Introduction to Partial Differential Equations
Exercise sheet 10

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30. Special solutions of the heat equation

(a) Solutions of PDEs that are constant in the time variable are called "steady-state" solutions. Describe steady-state solutions of the inhomogeneous heat equation. *(1 point)*

(b) Consider the heat equation $\dot{u} - \Delta u = 0$ on $\mathbb{R}^n \times \mathbb{R}^+$ with smooth initial condition $u(x, 0) = h(x)$. Suppose that the Laplacian of h is a constant. Show that there is a solution whose time derivative is constant. *(1 point)*

(c) Consider "translational solutions" to the heat equation on $\mathbb{R} \times \mathbb{R}^+$ (ie $n = 1$). These are solutions of the form $u(x, t) = F(x - bt)$. Find all such solutions. *(2 points)*

(d) If u is a solution to the heat equation, show for every $\lambda \in \mathbb{R}$ that $u_\lambda(x, t) := u(\lambda x, \lambda^2 t)$ is also a solution to the heat equation. *(2 points)*

31. The Fourier transform

In this question we expand on some details from Section 4.1. Recall that the Fourier transform of a function $h(x) : \mathbb{R}^n \to \mathbb{R}$ is defined to be a function $\hat{h}(k) : \mathbb{R}^n \to \mathbb{R}$ given by

$$\hat{h}(k) = \int_{\mathbb{R}^n} e^{-2\pi i k \cdot x} h(x)\, dx.$$

Lemma 4.3 shows that it is well-defined for Schwartz functions.

(a) Give the definition of a Schwartz function. *(1 point)*

(b) Argue that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \exp(-x^2)$ is a Schwartz function. *(2 points)*

(c) Consider

$$I^2 = \left(\int_{\mathbb{R}} e^{-x^2}\, dx \right)^2 = \left(\int_{\mathbb{R}} e^{-x^2}\, dx \right) \left(\int_{\mathbb{R}} e^{-y^2}\, dy \right) = \int_{\mathbb{R}^2} e^{-x^2 - y^2}\, dx\, dy.$$

By changing to polar coordinates, compute this integral. *(1 point)*

(d) Prove the rescaling law for Fourier transforms: if $h(x) = g(ax)$ then *(1 point)*

$$\hat{h}(k) = |a|^{-n} \hat{g}(a^{-1} k).$$

(e) Prove the shift law for Fourier transforms: if $h(x) = g(x - a)$ then *(1 point)*

$$\hat{h}(k) = e^{-2\pi i a \cdot k} \hat{g}(k).$$

(f) Show that δ is a tempered distribution. *(2 points)*

(g) Compute the Fourier transform of δ. *(2 points)*

(h) Try to compute the Fourier transform of 1. What is the difficulty?

(2 points)

32. One step at a time

Prove the following identity for the fundamental solution in one dimension ($n = 1$):

$$\Phi(x, s+t) = \int_{\mathbb{R}} \Phi(x-y, t)\Phi(y, s)\, dy.$$

(2 points)

Hint. You may use without proof that

$$\int_{\mathbb{R}} \exp(-A + By - Cy^2)\, dy = \sqrt{\frac{\pi}{C}} \exp\left(\frac{B^2}{4C} - A\right).$$