# Introduction to Partial Differential Equations
## Exercise sheet 14

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**42. Method of Descent**

In this exercise we will apply the method of descent to solve the wave equation on $\mathbb{R}^2$ for a particular set of initial conditions. The idea is to help you understand the key ideas and notation of the method. It is a combination of results from Sections 5.1–5.

Consider the wave equation on $\mathbb{R}^2$ with initial conditions

$$\partial_t^2 u - \Delta u = 0 \text{ on } (x,t) \in \mathbb{R}^2 \times (0,\infty),$$
$$u(x,0) = g(x) = \chi_{[0,\infty)}(x_1), \qquad \partial_t u(x,0) = h(x) = 0.$$

**(a)** Suppose $u$ is a solution of the wave equation for $n = 2$. Why does $(x_1, x_2, x_3, t) \mapsto u(x_1, x_2, t)$ solve the wave equation on $\mathbb{R}^3$? (note, the Laplacians are different in different dimensions). *(1 point)*

**(b)** Conversely, prove that a solution $\bar{u}$ to the 3-dimensional wave equation that does not depend on $x_3$ gives a solution to the 2-dimensional wave equation. *(1 point)*

**(c)** By (a) and (b), we now must solve a wave equation on $\mathbb{R}^3$. The key to solving the 3-dimensional wave equation is to consider the (spatial-)spherical means

$$U(x,t,r) = \frac{1}{4\pi r^2} \int_{\partial B(x,r)} \bar{u}(z,t)\, d\sigma(z),$$

and likewise let $G$ and $H$ be the spherical means of $\bar{g}$ and $\bar{h}$ respectively. Explain why $\bar{g}(x_1, x_2, x_3) = \chi_{[0,\infty)}(x_1)$ and $\bar{h} = 0$ (or give the definition of bar). Show that

$$G(x,r) = \begin{cases} 0 & \text{for } x_1 \leq -r \\ \frac{1}{2}\frac{x_1 + r}{r} & \text{for } |x_1| \leq r \\ 1 & \text{for } r \leq x_1 \end{cases} \qquad \text{and } H(x,r) = 0.$$

You may use the following geometric fact: for $-R < a < b < R$, the surface area of the part of the sphere $\partial B(0,R)$ with $a < x_1 < b$ is $2\pi R(b-a)$. *(4 points)*

**(d)** We know by Lemma 5.2 that $U$ obeys the Euler-Poisson-Darboux equation. Let $\tilde{U}(x,t,r) := rU(x,t,r)$. Show that $\tilde{U}$ obeys the following PDE

$$\partial_t^2 \tilde{U} - \partial_r^2 \tilde{U} = 0 \text{ on } (t,r) \in [0,\infty) \times [0,\infty),$$
$$\tilde{U}(x,0,r) = rG(x,r), \qquad \partial_t \tilde{U}(x,0,r) = rH(x,r).$$

Note that there are no $x$-derivatives in this PDE, so we can think of it as a family of PDEs in the variables $r, t$ parametrised by $x$. *(2 points)*

**(e)** Thus we see that $\tilde{U}$ obeys the 1-dimensional wave equation on the *half-line* $r \in [0,\infty)$. This is solved by a trick using reflection, and the formula is at the end of Section 5.2. We only need the solution for small $r$, so it is enough to consider the case $0 \leq r \leq t$. In this case, show

$$\tilde{U}(x,t,r) = \begin{cases} 0 & \text{for } x_1 \leq -(t+r) \\ \frac{1}{4}(x_1+t+r) & \text{for } -(t+r) \leq x_1 \leq -(t-r) \\ \frac{1}{2}r & \text{for } |x_1| \leq t-r \\ \frac{1}{4}(x_1-t+3r) & \text{for } t-r \leq x_1 \leq t+r \\ r & \text{for } x_1 \geq t+r. \end{cases}$$

*(4 points)*

**(f)** Recover $\bar{u}$ from $\tilde{U}$ using a certain property of spherical means. *(3 points)*

Observe that $\bar{u}$ does not depend on $x_3$. So by part (b) we have a solution to the 2-dimensional wave equation:

$$u(x_1,x_2,t) = \begin{cases} 0 & \text{for } x_1 < -t \\ 0.25 & \text{for } x_1 = -t \\ 0.5 & \text{for } -t < x_1 < t \\ 0.75 & \text{for } x_1 = t \\ 1 & \text{for } x_1 > t. \end{cases}$$

This solution has jump discontinuities, but this is unsurprising since the initial conditions also had them.

**Solution.**

**(a)** Recall the definition $\bar{u}(x_1,x_2,x_3,t) = u(x_1,x_2,t)$. In words, the bar function is the same formula but considered in a higher dimensional space. This explains $\bar{g}$ and $\bar{h}$. So it is constant in the $x_3$ dimension and $\partial_3 \bar{u} = 0$.

If we write out the wave equation on $\mathbb{R}^3$ fully

$$(\partial_t^2 - \partial_1^2 - \partial_2^2 - \partial_3^2)\bar{u} = (\partial_t^2 - \partial_1^2 - \partial_2^2)u - \partial_3^2\bar{u} = 0.$$

**(b)** By the same reasoning

$$(\partial_t^2 - \partial_1^2 - \partial_2^2)\bar{u} = \partial_3^2\bar{u} = 0.$$

Hence we get the solution $u(x_1,x_2,t) = \bar{u}(x_1,x_2,0,t)$. The choice $x_3 = 0$ is not significant, because $\bar{u}$ is constant in $x_3$. Any other choice gives the same thing.

**(c)** By definition

$$G(x,r) = \frac{1}{4\pi r^2}\int_{\partial B_3(x,r)} \chi_{[0,\infty)}(z_1)\, d\sigma(z).$$

We use $B_3$ here to make clear this is a ball in 3-dimensional space. If this ball lies entirely in the space with $z_1 \geq 0$ then the integrand is always 1 and the integral is just the surface area of the sphere. This occurs if $x_1$ (the first coordinate of the centre of the ball) is greater than the radius $r$. Likewise, if $x_1 < -r$ then the integrand is always zero.

So it remains to handle the case $-r \leq x \leq r$. The integral is

$$G(x,r) = \frac{1}{4\pi r^2}\int_{\partial B_3(x,r)\cap\{0\leq x_1\leq r\}} 1\, d\sigma(z) = \frac{1}{4\pi r^2} \times 2\pi r(r+x_1) = \frac{1}{2}\frac{x_1+r}{r}.$$

**(d)** We just differentiate

$$\partial_t^2(rU) - \partial_r^2(rU) = r\partial_t^2 U - \partial_r(U + r\partial_r U) = r\partial_t^2 U - 2\partial_r U - r\partial_r^2 U.$$

This is $r$ multiplied by the Euler-Poisson-Darboux equation for $n = 3$. Since $U$ solves this equation, we get 0 on the right hand side. For the initial conditions $\tilde{U}(x,0,r) = rU(x,0,r) = rG(x,r)$ and likewise for $H$.

**(e)** The solution of the wave equation on the half line, for $0 \le r \le t$ is

$$\begin{aligned}\tilde{U}(x,t,r) &= \frac{1}{2}\left[\tilde{G}(x,t+r) - \tilde{G}(x,t-r)\right] + \frac{1}{2}\int_{t-r}^{t+r} \tilde{H}(x,y)\,dy \\ &= \frac{1}{2}\left[(t+r)G(x,t+r) - (t-r)G(x,t-r)\right] + \frac{1}{2}\int_{t-r}^{t+r} 0\,dy,\end{aligned}$$

since $\tilde{H} = rH = 0$. Under the assumption $0 \le r \le t$, you can see that $-(t+r) \le -(t-r) \le 0 \le (t-r) \le (t+r)$. Thus there are five cases intervals to consider

$$\tilde{U}(x,t,r) = \begin{cases} \text{for } x_1 \le -(t+r): & 0 - 0 \\ \text{for } -(t+r) \le x_1 \le -(t-r): & \frac{1}{4}(x_1+t+r) - 0 \\ \text{for } |x_1| \le t-r: & \frac{1}{4}(x_1+t+r) - \frac{1}{4}(x_1+t-r) \\ \text{for } t-r \le x_1 \le t+r: & \frac{1}{2}(t+r) - \frac{1}{4}(x_1+t-r) \\ \text{for } x_1 \ge t+r: & \frac{1}{2}(t+r) - \frac{1}{2}(t-r) \end{cases}.$$

**(f)** We know that a function is equal to the limit of its spherical mean as the radius goes to zero, $\bar{u}(x,t) = \lim_{r\to 0} U(x,t,r) = \lim_{r\to 0} r^{-1}\tilde{U}(x,t,r)$. The first, third, and fifth cases of $\tilde{U}$ give

$$\bar{u}(x_1,x_2,x_3,t) = \begin{cases} 0 & \text{for } x_1 < -t \\ \frac{1}{2} & \text{for } -t < x_1 < t \\ 1 & \text{for } x_1 > t. \end{cases}$$

To find the value of $\bar{u}(x,t)$ for $x_1 = -t$, we see that this sits in the second case of $\tilde{U}$. We get

$$\lim_{r\to 0}\frac{1}{4}\frac{x_1+t+r}{r} = \lim_{r\to 0}\frac{1}{4}\frac{0+r}{r} = \frac{1}{4}.$$

Similarly $\bar{u}(t,x_2,x_3,t) = \frac{3}{4}$.

This behaviour at the jump discontinuities is typical for these averaging methods, it gives the value at the jump as the average of the two sides of the discontinuity. Properly, when we have functions which are not twice continuously differentiable we should use distributions and weak solutions. In that context, the value of the function at the jump is not significant, it is just an artifact of using spherical means.

**43. Wave energy modes**

Let us work with $n = 1$ for simplicity. If we apply the Fourier transform to the wave equation, we arrive at

$$\frac{\partial^2 \hat{u}}{\partial t^2} + 4\pi^2 k^2 \hat{u} = 0.$$

This leads us to define the *spectral energy density*

$$2\mathcal{E}(k,t) := \left|\frac{\partial \hat{u}}{\partial t}\right|^2 + 4\pi^2 k^2 \left|\hat{u}\right|^2.$$

**(a)** Through calculation, show that $\mathcal{E}$ is constant in $t$. Note that $\hat{u}$ is a complex valued function and it is important to respect complex conjugation: $|g|^2 = g\bar{g}$. *(1 point)*

**(b)** Consider a complex valued function $g$ and its conjugate $h = \bar{g}$. Prove that $\hat{h}(k) = \overline{\hat{g}(-k)}$. *(1 point)*

**(c)** Using an equation on page 73 for the script, establish Plancherel's theorem *(1 point)*

$$\int_{\mathbb{R}} |h|^2 \, dx = \int_{\mathbb{R}} |\hat{h}|^2 \, dk.$$

**(d)** Use this to show that the energy of a wave is constant. *(2 points)*

**Solution.**

**(a)**

$$\begin{aligned}
2\frac{\partial \mathcal{E}}{\partial t} &= \frac{\partial}{\partial t}\left[\frac{\partial \hat{u}}{\partial t}\overline{\frac{\partial \hat{u}}{\partial t}} + 4\pi^2 k^2 \hat{u}\overline{\hat{u}}\right] \\
&= \frac{\partial^2 \hat{u}}{\partial t^2}\overline{\frac{\partial \hat{u}}{\partial t}} + \frac{\partial \hat{u}}{\partial t}\overline{\frac{\partial^2 \hat{u}}{\partial t^2}} + 4\pi^2 k^2 \frac{\partial \hat{u}}{\partial t}\overline{\hat{u}} + 4\pi^2 k^2 \hat{u}\overline{\frac{\partial \hat{u}}{\partial t}} \\
&= \left[\frac{\partial^2 \hat{u}}{\partial t^2} + 4\pi^2 k^2 \hat{u}\right]\overline{\frac{\partial \hat{u}}{\partial t}} + \frac{\partial \hat{u}}{\partial t}\overline{\left[\frac{\partial^2 \hat{u}}{\partial t^2} + 4\pi^2 k^2 \hat{u}\right]} = 0.
\end{aligned}$$

**(b)**

$$\hat{h}(k) = \int_{\mathbb{R}} \overline{g(x)} e^{-2\pi i k\cdot x}\, dx = \overline{\int_{\mathbb{R}} g(x) e^{2\pi i k\cdot x}\, dx} = \overline{\int_{\mathbb{R}} g(x) e^{-2\pi i(-k)\cdot x}\, dx} = \overline{\hat{g}(-k)}.$$

**(c)** Begin with the equation

$$\int_{\mathbb{R}} \hat{u} v = \int_{\mathbb{R}} u \hat{v}.$$

The trick is to choose $v = \overline{\hat{u}}$. By the previous part

$$\hat{v}(k) = \mathcal{F}\left[\overline{\hat{u}}\right](k) = \overline{\mathcal{F}\left[\hat{u}\right](-k)} = \overline{\mathcal{F}^{-1}\left[\hat{u}\right](k)} = \overline{u(k)},$$

using the Fourier inversion theorem that $\mathcal{F}^{-1}[u](k) = \mathcal{F}[u](-k)$. Substitution of $v$ and $\hat{v}$ yields the theorem.

**(d)** We observe that the energy is given by

$$e(t) = \frac{1}{2}\int_{\mathbb{R}} \left|\frac{\partial u}{\partial t}\right|^2 + \left|\frac{\partial u}{\partial x}\right|^2 \, dx = \frac{1}{2}\int_{\mathbb{R}} \left|\mathcal{F}\left[\frac{\partial u}{\partial t}\right]\right|^2 + \left|\mathcal{F}\left[\frac{\partial u}{\partial x}\right]\right|^2 \, dk$$
$$= \frac{1}{2}\int_{\mathbb{R}} \left|\frac{\partial \hat{u}}{\partial t}\right|^2 + |2\pi i k\hat{u}|^2 \, dk = \int_{\mathbb{R}} \mathcal{E}(k,t)\, dk.$$

The right hand side is constant in $t$ by part (a).