

```
a1dotneu = -2*a1tilde*(a2tilde+16);
a2dotneu = -(a2tilde+16)*(a1tilde + a2tilde);
bdotneu = -(2(b-2)^2 + (a2tilde-8+16)*(b-2) + 8);
a1tilde = y1;
a2tilde = y1*y2;
b = y1*y3;
y1dot = a1dotneu;
y2dot = Simplify[(a2dotneu - y1dot*y2)/y1];
y3dot = Simplify[(bdotneu - y1dot*y3)/y1];
y1 = 0;
Solve[y2dot, y3dot == 0,0, y2,y3]
```

With that we get the vector field

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{pmatrix} = \begin{pmatrix} -2y_1(16 + y_1 y_2) \\ (y_2 - 1)(16 + y_1 y_2) \\ -2y_3(y_1 y_3 - 16) + y_2(2 + y_1 y_3) \end{pmatrix}$$

With setting $y_1 = 0$ we get $\dot{y}_1 = 0, \dot{y}_2 = 16(y_2 - 1), \dot{y}_3 = 2y_2 + 32y_3$. Therefore, our only root in this chart is $(0, 1, -\frac{1}{16})$.

We will now do the same in ϕ_2. Here we get the coordinates

$$\begin{aligned}\tilde{a}_1 &= y_1 y_2\\ \tilde{a}_2 &= y_2\\ \beta &= y_3 y_2\end{aligned}$$

That means we get the derivatives

$$\begin{aligned}\dot{y}_1 &= \frac{\dot{\tilde{a}}_1(y) - y_1\dot{y}_2}{y_2}\\ \dot{y}_2 &= \dot{\tilde{a}}_2\\ \dot{y}_3 &= \frac{\dot{\beta}(y) - y_3\dot{y}_2}{y_2}\end{aligned}$$

We now use Mathematica to get the explicit forms

```
a1dotneu = -2*a1tilde*(a2tilde+16);
a2dotneu = -(a2tilde+16)*(a1tilde + a2tilde);
bdotneu = -(2(b-2)^2 + (a2tilde-8+16)*(b-2) + 8);
a2tilde = y2;
a1tilde = y1*y2;
b = y3*y2;
y2dot = Simplify[a2dotneu];
y1dot = Simplify[(a1dotneu - y2dot*y1)/y2];
y3dot = Simplify[(bdotneu - y2dot*y3)/y2];
y2 = 0;
Solve[y1dot, y3dot == 0,0, y1,y3]
```

The vector field has now in this chart the form

$$\begin{pmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\end{pmatrix} = \begin{pmatrix}(y_1-1)y_1(16+y_2)\\ -(y_1+1)y_2(16+y_2)\\ 2+(16+y_1(16+y_2))y_3 - 2y_2y_3^2\end{pmatrix}$$

By setting $y_2 = 0$ we get $\dot{y}_1 = 16(y_1-1)y_1, \dot{y}_2 = 0, \dot{y}_3 = 2 + (16+16y_1)y_3$. That yields in the roots $(0,0,-\frac{1}{8})$ and $(1,0,-\frac{1}{16})$.
Now we will consider the final chart ϕ_3. We have the coordinates

$$\begin{aligned}\tilde{a}_1 &= y_1 y_3\\ \tilde{a}_2 &= y_2 y_3\\ \beta &= y_3\end{aligned}$$

and the derivatives

$$\dot{y}_1 = \frac{\dot{\tilde{a}}_1(y) - y_1\dot{y}_3}{y_3}$$
$$\dot{y}_2 = \frac{\dot{\tilde{a}}_2(y) - y_2\dot{y}_3}{y_3}$$
$$\dot{y}_3 = \dot{\beta}(y)$$

With the following Mathematica code we receive the new vector field

```
a1dotneu = -2*a1tilde*(a2tilde+16);
a2dotneu = -(a2tilde+16)*(a1tilde + a2tilde);
bdotneu = -(2(b-2)^2 + (a2tilde-8+16)*(b-2) + 8);
a1tilde = y1*y3;
a2tilde = y2*y3;
b = y3;
y3dot = bdotneu;
y1dot = Simplify[(a1dotneu - y3dot*y1)/y3];
y2dot = Simplify[(a2dotneu - y3dot*y2)/y3];
y3 = 0;
Solve[y1dot, y2dot == 0,0, y1,y2]
```

which has the form

$$\begin{pmatrix}\dot{y}_1\\ \dot{y}_2\\ \dot{y}_3\end{pmatrix} = \begin{pmatrix} -y_1(32 - 2y_3 + y_2(2+y_3))\\ -2y_2(8+y_2-y_3) - y_1(16+y_2y_3)\\ -8 - 2(y_3-2)^2 - (y_3-2)(y_2y_3+8)\end{pmatrix}$$

By again setting now $y_3 = 0$ we get $\dot{y}_1 = -y_1(32+2y_2), \dot{y}_2 = -16y_1 - 2y_2(8+y_2), \dot{y}_3 = 0$ and therefore the roots $(-16,-16,0)$, $(0,-8,0)$, $(0,0,0)$. Note that we will not consider $(0,0,0)$, since it is not a direction in the projective space.

So now we know every critical point of our vector field after blowing up. We will now again try to use the theorem about hyperbolic fixed points to show that there is only direction to flow out of the critical points. We start with the points found in ϕ_1:

First we will calculate the Jacobi matrix of $F \circ \phi_1$

$$\nabla(F\circ\phi_1) = \begin{pmatrix} -2y_1y_2 - 2(16+y_1y_2) & -2y_1^2 & 0\\ (y_2-1)y_2 & 16 + y_1(y_2-1) + y_1y_2 & 0\\ y_2y_3 - 2y_3^2 & 2 + y_1y_3 & y_1y_2 - 2y_1y_3 - 2(y_1y_3 - 16)\end{pmatrix}$$

If we now look at the value of our Jacobi matrix, we get

$$\begin{pmatrix} -32 & 0 & 0\\ 0 & 16 & 0\\ -\frac{9}{128} & 2 & 32\end{pmatrix}$$

One can easily see that the eigenvalues are $\lambda_1 = -32, \lambda_2 = 16, \lambda_3 = 32$ with the corresponding eigenvectors $v_1 = \begin{pmatrix} \frac{8192}{9} \\ 0 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 8 \\ -1 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Now since we have a hyperbolic fixed point we can separate the stable and the unstable manifold which are defined as

$$W_s(x_0) = \{x \in V | t \to \phi_f(t,x) \text{ exists for a } t \in [0,\infty) \text{ and } \lim_{t\to\infty} \phi_f(t,x) = x_0\}$$
$$W_u(x_0) = \{x \in V | t \to \phi_f(t,x) \text{ exists for a } t \in (-\infty,0] \text{ and } \lim_{t\to-\infty} \phi_f(t,x) = x_0\}$$

We can only flow out of the exceptional fibre if the coordinate in which we use the chart is non zero, which is only the case for the stable manifold. Since this manifold is one dimensional, there is only one unique way out of the exceptional fibre in this root.
Now we will look at the roots found in ϕ_2. Our Jacobi matrix has the form

$$\nabla(F \circ \phi_2) = \begin{pmatrix} (2y_1-1)(16+y_2) & y_1(1-y_1) & 0 \\ -y_2(16+y_2) & -(y_1+1)(16+2y_2) & 0 \\ (16+y_2)y_3 & y_1y_3 - 2y_3^2 & 16 + y_1(16+y_2) - 4y_2y_3 \end{pmatrix}$$

If we look at the Jacobi matrix at the roots of our vector field, we get

$$\nabla(F \circ \phi_2)(1,0,-\frac{1}{16}) = \begin{pmatrix} 16 & 0 & 0 \\ 0 & -32 & 0 \\ -1 & -\frac{9}{128} & 32 \end{pmatrix}$$

We can easily see that the eigenvalues are the same as in the first chart and the root also has a similar form suggesting that we look at the same point in different charts. We get the eigenvectors $v_1 = \begin{pmatrix} 0 \\ \frac{8192}{9} \\ 1 \end{pmatrix}$ for $\lambda_1 = -32$ and $v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ for $\lambda_2 = 32$ and $v_3 = \begin{pmatrix} 16 \\ 0 \\ 1 \end{pmatrix}$ for $\lambda_3 = 16$ yielding the same results as the root in chart 1. Now we examine the second root in chart 2

$$\nabla(F \circ \phi_2)(0,0,-\frac{1}{8}) = \begin{pmatrix} -16 & 0 & 0 \\ 0 & -16 & 0 \\ -2 - \frac{1}{32} & 16 & \end{pmatrix}$$

yielding the eigenvalues and eigenvectors $\lambda_1 = -16$ with $v_1 = \begin{pmatrix} 16 \\ 0 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -\frac{1}{64} \\ 1 \\ 0 \end{pmatrix}$, $\lambda_2 = 16, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Note that $\lambda_1 = -16$ is a double root of the characteristic polynomial,

which is why the Eigenspace is two dimensional. We see that y_2 only occurs in the stable manifold, which is two dimensional. So there are two ways in which we can leave the exceptional fibre. So we don't have the result we searched.
We will now also examine the roots in the third chart

$$\nabla(F\circ\phi_3) = \begin{pmatrix} -32+2y_3-y_2(2+y_3) & -y_1(2+y_3) & -y_1(y_2-2) \\ y_2y_3-16 & -2y_2-2(8+y_2-y_3)-y_1y_3 & 2y_2-y_1y_2 \\ 0-(y_3-2)y_3 & -8-(4+y_2)(y_3-2)-y_2y_3 & \end{pmatrix}$$

Looking at the first root we get

$$\nabla(F\circ\phi_3)(-16,-16,0) = \begin{pmatrix} 0 & 32 & -288 \\ -16 & 48 & -288 \\ 0 & 0 & -32 \end{pmatrix}$$

We see that the eigenvalues and eigenvectors are $\lambda_1 = -32, v_1 = \begin{pmatrix} 9 \\ 9 \\ 2 \end{pmatrix}, \lambda_2 = 32, v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \lambda_3 = 16, v_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. We also see that our fixed point is hyperbolic and that the component 3 only occurs in the stable manifold, which is one dimensional. Therefore, there is only one way out of the exceptional fibre in this root.
We will now examine the final root

$$\nabla(F\circ\phi_3)(0,-8,0) = \begin{pmatrix} -16 & 0 & 0 \\ -16 & 16 & -16 \\ 0 & 0 & -16 \end{pmatrix}$$

We see that the eigenvalues are $\lambda_1 = -16, v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \lambda_3 = 16, v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.
We note that the fixed point is hyperbolic and that the first eigenvalue is a double eigenvalue, hence the eigenspace is two dimensional. The third coordinate occurs only in the first space, therefore there are two ways out of the exceptional fibre. We see that the algebraic blow up didn't help us to show uniqueness of the maximal integral curves of our vector field since there are still fixed points where our integral curves can leave the space in more than one way.

6 Conclusion

In this work we have constructed a vector field that describes the Wente family under the Whitham deformation and we showed that the Whitham deformation leaves the Wente family invariant, since our vector field F was real valued.
After that we tried to show that the vector field we contructed has only one maximal integral curve, which we coulnd't accomplish. Now in two next steps one could try to show that every integral curve of the Wente family flows into the root of F $(0, 16, 2)$ we found in chapter 5. One could also try to prove that the Wente family only consists of one integral curve like we tried to prove by using the condition that the meromorphic differentials Θ_{b_i} have purely imaginary periods.

7 References

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