Chapter 4: Heat Equation.

$u : \Omega \times (0,T) \to \mathbb{R}$ $\qquad \Omega \subseteq \mathbb{R}^n$

$$\frac{\partial u}{\partial t} - \Delta u = 0 \quad , \; f$$

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_n^2}$$

$$a_{ij} = \begin{pmatrix} -1 & & & 0 \\ & \ddots & & \\ 0 & & -1 & \\ & & & 0 \end{pmatrix} \qquad -1, \dots, -1, 0$$

$$\frac{\partial}{\partial r} S = \frac{1}{n\omega_n r^{n-1}} \int_{B(x,r)} \Delta u(y)\,dy$$

approx for small r $\qquad$ $?/x$

$$\approx \frac{1}{n\omega_n r^{n-1}} \Delta u(x)\, \omega_n r^n = \frac{r}{n} \Delta u(x)$$

$$\frac{\partial S}{\partial r} \approx \frac{S(r) - u(x)}{r}$$

$$\Delta u(x) \approx \frac{n}{r^2}\left(S(r) - u(x)\right)$$

$$\frac{\partial u}{\partial t} \approx f(x,t) + \frac{n}{r^2}\left(\underbrace{S(r) - u(x)}\right)$$

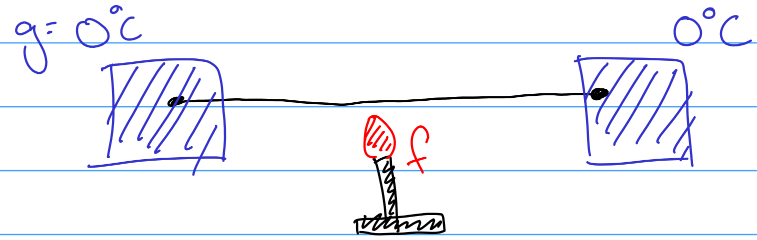
adding or subtracting heat

if $u(x)$ is less $S(r) \Rightarrow \frac{\partial u}{\partial t} > 0$

if $u(x)$ is more $S(r) \Rightarrow \frac{\partial u}{\partial t} < 0$

$$\dot{u} - \Delta u = f \quad \text{on } \mathbb{R}^n \times (0,T)$$
$$u(x,0) = h(x) \quad \text{on } \mathbb{R}^n$$
Cauchy problem

$$\dot{u} - \Delta u = f \quad \text{on } \Omega \times (0,T)$$
$$u = g \quad \text{on } \partial\Omega \times (0,T)$$
$$u(x,0) = h(x) \quad \text{on } \Omega \times \{0\}$$
Dirichlet problem.



The terminal problem

$$u_m(x,t) = e^{m^2(T-t)} \sin(mx)$$

$$u_m(x,T) = e^0 \sin(mx) = \sin(mx)$$

$$u_m(x,0) = e^{m^2 T} \sin(mx)$$

Monster sol$^{\underline{n}}$. to Cauchy problem

Is there a sol$^{\underline{n}}$ that is analytic in x?

$$u(x,t) = \sum_{l=0}^{\infty} g_l(t)\, x^l$$

$$\frac{\partial u}{\partial t} = \sum_{l=0}^{\infty} g_l'\, x^l \qquad \frac{\partial^2 u}{\partial x^2} = \sum_{l=2} g_l \cdot l(l-1)\, x^{l-2}$$
$$= \sum_{l=0} g_{l+2}\, (l+2)(l+1)\, x^l$$

$$0 = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \sum_{l=0}^{\infty} \left[g_l' - g_{l+2}\, (l+2)(l+1) \right] x^l$$

we get a sol^n if $\frac{\partial g_\ell}{\partial t} = (\ell+1)(\ell+2)\, g_{\ell+2}$

$$g_{\ell+2} = \frac{1}{(\ell+2)(\ell+1)} \frac{\partial}{\partial t} g_\ell$$

g_0, g_1 then $g_2 = \frac{1}{2\times 1} g_0'$

$$g_4 = \frac{1}{4\times 3} g_2' = \frac{1}{4\times 3}\frac{1}{2\times 1} g_0''$$

$$g_{2\ell} = \frac{1}{(2\ell)!} g_0^{(\ell)}$$

Monster $g_0 = g = \exp(-t^{-2}) \qquad g_1 \equiv 0$

$$g' = 2t^{-3} \exp(-t^{-2})$$

$$g'' = -6t^{-4} \exp(\;) + 4t^{-6} \exp(\;)$$

$$= \left[-6t^{-4} + 4t^{-6}\right] \exp(-t^{-2})$$

$$g^{(\ell)} = t^{-\ell} p_\ell(t^{-2}) \exp(-t^{-2})$$

$p_0 = 1$

$$g^{(\ell+1)} = -\ell t^{-\ell-1} p_\ell \exp + t^{-\ell}(-2t^{-3}) p_\ell' \exp + t^{-\ell} p_\ell (+2t^{-3}) \exp$$

$$= t^{-(\ell+1)} \underbrace{\left[-\ell p_\ell - 2t^{-2} p_\ell' + 2t^{-2} p_\ell\right]}_{p_{\ell+1}(t^{-2})} \exp(-t^{-2})$$

$$p_{\ell+1}(z) = -\ell p_\ell(z) - 2z\, p_\ell'(z) + 2z\, p_\ell(z).$$

We claim that the coefficient z^k in $p_\ell(z)$ is bounded by

$$\frac{\ell!\,7^\ell}{2^k\,k!}$$

$C_{\ell+1}^k$ is coeff of z^k in $p_{\ell+1}$

$$p_{\ell+1}(z) = -\ell p_\ell(z) - 2z\,p_\ell'(z) + 2z\,p_\ell(z).$$

$$C_{\ell+1}^k = -\ell\, C_\ell^k \quad - 2kC_\ell^k \qquad + 2C_\ell^{k-1}$$

$p_\ell = \dots + C_\ell^k z^k + \dots \qquad z\,p_\ell = \dots + C_\ell^{k-1} z^{k-1+1} + \dots$

$p_\ell' = \dots + kC_\ell^k z^{k-1} + \dots$

$z\,p_\ell' = \dots + kC_\ell^k z^k + \dots$

$$|C_{\ell+1}^k| \leq \ell\,|C_\ell^k| + 2k\,|C_\ell^k| + 2\,|C_\ell^{k-1}|$$

$$\leq \ell\,\frac{7^\ell \ell!}{2^k k!} + 2k\,\frac{7^\ell \ell!}{2^k k!} + 2\,\frac{7^\ell \ell!}{2^{k-1}(k-1)!}\,\frac{2}{2}\,\frac{k}{k}$$

$$= \frac{7^\ell \ell!\,[\ell + 2k + 4k]}{2^k k!}$$

$$\leq \frac{7^\ell \ell!\,[\ell + 2\ell + 4\ell]}{2^k k!} = \frac{7^\ell \ell!\,7\ell}{2^k k!}$$

$$\leq \frac{7^{\ell+1}(\ell+1)!}{2^k k!}$$

$$u(x,t) = \sum g_\ell(t)\, x^\ell = \sum_{\ell=0}^{\infty} \frac{g^{(\ell)}(t)}{(2\ell)!} x^{2\ell}$$

$$= \sum_{\ell=0}^{\infty} \frac{t^{-\ell} p_\ell(t^{-2}) \overbrace{\exp(-t^{-2})}^{g(t)}}{(2\ell)!} x^{2\ell}$$

$$= \sum \frac{g(t)}{(2\ell)!\, t^\ell} x^{2\ell} \sum_{k=0}^{\ell} C_\ell^k\, t^{-2k}$$

$$|u(x,t)| \leq \sum \frac{g(t)}{(2\ell)!\, t^\ell} x^{2\ell} \sum_{k=0}^{\ell} |C_\ell^k|\, t^{-2k}$$

$$\leq \sum \frac{g(t)}{(2\ell)!\, t^\ell} x^{2\ell} \sum_{k=0}^{\ell} \frac{7^\ell\, \ell!}{2^k\, k!\, t^{2k}}$$

$$\leq \sum \frac{g(t)\, 7^\ell\, \ell!}{(2\ell)!} \left(\frac{x^2}{t}\right)^\ell \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1}{2t}\right)^k$$

$$= \sum \frac{\exp(-t^{-2})\, \ell!}{(2\ell)!} \left(\frac{7x^2}{t}\right)^\ell \exp\left(\frac{1}{2t}\right)$$

$$\leq \exp\left(\frac{1}{2t} - \frac{1}{t^2}\right) \sum_{\ell=0}^{\infty} \frac{1}{4^\ell\, \ell!} \left(\frac{7x^2}{t}\right)^\ell$$

$$\frac{\ell!}{(2\ell)!} = \frac{\cancel{\ell} \times \cancel{(\ell-1)} \times (\ell-2) \cdots \times 1}{\underset{2}{\cancel{2\ell}} \times (2\ell-1) \times \underset{2}{\cancel{(2\ell-2)}} \times \cdots \times 1} = \frac{1}{2^\ell\, (2\ell-1)(2\ell-3) \cdots \times 1}$$

$$\leq \frac{1}{2^\ell\, (2\ell-2)(2\ell-4) \cdots \times 2} = \frac{1}{2^\ell\, 2^\ell\, \ell!}$$

$$= \exp\left(\frac{1}{2t} - \frac{1}{t^2}\right) \underbrace{\sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left(\frac{7x^2}{4t}\right)^\ell}_{\exp\left(\frac{7x^2}{4t}\right)}.$$