Introduction to Partial Differential Equations
Exercise sheet 11

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33. The distribution of heat

Consider the fundamental solution of the heat equation $\Phi(x,t)$ given in Definition 4.5.

(a) Show that this extends to a smooth function on $\mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\}$. *(2 points)*

(b) Verify that this obeys the heat equation on $\mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\}$. *(2 points)*

We want to show that $\varphi \mapsto H(\varphi) = \int_{\mathbb{R}^n \times \mathbb{R}} \Phi(x,t)\varphi(x,t)\, dx\, dt$ is a distribution. Clearly it is linear. Fix a set $K \subset \mathbb{R}^n \times \mathbb{R}$ and let $\varphi \in C_0^\infty(K)$.

(c) Why must there be a constant $T > 0$ with

$$H(\varphi) = \int_0^T \int_{\mathbb{R}^n} \Phi(x,t)\varphi(x,t)\, dx\, dt \ \ ?$$

(1 point)

(d) Conclude with the help of Lemma 4.6 and Theorem 4.7 that

$$|H(\varphi)| \leq T\, \|\varphi\|_{K,0}.$$

Hence H is a continuous linear functional. *(2 points)*

Finally, we want to show that (in the sense of distributions) $(\partial_t - \Delta)H = \delta$.

(e) Extend Theorem 4.7 to show that

$$\int_{\mathbb{R}^n} \Phi(x-y,t)h(y,s)\, dy \to h(x,s)$$

as $t \to 0$, uniformly in s. *(1 point)*

(f) Hence show that

$$\int_\varepsilon^\infty \int_{\mathbb{R}^n} \Phi(-\partial_t\varphi - \Delta\varphi)\, dy\, dt \to \varphi(0,0)$$

as $\varepsilon \to 0$. *(3 points)*

(g) Prove that as $\varepsilon \to 0$

$$\int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,t)h(y,t)\, dy\, dt \to 0$$

(2 points)

Together these integrals show that

$$(\partial_t - \Delta)H(\varphi) = \left(\int_0^\varepsilon + \int_\varepsilon^\infty\right) \int_{\mathbb{R}^n} \Phi(-\partial_t\varphi - \Delta\varphi)\, dy\, dt = \varphi(0,0) = \delta(\varphi)$$

for all test functions φ. Therefore $(\partial_t - \Delta)H = \delta$ as claimed.

34. Heat death of the universe

First a corollary to Theorem 4.7:

(a) Suppose that $h \in C_b(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and u is defined as in Theorem 4.7. Show

$$\sup_{x \in \mathbb{R}^n} |u(x,t)| \leq \frac{1}{(4\pi t)^{n/2}} \|h\|_{L^1}.$$

(2 points)

The above corollary shows how solutions to the heat equation on $\mathbb{R}^n \times \mathbb{R}^+$ with such initial conditions behave: they tend to zero as $t \to \infty$. Physically this is because if $h \in L^1$ then there is a finite amount of total heat, which over time becomes evenly spread across the space.

On open and bounded domains $\Omega \subset \mathbb{R}^n$ we can have different behaviour, due to the boundary conditions holding the temperature steady. In this question we determine the long time behaviour of solutions u to the heat equation on open and bounded sets Ω with $u(x,t) = 0$ on $\partial\Omega \times \mathbb{R}^+$ and $u(x,0) = h(x)$.We claim $u \to 0$ as $t \to \infty$.

(b) Let l_m be the function from Theorem 4.7 that solves heat equation on $\mathbb{R}^n$ with $l_m(x,0) = mk(x)$ for m a constant and $k : \mathbb{R}^n \to [0,1]$ a smooth function of compact support such that $k|_\Omega \equiv 1$. Why must k exist? Why does $l_m \to 0$ as $t \to \infty$? What boundary conditions on Ω does it obey? *(3 points)*

(c) Use the monotonicity property to show that u tends to zero. *(2 points)*

Hint. Consider $a = \sup_{x \in \Omega} |u(x,0)|$.