Introduction to Partial Differential Equations
Exercise sheet 13

Ross Ogilvie 25th November, 2024

38. Distinct characteristics

(a) Show that the a smooth function $u = u(\zeta, \eta) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a solution to $\partial_\zeta\partial_\eta u = 0$ exactly when it is of the form $u(\zeta, \eta) = F(\zeta) + G(\eta)$, for smooth functions $F, G : \mathbb{R} \to \mathbb{R}$.

(2 points)

(b) Under the parameterisation $\zeta = x + t, \eta = x - t$, show that u obeys the one dimensional wave equation $(\partial_t^2 - \partial_x^2)u = 0$ exactly when it solves the PDE in (a). *(3 points)*

(c) From parts (a) and (b), derive D'Alembert's formula. *(2 points)*

Solution.

(a) Suppose u is a solution. Integrating once, we see that $\partial_\eta u = g(\eta)$, because for each value of η, $\partial_\eta u$ must be constant in ζ. Integrating again gives $u = F(\zeta) + \int g(\eta)\, d\eta =: F(\zeta) + G(\eta)$. The converse is immediate.

(b) Using the chain rule we compute the operator under the change of variables. Note $x = \frac{1}{2}(\zeta + \eta)$ and $y = \frac{1}{2}(\zeta - \eta)$.

$$\frac{\partial}{\partial\zeta} = \frac{\partial x}{\partial\zeta}\frac{\partial}{\partial x} + \frac{\partial t}{\partial\zeta}\frac{\partial}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial t}$$

$$\frac{\partial}{\partial\eta} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2}\frac{\partial}{\partial t}$$

$$\partial_\zeta\partial_\eta = \frac{1}{4}(\partial_x - \partial_t)(\partial_x + \partial_t) = \frac{1}{4}(\partial_x\partial_x - \partial_x\partial_t + \partial_t\partial_x - \partial_t\partial_t) = \frac{1}{4}(\partial_x\partial_x - \partial_t\partial_t).$$

We see then that the operator $\partial_t\partial_t - \partial_x\partial_x$ is just a rescaling of $\partial_\zeta\partial_\eta$.

(c) We are asked to solve the problem of Theorem 5.1:

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0 \\ u(x, 0) = g(x) \\ \partial_t u(x, 0) = h(x), \end{cases}$$

where g is twice continuously differentiable and f is only once continuously differentiable.

From (a), we know that the equation is simpler in the (ζ, η) coordinates, where $u(\zeta, \eta) = F(\zeta) + G(\eta)$ solves the wave equation. The functions F and G are only defined up to a constant between them (ie $u = (F - C) + (G + C)$ also), so without loss of generality choose $G(0) = 0$.

When $t = 0$ that corresponds to $x = \zeta = \eta$. So the initial conditions say $F(\zeta) + G(\zeta) = g(\zeta)$ and $\partial_t u|_{t=0} = (\partial_\zeta - \partial_\eta)u|_{\zeta=\eta} = F'(\zeta) - G'(\zeta) = h(\zeta)$. Integrating the latter gives

$$\int_0^\zeta h(y)\, dy = \int_0^\zeta F'(y) - G'(y)\, dy = F(\zeta) - G(\zeta) - (F(0) - G(0)).$$

Now we have two linear equations for F and G, so solving gives

$$F(\zeta) = \frac{1}{2}\left[g(\zeta) + F(0) - G(0) + \int_0^\zeta h(y)\, dy\right],\; G(\zeta) = \frac{1}{2}\left[g(\zeta) - F(0) + G(0) - \int_0^\zeta h(y)\, dy\right].$$

Changing the variable in G back to η and summing gives:

$$\begin{aligned} u &= F(\zeta) + G(\eta) \\ &= \frac{1}{2}\left[g(\zeta) + \int_0^\zeta h(y)\,dy\right] + \frac{1}{2}\left[g(\eta) - \int_0^\eta h(y)\,dy\right] \\ &= \frac{1}{2}\left[g(\zeta) + g(\eta)\right] + \frac{1}{2}\int_\eta^\zeta h(y)\,dy. \end{aligned}$$

Finally, changing back to (x, t) coordinates gives the desired formula.

39. Faster!

How should you modify D'Alembert's formula for this situation?

$$\begin{cases} \partial_t^2 u - a^2 \partial_x^2 u = 0 \\ u(x, 0) = g(x) \\ \partial_t u(x, 0) = h(x), \end{cases}$$

Solve this for the initial data $a = 2$, $g(x) = \sin(x)$ and $h(x) = 1$. *(5 points)*

Solution. One can rescale one of the coordinates to compensate for the factor of a^2. Namely, let $\tau = at$. Because $t = 0$ when $\tau = 0$, the first initial condition is unchanged. The second initial condition however reads $a\partial_\tau u(x, 0) = h(x)$. Using the formula for the solution to this new initial value problem for the wave equation, but then further making the substitution $\tau = at$, gives

$$u(x, t) = \frac{1}{2}\left[g(x + at) + g(x - at)\right] + \frac{1}{2}\int_{x-at}^{x+at} \frac{1}{a} h(y)\,dy.$$

With the given initial data

$$\begin{aligned} u(x, t) &= \frac{1}{2}\left[\sin(x + 2t) + \sin(x - 2t)\right] + \frac{1}{4}\int_{x-2t}^{x+2t} 1\,dy \\ &= \frac{1}{2}\left[\sin(x + 2t) + \sin(x - 2t)\right] + t. \end{aligned}$$

40. Plane Waves

Suppose that $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a solution to the following modified wave equation:

$$\frac{\partial^2 u}{\partial t^2} - \sum_{j=1}^{n} c_j^2 \frac{\partial^2 u}{\partial x_j^2} = 0\,, \tag{$*$}$$

where $c_1, \ldots, c_n > 0$ are constants.

(a) Let $\alpha \in \mathbb{R}^n$ be a unit vector $\|\alpha\| = 1$, $\mu \in \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ a twice continuously differentiable function. Show that

$$u(x,t) := F(\alpha \cdot x - \mu t)$$

is a solution of $(*)$ exactly when

$$\mu^2 = \sum_{j=1}^{n} \alpha_j^2 c_j^2$$

or F is linear. Solutions of $(*)$ with this form are called *plane waves.* *(2 points)*

(b) For the solutions in (a), examine whether the following property holds for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$:

$$u(x,t) = u(x - \mu t\alpha, 0).$$

Interpret this equation in terms of direction and speed. *(3 points)*

Solution.

(a) We apply the chain rule to differentiate F:

$$\frac{\partial^2 u}{\partial t^2} - \sum_{j=1}^{n} c_j^2 \frac{\partial^2 u}{\partial x_j^2} = (-\mu)^2 F'' - \sum_{j=1}^{n} c_j^2 (\alpha_j)^2 F'' = \left(\mu^2 - \sum_{j=1}^{n} c_j^2 \alpha_j^2\right) F''.$$

Clearly this is zero only if the relation between μ and α holds or if $F'' = 0$.

(b) This property does hold, because of the normalistion condition $|\alpha| = \alpha \cdot \alpha = 1$:

$$u(x,t) = F(\alpha \cdot x - \mu t) = F(\alpha \cdot (x - \mu t\alpha)) = F(\alpha \cdot (x - \mu t\alpha) - \mu 0) = u(x - \mu t\alpha, 0).$$

This shows that plane waves are constant along the planes $x \cdot \alpha = \text{const.}$. If we consider a line parallel, then the problem is reduced to the one dimensional wave equation with speed μ. Hence we say the wave is moving in the direction α.

There are other sorts basic waves; spherical waves and standing waves are two important examples. In three dimensions, if a solution only depends on $r = |x|$ then the wave equation becomes

$$0 = \partial_t^2 u - \partial_r^2 u - \frac{2}{r}\partial_r u = \frac{1}{r}(\partial_t^2 - \partial_r^2)(ru).$$

This is again a one dimensional wave equation, solved by $u(r,t) = r^{-1}F(r-t) + r^{-1}G(r+t)$. The interpretation here is that there are inward and outward moving spheres, but the amplitude is diminished/concentrated as the radius is changed.

A standing wave is one whose peaks do not move in space, it only oscillates in time. Simple standing waves separate into the form $u(x,t) = \tilde{u}(x)\sin(\omega t)$. The profile of the wave (the $\tilde{u}$ part) is governed by the equation

$$0 = (\partial_{tt} - \triangle)u = (-\omega^2 \tilde{u} - \triangle \tilde{u})\sin(\omega t).$$

Alternatively, this arises from taking the Fourier transformation in t, namely $\hat{u}(x,\omega) = \int u(x,t)e^{-i\omega t}\, dt$, and considering solutions with a constant frequency ω.

41. Electromagnetic Waves

In physics, electrical and magnetic fields are modelled as time-dependent vector fields, which mathematically are smooth functions $E, B : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$. Through a series of experiments in the 18th and 19th centuries, the existence and properties of these fields were discovered. Importantly, it was discovered that the two phenomena were connected (both magnets and static electricity had been known since antiquity). In 1861 James Clerk Maxwell published a series of papers summarising electromagnetic theory, including a collection of 20 differential equations. Over time these were further reduced to the following four (by Heaviside 1884 using vector notation), called *Maxwell's Equations*:

$$\nabla \cdot E = \frac{1}{\varepsilon_0}\rho \qquad \nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \cdot B = 0 \qquad \nabla \times B = \mu_0 J + \varepsilon_0 \mu_0 \frac{\partial E}{\partial t}.$$

As is usual, the ∇ operator acts on the spatial coordinates x, and the $\times$ denotes the cross product of $\mathbb{R}^3$. The constants ε_0, the electrical permittivity, and μ_0, the magnetic permeability, are approximately $\varepsilon_0 \approx 8,854 \cdot 10^{-12} \frac{\mathrm{A \cdot s}}{\mathrm{V \cdot m}}$ and $\mu_0 \approx 1,257 \cdot 10^{-6} \frac{\mathrm{V \cdot s}}{\mathrm{A \cdot m}}$ (V=Volt, s=Seconds, A=Ampere and m=Metre) in a vacuum. Electrical charges are included via the charge density ρ and electric currents are the movements of charges, $J := v\rho$ for a velocity field v.

The two equations with divergence were formulated by Gauss, based on known inverse-square force laws, the curl of the electric field is due to Faraday, and the curl of the magnetic field is due to Ampère. The last term in Ampère's law that has the time-derivative of the electrical field was an addition of Maxwell. With this correction, he was able to derive the equations for electromagnetic waves, as you will now do.

(a) Let E und B be solutions to Maxwell's equations in the absence of electric charges, $\rho = 0, J = 0$. Show that they each satisfy a modified wave equation (Question 40). You may use without proof the identity $\nabla \times (\nabla \times f) = \nabla(\nabla \cdot f) - \triangle f$ for smooth functions $f : \mathbb{R}^3 \to \mathbb{R}^3$. *(3 points)*

(b) Predict the speed of these waves. *(2 Bonus Points)*

(c) Argue that Ampère's law in its original form $\nabla \times B = \mu_0 J$ violates the conservation of charge ρ under some conditions. Refer to Exercise Sheet 5 for the definition of a conservation law. Thereby derive Maxwell's additional term. *(3 Bonus Points)*

Solution.

(a) Suppose we have solutions E, B. As suggested by the hint, we take the curl of the curl equations. Because curl is a linear operator (and derivatives commute) we may write

$$\nabla \times \nabla \times E = -\frac{\partial}{\partial t} \nabla \times B = -\frac{\partial}{\partial t} \varepsilon_0 \mu_0 \frac{\partial E}{\partial t} = -\varepsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2}.$$

On the other hand, we know the twice curl of E is $\nabla(\nabla \cdot E) - \triangle E = \nabla(0) - \triangle E$, using Gauss' law of electric fields. Rearranging we get a modified wave equation:

$$\frac{\partial^2 E}{\partial t^2} = \frac{1}{\varepsilon_0 \mu_0} \triangle E.$$

and likewise for B.

(b) We expect that the speed is given by μ as in Question 41(b), and this can be calculated from the coefficients c_j and the direction α. In this case, the coefficients are the same in each coordinate direction, so factor out:

$$\mu = \sqrt{\sum a_j^2 c^2} = c|\alpha| = c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \approx 299\,800\,000\, ms^{-1}.$$

This is the speed of light. The speed of light had first been calculated nearly 200 years earlier by Romer using astronomical observations of Jupiter and its moons, and would in 1862 measured with less than 1% error. The electrical constant had been determined only 5 years earlier with experiments with capacitors by Weber and Kohlrausch. The magnetic constant is fixed by the definition to be $4\pi \cdot 10^{-7}$. The measurements were good enough in Maxwell's day to see that these were close, and on this basis Maxwell hypothesised light was an electromagnetic wave.

(c) We saw in Question 12 that a quantity, be it mass or in this case electrical charge, is conserved when the change of density is equal to the negative of the divergence of the flow (using the divergence theorem, the divergence of the flow is the amount of substance leaving a small ball around that point). Symbolically, $\partial_t \rho = -\nabla(v\rho) = \nabla J$. If we take the divergence of Ampère's version we have

$$0 = \nabla \cdot (\nabla \times B) = \mu_0 \nabla \cdot J.$$

This is only true when the charge density ρ is constant. As Ampère's experiment used two wires with constant currents, this was true in his experiment.

But in general we should add another term $\nabla \times B = \mu_0 J + G$. Applying the divergence now, we see that

$$\nabla \cdot G = -\mu_0 \nabla \cdot J = \mu_0 \frac{\partial}{\partial t} \rho = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \nabla \cdot E.$$

Hence we conclude that $G = \mu_0 \varepsilon_0 \partial_t E + \nabla \times g$. Taking the simplest possibility, $g = 0$, gives Maxwell's correction.

Note that we shows that each component of the electric and magnetic fields solve the wave equation, but this is a necessary condition. Faraday's law show that there is a dependence between the two fields. And both fields must have zero divergence, which creates a dependence directly between the components. For example, consider if all of E_i are plane waves travelling in the x_3 direction, so E depends only on x_3. Then $\nabla \cdot E = 0$ implies $E_3 = 0$. The relations between the components is polarization. For example, a solution such as $E_1 = E_1(x_3 - ct), E_2 = E_3 = 0$ is a wave travelling in the x_3 direction, but polarized in the x_1 direction.