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27. To be or not to be

Consider the Dirichlet problem for the Laplace equation $\Delta u = 0$ on Ω with u = g on $\partial \Omega$, where $\Omega \subset \mathbb{R}^n$ is an open and bounded subset and g is a continuous function. We know from the weak maximum principle that there is at most one solution. In this question we see that for some domains, existence is not guaranteed.

- (a) Consider $\Omega = B(0,1) \setminus \{0\}$, so that the boundary $\partial \Omega = \partial B(0,1) \cup \{0\}$ consists of two components. We write $g(x) = g_1(x)$ for $x \in \partial B(0,1)$ and $g(0) = g_2$. Show that there does not exist a solution for $g_1(x) = 0$ and $g_2 = 1$.
 - Hint. Use Lemma 3.23, even if we haven't reached it in lectures yet. (3 points)
- (b) Generalise this: What are the necessary and sufficient conditions on g for the Dirichlet problem to have a solution on this domain? (3 points)
- (c) Generalise again: What can you say about the Dirichlet problem for bounded domains whose boundaries have isolated points?

 (1 point)

28. Do nothing by halves

Let $H_1^+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 > 0\}$ be the upper half-space and $H_1^0 = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 = 0\}$ the dividing hyperplane. We call $R_1(x) = (-x_1, x_2, \dots, x_n)$ reflection in the plane H^0 .

(a) A reflection principle for harmonic functions Let $u \in C^2(\overline{H_1^+})$ be a harmonic function that vanishes on H_1^0 . Show that the function $v : \mathbb{R}^n \to \mathbb{R}$ defined through reflection

$$v(x) = \begin{cases} u(x) & \text{for } x_1 \ge 0\\ -u(R_1(x)) & \text{for } x_1 < 0 \end{cases}$$

is harmonic. (3 points)

(b) Green's function for the half-space Show that Green's function for H_1^+ is

$$G(x,y) = \Phi(x-y) - \Phi(R_1(x) - y).$$

(2 points)

(c) Green's function for the half-ball Compute the Green's function for B^+ . (3 points) Hint. Make use of both the Green's function for the ball and part (b).

29. Teach a man to fish

(a) Using the Green's function of H_1^+ from the previous question, derive the following formal integral representation for a solution of the Dirichlet problem $\Delta u = 0$ in $H_1^+, u|_{H_1^0} = g$

$$u(x) = \frac{2x_1}{n\omega_n} \int_{H_1^0} \frac{g(z)}{|x - z|^n} \, \mathrm{d}\sigma(z)$$

Here, 'formal' means that you do not need to prove that the integrals are finite/well-defined.

(3 points)

- (b) Show that if g is periodic (that is, there is some vector $L \in \mathbb{R}^{n-1}$ with g(x+L) = g(x) for all $x \in \mathbb{R}^{n-1}$) then so is the solution. (2 points)
- (c) Now consider the plane n=2 with g function with compact support. Approximate the value of u(x) for large |x|. Feel free to modify this question as you see fit, what interesting things can you say about the growth of u? (Bonus Points as deserved)