Introduction to Partial Differential Equations
Exercise sheet 12

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35. Some like it hot

Find the solution $u : (0, \pi) \times \mathbb{R}^+ \to \mathbb{R}$ of the initial and boundary value problem: *(6 points)*

$$\begin{cases} \dot{u} - 7\partial_{xx}u = 0 & \text{for } x \in (0, \pi),\ t > 0 \\ u(0, t) = u(\pi, t) = 0 & \text{for } t > 0 \\ u(x, 0) = 3\sin(2x) - 6\sin(5x) & \text{for } x \in (0, \pi). \end{cases}$$

Solution. Firstly, this isn't quite the heat equation, but we can rescale time to absorb the factor 7, namely $s = 7t$. We have just calculated the heat kernel for an interval, so let's use it! By Theorem 4.16

$$\begin{aligned} u(x, s) &= 0 - 0 + \int_{[0,\pi]} [3\sin(2y) - 6\sin(5y)]\, H_{[0,\pi]}(x, y, s)\, \mathrm{d}y \\ &= 0 - 0 + \int_{[0,\pi]} [3\sin(2y) - 6\sin(5y)]\, \frac{1}{\pi} H_{[0,1]}\left(\tfrac{x}{\pi}, \tfrac{y}{\pi}, \tfrac{s}{\pi^2}\right)\, \mathrm{d}y \\ &= \int_0^\pi [3\sin(2y) - 6\sin(5y)]\, \frac{1}{\pi} \sum_{k=1}^\infty e^{-k^2 s} 2\sin(kx)\sin(ky)\, \mathrm{d}y \\ &= \frac{2}{\pi} \sum_{k=1}^\infty e^{-k^2 s} \sin(kx) \left[\int_0^\pi 3\sin(2y)\sin(ky)\, \mathrm{d}y - \int_0^\pi 6\sin(5y)\sin(ky)\, \mathrm{d}y \right]. \end{aligned}$$

Now it is relatively easy to see, by applying integration by parts twice, that $\int_0^\pi \sin mz \sin nz$ is zero if $m \neq n$ and is $\pi/2$ if they are equal. Thus all but two terms of the sum are zero.

$$\begin{aligned} u(x, s) &= \frac{2}{\pi} e^{-4s} \sin(2x)\, 3\frac{\pi}{2} - \frac{2}{\pi} e^{-25s} \sin(5x)\, 6\frac{\pi}{2} \\ &= 3e^{-4s}\sin(2x) - 6e^{-25s}\sin(5x) \\ u(x, t) &= 3e^{-28t}\sin(2x) - 6e^{-175t}\sin(5x) \end{aligned}$$

The temperature falls very quickly, so make sure you have a jacket. We can also check our solution:

$$\begin{aligned} \dot{u}(x, t) &= -28 \cdot 3e^{-28t}\sin(2x) + 175 \cdot 6e^{-175t}\sin(5x) \\ \partial_{xx}u(x, t) &= -4 \cdot 3e^{-28t}\sin(2x) + 25 \cdot 6e^{-175t}\sin(5x). \end{aligned}$$

36. Out of the frying pan, into the fire

Find the solution $u : (0, \pi) \times \mathbb{R}^+ \to \mathbb{R}$ of the initial and boundary value problem:

$$\begin{cases} \dot{u} - \partial_{xx}u = 0 & \text{for } x \in (0, \pi),\ t > 0 \\ u(0, t) = u(\pi, t) = 0 & \text{for } t > 0 \\ u(x, 0) = x^2(\pi - x) & \text{for } x \in (0, \pi). \end{cases}$$

Further, show that your solution obeys $\int_0^\pi u(x,t)\,\mathrm{d}x = 8\sum_{k\text{ odd}} \frac{1}{k^4} e^{-k^2 t}$. *(7 points)*

Solution. We will again use the heat kernel and the representation formula, except this the solution will not be elementary:

$$u(x,t) = 0 - 0 + \int_{[0,\pi]} y^2(\pi - y)\,\frac{1}{\pi} H_{[0,1]}\left(\tfrac{x}{\pi}, \tfrac{y}{\pi}, \tfrac{t}{\pi^2}\right)\,\mathrm{d}y$$
$$= \frac{2}{\pi}\sum_{k=1}^{\infty} e^{-k^2 t}\sin(kx)\int_{[0,\pi]} y^2(\pi - y)\,\sin(ky)\,\mathrm{d}y.$$

The integral can be computed by repeated integration by parts and is $-2\pi k^{-3}(1 + 2\cos k\pi)$. Hence

$$u(x,t) = -4\sum_{k=1}^{\infty} k^{-3}(1 + 2\cos k\pi)\,e^{-k^2 t}\sin(kx).$$

This is the solution. We can easily compute its value numerically because the series is so rapidly convergent. But we see that it is quite difficult to understand the overall shape. For example, what it the highest temperature of the rod at any given time? Despite the initial condition being positive, it is even not immediately clear that the solution is even positive (though it is, this follows because it is an alternating series and so is bounded between its partial sums). Let's split this into even and odd terms:

$$u(x,t) = 4\sum_{k\text{ odd}}^{\infty} k^{-3}\,e^{-k^2 t}\sin(kx) - 12\sum_{k\text{ even}}^{\infty} k^{-3}\,e^{-k^2 t}\sin(kx).$$

Because $\int_0^\pi \sin kx\,\mathrm{d}x$ is zero if k is even and $2/k$ if k is odd, the result about the total heat in the domain follows. Notice that the total amount of heat is not conserved in this situation, because the ends of the rod are being kept at a constant temperature of zero and so heat is escaping through these ends.

37. Method of images for the heat kernel on $[0,1]$

(a) Show the following formula for theta functions

$$\Theta(\tfrac{z}{2}, \pi i t) = 1 + \sum_{k=1}^{\infty} e^{-\pi^2 t k^2}\,2\cos(\pi k z),$$

and therefore that

$$H_{[0,1]}(x,y,t) = \frac{1}{2}\Theta(\tfrac{x-y}{2}, \pi i t) - \frac{1}{2}\Theta(\tfrac{x+y}{2}, \pi i t)$$

as claimed in the script. *(2 points)*

(b) Let $\mathcal{A}$ be the space of all continuous functions on $\mathbb{R}$ with the following properties:

$$f(n+x) = \begin{cases} f(x) & \text{for even } n \in 2\mathbb{Z} \text{ and } x \in \mathbb{R} \\ -f(1-x) & \text{for odd } n \in 2\mathbb{Z}+1 \text{ and } x \in \mathbb{R}. \end{cases}$$

Show that the functions in $\mathcal{A}$ vanish at $\mathbb{Z}$ and that $\mathcal{A}$ contains all continuous odd and periodic functions with period 2. *(1 point)*

(c) Show that for any Schwartz function f on $\mathbb{R}$ the following series converges to a smooth function $\tilde{f}$ in $\mathcal{A}$: *(2 points)*

$$\tilde{f}(x) = \sum_{n\in\mathbb{Z}} f(2n+x) - \sum_{n\in\mathbb{Z}} f(2n-x).$$

(d) Conclude that the following sum is a heat kernel of $[0,1]$: *(2 points)*

$$\sum_{n\in\mathbb{Z}} \Phi(x+2n-y,t) - \sum_{n\in\mathbb{Z}} \Phi(x+2n+y,t).$$

(e) Using Poisson's summation formula

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inx}$$

and part (a), show the relation

$$H_{[0,1]}(x,y,t) = \sum_{n\in\mathbb{Z}} \Phi(x+2n-y,t) - \sum_{n\in\mathbb{Z}} \Phi(x+2n+y,t),$$

where the left hand side the heat kernel in terms of theta functions as given in the lecture script. Thus the method of images gives the same heat kernel as the Fourier series method (of course, the heat kernel is unique). *(2 bonus points)*

Solution.

(a) We begin with a theta function:

$$\begin{aligned}\Theta(\tfrac{z}{2},\pi it) &= \sum_{k\in\mathbb{Z}} \exp(\pi ikz + (\pi i)^2tk^2)\\ &= \sum_{k\in\mathbb{Z}} e^{-\pi^2tk^2}\exp(\pi ikz)\\ &= 1 + \sum_{k=1}^{\infty} e^{-\pi^2tk^2}\Big[\exp(\pi ikz)+\exp(-\pi ikz)\Big]\\ &= 1 + \sum_{k=1}^{\infty} e^{-\pi^2tk^2}2\cos(\pi kz).\end{aligned}$$

Taking the difference with $z = x-y$ and $z = x+y$ gives the the result.

(b) The functions of $\mathcal{A}$ are clearly periodic with period 2, since for $n=2$ we have the relation $f(2+x) = f(x)$. For $n=1$ we have that $f(1+x) = -f(1-x) = -f(-(1+x))$, which shows that these functions are odd. Therefore it vanishes at 0 and all even integers. Setting $x=0$ also gives $f(1+0) = -f(1-0)$ showing it vanishes at 1, and hence all odd integers also.

Now take any odd function f with period 2. Then clearly we have $f(n+x) = f(x)$ for any even integer n; this is the definition of 'period 2'. For an odd integer n:

$$f(n+x) = f(1+x) = -f(-(1+x)) = -f(-1-x) = -f(1-x).$$

This shows $f\in\mathcal{A}$.

(c) Just consider one sum. Because f is a Schwartz function, it is straightforward to prove the bound $f(x) \leq C(1+x^2)^{-1}$. This is a sufficient to show that the sum exists for all x. Moreover, we have uniform convergence on compact subsets, and so the resulting function is also smooth.

By shifting the summation indices we see that it has period 2 also. It remains to show that it is an odd function

$$\tilde{f}(-x) = \sum_{n\in\mathbb{Z}} f(2n-x) - \sum_{n\in\mathbb{Z}} f(2n+x) = -\tilde{f}(x).$$

(d) We have to check that it fits Definition 4.14 with $\Omega = (0,1)$. This sum is the transformation of part (c) applied to $y \mapsto \Phi(x-y,t)$, with the index of the sum negated. For positive time the fundamental solution is smooth and Schwartz. The sums are therefore smooth too and defined on all of $x, y \in \mathbb{R}$. Because they belong to $\mathcal{A}$, it is zero at $y \in \mathbb{Z}$. But this implies it vanishes on $\partial\Omega = \{0,1\}$.

Now we need to verify condition (i). It is a sum of fundamental solutions, so it solves the heat equation for $t > 0$. Consider

$$H_{[0,1]}(x,y,t) - \Phi(x-y,t) = \sum_{n\in\mathbb{Z}, n\neq 0} \Phi(x+2n-y,t) - \sum_{n\in\mathbb{Z}} \Phi(x+2n+y,t).$$

We know that $\Phi(z,t)$ extends continuously with value 0 as $t \to 0$ if $z \neq 0$ (at $(0,0)$ there is a singularity). If $x \in \Omega = (0,1)$ and $y \in \overline{\Omega} = [0,1]$ then $x-y \in (-1,1)$ and so $x+2n-y \neq 0$ for $n \neq 0$. Similarly $x+y \in (0,2)$ so the singularities are avoided in the second sum too.

(e) The Fourier transform of the fundamental solution is

$$\hat{\Phi}(k,t) = \exp(-4\pi^2 k^2 t).$$

Now use the hint applied to a sum of fundamental solutions

$$\sum_{n\in\mathbb{Z}} \Phi(z+n,t) = \sum_{n\in\mathbb{Z}} e^{-4\pi^2 n^2 t} e^{2\pi i n z} = \sum_{n\in\mathbb{Z}} e^{i\pi^2 n^2 (4it)} e^{2\pi i n z} = \Theta(z, 4\pi i t).$$

Now we use the theta function equation of the heat kernel

$$\begin{aligned} H_{(0,1)}(x,y,t) &= \frac{1}{2}\Theta(\tfrac{x-y}{2}, \pi i t) - \frac{1}{2}\Theta(\tfrac{x+y}{2}, \pi i t) \\ &= \sum_{n\in\mathbb{Z}} \frac{1}{2}\Phi(\tfrac{x-y}{2}+n, \tfrac{t}{4}) - \sum_{n\in\mathbb{Z}} \frac{1}{2}\Phi(\tfrac{x+y}{2}+n, \tfrac{t}{4}) \\ &= \sum_{n\in\mathbb{Z}} \Phi(x-y+2n,t) - \sum_{n\in\mathbb{Z}} \Phi(x+y+2n,t), \end{aligned}$$

using the rescaling property of the fundamental solution.