4.1 Spectal Theory and Fourier Transform

$$u(x,t) = \underbrace{e^{-t}}\underbrace{\sin x}$$ separable sol^{n}

$$u(x,t) = \varphi(t)\, h(x) \qquad \varphi(0)=1$$

$$u(x,0) = h(x)$$

$$0 = \partial_t u - \Delta u = (\partial_t \varphi) h - \varphi \Delta h$$

$$\frac{\partial_t \varphi(t)}{\varphi(t)} = \frac{\Delta h(x)}{h(x)} = -\lambda$$

$\partial_t \varphi = -\lambda \varphi$ and $-\Delta h = \lambda h$ $\quad Lx = \lambda x$

$\varphi(t) = e^{-\lambda t}$ $\quad$ eigenfunction equation of $-\Delta$

$$-\frac{\partial^2 \sin x}{\partial x^2} = -(-\sin x) = \sin x \qquad \lambda = 1$$

$$h(x) = \underbrace{\sin x}_{h_1} + \underbrace{\cos 2x}_{h_2} \qquad -\Delta \cos(2x) = 4\cos(2x)$$

$$u_1 = e^{-t} \sin x \qquad u_2 = e^{-4t} \cos 2x$$

$$u = u_1 + u_2$$

Can every initial condition be written as a sum of eigenfunctions of $-\Delta$?

I claim $e^{2\pi i k\cdot x}$ is an eigenfunction of $-\Delta$ with $\lambda = 4\pi^2|k|^2$

Recall $e^{it} = \cos t + i \sin t$ (Euler)

$k \cdot x = k_1 x_1 + k_2 x_2 + \dots + k_n x_n$

$$\partial_j e^{2\pi i k\cdot x} = e^{2\pi i k\cdot x}(2\pi i k_j)$$

$$\partial_j^2 e^{2\pi i k\cdot x} = e^{2\pi i k\cdot x}(2\pi i k_j)^2 = e^{2\pi i k\cdot x} 4\pi^2(-1) k_j^2$$

$$-\Delta e^{2\pi i k\cdot x} = e^{2\pi i k\cdot x}\left[4\pi^2 k_1^2 + \dots + 4\pi^2 k_n^2\right]$$

$$= \underbrace{4\pi^2|k|^2}_{\lambda} e^{2\pi i k\cdot x}$$

$$\langle f, g\rangle_{L^2} = \int f \bar{g} \qquad \langle f, f\rangle_{L^2} = \int f \bar{f} = \int |f|^2 = \|f\|_{L^2}$$

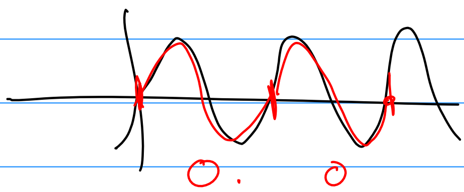
$$\langle e^{2\pi i k_1\cdot x}, e^{2\pi i k_2 \cdot x}\rangle$$

$$= \int_{\mathbb{R}^n} e^{2\pi i k_1\cdot x}\, \overline{e^{2\pi i k_2\cdot x}}\, dx = \int e^{2\pi i k_1\cdot x} e^{-2\pi i k_2\cdot x}\, dx$$

$$= \int_{\mathbb{R}^n} e^{2\pi i (k_1 - k_2)\cdot x}\, dx \text{ "=" } 0.$$

$$|e^{it}| = |\cos t + i \sin t| = \sqrt{\cos^2 + \sin^2} = 1$$

$$\sin\left[2\pi (k_1 - k_2)\cdot x\right]$$



In finite dimensions

$$h = \sum \langle h, e_k\rangle e_k$$

eg $h = \binom{1}{2}$ $\langle h, e_1\rangle = 1$

$\langle h, e_2\rangle = 2$

$h = 1 e_1 + 2 e_2$

$$h \text{ "="} \int \langle h, e^{2\pi i k\cdot x}\rangle e^{2\pi i k\cdot x} \, dk$$

Definition 4.1 The Fourier transform of $h: \mathbb{R}^n \to \mathbb{R}$ is defined to be

$$\hat{h}(k) = \mathcal{F}[h](k) = \int_{\mathbb{R}^n} h(x) e^{-2\pi i k\cdot x} \, dx$$

$$\text{"="} \langle h, e^{2\pi i k\cdot x}\rangle_{L^2}$$

Example Gaussian  $e^{-|\pi x|^2} = e^{-\pi^2 x_1^2} e^{-\pi^2 x_2^2} \cdots e^{-\pi^2 x_n^2}$

$$\int_{\mathbb{R}^n} e^{-\pi^2 |x|^2} e^{-2\pi i k\cdot x} \, dx$$

$|x-k|^2 = x\cdot x - 2x\cdot k + k\cdot k$

$$= \int_{\mathbb{R}^n} e^{-|k|^2+|k|^2 - 2\pi i k\cdot x - \pi^2 x\cdot x} \, dx$$

$$= \int_{\mathbb{R}^n} e^{-|k|^2 - (ik+\pi x)\cdot(ik+\pi x)} \, dx$$

$$= e^{-|k|^2} \int_{\mathbb{R}^n} e^{-|ik+\pi x|^2} \, dx$$

$y = ik + \pi x$

$dy = \pi^n \, dx \Leftarrow \mathbb{R}^n$ Jacobi

$$= \pi^{-n} e^{-|k|^2} \underbrace{\int_{ik+\mathbb{R}^n} e^{-|y|^2} \, dy}_{\pi^{n/2}}$$

$$= \pi^{-n/2} e^{-|k|^2}$$

$$\mathcal{F}\left[e^{-\pi^2|x|^2}\right](k) = \pi^{-n/2} e^{-|k|^2}$$

$$\mathcal{F}\left[e^{-a|x|^2}\right](k) = \left(\frac{\pi}{a}\right)^{n/2} e^{-\frac{1}{a}|\pi k|^2}$$

Definition 4.2 Schwartz $\mathcal{S}$ smooth complex valued $f: \mathbb{R}^n \to \mathbb{C}$

$$\rho_{\ell,\alpha} = \sup_{\mathbb{R}^n} \left(|x|^{2\ell} |\partial^\alpha f| \right)$$

such that $\rho_{\ell,\alpha}(f)$ is finite $\forall \ell, \alpha$.

Also $(1+|x|^2)^\ell$

Schwartz are the largest subset of $L^1(\mathbb{R}^n)$ that is closed under differentiation and multiplication by a polynomial

$$\|h\|_{L^1} = \int_{\mathbb{R}^n} |h|\, dx = \underbrace{\int_{B(0,1)} |h|\, dx}_{< \infty} + \underbrace{\int_{\mathbb{R}^n \setminus B(0,1)} |h|\, dx}$$

$$= \int_1^\infty \int_{\partial B(0,r)} |h|\, d\sigma\, dr \leq \int_1^\infty \sup_{\partial B(0,r)} |h| \, n\omega_n r^{n-1}\, dr$$

$$= \int_1^\infty r^{-2} n\omega_n \underbrace{r^{n+1} \sup_{\partial B(0,r)} |h|}_{\rho_{\ell,0}(h)} dr \qquad \text{for } 2\ell > n+1$$

$$\leq n\omega_n \rho_{\ell,0}(h) \int_1^\infty r^{-2}\, dr$$

$$= n\omega_n \rho_{\ell,0}(h) \underbrace{\left[-r^{-1}\right]_1^\infty}_{0+1} < \infty.$$

Lemma 4.3 The Fourier transform of a Schwartz function is Schwartz.

$$\mathcal{F}[\partial_j h](k) = 2\pi i k_j \hat{h}(k) \quad \text{and} \quad \mathcal{F}[-2\pi i x_j h](k) = \partial_j \hat{h}(k)$$

proof

Step 1. If $h \in L^1 \Rightarrow \hat{h} \in C_b$.

$h \in C_0^\infty(\mathbb{R}^n, \mathbb{C}) \subset L^1$

$$|\hat{h}(k)| \leq \int_{\mathbb{R}^n} |h(x)| \underbrace{|e^{-2\pi i k \cdot x}|}_{1} dx = \int |h(x)| dx = \|h\|_{L^1}$$

$$\boxed{\|\hat{h}\|_\infty \leq \|h\|_{L^1}}$$ bounded

$$\hat{h}(k) = \int_{\operatorname{supp} h} h(x) e^{-2\pi i k \cdot x} dx$$ cts

$$\mathcal{F} : \underset{\|\cdot\|_{L^1}}{C_0^\infty(\mathbb{R}^n, \mathbb{C})} \to \underset{\|\cdot\|_\infty}{C_b(\mathbb{R}^n, \mathbb{C})}$$

Since C_0^∞ is dense in L^1 $\mathcal{F}$ extends by limits to a cts

$$\mathcal{F} : L^1(\mathbb{R}^n, \mathbb{C}) \to C_b(\mathbb{R}^n, \mathbb{C}).$$

$$\mathbin{\rotatebox[origin=c]{90}{$\subset$}}$$
$$\mathcal{S}$$

Step 2. prove $\mathcal{F}[\partial_j h]$ formula.

$[-R, R]^n$

$$\mathcal{F}[\partial_j h] = \int_{\mathbb{R}^n} \frac{\partial h}{\partial x_j} e^{-2\pi i k \cdot x} dx$$

$$= 0 - \int_{\mathbb{R}^n} h(x) \frac{\partial}{\partial x_j}\left[e^{-2\pi i k \cdot x}\right] dx$$

$$= 2\pi i k_j \underbrace{\int_{\mathbb{R}^n} h(x)\, e^{-2\pi i k\cdot x}\, dx}_{\hat{h}}$$

Repeating $\qquad k_{(1,1,2)} = k_1 k_2 k_3^2$

$$\underset{\uparrow\atop C_b}{\mathcal{F}[\partial^\alpha h]} = (2\pi i)^{|\alpha|} \underbrace{k_\alpha}_{\text{any polynomial}} \hat{h}(k)$$

$$\rho_{\ell,0}(\hat{h}) < \infty.$$

Step 3. $\mathcal{F}\{-2\pi i x_j h\} = \partial_j \hat{h}$

$$\frac{\partial \hat{h}}{\partial k_j} = \frac{\partial}{\partial k_j} \int_{\mathbb{R}^n} h(x)\, e^{-2\pi i k\cdot x}\, dx$$

$$= \int_{\mathbb{R}} \underline{h(x)(-2\pi i x_j)}\, e^{-2\pi i k\cdot x}\, dx$$

$$= \mathcal{F}[-2\pi i x_j h]$$

ie $\dfrac{\partial \hat{h}}{\partial k_j} \in C$. $\qquad$ ie $\hat{h} \in C^\infty$

$\hat{h} \in S$. $\qquad \square$

$$\mathcal{F}[\partial_j^2 u] = 2\pi i k_j \mathcal{F}[\partial_j u] = (2\pi i k_j)^2 \mathcal{F}[u]$$

$$= -4\pi^2 k_j^2 \hat{u}$$

$$\mathcal{F}[\Delta u] = \mathcal{F}[\partial_1^2 u + \dots + \partial_n^2 u]$$

$$= -4\pi^2 k_1^2 \hat{u} - 4\pi^2 k_2^2 \hat{u} - \dots - 4\pi^2 k_n^2 \hat{u}$$

$$= -4\pi^2 |k|^2 \hat{u}$$

$$\mathcal{F}[\partial_t u] = \int_{\mathbb{R}^n} \partial_t u(x,t)\, e^{-2\pi i k \cdot x}\, dx$$

not over t.

$$= \partial_t \hat{u}$$

$$\mathcal{F}[0] = 0$$

$$0 = \partial_t u - \Delta u$$

$$0 = \partial_t \hat{u} + 4\pi^2 |k|^2 \hat{u} \qquad \hat{u}(k,t)$$

$$\partial_t \hat{u} = -4\pi^2 |k|^2 \hat{u} \qquad \partial_t y = cy \Rightarrow y = e^{ct} y(0)$$

$$\hat{u}(k,t) = e^{-4\pi^2 |k|^2 t}\, \hat{u}(k,0)$$

$$= \underbrace{e^{-4\pi^2 |k|^2 t}}_{\text{fourier}}\; \underbrace{\hat{h}(k)}_{\text{fourier}}$$

Lemma 4.4 $u, v \in S$

$$\mathcal{F}[u * v] = \hat{u}\,\hat{v} \qquad \mathcal{F}[uv] = \hat{u} * \hat{v}$$

proof

$$\mathcal{F}[u*v](k) = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} u(x-y)\,v(y)\,dy \right) e^{-2\pi i k\cdot x}\,dx$$

$$= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} u(x-y) e^{-2\pi i k\cdot x}\,dx \right) v(y)\,dy$$

$z = x-y \qquad dz = dx$

$$= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} u(z) e^{-2\pi i k\cdot z}\, \underbrace{e^{-2\pi i k\cdot y}}\,dz \right) v(y)\,dy$$

$$= \int_{\mathbb{R}^n} \underbrace{\hat{u}(k)}\, e^{-2\pi i k\cdot y}\, v(y)\,dy$$

$$= \hat{u}(k)\,\hat{v}(k).$$

Second half of proof after Lemma 4.7 $\square$

$$\hat{u}(k,t) = \underbrace{e^{-4\pi^2|k|^2 t}}_{\text{fourier}} \underbrace{\hat{h}(k)}_{\text{fourier}}$$

$$\mathcal{F}[u] = \mathcal{F}\left[\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}\right] \mathcal{F}[h]$$

$$\Longrightarrow \quad u(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} * h .$$