4.5 The heat kernel of (0,1) $\subseteq \mathbb{R}$

$e^{2\pi i k x}$ $\quad$ $\cancel{\cos(2\pi k x)}$ $\quad$ $\underline{\sin(2\pi k x)}$ $\quad$ $\lambda = 4\pi^2 k^2$

The heat kernel is zero on the $\partial\Omega = \{0,1\}$

$$\sin(2\pi k x) \quad = 0 \text{ at } x=0$$
$$= 0 \text{ at } x=1 \quad \text{if } k \in \tfrac{1}{2}\mathbb{Z}$$

$$h_k(x) = \sqrt{2}\,\sin(2\pi k x) \qquad k \in \tfrac{1}{2}\mathbb{N}^+$$

$L^2((0,1))$.

$$\langle h_k(x), h_\ell(x)\rangle_{L^2((0,1))} = \int_{(0,1)} 2\sin(2\pi k x)\sin(2\pi \ell x)\,dx$$

$$= \int_{(0,1)} \cos 2\pi(k-\ell)x - \underbrace{\cos 2\pi\underbrace{(k+\ell)}_{>0}x}_{\text{integrates to } 0} \quad dx$$

$$= \begin{cases} k \neq \ell & 0 \\ k = \ell & 1 \end{cases}$$

$$u_k(x,t) = e^{-4\pi^2 k^2 t}\,h_k(x)$$

is a sol^n to homog heat
is zero at $x=0,1$
is $h_k(x)$ at $t=0$

We make an ansatz $H_{(0,1)}(x,y,t) = \sum_{k\in\frac{1}{2}\mathbb{N}^+} a_k(x,t)\,h_k(y)$.

Use representation formula to u_k

$$u_k(x,t) = \iint_\Omega H \underbrace{(\partial_t - \Delta) u_k}_{0} - \iint_{\partial\Omega} \underbrace{u}_{0} \nabla H \cdot N + \int_\Omega H \underbrace{u(y,0)}_{h_k} dy.$$

$$= 0 - 0 + \int_\Omega \left[\sum_\ell a_\ell(x,t) h_\ell(y)\right] h_k(y)\, dy.$$

$$= \sum_\ell a_\ell(x,t) \int_{(0,1)} h_\ell(y) h_k(y)\, dy$$

$$= a_k(x,t)$$

Hence

$$H_{(0,1)}(x,y,t) = \sum_{k \in \frac{1}{2}\mathbb{N}^+} u_k(x,t) h_k(y)$$

$$= \sum_{n=1}^{\infty} 2 e^{-\pi^2 n^2 t} \sin(\pi n x) \sin(\pi n y) .$$

$$= \frac{1}{2} \Theta\left(\frac{x-y}{2}, \pi i t\right) - \frac{1}{2} \Theta\left(\frac{x+y}{2}, \pi i t\right)$$

$$\Theta(z,\tau) = \sum_{k \in \mathbb{Z}} \exp\left(2\pi i k x + \pi i \tau k^2\right)$$

Jacobi theta function

Heat kernel for (a,b)

$$\Phi(x-y,t) = \frac{1}{r^n}\Phi\left(\frac{x}{r}-\frac{y}{r},\frac{t}{r^2}\right)$$

$$= \frac{1}{r^n}\Phi\left(\frac{x-a}{r}-\frac{y-a}{r},\frac{t}{r^2}\right)$$

$$H_{(a,b)}(x,y,t) = \frac{1}{b-a}H_{(0,1)}\left(\frac{x-a}{b-a},\frac{y-a}{b-a},\frac{t}{(b-a)^2}\right)$$

eg $x=a \longmapsto \frac{a-a}{b-a}=0$

$x=b \longmapsto \frac{b-a}{b-a}=1.$

A box $\Omega=(a_1,b_1)\times(a_2,b_2)\cdots\times(a_n,b_n)\subset\mathbb{R}^n$

We know $H_\Omega = H_{(a_1,b_2)}H_{(a_2,b_2)}\cdots H_{(a_n,b_n)}$

Cor 4.20

Any soln to homog heat eqn is a smooth function and for fixed t analytic function in x.

proof $(x,t)\in\Omega_t$ assume $\Omega=(0,1)^n$

$$u(x,t)=0-\int_0^t\int_{\partial(0,1)^n} u(y,s)\,\nabla \overset{g}{H}^{(x,t)}\cdot N\,d\sigma\,ds + \int_{(0,1)^n} u(y,0)\,H_{(0,1)}^{(x,t)}\,dy.$$

For fixed $t>0$, $H_{(0,1)}$ is analytic in x.

$$H_{(0,1)} = \sum_{n=1}^{\infty} 2e^{-\pi^2n^2t}\sin(\pi n x)\sin(\pi n y).$$

$$= \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} 2e^{-\pi^2n^2t}\frac{(\pi n)^{2m+1}(-1)^m}{(2m+1)!}\sin\pi n y\right)x^{2m+1}$$

At $t=0$

$$H_{(0,1)} = \sum_{n=1}^{\infty} 2e^{-\pi^2 n^2 t} \sin(\pi n x)\sin(\pi n y).$$

$$H^0 = H_{(0,1)}(x,y,0) = \sum 2\sin(\pi n x)\sin(\pi n y)$$

$$H^1 = (\partial_t H)(x,y,0) = \sum 2(-\pi^2 n^2)\cdot 1 \cdot \sin(\pi n x)\sin(\pi n y)$$

H is not analytic at $t=0$ in t. $\square$

Chapter 4.

Fourier analysis for Schwartz functions
extended it to tempered distributions

heat eqn for u $\xrightarrow{\mathcal{F}}$ ODE of $\hat{u}$ in t $\xrightarrow{\text{solve}}$ $\hat{u} = \dots$

$\xrightarrow{\mathcal{F}^{-1}}$ get u.

This gave us soln to the Cauchy.

Maximum principle $\Rightarrow$ uniqueness of Dirichlet problem
& non-monster Cauchy.

Existence through heat kernels.

$H_{(0,1)}$ found by Fourier analysis $\square$