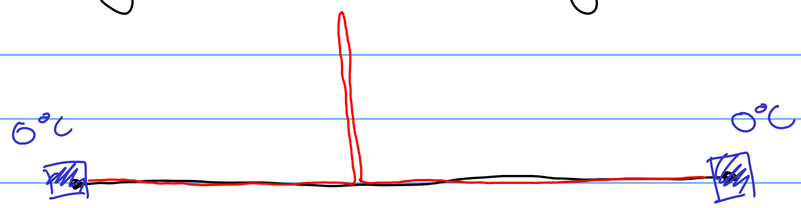
4.4 Heat Kernels

A temperature spike "hot particle" at x and boundary domain should stay $0°C$.



Def 4.14

For open bounded $\Omega \subseteq \mathbb{R}^n$

$$H_\Omega : \Omega \times \Omega \times \mathbb{R}^+ \to \mathbb{R}$$

(i) For every $x \in \Omega$ $\quad (y,t) \mapsto H_\Omega(x,y,t) - \Phi(x-y,t)$ is a homog sol^n to heat eqn and extends to $t=0$ continuously and is zero there.

(ii) For $(x,t) \in \Omega \times \mathbb{R}^+$ $\quad y \mapsto H_\Omega(x,y,t)$ extends cts to 0 on $\partial\Omega$

Lemma 4.15 On bounded domain Ω the heat kernel is unique

proof

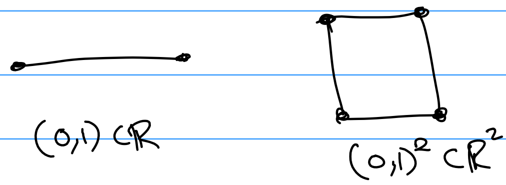
$V(y,t) = H_\Omega(x,y,t) - \Phi(x-y,t)$ for any $x \in \Omega$

$$\begin{cases} (\partial_t - \Delta) V = 0 & \text{on } \Omega_T \\ V = -\Phi(x-y,t) & \text{on } \partial\Omega \times [0,T] \\ V(y,0) = 0 & t = 0 \end{cases}$$

has a unique sol^n. $\square$

If you know H_Ω $\Omega \subseteq \mathbb{R}^n$ and $H_{\Omega'}$ $\Omega' \subset \mathbb{R}^m$
there is a nice formula for heat kernel $\Omega \times \Omega' \subseteq \mathbb{R}^{n+m}$.

eg $(0,1) \subseteq \mathbb{R} \rightsquigarrow (0,1)^n \subset \mathbb{R}^n$



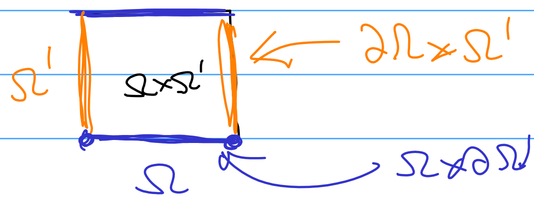
Lemma 4.16

$$H_{\Omega\times\Omega'}\big((x,x'),(y,y'),t\big) = H_\Omega(x,y,t)\,H_{\Omega'}(x',y',t)$$

<u>proof</u>

(ii) is immediate if $y \in \partial\Omega$ or $y' \in \partial\Omega'$ then $H_{\Omega\times\Omega'} = 0$.

$$\partial(\Omega\times\Omega') = (\partial\Omega\times\Omega') \cup (\Omega\times\partial\Omega')$$



$$\partial_t H_{\Omega\times\Omega'} = \partial_t\left(H_\Omega H_{\Omega'}\right) = \underline{\partial_t H_\Omega\, H_{\Omega'}} + \underline{H_\Omega\, \partial H_{\Omega'}}$$

$$\Delta_{\mathbb{R}^{n+m}} = \underbrace{\frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_n^2}} + \underbrace{\frac{\partial^2}{\partial {y'_1}^2} + \cdots + \frac{\partial^2}{\partial {y'_m}^2}}$$

$$= \Delta_{\mathbb{R}^n} + \Delta_{\mathbb{R}^m}$$

$$\Delta_{\mathbb{R}^{n+m}} H_{\Omega\times\Omega'} = \Delta_{\mathbb{R}^n}\left(H_\Omega H_{\Omega'}\right) + \Delta_{\mathbb{R}^m}\left(H_\Omega H_{\Omega'}\right)$$

$$= \underline{\left(\Delta_{\mathbb{R}^n} H_\Omega\right) H_{\Omega'}} + \underline{H_\Omega\left(\Delta_{\mathbb{R}^m} H_{\Omega'}\right)}$$

$$\left(\partial_t - \Delta_{\mathbb{R}^{n+m}}\right) H_{\Omega\times\Omega'} = \underline{\left(\partial_t H_\Omega - \Delta_{\mathbb{R}^n} H_\Omega\right)} H_{\Omega'} + \underline{H_\Omega\left(\partial_t H_{\Omega'} - \Delta_{\mathbb{R}^m} H_{\Omega'}\right)}$$

$$= 0\, H_{\Omega'} + H_\Omega\, 0$$

$$= 0$$

What is the fundamental sol^n for $\mathbb{R}^{n+m}$?

$|(x,x')|^2 = |x|^2 + |x'|^2$
pythag.

$$\Phi_{\mathbb{R}^{n+m}} = \frac{1}{(4\pi t)^{\frac{n+m}{2}}} \exp\left(-\frac{|x|^2+|x'|^2}{4t}\right)$$

$$= \underline{\frac{1}{(4\pi t)^{n/2}}}\ \underline{\frac{1}{(4\pi t)^{m/2}}}\ \underline{\exp\left(-\frac{|x|^2}{4t}\right)}\ \underline{\exp\left(-\frac{|x'|^2}{4t}\right)}$$

$$= \Phi_{\mathbb{R}^n}(x,t)\ \Phi_{\mathbb{R}^m}(x,t).$$

$$H_{\Omega\times\Omega'} - \Phi_{\mathbb{R}^{n+m}} = H_\Omega H_{\Omega'} - \Phi_{\mathbb{R}^n}\Phi_{\mathbb{R}^m}$$

$$= (\underbrace{H_\Omega - \Phi_{\mathbb{R}^n}}_{0})(\underbrace{H_{\Omega'} - \Phi_{\mathbb{R}^m}}_{0})$$

$$+ \Phi_{\mathbb{R}^n}(\underbrace{H_{\Omega'} - \Phi_{\mathbb{R}^m}}_{0}) + (\underbrace{H_\Omega - \Phi_{\mathbb{R}^n}}_{0.})\Phi_{\mathbb{R}^m}$$

As $t \to 0$

extends continuously with value 0.

$\boxed{\cdot}$.

Start with Green's second formula $v(y,s) = H_\Omega(x,y,t-s)$
treat x,t,s as additional parameters.

$$\int_0^{t-\varepsilon}\int_\Omega H(x,y,t-s)\,\Delta_y u(y,s) - \Delta_y H(x,y,t-s)\,u(y,s)\,dy\,ds$$

$$= \int_0^{t-\varepsilon}\int_{\partial\Omega}\left[\cancel{H(x,y,t-s)\nabla_y u(y,s)} - \nabla_y H(x,y,t-s)\,u(y,s)\right]\cdot N(y)\,d\sigma(y)\,ds$$

we need $\int_{(0,T)}\int_\Omega H\left(\partial_s u - \Delta u\right)$

$$\int_0^{t-\varepsilon}\int_\Omega H(x,y,t-s)\,\partial_s u(y,s)\,dy\,ds.$$

$$= \int_0^{t-\varepsilon}\int_\Omega \partial_s\big(H(x,y,t-s)\,u(y,s)\big) + \partial_s H(x,y,t-s)\,u(u,s)\,dy\,ds$$

$$= \int_\Omega H(x,y,t-s)\,u(y,s)\,dy\,\Big|_{s=0}^{s=t-\varepsilon} + \iint \partial_s H(x,y,t-s)\,u(u,s)\,dy\,ds$$

$$= \int_\Omega H(x,y,\varepsilon)\,u(y,t-\varepsilon)\,dy - \int_\Omega H(x,y,t)\,u(y,0)\,dy + \iint \dots$$

Subtract them!

$$\int_0^{t-\varepsilon}\int_\Omega H(x,y,t-s)\left[\partial_s u(y,s) - \Delta_y u(y,s)\right]dy\,ds$$

$$+\iint\left[\cancel{-\partial_s H(x,y,t-s)} + \Delta_y H(x,y,t-s)\right]u(y,s)\,dy\,ds$$

$$= \int_0^{t-\varepsilon}\int_{\partial\Omega} + \nabla_y H(x,y,t-s)\,u(y,s)\cdot N(y)\,d\sigma(y)\,ds$$

$$+ \int_\Omega H(x,y,\varepsilon)\,u(y,t-\varepsilon)\,dy - \int_\Omega H(x,y,t)\,u(y,0)\,dy$$

$$\int_0^{t-\varepsilon}\int_\Omega H(x,y,t-s)\left[\partial_s u(y,s) - \Delta_y u(y,s)\right] dy\, ds$$

$$= \int_0^{t-\varepsilon}\int_{\partial\Omega} \nabla_y H(x,y,t-s)\, u(y,s) \cdot N(y)\, d\sigma(y)\, ds$$
$$+ \int_\Omega H(x,y,\varepsilon)\, u(y,t-\varepsilon)\, dy - \int_\Omega H(x,y,t)\, u(y,0)\, dy$$

Consider

$$\lim_{\varepsilon\to 0} \int_\Omega H(x,y,\varepsilon)\, u(y,t-\varepsilon)\, dy$$

$$= \lim_{\varepsilon\to 0} \int_\Omega \underbrace{\left[H(x,y,\varepsilon) - \Phi(x-y,\varepsilon)\right]}_{\to 0} u(y,t-\varepsilon) + \underbrace{\Phi(x-y,\varepsilon) u(y,t-\varepsilon)}_{\text{Thm 4.7 (iii)}}\, dy$$

$$= u(x, t-0) = u(x,t)$$

Rearrange

$$u(x,t) = \int_0^{t-\varepsilon}\int_\Omega H(x,y,t-s)\overbrace{\left[\partial_s u(y,s) - \Delta_y u(y,s)\right]}^{f} dy\, ds$$

$$- \int_0^{t-\varepsilon}\int_{\partial\Omega} \nabla_y H(x,y,t-s)\, \overbrace{u(y,s)}^{g} \cdot N(y)\, d\sigma(y)\, ds + \int_\Omega H(x,y,t)\, \overbrace{u(y,0)}^{h}\, dy$$

Representation for functions adapted to heat equation.

$$(\partial_t - \Delta) u = f$$
$$u = g \quad \text{on } \partial\Omega\times(0,T)$$
$$u = h \quad \text{on } \Omega\times\{t=0\}$$

Lemma 4.18 Symmetry in x and y.

proof. $u(y,s) = H(z,y,s)$

$$H(z,x,t) = \int_0^{t-\varepsilon}\int_\Omega H(x,y,t-s)\left[\cancel{\partial_s u(y,s) - \Delta_y u(y,s)}\right] dy\, ds$$

$$- \int_0^{t-\varepsilon}\int_{\partial\Omega} \nabla_y H(x,y,t-s)\, \overbrace{\cancel{u(y,s)}}^{g} \cdot N(y)\, d\sigma(y)\, ds + \int_\Omega H(x,y,t)\, \overbrace{u(y,0)}^{h}\, dy$$

$$= \lim_{\varepsilon\to 0}\int_\Omega H(x,y,t)\, H(z,y,\varepsilon)\, dy.$$

thm 4.7 (iii)

$$= H(x,z,t)$$

$\square$

We would also need regularity of homog sol^n to heat equation

$$H = \underbrace{(H - \Phi)}_{\text{smooth}} + \Phi$$