

13. Is this an applied math course?

In economics, the Black-Scholes equation is a PDE that describes the price V of an (European-style) option under some assumptions about the risk and expected return, as a function of time t and current stock price S . The equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS \frac{\partial V}{\partial S},$$

where r and σ are constants representing the interest rate and the stock volatility respectively. Describe the order of this equation, and whether it is elliptic, parabolic, and/or hyperbolic.

(3 points)

14. Around and around

Consider the unit circle $C = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$. In this question we will evaluate the integral

$$\int_C y \, d\sigma \quad (*)$$

in two different ways, demonstrating that it does not depend on the choice of parametrisation.

- (a) Consider a regular parameterisation Φ of a subset A and a continuous function f on A . Why (or under what conditions) is the integral unchanged by removing a point from A :

(1 bonus point)

$$\int_A f \, d\sigma = \int_{A \setminus \{a\}} f \, d\sigma.$$

Therefore in order to compute the submanifold integration $(*)$ it is enough to use parametrisation that cover all but finitely many points of C .

- (b) Consider the regular parametrisation $\Phi : (0, 2\pi) \rightarrow C$ given by $t \mapsto (\cos t, \sin t)$. Compute the integral $(*)$ using this parametrisation.

(2 points)

- (c) Consider upper and lower halves of the circle: $U_1 = \{(x, y) \in C \mid y > 0\}$ and $U_2 = \{(x, y) \in C \mid y < 0\}$. There are obvious parametrisations $\Phi_i : (-1, 1) \rightarrow U_i$ given by $\Phi_1(x) = (x, +\sqrt{1-x^2})$ and $\Phi_2(x) = (x, -\sqrt{1-x^2})$. Compute $(*)$ using these parametrisations.

(2 points)

- (d) Compute this integral using the divergence theorem.

(2 points)

15. The Proof is Left as an Exercise for the Reader

Using Definitions 2.3 and 2.4, prove the following properties for the submanifold integral: The following properties hold for $a, b \in \mathbb{R}$ and $f, g \in C(A)$.

- (i) Linearity: $\int_A af + bg \, d\sigma = a \int_A f \, d\sigma + b \int_A g \, d\sigma$.

(1 point)

- (ii) Order Preserving: if $f \leq g$ on A then $\int_A f \, d\sigma \leq \int_A g \, d\sigma$.

(1 point)

- (iii) Triangle Inequality: $|\int_A f \, d\sigma| \leq \int_A |f| \, d\sigma$.

(1 point)

- (iv) Transformation: If $P : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a euclidean motion (translation, reflection, rotation) and $s \in \mathbb{R}^+$ is a scaling factor then $\int_A f d\sigma = s^k \int_{(sP)^{-1}[A]} f \circ (sP) d\sigma$. (3 points)

16. The Black Spot

Consider the plane $\mathbb{R}^2$, a disc $B_r = \{x^2 + y^2 \leq r^2\}$ and the function $g(x, y) = \ln(x^2 + y^2)$.

- (a) By calculating $\nabla g \cdot N$ for the outward pointing normal $N = \frac{1}{r}(x, y)$, show that the value of the integral

$$\int_{\partial B_r} \nabla g \cdot N d\sigma$$

does not depend on the radius r .

(2 points)

- (b) Can you explain this fact using the divergence theorem?

Hint: Apply the divergence theorem to the region $A_{r,R} = B_R \setminus \overline{B_r}$ for two different radii $r < R$. (3 points)