Section 3.5 A PDE with no solutions.

Lemma 3.23 Let $\Omega = \mathbb{R}^n \setminus \{0\}$. The only twice-differentiable function $u : \Omega \to \mathbb{R}$ that satisfies the following inequality is $u \equiv 0$:

$$-|x|^2 \Delta u \geq u^2$$

Eg. • u harmonic then $0 \geq u^2 \Rightarrow u \equiv 0$

• $u = C x^n$ $\quad \Delta u = C n(n-1) x^{n-2}$

LHS $= -Cn(n-1)x^n$ $\quad$ RHS $C^2 x^{2n}$

$n = -3$ $\quad x^{-3}$ vs x^{-6}

Consider the PDE

$$\boxed{-|x|^2 \Delta u = u^2 + 1}$$

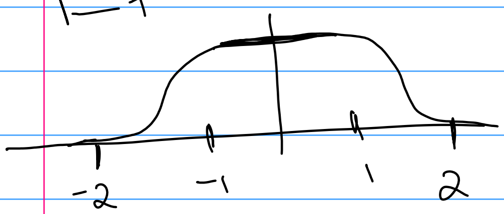
2^{nd} order, smooth analytic coefficients. $|x|^2 = (x_1)^2 + (x_2)^2 + \dots + (x_n)^2$

If we had a sol^n

$$-|x|^2 \Delta u = u^2 + 1 \geq u^2$$

$u \equiv 0$ but this is not a sol^n $\quad 0 \neq 1$ ※

proof

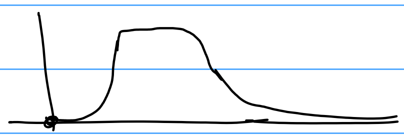


ψ_0 $\quad$ supp $\psi_0 \subseteq [-2, 2]$

but also

for $|t| \leq 1$ $\quad \psi_0 \equiv 1$

$\psi_R(r) = \psi_0(R \ln r)$



$\varphi_R(x) = \psi_R(|x|)$

$x \in \operatorname{supp} \varphi_R \iff R \ln|x| \leq 2 \iff \{e^{-2R} \leq |x| \leq e^{2R}\} = A_R$

$A'_R = \{ e^{-R} \le |x| \le e^R \}$ on A'_R, $\varphi_R \equiv 1$

$$I_R := \int_{A'_R} \frac{u^2}{|x|^2} = \int_{A'_R} \frac{u^2}{|x|^2} \varphi_R \le \int_{A_R} \frac{u^2}{|x|^2} \varphi_R =: J_R$$

$$J_R = \int_{A_R} \frac{u^2}{|x|^2} \varphi_R \le \int_{A_R} -\Delta u \ \varphi_R = \int_{A_R} -u \Delta \varphi_R$$

$$\int_{A_R} \Delta u \ \varphi_R - u \Delta \varphi_R = \int_{\partial A_R} [\underbrace{\varphi_R \nabla u}_{0} - \underbrace{u \nabla \varphi_R}_{0}] \cdot N \, d\sigma$$

$$= \int_{A_R} \frac{-u\sqrt{\varphi_R}}{|x|} \cdot \frac{|x| \Delta \varphi_R}{\sqrt{\varphi_R}} = \left\langle \frac{-u\sqrt{\varphi_R}}{|x|}, \frac{|x|\Delta\varphi_R}{\sqrt{\varphi_R}} \right\rangle_{L^2}$$

Cauchy-Schwarz $a \cdot b \le |a|\,|b|$ all inner products including L^2:

$$\langle f, g \rangle_{L^2} = \int fg \qquad \|f\|_{L^2} = \left(\int f^2 \right)^{1/2}$$

$$\le \left\| \frac{-u\sqrt{\varphi_R}}{|x|} \right\|_{L^2} \left\| \frac{|x|\Delta\varphi_R}{\sqrt{\varphi_R}} \right\|_{L^2}$$

$$= \left(\int_{A_R} \frac{u^2 \varphi_R}{|x|^2} \right)^{1/2} \left(\int_{A_R} \frac{|x|^2 (\Delta\varphi_R)^2}{\varphi_R} \right)^{1/2}$$

$$= J_R^{1/2} \left(\int_{A_R} \frac{|x|^2 (\Delta\varphi_R)^2}{\varphi_R} \right)^{1/2}$$

$$J_R / J_R^{1/2} \le \left(\int_{A_R} \frac{|x|^2 (\Delta\varphi_R)^2}{\varphi_R} \right)^{1/2}$$

$$J_R \leq \int_{A_R} \frac{|x|^2 (\Delta \varphi_R)^2}{\varphi_R} \quad \text{independent of } u$$

$$\varphi_R = \psi_0\left(R^{-1} \ln|x|\right) \quad \text{weird scaling}$$

$$\Delta \varphi_R = \psi_0''\left(R^{-1} \ln|x|\right) \frac{1}{R^2 |x|^2}$$

$$J_R \leq \int_{A_R} \frac{\left[\psi_0''(R^{-1}\ln|x|)\right]^2}{R^4 |x|^2 \psi_0(R^{-1}\ln|x|)} \, dx.$$

$$= \int_0^{2\pi} \int_{e^{-2R}}^{e^{2R}} \frac{\left[\psi_0''(R^{-1}\ln r)\right]^2}{R^4 r^2 \psi_0(R^{-1}\ln r)} \, r \, dr \, d\theta$$

$$= \frac{2\pi}{R^4} \int_{e^{-2R}}^{e^{2R}} \frac{\left[\psi_0''(R^{-1}\ln r)\right]^2}{r \, \psi_0(R^{-1}\ln r)} \, dr$$

$t = R^{-1} \ln r$

$dt = R^{-1} \frac{dr}{r}$

$r = e^{2R}$ $\quad t = R^{-1} 2R = 2$

$r = e^{-2R}$ $\quad t = -2$

$$= \frac{2\pi}{R^4} \int_{-2}^{2} \frac{\left[\psi_0''(t)\right]^2}{\psi_0(t)} \, R \, dt$$

$$= \frac{2\pi}{R^3} \int_{-2}^{2} \frac{\left[\psi_0''(t)\right]^2}{\psi_0(t)} \, dt$$

There is a choice of ψ_0 such that the integral is finite

$$= 2\pi C R^{-3}$$

$$I_R = \int_{A'_R} \frac{u^2}{|x|^2} \leq 2\pi C R^{-3}$$

Choose any point $x \in \Omega$.
Choose S so that $A'_S \ni x$

For any $S < R$ $0 \le I_S \le I_R \le 2\pi C R^{-3}$

$\Rightarrow I_S = 0 \Rightarrow \frac{u^2}{|x|^2} = 0$ on $A'_S \Rightarrow u(x) = 0$ $\square$