Section 4.2 Fundamental Solution

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} *_x h(x)$$

Laplace equation $-\Delta u = f$ had sol^n $u = \Phi * f$

Definition 4.5 Fundamental sol^n of heat equation

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

- Solves the heat equation except at $(0,0)$ (exercise)
- $\Phi(ax, a^2 t) = a^{-n} \Phi(x,t)$
- Lemma 4.6 for $t>0$ $\int_{\mathbb{R}^n} \Phi(x,t)\,dx = 1$

proof

$$\int_{\mathbb{R}^n} \Phi(x,t)\,dx = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}\,dx$$

$y_i = \frac{x_i}{2\sqrt{t}}$ $dy = \frac{1}{(2\sqrt{t})^n} dx$

$|y|^2 = \frac{|x|^2}{4t}$

$$= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|y|^2} (\sqrt{4t})^n\,dy$$

$$= \frac{\cancel{(4t)}^{n/2}}{(\cancel{4}\pi \cancel{t})^{n/2}} \int_{\mathbb{R}^n} e^{-|y|^2}\,dy$$

$$= \frac{1}{\pi^{n/2}} \pi^{n/2}$$

$$= 1.$$

$\square$

Theorem 4.7

For $h \in C_b(\mathbb{R}^n, \mathbb{R})$ define $u(x,t) = \Phi(x,t) *_x h(x)$

$$= \int_{\mathbb{R}^n} \Phi(x-y,t)\, h(y)\, dy.$$

(i) $u \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+)$ $t>0$

(ii) $\dot{u} - \Delta u = 0$ on $\mathbb{R}^n \times \mathbb{R}^+$

(iii) u extends continuously to $\mathbb{R}^n \times [0,\infty)$ with $\lim_{t\to 0} u(x,t) = h(x)$.

In other words u is a smooth sol^n to

$$\left.\begin{aligned} &\dot{u} - \Delta u = 0 && \text{on } \mathbb{R}^n \times \mathbb{R}^+ \\ \text{and } &u(x,0) = h(x) && \text{on } \mathbb{R}^n \end{aligned}\right\} \text{Cauchy problem.}$$

$\partial(f * g) = (\partial f) * g$

Proof

$$(\partial_t - \Delta) u = \int_{\mathbb{R}^n} (\partial_t - \Delta_x)\left(\Phi(x-y,t) h(y)\right) dy.$$

$$= \int_{\mathbb{R}^n} \underbrace{\left[(\partial_t - \Delta_x)\Phi(x-y,t)\right]}_{0.} h(y)\, dy.$$

$$= 0.$$

(iii) Choose a compact set $K \subset \mathbb{R}^n$.

h is cts $\Rightarrow$ uniformly cts on K

$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x, y \in K$ if $|x-y| < \delta \Rightarrow |h(x) - h(y)| < \varepsilon$

$$\int_{\mathbb{R}^n \setminus B(0,\delta)} \Phi(y,t)\, dy = \int_{\mathbb{R}^n \setminus B(0,\delta)} t^{-n/2}\, \Phi\Big(\underbrace{\frac{y}{\sqrt{t}}}_{z}, 1\Big)\, dy.$$

$z = \frac{1}{\sqrt{t}} y \qquad dz = t^{-n/2}\, dy$

$y \notin B(0,\delta) \qquad z \notin B(0, \delta/\sqrt{t})$

$$= \int_{\mathbb{R}^n \setminus B(0, \delta/\sqrt{t})} \cancel{t^{-n/2}}\, \Phi(z,1)\, \cancel{t^{n/2}}\, dz$$

When t is small $\delta/\sqrt{t}$ is big so $B(0, \delta/\sqrt{t})$ is big

$\int_{\mathbb{R}^n} \Phi(z,1)\,dz = 1.$ As $t \to 0$ $\int_{B(0,\delta/\sqrt{t})} \Phi(z,1)\,dz \to 1.$

$\int_{\mathbb{R}^n \setminus B(0,\delta/\sqrt{t})} \Phi \to 0.$

$\exists T$ so that $0 < t < T$ $\int_{\mathbb{R}^n \setminus B(0,\delta/\sqrt{t})} \Phi(z,1)\,dz < \varepsilon.$

Consider $x \in K$

$$|u(x,t) - h(x)| = \Big| \int_{\mathbb{R}^n} \Phi(x-y,t) h(y)\,dy - h(x) \overbrace{\int_{\mathbb{R}^n} \Phi(x-y,t)\,dy}^{1} \Big|$$

$$= \Big| \int_{\mathbb{R}^n} \Phi(x-y,t) \big[h(y) - h(x)\big]\,dy \Big|$$

$$\le \int_{\mathbb{R}^n} \Phi(x-y,t)\, |h(y) - h(x)|\,dy.$$

$$= \int_{B(x,\delta)} \Phi(x-y,t)\, \underbrace{|h(y) - h(x)|}_{<\varepsilon}\,dy$$

$$+ \int_{\mathbb{R}^n \setminus B(x,\delta)} \Phi(x-y,t)\, \underbrace{|h(y) - h(x)|}_{\substack{\le |h(y)| + |h(x)| \\ \le 2\|h\|_\infty}}\,dy$$

$z = x - y$

$$< \varepsilon \underbrace{\int_{B(x,\delta)} \Phi(x-y,t)\,dy}_{<1} + 2\|h\|_\infty \underbrace{\int_{\mathbb{R}^n \setminus B(0,\delta)} \Phi(z,t)\,dz}_{<\varepsilon \quad \text{when } t<T}$$

$$< \varepsilon + 2\|h\|_\infty \varepsilon = \varepsilon(1 + 2\|h\|_\infty) = \varepsilon'$$

$u \to h$ uniformly on $x \in K$ as $t \to 0$. $\square$

"magic formula of Fourier analysis" $u, v \in S$

$$\int_{\mathbb{R}^n} \hat{u}(k) v(k) e^{2\pi i k\cdot x}\,dk = \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} u(y) e^{-2\pi i k\cdot y}\,dy\right) v(k) e^{2\pi i k\cdot x}\,dk$$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(y)\, v(k)\, e^{2\pi i k\cdot \underbrace{(x-y)}_{-z}}\,dy\,dk$$

$z = y - x \qquad y = z + x \qquad dy = dz$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(z+x)\, v(k)\, e^{-2\pi i k\cdot z}\,dz\,dk$$

$$= \int_{\mathbb{R}^n} u(z+x) \underbrace{\left[\int_{\mathbb{R}^n} v(k)\, e^{-2\pi i z\cdot k}\,dk\right]}_{\hat{v}(z)} dz$$

$$= \int_{\mathbb{R}^n} u(z+x)\, \hat{v}(z)\,dz.$$

The trick to get the inverse Fourier transform is to put

$$\hat{v}(x) = \Phi(x,\varepsilon) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}$$

$$v(k) = e^{-4\pi^2|k|^2\varepsilon}$$

$$\int_{\mathbb{R}^n} \hat{u}(k)\, e^{-4\pi^2|k|^2\varepsilon}\, e^{2\pi i k\cdot x}\, dk = \int_{\mathbb{R}^n} u(\underbrace{z+x}_{y})\, \Phi(z,\varepsilon)\, dz$$

Thm 4.7

$$= \int_{\mathbb{R}^n} u(y)\, \Phi(y-x,\varepsilon)\, dy.$$

By 4.7(iii) as $\varepsilon \to 0$ RHS $\to u(x)$

$$u(x) = \int_{\mathbb{R}^n} \hat{u}(k)\, e^{2\pi i k\cdot x}\, dk.$$

$$\mathcal{F}^{-1}[\hat{u}](x) = \int_{\mathbb{R}^n} \hat{u}(k)\, e^{2\pi i k\cdot x}\, dk.$$

only difference is $\pm$.

$$\mathcal{F}[uv] = \hat{u} * \hat{v}$$

$uv = \mathcal{F}^{-1}[\hat{u} * \hat{v}]$ basically $\mathcal{F}[\hat{u} * \hat{v}]$

$$\int_{\mathbb{R}^n} \hat{u}(k)\, v(k)\, dk = \int_{\mathbb{R}^n} u(z)\, \hat{v}(z)\, dz.$$

$$F_{\hat{u}}(v) = F_u(\hat{v})$$

We should define $\hat{F}(\phi) = F(\hat{\phi})$

The issue is $\hat{\phi}$ might not be a test function; is not fixable.

We have to find special distributions where this is valid, we need to find distributions that can act on Schwartz functions

Definition 4.8

ϕ_m a sequence of test function $\phi_m \to 0$ in S

$\Big[\lim_{m\to\infty} \rho_{\ell,\alpha}(\phi_m) = 0$ for all ℓ, α

D test

D' distribution

Recall $\rho_{\ell,\alpha}(\phi) = \sup_{\mathbb{R}^n} |x|^{2\ell} |\partial^\alpha \phi|$. $\Big]$

We call a distribution F "tempered" or $F \in S'$ if for all $\phi_m \in D$ $\phi_m \to 0$ in S $\Rightarrow \lim_{m\to\infty} F(\phi_m) = 0$

A tempered distribution acts on $\psi \in S$, $\phi_m \to \psi$ in S $\phi_m \in D$

$$F(\psi) = \lim_{m\to\infty} F(\phi_m).$$

Note if $\tilde{\phi}_m \to \psi$ in S.

$$\lim_{m\to\infty} F(\tilde{\phi}_m) = \lim_{m\to\infty} F(\tilde{\phi}_m - \phi_m + \phi_m) = \underbrace{\lim_{m\to\infty} F(\underbrace{\tilde{\phi}_m - \phi_m}_{\to 0})}_{0} + \lim F(\phi_m)$$

If $F \in S'$ we define $\hat{F}(\phi) = F(\underbrace{\hat{\phi}}_{\in S})$

What are examples of tempered distributions?

- Distributions associated to polynomials $x, x^2, \dots$
- " " $\sin x, \cos x$
- Dirac distributions

$$\mathcal{F}[1] = \delta$$

$$\partial_t u - \Delta u = f(x,t) \quad \text{inhomogeneous}$$

$$\Big\downarrow \mathcal{F}$$

$$\partial_t \hat{u} - (2\pi i)^2 |k|^2 \hat{u} = \hat{f}(k,t)$$

$$\partial_t \hat{u} + 4\pi^2 |k|^2 \hat{u} = \hat{f}$$

$$\underbrace{e^{4\pi^2|k|^2 t} \partial_t \hat{u} + 4\pi^2 |k|^2 e^{4\pi^2|k|^2 t} \hat{u}}_{= \partial_t\left[e^{4\pi^2|k|^2 t} \hat{u}\right]} = e^{4\pi^2|k|^2 t} \hat{f}$$

$$e^{4\pi^2|k|^2 t} \hat{u} = \int_0^t e^{4\pi^2|k|^2 s} \hat{f}(k,s)\, ds + \hat{h}(k)$$

$$\hat{u}(k,t) = \underbrace{e^{-4\pi^2|k|^2 t} \hat{h}(k)}_{\text{sol}^n \text{ to homog}} + \underbrace{\int_0^t e^{-4\pi^2|k|^2 (t-s)} \hat{f}(k,s)\, ds}_{\text{new.}}$$

$$\Big\downarrow \mathcal{F}^{-1}$$

$$u(x,t) = \mathcal{F}^{-1}\left[e^{-4\pi^2|k|^2 t}\right] * h(x) + \int_0^t \mathcal{F}^{-1}\left[\quad\right] ds.$$

for $f \in C_b^2(\mathbb{R}^n \times [0,\infty), \mathbb{R})$

we assert that, the follow is an inhomogeneous $sol^{\underline{n}}$. (Thm 4.9)

$$u(x,t) = \int_0^t \Phi(x, t-s) * f(x,s)\, ds$$

$$= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\, dy\, ds.$$

proof

$$u_\varepsilon(x,t) = \int_0^{t-\varepsilon} \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\, dy\, ds.$$

$$\partial_t u_\varepsilon = \int_{\mathbb{R}^n} \Phi(x-y, t-(t-\varepsilon)) f(y, t-\varepsilon)\, dy$$

$$+ \int_0^{t-\varepsilon} \int_{\mathbb{R}^n} \partial_t \Phi(x-y, t-s) f(y,s)\, dy\, ds.$$

$$\Delta u_\varepsilon = \int_0^{t-\varepsilon} \int_{\mathbb{R}^n} \Delta \Phi(x-y, t-s) f(y,s)\, dy\, ds.$$

$$(\partial_t - \Delta) u_\varepsilon = \int_{\mathbb{R}^n} \Phi(x-y, t-(t-\varepsilon)) f(y, t-\varepsilon)\, dy$$

$$+ \int_0^{t-\varepsilon} \int_{\mathbb{R}^n} \underbrace{(\partial_t - \Delta) \Phi(x-y, t-s) f(y,s)}_{0.}\, ds.$$

$$= \int_{\mathbb{R}^n} \Phi(x-y, \varepsilon) f(y, t-\varepsilon)\, dy$$

Thm 4.7(iii) implies $\lim_{\varepsilon \to 0} (\partial_t - \Delta) u_\varepsilon = f(y, t-0) = f.$

$$\Rightarrow (\partial_t - \Delta)\left(\lim_{\varepsilon \to 0} u_\varepsilon\right) = (\partial_t - \Delta) u \quad \square.$$

Consider now

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, h(y)\, dy. \quad \leftarrow u_1$$

$$+ \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\, dy\, ds. \leftarrow u_2$$

PDE:

$$(\partial_t - \Delta) u = (\partial_t - \Delta) u_1 + (\partial_t - \Delta) u_2$$

$$= 0 + f. = f$$

Initial Cond

$$u(x,0) = h(x) + 0 = h$$

Solves Cauchy problem on $\mathbb{R}^n$ for inhomogeneous heat equation