Introduction to Partial Differential Equations
Exercise sheet 3

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7. Royale with Cheese

Recall Burgers' equation from Example 1.5 of the lecture script:

$$\dot{u} + u\partial_x u = 0,$$

for $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. In this question we will apply the method of characteristics to solve this equation for the initial condition $g(x) = 2x$.

(a) According to Theorem 1.4, there is a unique C^1 solution to this initial value problem, at least when t is small. For how long does the theorem guarantee that the solution exists uniquely? *(1 point)*

(b) Suppose that u is a solution to this equation and suppose that $(x(s), t(s))$ is a path in the domain of u. What is the s derivative of u along this path? What constraints should we place on the derivatives of x and t? *(2 points)*

(c) On an (x, t)-plane draw the characteristics. *(1 point)*

(d) Finally, derive the following solution to the initial value problem: *(2 points)*

$$u(x,t) = \frac{2x}{1+2t}.$$

8. Linear Partial Differential Equations Consider a PDE of the form $F(\nabla u(x), u(x), x) = 0$. Suppose that F is linear in the derivatives and has continuously differentiable coefficients. That is, it can be written in the form

$$F(p, z, x) = b(z, x) \cdot p + c(z, x)$$

with b and c continuously differentiable. Show that the characteristic curves $(x(s), z(s))$ for $z(s) := u(x(s))$ can be described by ODEs that are independent of $p(s) := \nabla u(x(s))$. *(4 points)*

9. Solving PDEs Solve the initial value problems of the following PDEs using the method of characteristics. You may assume that g is continuously differentiable on the corresponding domain.

(a) $x_2\partial_1 u - x_1\partial_2 u = u$ on the domain $x_1, x_2 > 0$, with initial condition $u(x_1, 0) = g(x_1)$. *(3 points)*

(b) $x_1\partial_1 u + 3x_2\partial_2 u + \partial_3 u = 2u$ on $x_1, x_2 \in \mathbb{R}$, $x_3 > 0$, with initial condition $u(x_1, x_2, 0) = g(x_1, x_2)$. *(3 points)*

(c) $u\partial_1 u + \partial_2 u = 1$ on the domain $x_1, x_2 > 0$, with initial condition $u(x_1, x_1) = \frac{1}{2}x_1$. *(4 points)*