Chapter 5

Wave Equation

In this final chapter we consider the homogeneous and inhomogeneous wave equation $\frac{\partial^2 u}{\partial t^2} - \triangle u = 0$ and $\frac{\partial^2 u}{\partial t^2} - \triangle u = f$ on open subsets of $\mathbb{R}^n \times \mathbb{R}$. The wave equation is a linear second order PDE. The coefficient matrix for the second derivatives has one positive and n negative eigenvalues and is neither definite nor semi definite. In the second chapter we introduced this differential equation as the simplest hyperbolic differential equation. The method which solves this PDE is completely different from the methods which solve the Laplace equation or the heat equation. The wave equation describes phenomena which propagate with finite speed through space time. The example of electrodynamic waves motivated the investigation of this equation. Later these methods were generalised to non linear hyperbolic equations in order to describe gravitational waves.

5.1 D'Alembert's Formula for $n = 1$

First we solve the following initial value problem:

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 && && \text{for} \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+ \\ &\quad u(x,0) = g(x) && \frac{\partial u}{\partial t}(x,0) = h(x) && \text{for} \quad x \in \mathbb{R}, \end{aligned}$$

for given functions g and h on $\mathbb{R}$. If we consider this PDE as an ODE on the vector space of smooth functions on $\mathbb{R}$, then the theory of ODEs suggests that we should fix the initial value $u(\cdot, t_0)$ and the first derivative $\frac{\partial u}{\partial t}(\cdot, t_0)$ with respect to t for a given initial time t_0. Here the initial values g and h are exactly of this form for $t_0 = 0$. So we expect that this initial value problem should have exactly one solution.

For $n = 1$ we may factorise the wave operator (also called D'Alembert's operator)

$$\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right).$$

If u solves the homogeneous wave equation, then $v(x,t) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u(x,t)$ solves $\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} = 0$. This is the transport equation with constant coefficient with the unique solution

$$v(x,t) = a(x-t) \quad \text{with} \quad a(x) = v(x,0).$$

So the solution $u(x,t)$ of the wave equation solves the first order linear PDE

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = a(x-t).$$

This is an inhomogeneous transport equation with constant coefficients with the solution

$$u(x,t) = \int_0^t a(x + (t-s) - s)ds + b(x+t) = \frac{1}{2}\int_{x-t}^{x+t} a(y)dy + b(x+t)$$

with $b(x) = u(x,0)$. The initial values $u(x,0) = g(x)$ and $\frac{\partial u}{\partial t}(x,0) = h(x)$ yields

$$b(x) = g(x) \quad \text{and} \qquad a(x) = v(x,0) = \frac{\partial u}{\partial t}(x,0) - \frac{\partial u}{\partial x}(x,0) = h(x) - g'(x).$$

If we insert this in our solutions, then we obtain

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} \left(h(y) - g'(y)\right) dy + g(x+t)$$

Hence the solution of the initial value problem of the wave equation is given by

$$u(x,t) = \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy.$$

In this way we derived with the help of the homogeneous and inhomogeneous transport equation the D'Alembert's Formula:

Theorem 5.1 (D'Alembert's Formula)**.** *If $g : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable and $h : \mathbb{R} \to \mathbb{R}$ continuously differentiable, then*

$$u(x,t) = \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy$$

is a twice continuously differentiable function on $\mathbb{R} \times \mathbb{R}_0^+$, which is the unique solution of the initial value problem

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0 \qquad \text{for} \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+,$$

$$u(x,0) = g(x) \ \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = h(x) \qquad \text{for} \quad x \in \mathbb{R}.$$

D'Alembert's Formula shows that for $n = 1$ the general solution of the wave equation takes the form

$$u(x,t) = F(x+t) + G(x-t).$$

Conversely, every function of this form is a solution of the wave equation if F and G are twice differentiable. This fact is a consequence of the factorisation of the wave operator into the two first order linear PDEs'

$$\frac{\partial F}{\partial t} - \frac{\partial F}{\partial x} = 0 \quad \text{and} \quad \frac{\partial G}{\partial t} + \frac{\partial G}{\partial x} = 0$$

whose solutions are differentiable functions of the form $F(x+t)$ and $G(x-t)$.

We interpret the fact, that the value of the solution at (x,t) depends only on the values of g at $x \pm t$ and the values of h at points in the interval $[x-t, x+t]$ as the bound 1 on the length of the speed of propagation, since the straight lines from all these points to (x,t) propagate with speed of length not larger than 1. If we insert instead of h an anti-derivative H, then the value $u(x,t)$ of the solution at (x,t) depends only on the values of g and H at $x \pm t$ and the propagation speed has length 1. Furthermore, the solution is k-times differentiable, if g and H are k times differentiable, or equivalently if g is k times differentiable and h is $(k-1)$ times differentiable. So the regularity of the solution does not improve with time, as it does for solutions of the heat equation.

Let us use a reflection in order to derive the solution of the following initial value problem, which will show up later in this chapter:

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) &= 0 \quad \text{for} \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, & u(0,t) &= 0 & \text{for} \quad & t \in \mathbb{R}^+_0, \\
u(x,0) = g(x) \quad & \text{and} & \frac{\partial u}{\partial t}(x,0) &= h(x) & \text{for} \quad & x \in \mathbb{R}^+.
\end{aligned}$$

The functions u, g and h extend to odd functions on the whole space $\mathbb{R} \times \mathbb{R}^+_0$:

$$\tilde{u}(x,t) = \begin{cases} u(x,t) & \text{for } x \geq 0, \\ -u(-x,t) & \text{for } x \leq 0, \end{cases}$$

$$\tilde{g}(x) = \begin{cases} g(x) & \text{for } x \geq 0, \\ -g(-x) & \text{for } x \leq 0, \end{cases} \qquad \tilde{h}(x) = \begin{cases} h(x) & \text{for } x \geq 0, \\ -h(-x) & \text{for } x \leq 0. \end{cases}$$

For any solution $\tilde{u}$ of the initial value problem

$$\begin{aligned}
\frac{\partial^2 \tilde{u}}{\partial t^2} - \frac{\partial^2 \tilde{u}}{\partial x^2} &= 0 & & \text{for} \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\
\tilde{u}(x,0) = \tilde{g}(x) \quad & \text{and} \quad \frac{\partial \tilde{u}}{\partial t}(x,0) = \tilde{h}(x) & & \text{for} \quad x \in \mathbb{R},
\end{aligned}$$

the function $(x,t) \mapsto -\tilde{u}(-x,t)$ is another solution. Due to the uniqueness of the solution both solutions coincide: $\tilde{u}(-x,t) = -\tilde{u}(x,t)$. In particular for the unique solution of the first initial value problem the solution $\tilde{u}$ of the former initial value problem extends to a solution of the latter initial value problem on $\mathbb{R} \times \mathbb{R}_0^+$. Because the functions $\tilde{u}$, $\tilde{g}$ and $\tilde{h}$ are odd with respect to x, for $x \leq t$ the first integral on the right hand side vanishes:

$$\int_{x-t}^{x+t} \tilde{h}(y)dy = \int_{x-t}^{t-x} \tilde{h}(y)dy + \int_{t-x}^{t+x} \tilde{h}(y)dy.$$

Hence the solution of the former initial value problem is given by

$$u(x,t) = \begin{cases} \frac{1}{2}\left(g(x+t) + g(x-t) + \int_{x-t}^{x+t} h(y)dy\right) & \text{for } 0 \leq t \leq x \\ \frac{1}{2}\left(g(t+x) - g(t-x) + \int_{t-x}^{t+x} h(y)dy\right) & \text{for } 0 \leq x \leq t. \end{cases}$$

Note that the waves propagating towards the boundary at $x = 0$ are reflected at the boundary and propagate back.

5.2 Spherical Means of the Wave Equation

We shall derive now the PDE which is obeyed by the spherical means of the wave equation. This PDE is similar to the one–dimensional wave equation, which we shall solve later. This will lead to the general solution of the initial value problem of the wave equation in all dimensions. This initial value problem is the search for the solution u of the wave equation $\frac{\partial^2 u}{\partial t^2} - \triangle u = 0$ on $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$ which obeys $u(x,0) = g(x)$ and $\frac{\partial u}{\partial t}(x,0) = h(x)$. We define for all $x \in \mathbb{R}^n, t \geq 0, r > 0$

$$U(x,r,t) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} u(y,t)d\sigma(y) = \fint_{\partial B(x,r)} u(y,t)d\sigma(y).$$

Here the symbol $\fint$ denotes the mean on the domain Ω, i.e. the integral over the domain Ω divided by the integral of the function 1 over the domain Ω. Analogously we define

$$G(x,r) = \fint_{\partial B(x,r)} g(y)d\sigma(y) \quad \text{and} \qquad H(x,r) = \fint_{\partial B(x,r)} h(y)d\sigma(y).$$

Lemma 5.2. *If $u \in C^m(\mathbb{R}^n \times \mathbb{R}_0^+)$ is a m-times continuously differentiable solution of the initial value problem (with continuous partial derivatives of order $\leq m$ on $\mathbb{R}^n \times \mathbb{R}_0^+$)*

$$\frac{\partial^2 u}{\partial t} - \triangle u = 0 \qquad \textit{on} \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \qquad \textit{with}$$

$$u(x,0) = g(x) \qquad \textit{and} \quad \frac{\partial u}{\partial t}(x,0) = h(x),$$

then the spherical mean $U(x,r,t)$ is for fixed $x \in \mathbb{R}^n$ a m-times differentiable function on $(r,t) \in \mathbb{R}^+ \times \mathbb{R}^+$, which solves the following initial value problem of the Euler-Poisson-Darboux Equation (with continuous partial derivatives of order $\leq m$):

$$\frac{\partial^2 U}{\partial t^2}(x,r,t) - \frac{\partial^2 U}{\partial r^2}(x,r,t) - \frac{n-1}{r}\frac{\partial U}{\partial r}(x,r,t) = 0 \quad \textit{on} \quad (r,t) \in \mathbb{R}^+ \times \mathbb{R}^+$$

$$U(x,r,0) = G(x,r) \quad \textit{and} \quad \frac{\partial U}{\partial t}(x,r,0) = H(x,r)$$

Proof. By a substitution the domain of the integral becomes independent of t and r:

$$U(x,r,t) = \frac{1}{n\omega_n}\int_{\partial B(0,1)} u(ry+x,t)d\sigma(y).$$

Hence we may calculate the derivative

$$\begin{aligned}\frac{\partial U}{\partial r}(x,r,t) &= \frac{1}{n\omega_n}\int_{\partial B(0,1)} \nabla u(x+ry,t)\cdot y d\sigma(y)\\ &= \frac{r}{n\omega_n}\int_{B(0,1)} \triangle u(x+ry,t)d^n y\\ &= \frac{r}{n}\int_{B(x,r)} \triangle u(y,t)d^n y.\end{aligned}$$

The first line determines the first derivative of the spherical mean with respect to r for continuouls differentiable u and implies $\lim\limits_{r\to 0}\frac{\partial U}{\partial r}(x,r,t) = 0$ by continuity of ∇u. The second line prepares higher derivatives of the spherical mean and evaluates the divergence of $y \mapsto \nabla u(ry+x,t)$ as $r\triangle u(ry+x,t)$. Analogously we get

$$\begin{aligned}\frac{\partial^2 U}{\partial r^2}(x,r,t) &= \frac{\partial}{\partial r}\frac{1}{n\omega_n r^{n-1}}\int_{B(x,r)} \triangle u(y,t)d^n y\\ &= \frac{1-n}{n\omega_n r^n}\int_{B(x,r)} \triangle u(y,t)d^n y + \frac{1}{n\omega_n r^{n-1}}\int_{\partial B(x,r)} \triangle u(y,t)d\sigma(y).\\ &= \left(\frac{1}{n}-1\right)\int_{B(x,r)} \triangle u(y,t)d^n y + \int_{\partial B(x,r)} \triangle u(y,t)d\sigma(y).\end{aligned}$$

In particular we have $\lim_{r\to 0}\frac{\partial^2 U}{\partial r^2}(x,r,t) = \frac{1}{n}\triangle u(x,t)$. Furthermore, the partial derivative of the mean over a ball of radius r with respect to r is equal to $\frac{n}{r}$ times the corresponding spherical mean minus $\frac{n}{r}$ times this mean:

$$\begin{aligned}\frac{\partial}{\partial r}\frac{1}{\omega_n r^n}\int_{B(x,r)} u(y,t)d^n y &= -\frac{n}{\omega_n r^{n+1}}\int_{B(x,r)} u(y,t)d^n y + \frac{1}{\omega_n r^n}\int_{\partial B(x,r)} u(y,t)d\sigma(y)\\ &= -\frac{n}{r}\int_{B(x,r)} u(y,t)d^n y + \frac{n}{r}\int_{\partial B(x,r)} u(y,t)d\sigma(y).\end{aligned}$$

These formulas allow to calculate all partial derivatives of the spherical means in terms of the mean of powers of the Laplace operator applied to the original functions over spheres and balls. On the other hand we also have

$$\frac{\partial}{\partial r} r^{n-1}\frac{\partial U}{\partial r}(x,r,t) = \frac{\partial}{\partial r}\frac{1}{n\omega_n}\int_{B(x,r)}\triangle u(y,t)d^n y = \frac{1}{n\omega_n}\int_{\partial B(x,r)}\frac{\partial^2 u}{\partial t^2}(y,t)d\sigma(y) = r^{n-1}\frac{\partial^2 U}{\partial t^2}(x,r,t).$$

This implies
$$r^{n-1}\frac{\partial^2 U}{\partial t^2} = (n-1)r^{n-2}\frac{\partial U}{\partial r} + r^{n-1}\frac{\partial^2 U}{\partial r^2}.$$
□

5.3 Solution in Dimension 3

We shall see that for odd dimensions the spherical means of solutions of the wave equation can be transformed into solutions of the one–dimensional wave equation, but not for even dimensions. For this reason we shall next solve the initial value problem of the wave equation in three dimensions. In this section we consider for any $x \in \mathbb{R}^3$ the following initial value problem for the spherical means of a solution of the wave equation:

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial r^2} - \frac{2}{r}\frac{\partial U}{\partial r} = 0 \quad \text{on} \quad (x,r,t) \in \{x\} \times \mathbb{R}^+ \times \mathbb{R}^+$$

$$U = G \quad \text{and} \quad \frac{\partial U}{\partial t} = H \quad \text{on} \quad (x,r,t) \in \{x\} \times \mathbb{R}^+ \times \{0\}.$$

The substitution $\tilde{U} = rU$ transforms the former initial value problem into the following:

$$\frac{\partial^2 \tilde{U}}{\partial t^2} - \frac{\partial^2 \tilde{U}}{\partial r^2} = 0 \text{ on } (x,r,t) \in \{x\} \times \mathbb{R}^+ \times \mathbb{R}^+, \qquad \tilde{U}(x,0,t) = 0 \text{ for } t \in \mathbb{R}^+_0,$$

$$\tilde{U}(x,r,0) = \tilde{G}(x,r) = rG(x,r) \text{ and } \frac{\partial \tilde{U}}{\partial t}(x,r,0) = \tilde{H}(x,r) = rH(x,r) \text{ for } r \in \mathbb{R}^+.$$

We solved this initial value problem in the first section. The solution is

$$\tilde{U}(x,r,t) = \frac{1}{2}\left(\tilde{G}(x,r+t) - \tilde{G}(x,t-r)\right) + \frac{1}{2}\int_{-r+t}^{r+t} \tilde{H}(x,s)ds \quad \text{for } 0 \leq r \leq t.$$

The continuity of $u(x,t)$ implies

$$u(x,t) = \lim_{r \downarrow 0} U(x,r,t) = \lim_{r \downarrow 0} \frac{\tilde{U}(x,r,t)}{r}.$$

Therefore we obtain for all $x \in \mathbb{R}^3, t > 0$.

$$\begin{aligned}
u(x,t) &= \lim_{r\downarrow 0} \tfrac{1}{2}\left(\tfrac{\tilde{G}(x,t+r)-\tilde{G}(x,t)}{r} + \tfrac{\tilde{G}(x,t-r)-\tilde{G}(x,t)}{-r}\right) && + \lim_{r\downarrow 0} \tfrac{1}{2r}\int_{t-r}^{t+r} \tilde{H}(x,s)ds \\
&= \frac{\partial \tilde{G}(x,t)}{\partial t} && + \tilde{H}(x,t) \\
&= \frac{\partial}{\partial t} t \int_{\partial B(x,t)} g(y)d\sigma(y) && + t\int_{\partial B(x,t)} h(y)d\sigma(y) \\
&= \frac{\partial}{\partial t} t \int_{\partial B(0,1)} g(x+tz)d\sigma(z) && + t\int_{\partial B(x,t)} h(y)d\sigma(y) \\
&= \int_{\partial B(0,1)} \nabla_y g(x+tz)\cdot tz d\sigma(z) && + \int_{\partial B(x,t)} \left(th(y)+g(y)\right)d\sigma(y) \\
&= \int_{\partial B(x,t)} \left(th(y)+g(y)\right)d\sigma(y) && + \int_{\partial B(x,t)} \nabla_y g(y)\cdot(y-x)d\sigma(y)
\end{aligned}$$

The last line is Krichoff's Formula for the solution of the initial value problem of the three dimensional wave equation.

5.4 Solution in Dimension 2

In two dimensions the Euler-Poisson-Darboux equations cannot be transformed into the one–dimensional wave equation. We present another method and transform the initial value problem of the two–dimensional wave equation into an initial value problem of the three–dimensional wave equation, by choosing initial vales which depend only on the coordinates x_1 and x_2 and not on the coordinate x_3: Let $\bar{u}(x,t)$ be on $(x,t) \in \mathbb{R}^3 \times \mathbb{R}^+$ the solution of the initial value problem

$$\frac{\partial^2 \bar{u}(x,t)}{\partial t^2} - \triangle \bar{u}(x,t) = 0 \qquad \text{for} \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}^+$$

$$\bar{u}(x,0) = g(\bar{x}) \quad \text{and} \qquad \frac{\partial \bar{u}}{\partial t}(x,0) = h(\bar{x}) \qquad \text{for} \quad x \in \mathbb{R}^3.$$

Here we set $\bar{x} = (x_1, x_2)$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. We observe that the mean over $\partial B(x,r)$ of a function f depends only on $\bar{x}$, if f depends only on $\bar{x}$:

$$\frac{\partial}{\partial x_3} \int_{\partial B(x,r)} f(y) d\sigma(y) = \frac{\partial}{\partial x_3} \int_{\partial B(0,r)} f(x+y) d\sigma(y) = \int_{\partial B(0,r)} \frac{\partial f(x+y)}{\partial x_3} d\sigma(y) = 0.$$

This shows that $\bar{u}(x,t)$ is of the form $u(\bar{x},t)$ and the latter function $u(\bar{x},t)$ yields the desired solution of the two–dimensional initial value problem. Let us now calculate this function. The function $\gamma(z) = \sqrt{r^2 - (z-\bar{x})^2}$ on the two–dimensional ball $B(\bar{x},r)$ yields by the formula $\Psi(z) = (z, \pm\gamma(z))$ a parametrisations of both hemispheres of the boundary of the three–dimensional ball $B((\bar{x},0),r)$ by the two–dimensional ball $B(\bar{x},r)$ as in Definition 2.4. The 3×2-matrix $\Psi'(z)$ has the same form $\Psi'(z) = \left(\begin{smallmatrix} \mathbb{1}_{\mathbb{R}^2} \\ \pm\nabla^T\gamma(z) \end{smallmatrix}\right)$ as Φ' after Lemma 2.5 with $\mathsf{O} = \mathbb{1}$ and $g = \pm\gamma$. Hence the determinant of $(\Psi'(z))^T\Psi(z)$ is $1 + (\nabla\gamma(z))^2$. So the intgerals over both hemispheres is equal to the integral over $B(\bar{x},r)$ with the measure $d\sigma(z, \pm\gamma(z)) = \sqrt{1 + (\nabla\gamma(z))^2} dz^2$:

$$\fint_{\partial B(x,r)} g(\bar{y}) d\sigma(y) = \frac{1}{4\pi r^2} \int_{\partial B(x,r)} g(\bar{y}) d\sigma(y) = \frac{2}{4\pi r^2} \int_{B(\bar{x},r)} g(z) \sqrt{1 + (\nabla\gamma(z))^2} d^2 z$$

$$\text{with} \quad \sqrt{1 + (\nabla\gamma(z))^2} = \sqrt{\frac{r^2 - (z-\bar{x})^2 + (z-\bar{x})^2}{r^2 - (z-\bar{x})^2}} = \frac{r}{\sqrt{r^2 - (z-\bar{x})^2}}.$$

Both hemispheres do not cover the boundary $\partial B((\bar{x},0),r)$ completely, but the missing equator is one–dimensional and has measure zero with respect to $d\sigma(y)$. This implies

$$\fint_{\partial B(x,r)} g(\bar{y}) d\sigma(y) = \frac{1}{2\pi r} \int_{B(\bar{x},r)} \frac{g(z)}{\sqrt{r^2 - (z-\bar{x})^2}} d^2 z = \frac{r}{2} \fint_{B(\bar{x},r)} \frac{g(z)}{\sqrt{r^2 - (z-\bar{x})^2}} d^2 z.$$

This gives finally the following formula for $u(\bar{x},t)$ on $(\bar{x},t)\in\mathbb{R}^2\times\mathbb{R}^+$:

$$\begin{aligned} u(\bar{x},t) &= \frac{\partial}{\partial t} t \int_{\partial B(x,t)} g(\bar{y})d\sigma(y) & &+ t\int_{\partial B(x,t)} h(\bar{y})d\sigma(y) \\ &= \frac{\partial}{\partial t}\frac{t^2}{2}\int_{B(\bar{x},t)} \frac{g(z)}{\sqrt{t^2-(z-\bar{x})^2}}d^2z & &+ \frac{t^2}{2}\int_{B(\bar{x},t)} \frac{h(z)}{\sqrt{t^2-(z-\bar{x})^2}}d^2y \\ &= \frac{\partial}{\partial t}\frac{t}{2}\int_{B(0,1)} \frac{g(\bar{x}+tz)}{\sqrt{1-z^2}}d^2z & &+ \frac{t^2}{2}\int_{B(\bar{x},t)} \frac{h(z)}{\sqrt{t^2-(z-\bar{x})^2}}d^2z \\ &= \frac{t}{2}\int_{B(\bar{x},t)} \frac{g(z)+th(z)+\nabla g(z)(z-\bar{x})}{\sqrt{t^2-(z-\bar{x})^2}}d^2z. \end{aligned}$$

The last line is Poisson's formula for the solution of the inital value problem of the two–dimensional wave equation. It shows that in two dimensions the propagation speed of solutions of the wave equation has all lengths bounded by 1. In fact the value $u(x,t)$ depends only on the values of g and h on the ball $B(\bar{x},t)$ and the straight lines connecting these points with (x,t) have all speeds whose lengths are bounded by 1.

This method of deriving the solution of the initial value problem in a lower dimension by transfoming the initial value problem into an initial value problem in the higher dimensional space, is called the method of descent. Here the initial values do not depend on some of the coordinates of the higher dimensional space. We have to show that the corresponding solutions do not depend on these coordinates as well. A natural question is, whether we may obtain the solution of the one–dimensional wave equation by this method of descent from Poisson's formula?

5.5 Solution in odd Dimensions

In any odd dimension we can transform the Euler-Poisson-Darboux equation into the one–dimensional wave equation. As a preparation we first show the following Lemma:

Lemma 5.3. *Let ϕ ba a $(k+1)$ times continuously differentiable function on $\mathbb{R}$. We obtain for all $k\in\mathbb{N}$*

(i) $\left(\frac{d}{dr}\right)^2\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1} r^{2k-1}\phi(r) = \left(\frac{1}{r}\frac{d}{dr}\right)^k r^{2k}\frac{d\phi}{dr}(r)$

(ii) $\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1} r^{2k-1}\phi(r) = \sum_{j=0}^{k-1}\beta_{k,j}r^{j+1}\frac{d^j\phi}{dr^j}(r)$ *with numbers $\beta_{k,j}$ $(j=0,\ldots,k-1)$, which do not depend on ϕ.*

(iii) $\beta_{k,0} = 1\cdot 3\cdot 5\cdots(2k-1) = \frac{(2k-1)!}{2\cdot 4\cdot\ldots\cdot(2k-2)} = \frac{(2k-1)!}{2^{(k-1)}(k-1)!}$

Proof. First we proof (i) by induction. For $k = 1$ we have

$$\tfrac{d^2}{dr^2} r\phi(r) = 2\tfrac{d\phi}{dr}(r) + r\tfrac{d^2\phi}{dr^2}(r) = \tfrac{1}{r}\tfrac{d}{dr} r^2 \tfrac{d\phi}{dr}(r).$$

Now we assume that the statement is true for $k \in \mathbb{N}$. Then we obtain for $k+1$:

$$\begin{aligned}(\tfrac{d}{dr})^2(\tfrac{1}{r}\tfrac{d}{dr})^k r^{2k+1}\phi(r) &= (\tfrac{d}{dr})^2(\tfrac{1}{r}\tfrac{d}{dr})^{k-1} r^{2k-1}((2k+1)\phi(r) + r\tfrac{d\phi}{dr}(r))\\ &= (\tfrac{1}{r}\tfrac{d}{dr})^k r^{2k}\tfrac{d}{dr}((2k+1)\phi(r) + r\tfrac{d\phi}{dr}(r))\\ &= (\tfrac{1}{r}\tfrac{d}{dr})^k((2k+2)r^{2k}\tfrac{d\phi}{dr}(r) + r^{2k+1}\tfrac{d^2\phi}{dr^2}(r)) = (\tfrac{1}{r}\tfrac{d}{dr})^{k+1} r^{2k+2}\tfrac{d\phi}{dr}(r).\end{aligned}$$

By the Leibniz rule every derivative in (ii) results in two contributions: The first deminishes of the power of r by one without changing the order of the derivative of ϕ. The other increases the order of the derivative of ϕ by one without changing the power of r. The highest power of r is $r^{2k-1-(k-1)} = r^k$ and the highest order of derivatives of ϕ is $(\frac{d}{dr})^{k-1}$. This implies (ii).

In the term with the coefficient $\beta_{k,0}$ all derivatives act only on the powers of r. The first derivative acts on r^{2k-1}, the second derivatives acts on r^{2k-3} and the last derivative acts on r^3. This shows (iii). □

Let the dimension $n = 2k+1 \geq 3$ be odd and let $u \in C^{k+1}(\mathbb{R}^{2k+1} \times \mathbb{R}_0^+)$ obey

$$\frac{\partial^2 u}{\partial t^2} - \triangle u = 0 \qquad \text{for} \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+$$

$$u(x,0) = g(x) \quad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = h(x) \qquad \text{for} \quad x \in \mathbb{R}^n.$$

We define

$$\tilde{U}(x,r,t) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} r^{2k-1} U(x,r,t)$$

$$\tilde{G}(x,r) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} r^{2k-1} G(x,r) \qquad \tilde{H}(x,r) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} r^{2k-1} H(x,r).$$

Lemma 5.4. *If $u \in C^{k+1}(\mathbb{R}^{2k+1} \times \mathbb{R}_0^+)$ solves the initial value problem of the wave equation, then $\tilde{U}(x,r,t)$ solves for any $x \in \mathbb{R}^{2k+1}$ the following initial value problem:*

$$\frac{\partial^2 \tilde{U}}{\partial t^2} - \frac{\partial^2 \tilde{U}}{\partial r^2} = 0 \text{ on } (x,r,t) \in \{x\} \times \mathbb{R}^+ \times \mathbb{R}^+ \qquad \tilde{U}(x,0,t) = 0 \text{ for } t \in \mathbb{R}_0^+$$

$$\tilde{U}(x,r,0) = \tilde{G}(x,r) \text{ and } \qquad \frac{\partial \tilde{U}}{\partial t}(x,r,0) = \tilde{H}(x,r) \text{ for } r \in \mathbb{R}^+.$$

Proof. Let $U(x,r,t)$ solve in the dimension $2k = n-1$ the corresponding initial value problem of the Euler-Poisson-Darboux equation:

$$\begin{aligned}
\frac{\partial^2 \tilde{U}}{\partial r^2}(x,r,t) &= \left(\frac{\partial^2}{\partial r^2}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} r^{2k-1}U(x,r,t) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k} r^{2k}\frac{\partial U}{\partial r}(x,r,t)\\
&= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left(r^{2k-1}\frac{\partial^2 U}{\partial r^2}(x,r,t) + 2kr^{2k-2}\frac{\partial U}{\partial r}(x,r,t)\right)\\
&= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} r^{2k-1}\frac{\partial^2 U}{\partial t^2}(x,r,t) = \frac{\partial^2 \tilde{U}}{\partial t^2}(x,r,t)
\end{aligned}$$

Due to the Lemmas 5.2 and 5.3 (iii) the values of $\tilde{U}(x,r,t)$ vanish for $r=0$. □

For any $(x,r,t)\in\mathbb{R}\times\mathbb{R}_0^+\times\mathbb{R}_0^+$ with $r\le t$ the solution of the initial value problem is

$$\tilde{U}(x,r,t) = \frac{1}{2}\left(\tilde{G}(x,t+r)-\tilde{G}(x,t-r)\right)+\frac{1}{2}\int_{t-r}^{t+r}\tilde{H}(x,s)ds.$$

We recall Lemma 5.3 (ii): $\tilde{U}(x,r,t) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} r^{2k-1}U(x,r,t) = \sum_{j=0}^{k-1}\beta_{k,j}r^{j+1}\frac{\partial^j U}{\partial r^j}(x,r,t)$. Lemma 5.2 implies $\lim_{r\to 0} r^{j+1}\frac{\partial^j U}{\partial r^j}(x,r,t)=0$ for all $j\in\mathbb{N}_0$. We conclude

$$u(x,t) = \lim_{r\to 0}U(x,r,t) = \lim_{r\to 0}\frac{\tilde{U}(x,r,t)}{\beta_{k,0}r}$$

Alltogether the solution of the initial value problem in odd dimensions is given by

$$u(x,t) = \tfrac{2^{k-1}(k-1)!}{(2k-1)!}\lim_{r\to 0}\left(\tfrac{\tilde{G}(x,t+r)-\tilde{G}(x,t-r)}{2r} + \fint_{t-r}^{t+r}\tilde{H}(x,s)ds\right) = \tfrac{2^{k-1}(k-1)!}{(2k-1)!}\left(\tfrac{\partial\tilde{G}}{\partial t}(x,t)+\tilde{H}(x,t)\right).$$

Theorem 5.5. *For odd $n\ge 3$ the solution of the initial value problem*

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2}-\triangle u = 0 && \textit{for}\quad (x,t)\in\mathbb{R}^n\times\mathbb{R}^+\\
&u(x,0)=g(x)\quad\textit{and}\quad \frac{\partial u}{\partial t}(x,0)=h(x) && \textit{for}\quad x\in\mathbb{R}^n
\end{aligned}$$

with $g\in C^{\frac{n+3}{2}}(\mathbb{R}^n)$ and $h\in C^{\frac{n+1}{2}}(\mathbb{R}^n)$ has at any $(x,t)\in\mathbb{R}^n\times\mathbb{R}_0^+$ the value

$$u(x,t) = \frac{2^{\frac{n-3}{2}}(\frac{n-3}{2})!}{(n-2)!}\left(\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} t^{n-2}\fint_{\partial B(x,t)} g(y)d\sigma(y) + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} t^{n-2}\fint_{\partial B(x,t)} h(y)d\sigma(y)\right).$$

Proof. First we consider the case $g = 0$. In this case Lemma 5.3 (i) implies

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= \frac{2^{\frac{n-3}{2}}(\frac{n-3}{2})!}{(n-2)!}\frac{\partial^2}{\partial t^2}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} t^{n-2}\fint_{\partial B(x,t)} h(y)d\sigma(y)\\
&= \frac{2^{\frac{n-3}{2}}(\frac{n-3}{2})!}{(n-2)!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-1}{2}} t^{n-1}\frac{\partial}{\partial t}\fint_{\partial B(x,t)} h(y)d\sigma(y)\\
&= \frac{2^{\frac{n-3}{2}}(\frac{n-3}{2})!}{(n-2)!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-1}{2}} t^{n-1}\frac{\partial}{\partial t}\frac{1}{n\omega_n}\int_{\partial B(0,1)} h(x+tz)d\sigma(z)\\
&= \frac{2^{\frac{n-3}{2}}(\frac{n-3}{2})!}{(n-2)!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-1}{2}} \frac{t^{n-1}}{n\omega_n}\int_{\partial B(0,1)} \nabla_x h(x+tz)\cdot z d\sigma(z)\\
&= \frac{2^{\frac{n-3}{2}}(\frac{n-3}{2})!}{(n-2)!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-1}{2}} \frac{1}{n\omega_n}\int_{\partial B(x,t)} \nabla_y h(y)\cdot N(y) d\sigma(y)\\
&= \frac{2^{\frac{n-3}{2}}(\frac{n-3}{2})!}{(n-2)!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-1}{2}} \frac{1}{n\omega_n}\int_{B(x,t)} \triangle h(y) d^n y\\
&= \frac{2^{\frac{n-3}{2}}(\frac{n-3}{2})!}{(n-2)!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} \frac{1}{n\omega_n t}\int_{\partial B(x,t)} \triangle h(y) d\sigma(y)\\
&= \frac{2^{\frac{n-3}{2}}(\frac{n-3}{2})!}{(n-2)!}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} t^{n-2}\fint_{\partial B(0,t)} \triangle_x h(x+y) d\sigma(y)\\
&= \triangle u(x,t).
\end{aligned}$$

Here we used the divergence theorem and polar coordinates for the calculation of the derivative of integrals over $B(x,t)$ with respect to t. If we replace h by g and $u(x,t)$ by $v(x,t)$ with $u(x,t) = \frac{\partial v}{\partial t}(x,t)$, then we obtain the solution in the case with $h = 0$. This shows that $v(x,t)$ and therefore also $u(x,t)$ solves the wave equation. The Lemmas 5.2 and 5.3 (iii) finally imply

$$\begin{aligned}
u(x,0) &= \lim_{t\to 0}\left(\frac{\partial}{\partial t}t\fint_{\partial B(x,t)} g(y)d\sigma(y) + t\fint_{\partial B(x,t)} h(y)d\sigma(y)\right) + \mathbf{O}(t) &&= g(x)\\
\frac{\partial u}{\partial t}(x,0) &= \lim_{t\to 0}\left(t\fint_{\partial B(x,t)} \triangle g(y)d\sigma(y) + \frac{\partial}{\partial t}t\fint_{\partial B(x,t)} h(y)d\sigma(y)\right) + \mathbf{O}(t) &&= h(x). \qquad \square
\end{aligned}$$

To the limit $t \downarrow 0$ only the lowest powers of t with $j = 0$ in Lemma 5.3 (ii) contribute and in the second formula we used that the integral solves the homogeneous wave equation. The solution $u(x,t)$ depends only on the values of g and h at elements of $\partial B(x,t)$. Therefore the speed of propagation has length 1. This is the content of Huygen's principle. However the formula for the solution of the initial value problem of the heat equation shows that the speed of propagation of solutions of the heat equation can be arbitrary large. Besides this difference there exists another difference to the heat equation: The solution u of the homogeneous wave equation is at (x,t) $\frac{n-1}{2}$ times less differentiable than g and $\frac{n-3}{2}$

times less differentiable than h. This is a general property of solutions of hyperbolic PDEs in contrast to parabolic PDEs: for a large class of initial values the solutions of the homogeneous heat equation are smooth independent of the regularity of the initial values.

5.6 Solution in even Dimensions

We again use the method of descent, and derive the solutions in the dimension n as special solutions in the dimension $n+1$ with initial values $g(x)$ and $h(x)$ on $\mathbb{R}^{n+1}$ not depending on x_{n+1} for $x = (x_1, \ldots, x_{n+1})$. By the general formula of the last section we see that in this case also the solution does not depend on x_{n+1}. Indeed if the partial derivative $\partial_{n+1} g(x)$ and $\partial_{n+1} h(x)$ vanish, then the same is true for

$$\fint_{\partial B(x,t)} g(y)d\sigma(y) \quad \text{and} \quad \fint_{\partial B(x,t)} h(y)d\sigma(y).$$

If the function f obeys $\frac{\partial f}{\partial x_{n+1}} = 0$, then we obtain

$$\frac{\partial}{\partial x_{n+1}} \fint_{\partial B(x,t)} f(y)d\sigma(y) = \fint_{\partial B(0,t)} \frac{\partial}{\partial x_{n+1}} f(x+y)d\sigma(y) = 0.$$

This holds for all $t \in \mathbb{R}^+$ and therefore also for all partial derivatives with respect to t. So the solution in dimension $n = 2k$ is obtained as the composition of the inclusion $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}, x \mapsto (x, 0)$ with the solution of the corresponding initial value problem in dimension $n+1$. As in the application of the method of descent to the descent from dimension three to dimension two we parameterise both hemispheres of $\partial B(x,t)$ by the maps $\Psi : B(\bar{x}, t) \to \partial B(x,t)$ with $z \mapsto (z, \pm\gamma(z))$ with $\gamma(z) = \sqrt{t^2 - (z - \bar{x})^2}$. The Jacobi matrix of this map is the $(n+1) \times n$ matrix $\left(\begin{smallmatrix} \mathbb{1}_{\mathbb{R}^n} \\ \pm(\nabla\gamma(z))^t \end{smallmatrix}\right)$ and the determinant of $(\Psi'(z))^T \Psi'(z)$ is again $1 + (\nabla_z \gamma(z))^2$. Hence we obtain

$$\begin{aligned}\fint_{\partial B(x,t)} \bar{f}(y)d\sigma(y) &= \frac{1}{(n+1)\omega_{n+1}t^n} \int_{\partial B(x,t)} \bar{f}(y)d\sigma(y) = \frac{2}{(n+1)\omega_{n+1}t^n} \int_{B(\bar{x},t)} f(z)\sqrt{1+(\nabla\gamma(z))^2} d^n z \\ &= \frac{2t}{(n+1)\omega_{n+1}t^n} \int_{B(\bar{x},t)} \frac{f(z)d^n z}{\sqrt{t^2 - (z-\bar{x})^2}} = \frac{2t\omega_n}{(n+1)\omega_{n+1}} \fint_{B(\bar{x},t)} \frac{f(z)d^n z}{\sqrt{t^2 - (z-\bar{x})^2}}.\end{aligned}$$

Theorem 5.6 (Solution in even dimension). *Let n be a positive even integer and $g \in C^{\frac{n+4}{2}}(\mathbb{R}^n)$ and $h \in C^{\frac{n+2}{2}}(\mathbb{R}^n)$. Then the solution of the initial value problem*

$$\frac{\partial^2 u}{\partial t^2} - \triangle u = 0 \qquad \textit{for} \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+$$

$$u(x,0) = g(x) \quad \textit{and} \quad \frac{\partial u}{\partial t}(x,0) = h(x) \qquad \textit{for} \quad x \in \mathbb{R}^n$$

takes at any $(x,t)\in\mathbb{R}^n\times\mathbb{R}^+_0$ *the following value:*

$$u(x,t)=\frac{1}{2^{\frac{n}{2}}\frac{n}{2}!}\left(\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}t^n\fint_{B(x,t)}\frac{g(z)d^nz}{\sqrt{t^2-(z-x)^2}}+\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}t^n\fint_{B(x,t)}\frac{h(z)d^nz}{\sqrt{t^2-(z-x)^2}}\right).$$

Proof. The foregoing formula yields with the function $f=1$ and for $x=0$ and $t=r$:

$$\begin{aligned}(n+1)\omega_{n+1}r^n&=\int_{\partial B(0,r)}d\sigma(y)=2r\int_{B(\bar{0},r)}\frac{d^nz}{\sqrt{r^2-z^2}}=2r\int_0^r\int_{\partial B(\bar{0},s)}\frac{d\sigma(z)}{\sqrt{r^2-s^2}}ds\\&=2r\int_0^r\frac{n\omega_ns^{n-1}ds}{\sqrt{r^2-s^2}}=2n\omega_nr^n\int_0^1(1-x^2)^{\frac{n-2}{2}}dx\end{aligned}$$

with the substitution $x=\frac{\sqrt{r^2-s^2}}{r}=\sqrt{1-(\frac{s}{r})^2}$, $rdx=-\frac{sds}{rx}$ and $s=r\sqrt{1-x^2}$. We calculate the remaining integral by inductive partial integration and insert the result:

$$\begin{aligned}&\int_0^1(1-x^2)^mdx=x(1-x^2)^m\Big|_{x=0}^1+2m\int_0^1(1-x^2)^{m-1}x^2dx\\&=-2m\int_0^1(1-x^2)^mdx+2m\int_0^1(1-x^2)^{m-1}dx=\frac{2m}{2m+1}\int_0^1(1-x^2)^{m-1}=\frac{(2^mm!)^2}{(2m+1)!}.\\&\frac{2^{\frac{n+1-3}{2}}\frac{n+1-3}{2}!}{(n+1-2)!}\frac{2\omega_n}{(n+1)\omega_{n+1}}=\frac{2^{\frac{n-2}{2}}\frac{n-2}{2}!}{(n-1)!}\frac{1}{n}\frac{(2\frac{n-2}{2}+1)!}{(2^{\frac{n-2}{2}}\frac{n-2}{2}!)^2}=\frac{1}{2^{\frac{n}{2}}\frac{n}{2}!}.\end{aligned}$$

□

By this formula the value of the solution at (x,t) depends on the values of g and h on $B(x,t)$ rather than the values on $\partial B(x,t)$ like in odd dimensions. Hence the length of the speed of propagation is bounded by 1, but not equal to 1. Furthermore the solution is at (x,t) $\frac{n}{2}$ times less differentiable as g and $\frac{n-2}{2}$ times less differentiable then h.

We close this section by showing that for any $k\in\mathbb{N}$ the solution in dimenson $n=2k-1$ is obtained by the method of descent from the solution in dimension $n+1=2k$. The initial values g and h are functions on $\mathbb{R}^n$ and again they define functions on $\mathbb{R}^{n+1}$ $\bar{g}(x)=g(\bar{x})$ and $\bar{h}(x)=h(\bar{x})$ with $\overline{(x_1,\ldots,x_{n+1})}=(x_1,\ldots,x_n)$ which do not depend on x_{n+1}. For a differentiable function f on $\mathbb{R}^n$ we have

$$\frac{\partial}{\partial x_{n+1}}\int_{B(x,t)}\frac{f(\bar{z})}{\sqrt{t^2-(z-x)^2}}d^{n+1}z=\int_{B(0,t)}\frac{\frac{\partial f}{\partial x_{n+1}}(\bar{x}+\bar{z})}{\sqrt{t^2-z^2}}d^{n+1}z=0.$$

Hence the solution of the corresponding initial value problem in dimension $n+1$ does not depend on x_{n+1} and again defines a solutions of the initial value problem in dimension n. The map $z\mapsto\bar{z}$ maps the ball $B(0,t)\subset\mathbb{R}^{n+1}$ onto the corresponding ball $B(\bar{0},t)\subset\mathbb{R}^n$. The preimage of $\bar{z}\in B(\bar{0},t)$ with respect to this map is $\{(\bar{z},z_{n+1})\mid z_{n+1}\in(-\sqrt{t^2-\bar{z}^2},\sqrt{t^2-\bar{z}^2})\}$. The substitution $y=\frac{z_{n+1}}{\sqrt{t^2-\bar{z}^2}}$ yields $dz_{n+1}=\sqrt{t^2-\bar{z}^2}dy$ and $\sqrt{t^2-\bar{z}^2-z_{n+1}^2}=\sqrt{(t^2-\bar{z}^2)\frac{t^2-\bar{z}^2-z_{n+1}^2}{t^2-\bar{z}^2}}=\sqrt{t^2-\bar{z}^2}\sqrt{1-y^2}$:

$$\int_{B(0,t)} \frac{\bar{f}(x+z)}{\sqrt{t^2-z^2}} d^{n+1}z = \int_{B(\bar{0},t)} \int_{-\sqrt{t^2-\bar{z}^2}}^{\sqrt{t^2-\bar{z}^2}} \frac{dz_{n+1}}{\sqrt{t^2-\bar{z}^2-z_{n+1}^2}} f(\bar{x}+\bar{z}) d^n\bar{z} =$$
$$= \int_{B(\bar{0},t)} \int_{-1}^{1} \frac{dy}{\sqrt{1-y^2}} f(\bar{x}+\bar{z}) d^n\bar{z} = \pi \int_{B(\bar{x},t)} f(\bar{z}) d^n\bar{z}.$$

So for odd $n \geq 3$ we indeed recover the formula for the solution in odd dimension:

$$u(x,t) = \frac{\pi}{2^{\frac{n+1}{2}} \frac{n+1}{2}! \omega_{n+1}} \left(\frac{\partial}{\partial t} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-1}{2}} \int_{B(x,t)} g(z) d^n z + \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-1}{2}} \int_{B(x,t)} h(z) d^n z \right)$$
$$= \frac{\pi n \omega_n}{2^{\frac{n+1}{2}} \frac{n+1}{2}! \omega_{n+1}} \left(\frac{\partial}{\partial t} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} t^{n-2} \int_{\partial B(x,t)} g(z) d^n z + \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} t^{n-2} \int_{\partial B(x,t)} h(y) d^n y \right),$$
$$\frac{\pi n \omega_n}{2^{\frac{n+1}{2}} \frac{n+1}{2}! \omega_{n+1}} = \frac{\pi}{2^{\frac{n+1}{2}} \frac{n-1}{2}!} \frac{2n\omega_n}{(n+1)\omega_{n+1}} = \frac{\pi}{2^{\frac{n+1}{2}} \frac{n-1}{2}!} \frac{(2^{\frac{n-3}{2}} \frac{n-3}{2}!)^2 (n-1)}{(n-2)! \frac{\pi}{2}} = \frac{2^{\frac{n-3}{2}} \frac{n-3}{2}!}{(n-2)!}.$$

Here we use the same formulas as for even n. However, for odd $n \geq 3$ the final integral is

$$\frac{(n+1)\omega_{n+1}}{2n\omega_n} = \int_0^1 (1-x^2)^{\frac{n}{2}-1} dx = \frac{n-2}{n-1} \int_0^1 (1-x^2)^{\frac{n}{2}-2} dx$$
$$= \frac{(n-2)!}{(2^{\frac{n-3}{2}} \frac{n-3}{2}!)^2 (n-1)} \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{(n-2)! \, [\arcsin]_0^1}{(2^{\frac{n-3}{2}} \frac{n-3}{2}!)^2 (n-1)} = \frac{(n-2)! \frac{\pi}{2}}{(2^{\frac{n-3}{2}} \frac{n-3}{2}!)^2 (n-1)}.$$

For $n = 1$ the first formula gives with $\omega_2 = \pi$ D'Alembert's Formula. In particular, the formula for the solution in dimension $m \in \mathbb{N}$ gives by the method of descent the formulas for the solutions in all dimensions less than m. The iterated application of the method of descent shows that the solutions in dimensions $n < m$ are obtained by considering solutions u in dimension m with initial values which depend only on the first n variables $x_1, \ldots, x_n$. These solutions also depend only on $x_1, \ldots, x_n$ and define the corresponding solutions as functions depending on $(x_1, \ldots, x_n, t) \in \mathbb{R}^n \times \mathbb{R}^+$.

5.7 Inhomogeneous Wave Equation

Duhamel's principle also applies to the initial value problem of the wave equation. We conceive the wave equation as a first order linear ODE on pairs of functions on $x \in \mathbb{R}^n$:

$$\frac{d}{dt} \begin{pmatrix} u(\cdot,t) \\ \frac{\partial u}{\partial t}(\cdot,t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \triangle & 0 \end{pmatrix} \begin{pmatrix} u(\cdot,t) \\ \frac{\partial u}{\partial t}(\cdot,t) \end{pmatrix} + \begin{pmatrix} 0 \\ f(\cdot,t) \end{pmatrix}.$$

So we may calculate the special solution of the inhomogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} - \triangle u = f \qquad \text{for} \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+$$
$$u(x,0) = 0 \quad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = 0 \qquad \text{for} \quad x \in \mathbb{R}^n$$

as an integeral of the family of solutions of the homogeneous wave equation whose initial values is given by the inhomogeneity: If $u(x,t,s)$ solves for any $s \in \mathbb{R}^+$

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \triangle u &= 0 && \text{for} \quad (x,t) \in \mathbb{R}^n \times (s,\infty) \\ u(x,s,s) = 0 \quad \text{and} \quad & \frac{\partial u}{\partial t}(x,s,s) = f(x,s) && \text{for} \quad x \in \mathbb{R}^n, \end{aligned}$$

then $u(x,t) = \int_0^t u(x,t,s)ds$ solves the former inhomogeneous wave equation since

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(x,t) &= \frac{\partial}{\partial t}\left(u(x,t,t) + \int_0^t \frac{\partial u}{\partial t}(x,t,s)ds\right) = \frac{\partial}{\partial t}\int_0^t \frac{\partial u}{\partial t}(x,t,s)ds = \\ &= \frac{\partial u}{\partial t}(x,t,t) + \int_0^t \frac{\partial^2 u}{\partial t^2}(x,t,s)ds = f(x,t) + \int_0^t \triangle u(x,t,s)ds = f(x,t) + \triangle u(x,t). \end{aligned}$$

Consequently the initial value problem of the inhomogeneous wave equation

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \triangle u &= f && \text{for} \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ u(x,0) = g(x) \quad \text{and} \quad & \frac{\partial u}{\partial t}(x,0) = h(x) && \text{for} \quad x \in \mathbb{R}^n \end{aligned}$$

is the sum of the former special solution with trivial initial value and the solution of the corresponding homogeneous initial value problem.

Finally we investigate how the presence determines the past. The wave equations is invariant with respect to time reversal $t \mapsto -t$. However, this transformation replaces $\frac{\partial u}{\partial t}$ by $-\frac{\partial u}{\partial t}$. Therefore the values $u(x,t)$ of the solution of the end value problem

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \triangle u &= f && \text{for} \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^- \\ u(x,0) = g(x) \quad \text{and} \quad & \frac{\partial u}{\partial t}(x,0) = h(x) && \text{for} \quad x \in \mathbb{R}^n \end{aligned}$$

are given by the values $u(x,-t)$ of the solution of the initial value problem with initial values g and $-h$ and inhomogeneity $(x,t) \mapsto f(x,-t)$. This means that we can derive both the future and the past from the presence. Both solutions fit together and form a solution $u(x,t)$ of the wave equation on $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ which is completely determined by its values $u(x,0)$ and $\frac{\partial u}{\partial t}(x,0)$ on $x \in \mathbb{R}^n$.

5.8 Energy Methods

Hyperbolic PDEs do not satisfy a maximum principle. A maximum in the interior of a domain can be only excluded by a second order PDE which ensures that the Hessian cannot be indefinite. This are exactely the elliptic PDEs and theire limiting cases as the parabolic PDEs. Indeed the methods of Theorem 3.13 applies to degenerate elliptic PDEs as well. However, energy methods apply to hyperbolic PDEs as well as to elliptic PDEs and we may prove the uniqeness of solutions with such methods:

Theorem 5.7 (uniqueness of the solutions of the wave equation)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then the following initial values porblem of the wave equation*

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \triangle u &= f && \text{on} \quad \Omega \times (0,T) \\ u(x,t) &= g(x,t) && \text{on} \quad \Omega \times \{t=0\} \qquad \text{and on} \quad \partial\Omega \times (0,T) \\ \frac{\partial u}{\partial t}(x,0) &= h(x) && \text{on} \quad \Omega \times \{t=0\} \end{aligned}$$

has a unique solution in $C^2(\Omega \times (0,T))$ with continuous extensions of $\partial^\alpha u$ to $\bar{\Omega} \times [0,T]$ for $|\alpha| \leq 2$.

Proof. The difference of two solutions solves the analogous homogeneous initial value problem with $f = g = h = 0$. For such a solution we define the energy as

$$e(t) = \frac{1}{2} \int_\Omega \left(\left(\frac{\partial u}{\partial t}(x,t) \right)^2 + (\nabla u(x,t))^2 \right) d^n x.$$

Then we calculate

$$\begin{aligned} \frac{de}{dt}(t) &= \int_\Omega \left(\frac{\partial^2 u}{\partial t^2}(x,t) \frac{\partial u}{\partial t}(x,t) + \frac{\partial \nabla u}{\partial t} u(x,t) \nabla u(x,t) \right) d^n x \\ &= \int_\Omega \frac{\partial u}{\partial t}(x,t) \left(\frac{\partial^2 u}{\partial t^2}(x,t) - \triangle u(x,t) \right) d^n x = 0. \end{aligned}$$

Here we applied once the divergence theorem to the vector field $\frac{\partial u}{\partial t} \nabla u$ which vanishes at $\partial\Omega \times [0,T]$ together with u and $\frac{\partial u}{\partial t}$. Initially the energy is zero $e(0) = 0$. Since the energy is non negative it stays zero for all positive times $t > 0$. This shows that u is constant and vanishes on $\Omega \times [0,T)$ since it vanishes initially. □

The proof gives the same conclusion if we assume that instead of $u(x,t)$ the normal derivative $\nabla u(x,t) \cdot N(x,t)$ is given on $\partial\Omega \times [0,T]$. Finally we give a simple proof that the length of the speed of propagation is bounded by 1.

Theorem 5.8. *If u is any solution of the homogeneous wave equation obeying $u = \frac{\partial u}{\partial t} = 0$ on $B(x_0,t_0)$ for $t = 0$, then u vanishes on the cone $\{(x,t) \mid |x - x_0| \leq t_0 - t, t > 0\}$.*

Proof. Again we calculate the time derivative of the energy

$$
\begin{aligned}
e(t) &= \frac{1}{2}\int_{B(x_0,t_0-t)} \left(\left(\frac{\partial u}{\partial t}(x,t)\right)^2 + (\nabla u(x,t))^2\right) d^n x \quad \text{as} \\
\frac{de}{dt}(t) &= \frac{1}{2}\frac{d}{dt}\int_0^{t_0-t}\int_{\partial B(x_0,s)} \left(\left(\frac{\partial u}{\partial t}(x,t)\right)^2 + (\nabla u(x,t))^2\right) d\sigma(x) ds \\
&= \int_{B(x_0,t_0-t)} \left(\frac{\partial^2 u}{\partial t^2}(x,t)\frac{\partial u}{\partial t}(x,t) + \frac{\partial \nabla u}{\partial t}(x,t)\nabla u(x,t)\right) d^n x \\
&- \frac{1}{2}\int_{\partial B(x_0,t_0-t)} \left(\left(\frac{\partial u}{\partial t}(x,t)\right)^2 + (\nabla u(x,t))^2\right) d\sigma(x) \\
&= \int_{B(x_0,t_0-t)} \frac{\partial u}{\partial t}(x,t)\left(\frac{\partial^2 u}{\partial t^2}(x,t) - \triangle u(x,t)\right) d^n x \\
&+ \int_{\partial B(x_0,t_0-t)} \left(\frac{\partial u}{\partial t}(x,t)\nabla u(x,t)\cdot N(x,t) - \frac{1}{2}\left(\frac{\partial u}{\partial t}(x,t)\right)^2 - \frac{1}{2}\left(\nabla u(x,t)\right)^2\right) d\sigma(x) \\
&= \int_{\partial B(x_0,t_0-t)} \left(\frac{\partial u}{\partial t}(x,t)\nabla u(x,t)\cdot N(x,t) - \frac{1}{2}\left(\frac{\partial u}{\partial t}(x,t)\right)^2 - \frac{1}{2}(\nabla u(x,t))^2\right) d\sigma(x).
\end{aligned}
$$

Since the outer normal has length one we derive

$$\frac{\partial u}{\partial t}(x,t)\nabla u(x,t)\cdot N(x,t) \le \frac{1}{2}\left(\frac{\partial u}{\partial t}(x,t)\right)^2 + \frac{1}{2}(\nabla u(x,t))^2$$

with $a = \nabla u(x,t)$ and $b = \frac{\partial u}{\partial t}(x,t)N(x,t)$ from the following inequality:

$$a\cdot b \le a\cdot b + \frac{1}{2}(a-b)\cdot(a-b) = \frac{1}{2}a^2 + \frac{1}{2}b^2.$$

So by $\dot{e}(t) \le 0$ the energy is monotonically decreasing. Because the energy is non–negative and vanishes initially it stays zero for all positive times in $t \in [0, t_0]$. This implies $u = 0$ on $\{(x,t) \mid |x - x_0| \le t_0 - t, t > 0\}$. □

By the invariance with respect to time reversal we can also deduce the vanishing of u on the cone $\{(x,t) \mid |x - x_0| < t_0 + t, t < 0\}$ from the vanishing of u and $\frac{\partial u}{\partial t} = 0$ on $(x,t) \in B(x_0,t_0) \times \{0\}$.