Introduction to Partial Differential Equations
Exercise sheet 11

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33. The distribution of heat

Consider the fundamental solution of the heat equation $\Phi(x,t)$ given in Definition 4.5.

(a) Show that this extends to a smooth function on $\mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\}$. *(2 points)*

(b) Verify that this obeys the heat equation on $\mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\}$. *(2 points)*

We want to show that $\varphi \mapsto H(\varphi) = \int_{\mathbb{R}^n \times \mathbb{R}} \Phi(x,t)\varphi(x,t)\ dx\,dt$ is a distribution. Clearly it is linear. Fix a set $K \subset \mathbb{R}^n \times \mathbb{R}$ and let $\varphi \in C_0^\infty(K)$.

(c) Why must there be a constant $T > 0$ with

$$H(\varphi) = \int_0^T \int_{\mathbb{R}^n} \Phi(x,t)\varphi(x,t)\ dx\,dt\ \ ?$$

(1 point)

(d) Conclude with the help of Lemma 4.6 and Theorem 4.7 that

$$|H(\varphi)| \le T\,\|\varphi\|_{K,0}.$$

Hence H is a continuous linear functional. *(2 points)*

Finally, we want to show that (in the sense of distributions) $(\partial_t - \Delta)H = \delta$.

(e) Extend Theorem 4.7 to show that

$$\int_{\mathbb{R}^n} \Phi(x-y,t)h(y,s)\ dy \to h(x,s)$$

as $t \to 0$, uniformly in s. *(1 point)*

(f) Hence show that

$$\int_\varepsilon^\infty \int_{\mathbb{R}^n} \Phi(-\partial_t\varphi - \Delta\varphi)\ dy\,dt \to \varphi(0,0)$$

as $\varepsilon \to 0$. *(3 points)*

(g) Prove that as $\varepsilon \to 0$

$$\int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,t)h(y,t)\ dy\,dt \to 0$$

(2 points)

Together these integrals show that

$$(\partial_t - \Delta)H(\varphi) = \left(\int_0^\varepsilon + \int_\varepsilon^\infty\right)\int_{\mathbb{R}^n} \Phi(-\partial_t\varphi - \Delta\varphi)\ dy\,dt = \varphi(0,0) = \delta(\varphi)$$

for all test functions φ. Therefore $(\partial_t - \Delta)H = \delta$ as claimed.

Solution.

(a) For $t > 0$, $\Phi(x,t)$ and all its derivatives have the form $t^{-k} q(x,t) \exp(-x^2/4t)$ for $k \in \frac{1}{2}\mathbb{N}_0$ and q a polynomial. As $t \to 0^+$ for $x \neq 0$, the exponential terms is dominant and forces the expression to zero. For $t < 0$ the heat kernel is identically zero, and so all its derivatives are zero and the limits as $t \to 0^-$ is zero. Thus we have the smooth extension $\Phi(x,0) = 0$ for $x \neq 0$.

(b) By direct calculation, for $t > 0$

$$\begin{aligned}
(4\pi)^{n/2}\partial_t \Phi &= -\frac{n}{2} t^{-n/2-1} e^{-\frac{|x|^2}{4t}} + \frac{|x|^2}{4} t^{-n/2-2} e^{-\frac{|x|^2}{4t}} \\
(4\pi)^{n/2}\partial_j \Phi &= -\frac{x_j}{2} t^{-n/2-1} e^{-\frac{|x|^2}{4t}} \\
(4\pi)^{n/2}\partial_j^2 \Phi &= -\frac{1}{2} t^{-n/2-1} e^{-\frac{|x|^2}{4t}} + \frac{x_j^2}{4} t^{-n/2-2} e^{-\frac{|x|^2}{4t}}.
\end{aligned}$$

The appropriate sum gives zero. We see that all derivatives of the function are zero for $t = 0, x \neq 0$. Therefore the heat equation holds there too.

(c) Because φ has compact support, it is zero outside $B(0,R) \times [-T,T]$ for some positive constants R and T. Additionally, we know that Φ is zero for $t < 0$. Therefore the integrand is zero outside $B(0,R) \times [0,T]$ and can be discarded.

(d) First just apply the estimate that bounds φ:

$$|H(\varphi)| \leq \int_0^T \int_{\mathbb{R}^n} \Phi(x,t) \|\varphi(x,t)\|_{K,0}\, dx\, dt = \|\varphi(x,t)\|_{K,0} \int_0^T \int_{\mathbb{R}^n} \Phi(x,t)\, dx\, dt.$$

So it remains to bound the integral of Φ over this region. Consider the function $g(t) := \int_{\mathbb{R}^n} \Phi(x,t)\, dx$. Lemma 4.6 says that $g(t) = 1$ for $t > 0$. Theorem 4.7(iii) says that $g(0) = 1$. This gives

$$\int_0^T \int_{\mathbb{R}^n} \Phi(x,t)\, dx\, dt = \int_0^T 1\, dt = T$$

as the constant.

(e) One only needs to modify one step in the proof of Theorem 4.7: $|h(y,s) - h(x,s)|$ is bounded by twice the supremum of h over space *and time* variables.

(f) We should try to apply integration by parts, in order to move the derivatives from φ to Φ, because we know what Φ is. However the boundary term does not necessarily vanish on the $t = \varepsilon$ plane.

$$\begin{aligned}
-\int_\varepsilon^\infty \int_{\mathbb{R}^n} \Phi \partial_t \varphi - \int_\varepsilon^\infty \int_{\mathbb{R}^n} \Phi \Delta \varphi &= -\left[\int_{\mathbb{R}^n} \Phi\varphi \Big|_{t=\varepsilon}^{t=\infty} - \int_\varepsilon^\infty \int_{\mathbb{R}^n} \partial_t \Phi \varphi \right] - \int_\varepsilon^\infty \int_{\mathbb{R}^n} \Delta \Phi \varphi \\
&= \int_{\mathbb{R}^n} \Phi(x,\varepsilon)\varphi(x,\varepsilon) + \int_\varepsilon^\infty \int_{\mathbb{R}^n} (\partial_t \Phi - \Delta \Phi)\varphi \\
&= \int_{\mathbb{R}^n} \Phi(x,\varepsilon)\varphi(x,\varepsilon) \\
&= \int_{\mathbb{R}^n} \Phi(0 - x,\varepsilon)\varphi(x,\varepsilon).
\end{aligned}$$

The second integral on the second line vanishes due to part (b). We can now apply the previous part to conclude that this limits to $\varphi(0,0)$. Notice the need for uniform convergence, because the second parameter of φ is also being changed by the limit $\varepsilon \to 0$.

(g) We can estimate the norm of h out

$$\left|\int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(x,t)h(x,t)\,dx\,dt\right| \leq \|h\|_\infty \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(x,t)\,dx\,dt. = \|h\|_\infty \int_0^\varepsilon dt = \|h\|_\infty \varepsilon.$$

Clearly this tends to zero.

34. Heat death of the universe

First a corollary to Theorem 4.7:

(a) Suppose that $h \in C_b(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and u is defined as in Theorem 4.7. Show

$$\sup_{x\in\mathbb{R}^n} |u(x,t)| \leq \frac{1}{(4\pi t)^{n/2}}\|h\|_{L^1}.$$

(2 points)

The above corollary shows how solutions to the heat equation on $\mathbb{R}^n \times \mathbb{R}^+$ with such initial conditions behave: they tend to zero as $t \to \infty$. Physically this is because if $h \in L^1$ then there is a finite amount of total heat, which over time becomes evenly spread across the space.

On open and bounded domains $\Omega \subset \mathbb{R}^n$ we can have different behaviour, due to the boundary conditions holding the temperature steady. In this question we determine the long time behaviour of solutions u to the heat equation on open and bounded sets Ω with $u(x,t) = 0$ on $\partial\Omega \times \mathbb{R}^+$ and $u(x,0) = h(x)$.We claim $u \to 0$ as $t \to \infty$.

(b) Let l_m be the function from Theorem 4.7 that solves heat equation on $\mathbb{R}^n$ with $l_m(x,0) = mk(x)$ for m a constant and $k : \mathbb{R}^n \to [0,1]$ a smooth function of compact support such that $k|_\Omega \equiv 1$. Why must k exist? Why does $l_m \to 0$ as $t \to \infty$? What boundary conditions on Ω does it obey? *(3 points)*

(c) Use the monotonicity property to show that u tends to zero. *(2 points)*

Hint. Consider $a = \sup_{x\in\Omega} |u(x,0)|$.

Solution.

(a) From the formula in Theorem 4.7 and the definition of the heat kernel

$$|u(x,t)| \leq \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} |h(y)|\,dy \leq \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} |h(y)|\,dy = \frac{1}{(4\pi t)^{n/2}}\|h\|_{L^1}.$$

(b) Since Ω is bounded, it is contained in a ball $B(0,R)$. Choose k to be a hat function that is identically 1 on $B(0,R)$ and zero outside $B(0,2R)$. We have shown how to construct such hat function in the tutorials.

$mk(x)$ is a smooth function of compact support, so it is continuous, bounded, and has finite L^1 norm. Therefore Part (a) applies to it.

We can see directly from the integral that l_m is non-negative, in particular on $\partial\Omega \times \mathbb{R}^+$. And at time zero, we know from Theorem 4.7 that $l_m(x,0) = mk(x) \equiv m$ on $x \in \Omega$.

(c) Let $a = \sup_{x\in\Omega} |u(x,0)| = \sup_{x\in\Omega} |h(x)|$. By definition then $l_{-a}(x,0) \le u(x,0) \le l_a(x,0)$ on $\Omega \times \{0\}$. On the parabolic boundary $(x,t) \in \partial\Omega \times \mathbb{R}^+$ we see that $l_{-a}(x,t) \le u(x,t) = 0 \le l_a(x,t)$. By the monotonicity property it follows that $l_{-a}(x,t) \le u(x,t) \le l_a(x,t)$ for all points. The squeeze theorem then shows that $u \to 0$ as $t \to \infty$.