Introduction to Partial Differential Equations
Exercise sheet 14

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42. Method of Descent

In this exercise we will apply the method of descent to solve the wave equation on $\mathbb{R}^2$ for a particular set of initial conditions. The idea is to help you understand the key ideas and notation of the method. It is a combination of results from Sections 5.1–5.

Consider the wave equation on $\mathbb{R}^2$ with initial conditions

$$\partial_t^2 u - \Delta u = 0 \text{ on } (x,t) \in \mathbb{R}^2 \times (0,\infty),$$
$$u(x,0) = g(x) = \chi_{[0,\infty)}(x_1), \qquad \partial_t u(x,0) = h(x) = 0.$$

(a) Suppose u is a solution of the wave equation for $n = 2$. Why does $(x_1, x_2, x_3, t) \mapsto u(x_1, x_2, t)$ solve the wave equation on $\mathbb{R}^3$? (note, the Laplacians are different in different dimensions). *(1 point)*

(b) Conversely, prove that a solution $\bar{u}$ to the 3-dimensional wave equation that does not depend on x_3 gives a solution to the 2-dimensional wave equation. *(1 point)*

(c) By (a) and (b), we now must solve a wave equation on $\mathbb{R}^3$. The key to solving the 3-dimensional wave equation is to consider the (spatial-)spherical means

$$U(x,t,r) = \frac{1}{4\pi r^2} \int_{\partial B(x,r)} \bar{u}(z,t)\, d\sigma(z),$$

and likewise let G and H be the spherical means of $\bar{g}$ and $\bar{h}$ respectively. Explain why $\bar{g}(x_1, x_2, x_3) = \chi_{[0,\infty)}(x_1)$ and $\bar{h} = 0$ (or give the definition of bar). Show that

$$G(x,r) = \begin{cases} 0 & \text{for } x_1 \leq -r \\ \frac{1}{2}\frac{x_1 + r}{r} & \text{for } |x_1| \leq r \\ 1 & \text{for } r \leq x_1 \end{cases} \qquad \text{and } H(x,r) = 0.$$

You may use the following geometric fact: for $-R < a < b < R$, the surface area of the part of the sphere $\partial B(0,R)$ with $a < x_1 < b$ is $2\pi R(b-a)$. *(4 points)*

(d) We know by Lemma 5.2 that U obeys the Euler-Poisson-Darboux equation. Let $\tilde{U}(x,t,r) := rU(x,t,r)$. Show that $\tilde{U}$ obeys the following PDE

$$\partial_t^2 \tilde{U} - \partial_r^2 \tilde{U} = 0 \text{ on } (t,r) \in [0,\infty) \times [0,\infty),$$
$$\tilde{U}(x,0,r) = rG(x,r), \qquad \partial_t \tilde{U}(x,0,r) = rH(x,r).$$

Note that there are no x-derivatives in this PDE, so we can think of it as a family of PDEs in the variables r, t parametrised by x. *(2 points)*

(e) Thus we see that $\tilde{U}$ obeys the 1-dimensional wave equation on the *half-line* $r \in [0,\infty)$. This is solved by a trick using reflection, and the formula is at the end of Section 5.2. We only need the solution for small r, so it is enough to consider the case $0 \leq r \leq t$. In this case, show

$$\tilde{U}(x,t,r) = \begin{cases} 0 & \text{for } x_1 \leq -(t+r) \\ \frac{1}{4}(x_1+t+r) & \text{for } -(t+r) \leq x_1 \leq -(t-r) \\ \frac{1}{2}r & \text{for } |x_1| \leq t-r \\ \frac{1}{4}(x_1-t+3r) & \text{for } t-r \leq x_1 \leq t+r \\ r & \text{for } x_1 \geq t+r. \end{cases}$$

(4 points)

(f) Recover $\bar{u}$ from $\tilde{U}$ using a certain property of spherical means. *(3 points)*

Observe that $\bar{u}$ does not depend on x_3. So by part (b) we have a solution to the 2-dimensional wave equation:

$$u(x_1,x_2,t) = \begin{cases} 0 & \text{for } x_1 < -t \\ 0.25 & \text{for } x_1 = -t \\ 0.5 & \text{for } -t < x_1 < t \\ 0.75 & \text{for } x_1 = t \\ 1 & \text{for } x_1 > t. \end{cases}$$

This solution has jump discontinuities, but this is unsurprising since the initial conditions also had them.

43. Wave energy modes

Let us work with $n = 1$ for simplicity. If we apply the Fourier transform to the wave equation, we arrive at

$$\frac{\partial^2 \hat{u}}{\partial t^2} + 4\pi^2 k^2 \hat{u} = 0.$$

This leads us to define the *spectral energy density*

$$2\mathcal{E}(k,t) := \left|\frac{\partial \hat{u}}{\partial t}\right|^2 + 4\pi^2 k^2 \left|\hat{u}\right|^2.$$

(a) Through calculation, show that $\mathcal{E}$ is constant in t. Note that $\hat{u}$ is a complex valued function and it is important to respect complex conjugation: $|g|^2 = g\bar{g}$. *(1 point)*

(b) Consider a complex valued function g and its conjugate $h = \bar{g}$. Prove that $\hat{h}(k) = \overline{\hat{g}(-k)}$. *(1 point)*

(c) Using an equation on page 73 for the script, establish Plancherel's theorem *(1 point)*

$$\int_{\mathbb{R}} |h|^2 \, dx = \int_{\mathbb{R}} |\hat{h}|^2 \, dk.$$

(d) Use this to show that the energy of a wave is constant. *(2 points)*