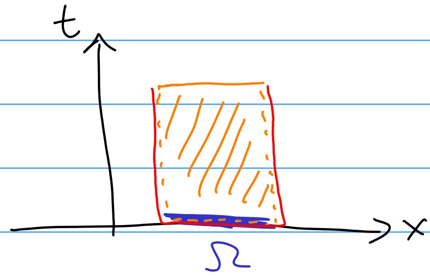
4.3 Maximum principle

Bounded $\Omega \subseteq \mathbb{R}^n$

$\Omega \times (0,T)$

Ω_T "parabolic cylinder" $= \Omega \times (0,T]$



$\partial \Omega_T = \overline{\Omega_T} \setminus \Omega_T$

Theorem 4.11 Weak maximum principle.

Ω open, bounded $\subset \mathbb{R}^n$

$u \in C^2(\Omega_T)$

Suppose $\partial_t u - \Delta u \leq 0$ on Ω_T

Then the max of u occurs on $\partial \Omega_T$.

Proof Similar to theorem 3.13

Step 1. Define $u_\varepsilon = u(x,t) - \varepsilon t$

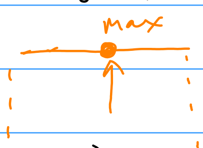
$$(\partial_t - \Delta) u_\varepsilon = \underbrace{(\partial_t - \Delta) u}_{\leq 0} - \underbrace{(\partial_t - \Delta)(\varepsilon t)}_{\varepsilon} < 0$$

Step 2. Suppose $(x_0, t_0) \in \Omega_T$ is a max

Case 1. $t_0 < T$

Because it max $\partial_t u_\varepsilon(x_0, t_0) = 0$

Case 2. $t_0 = T$



$\partial_t u_\varepsilon(x_0, t_0) \geq 0$

$\nabla_x u_\varepsilon(x_0, t_0) = 0$ looking at $x \mapsto u_\varepsilon(x, t_0)$ local max at x_0

$\text{Hess}_x\, u_\varepsilon(x_0, t_0)$ is negative semi def.

By Theorem 3.13

$$\Delta_x u_\varepsilon(x_0,t_0) \le 0$$

$$\underbrace{\partial_t u_\varepsilon(x_0,t_0)}_{\geq 0} - \underbrace{\Delta_x u_\varepsilon(x_0,t_0)}_{\geq 0} > 0 \quad \text{⚡}$$

Hence u_ε has no max on Ω_T.

Step 3.

$$\max_{\overline{\Omega}_T} u = \max_{\overline{\Omega}_T} (u_\varepsilon + \varepsilon t) \le \max_{\overline{\Omega}_T} u_\varepsilon + \underbrace{\max_{\overline{\Omega}_T} \varepsilon t}_{\varepsilon T}$$

$$= \max_{\partial \Omega_T} u_\varepsilon + \varepsilon T$$

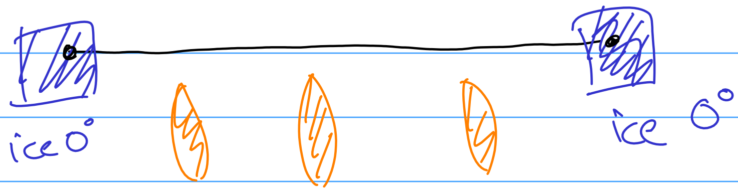
$$\le \max_{\partial \Omega_T} u + \varepsilon T$$

lim $\varepsilon \to 0$

$$\max_{\overline{\Omega}_T} u \le \max_{\partial \Omega_T} u \qquad \square$$

Dirichlet problem of heat equation Ω bounded

$$\partial_t u - \Delta u = f \quad \text{on } \Omega_T$$
$$u(x,t) = g(x,t) \quad \text{for } x \in \partial\Omega \text{ and } t \in (0,T) \quad \text{spatial}$$
$$u(x,0) = h(x) \quad \text{for } x \in \Omega \quad \text{initial}$$



$g = 0$.

Thm 4.12

The solution to the Dirichlet problem is unique.

proof.

If two sol^n u_1, u_2 Define $v = u_2 - u_1$

$$(\partial_t - \Delta) v = f - f = 0$$

on $\partial\Omega \times (0,T)$ $\quad v = g(x,t) - g(x,t) = 0$

on $\Omega \times \{0\}$ $\quad v = h(x) - h(x) = 0$.

So by weak max principle
min.

$$0 = \min_{\partial\Omega_T} v \leq v \leq \max_{\partial\Omega_T} v = 0 \Rightarrow v \equiv 0. \quad \square$$

Comparison principle

$$\partial_t u_1 - \Delta u_1 = f_1 \qquad \partial_t u_2 - \Delta u_2 = f_2$$

on $\partial\Omega \times (0,T)$ $\quad u_1 = g_1 \qquad u_2 = g_2$

on $\Omega \times \{0\}$ $\quad u_1 = h_1 \qquad u_2 = h_2$.

Assume $\underline{h_1 \geq h_2}$ $\quad \underline{g_1 \geq g_2}$ and $\underline{f_1 \geq f_2}$.

Wire 1 starts hotter than wire 2.

At all times the ends of wire 1 are kept hotter than the ends of wire 2.

We add more heat to wire 1 than wire 2.

$$\Rightarrow \quad u_1(x,t) \geq u_2(x,t) \qquad (x,t) \in \Omega_T$$

$V = u_2 - u_1$

$$(\partial_t - \Delta)V = f_2 - f_1 \leq 0$$

on $\partial\Omega \times (0,T)$ $\quad V = g_2 - g_1 \leq 0$

$\Omega \times \{0\}$ $\quad V = h_2 - h_1 \leq 0$

$V \leq 0$ by weak max principle $\Rightarrow$ $u_2 \leq u_1$.

Cauchy problem

$u \equiv 0$ $\qquad h=0$ $\qquad$ not have unique sol^{n}s.

$u =$ monster $\qquad h=0$

Theorem 4.13

Let u be a sol^n on $\mathbb{R}^n \times (0,T]$ of

$\dot{u} - \Delta u = 0 \qquad u(x,0) = 0$

that has the bound

$$|u(x,t)| \leq M e^{A|x|^2} \quad \text{for constants } M \text{ and } A.$$

Then $u \equiv 0$.

proof

Choose $a > A$ $\quad$ we will prove it on $t \in [0, \frac{1}{4a}]$

If we prove this then repeat on $[\frac{1}{4a}, \frac{2}{4a}]$ and $[\frac{2}{4a}, \frac{3}{4a}]$ and ...

Consider the family of functions

$$V_R(x,t) = \frac{Me^{-(a-A)R^2}}{(1-4at)^{n/2}} \exp\left(\frac{a|x|^2}{1-4at}\right)$$

- V_R is a sol^n to homogeneous heat equation
- Defined. $\quad t \in [0, 1/4a) \quad x \in \mathbb{R}^n$
- Positive

• On $x \in \partial B(0,R)$

$$v_R(x,t) = \frac{Me^{-(a-A)R^2}}{(1-4at)^{n/2}} \exp\left(\frac{aR^2}{1-4at}\right)$$

$$\geq \frac{Me^{-(a-A)R^2}}{(1-0)^{n/2}} \exp\left(\frac{aR^2}{1-0}\right)$$

$$= M e^{-aR^2 + AR^2} e^{aR^2}$$

$$= M e^{AR^2}$$

$$\geq |u(x,t)| \qquad \text{for } x \in \partial B(0,R)$$

$$-v_R \leq u \leq v_R \qquad \text{for } x \in \partial B(0,R)$$

$$-v_R < 0 \equiv u < v_R \qquad \text{at } t=0$$

$$\xrightarrow[\text{principle}]{\text{max}} \quad -v_R \leq u \leq v_R \quad \text{on all of } \overline{B(0,R)} \times \left[0, \tfrac{1}{4a}\right]$$

Choose any point $(x,t) \in \mathbb{R}^n \times \left(0, \frac{1}{4a}\right)$
for all $R \geq |x|$

$$|u(x,t)| \leq v_R(x,t) = \frac{Me^{-(a-A)R^2}}{(1-4at)^{n/2}} \exp\left(\frac{a|x|^2}{1-4at}\right)$$

take limit $R \to \infty$

$$|u(x,t)| \leq \frac{M \cdot 0}{(\ )^{n/2}} \exp(\ ) = 0.$$

ie $u \equiv 0$.

$\square$

Cor 4.10 is a formula for the solⁿ of Cauchy.
Does it obey this bound?

Suppose $|h(x)| \le M e^{A|x|^2}$ $\quad |f(x,t)| \le M e^{A|x|^2}$

$$\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right)$$

$$= \frac{2^{n/2}}{(4\pi(2t))^{n/2}} \exp\left(-2\frac{|x|^2}{8t}\right)$$

$$= 2^{n/2} \underbrace{\frac{1}{(4\pi(2t))^{n/2}} \exp\left(-\frac{|x|^2}{4(2t)}\right)}_{\Phi(x,2t)} \exp\left(-\frac{|x|^2}{4(2t)}\right)$$

If $t \le \frac{1}{16A} = T_0$ $\qquad \frac{1}{t} \ge 16A \Rightarrow -\frac{1}{t} \le -16A \Rightarrow \exp\left(-\frac{|x|^2}{8t}\right)$

$\le \exp\left(-\frac{|x|^2}{8} 16A\right)$

$$\le 2^{n/2} \Phi(x,2t) \exp(-2A|x|^2)$$

Term 1

$$\left|\int_{\mathbb{R}^n} \Phi(x-y,t) h(y)\, dy\right|$$

$$\le \int_{\mathbb{R}^n} 2^{n/2} \Phi(x-y,2t) \exp(-2A|x-y|^2) \quad M e^{A|y|^2} \quad dy$$

$$= 2^{n/2} M \int_{\mathbb{R}^n} \Phi(x-y,2t) \exp\left(-2A|x-y|^2 + A|y|^2\right) dy$$

$$-2A\left(|x|^2 - 2x\cdot y + |y|^2\right) + A|y|^2$$

$$= -A\left(2|x|^2 - 4x\cdot y + 2|y|^2 - |y|^2\right)$$

$$= -A\left(|2x|^2 - 2(2x)\cdot y + |y|^2 - 2|x|^2\right)$$

$$= -A|2x-y|^2 + 2A|x|^2$$

$$= 2^{n/2} M \int_{\mathbb{R}^n} \Phi(x-y, 2t) \underbrace{\exp\left(-A|2x-y|^2\right)}_{\leq 1} \exp\left(2A|x|^2\right) dy$$

$$\leq 2^{n/2} M e^{2A|x|^2} \underbrace{\int_{\mathbb{R}^n} \Phi(x-y, 2t)\, dy}_{1}$$

$$= 2^{n/2} M e^{2A|x|^2}$$

Term 2.

$$\left| \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\, dy\, ds \right|$$

$$\leq \int_0^t 2^{n/2} M e^{2A|x|^2}\, ds$$

$$= 2^{n/2} M e^{2A|x|^2}\, t$$

$$\leq 2^{n/2} M e^{2A|x|^2}\, T_0$$

$$|u(x,t)| \leq 2^{n/2} M e^{2A|x|^2} + 2^{n/2} M e^{2A|x|^2} T_0$$

$$= \underbrace{2^{n/2} M (1+T_0)}_{M'} e^{\overbrace{2A}^{A'}|x|^2}$$

Conclusion, the solution in Cor 4.10 is the unique non-monster soln of Cauchy problem.

Using a growth bound can only guarantee exist on a finite interval.

Exploding sol^n

$$u(x,t) = \frac{1}{(T-t)^{n/2}} \exp\left(\frac{|x|^2}{4(T-t)}\right)$$

$$u(x,0) = T^{-n/2} \exp\left(\frac{1}{4T} |x|^2\right)$$ exactly the growth bound above.

not defined at $t=T$.