Introduction to Partial Differential Equations
Exercise sheet 10

Ross Ogilvie 4th November, 2024

30. Special solutions of the heat equation

(a) Solutions of PDEs that are constant in the time variable are called "steady-state" solutions. Describe steady-state solutions of the inhomogeneous heat equation. *(1 point)*

(b) Consider the heat equation $\dot{u} - \Delta u = 0$ on $\mathbb{R}^n \times \mathbb{R}^+$ with smooth initial condition $u(x, 0) = h(x)$. Suppose that the Laplacian of h is a constant. Show that there is a solution whose time derivative is constant. *(1 point)*

(c) Consider "translational solutions" to the heat equation on $\mathbb{R} \times \mathbb{R}^+$ (ie $n = 1$). These are solutions of the form $u(x, t) = F(x - bt)$. Find all such solutions. *(2 points)*

(d) If u is a solution to the heat equation, show for every $\lambda \in \mathbb{R}$ that $u_\lambda(x, t) := u(\lambda x, \lambda^2 t)$ is also a solution to the heat equation. *(2 points)*

31. The Fourier transform

In this question we expand on some details from Section 4.1. Recall that the Fourier transform of a function $h(x) : \mathbb{R}^n \to \mathbb{R}$ is defined to be a function $\hat{h}(k) : \mathbb{R}^n \to \mathbb{R}$ given by

$$\hat{h}(k) = \int_{\mathbb{R}^n} e^{-2\pi i k \cdot x} h(x)\, dx.$$

Lemma 4.3 shows that it is well-defined for Schwartz functions.

(a) Give the definition of a Schwartz function. *(1 point)*

(b) Argue that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \exp(-x^2)$ is a Schwartz function. *(2 points)*

(c) Consider

$$I^2 = \left(\int_{\mathbb{R}} e^{-x^2}\, dx \right)^2 = \left(\int_{\mathbb{R}} e^{-x^2}\, dx \right) \left(\int_{\mathbb{R}} e^{-y^2}\, dy \right) = \int_{\mathbb{R}^2} e^{-x^2 - y^2}\, dx\, dy.$$

By changing to polar coordinates, compute this integral. *(1 point)*

(d) Prove the rescaling law for Fourier transforms: if $h(x) = g(ax)$ then *(1 point)*

$$\hat{h}(k) = |a|^{-n} \hat{g}(a^{-1} k).$$

(e) Prove the shift law for Fourier transforms: if $h(x) = g(x - a)$ then *(1 point)*

$$\hat{h}(k) = e^{-2\pi i a \cdot k} \hat{g}(k).$$

(f) Show that δ is a tempered distribution. *(2 points)*

(g) Compute the Fourier transform of δ. *(2 points)*

(h) Try to compute the Fourier transform of 1. What is the difficulty? *(2 points)*

32. One step at a time

Prove the following identity for the fundamental solution in one dimension ($n = 1$):

$$\Phi(x, s+t) = \int_{\mathbb{R}} \Phi(x-y, t)\Phi(y, s)\, dy.$$

(2 points)

Hint. You may use without proof that

$$\int_{\mathbb{R}} \exp(-A + By - Cy^2)\, dy = \sqrt{\frac{\pi}{C}} \exp\left(\frac{B^2}{4C} - A\right).$$