Introduction to Partial Differential Equations
Exercise sheet 9

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27. To be or not to be

Consider the Dirichlet problem for the Laplace equation $\Delta u = 0$ on Ω with $u = g$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^n$ is an open and bounded subset and g is a continuous function. We know from the weak maximum principle that there is at most one solution. In this question we see that for some domains, existence is not guaranteed.

(a) Consider $\Omega = B(0,1) \setminus \{0\}$, so that the boundary $\partial\Omega = \partial B(0,1) \cup \{0\}$ consists of two components. We write $g(x) = g_1(x)$ for $x \in \partial B(0,1)$ and $g(0) = g_2$. Show that there does not exist a solution for $g_1(x) = 0$ and $g_2 = 1$.

Hint. Use Lemma 3.23, even if we haven't reached it in lectures yet. *(3 points)*

(b) Generalise this: What are the necessary and sufficient conditions on g for the Dirichlet problem to have a solution on this domain? *(3 points)*

(c) Generalise again: What can you say about the Dirichlet problem for bounded domains whose boundaries have isolated points? *(1 point)*

Solution.

(a) Suppose for contradiction that u exists. From the definition of the problem, we know that u is a continuous function on the ball, but we do not know that u is harmonic at 0; the PDE $\Delta u = 0$ is only taken to hold on Ω. However, u is bounded on the ball, so Lemma 3.23 allows us to conclude that u is in fact harmonic on the ball $B(0,1)$. Thus u obeys $\Delta u = 0$ on all of $B(0,1)$. By the weak maximum principle, the only harmonic function on $B(0,1)$ which is identically zero on $\partial B(0,1)$ is $u \equiv 0$. That contradicts the boundary condition $u(0) = 1$.

(b) From Poisson's Representation Formula 3.21 that there is a unique solution v to the Dirichlet problem of the Laplace equation on the ball $B(0,1)$ with boundary condition $v|_{\partial B(0,1)} = g_1$. If this v has $v(0) = g_2$ then it is a solution to the Dirichlet problem on Ω. (Sufficient condition).

On the other hand, if $v(0) \neq g_2$ then there is no solution. For contradiction, suppose there were a solution u. Then $w = u - v$ is also harmonic on Ω and has $w|_{\partial B(0,1)} = g_2 - g_2 = 0$ but $w(0) \neq 0$. Applying the argument of part (a) to w gives a contradiction.

(c) Let Ω be a bounded domain with an isolated point x_0 in its boundary. Then $\Omega' = \Omega \cup \{x_0\}$ is also an open bounded domain. By Lemma 3.24 any solution to the Dirichlet problem on Ω is a solution to the Dirichlet problem on Ω'. Therefore there is a solution on Ω if and only if there is a solution v on Ω' with $v(x_0) = g(x_0)$.

28. Do nothing by halves

Let $H_1^+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 > 0\}$ be the upper half-space and $H_1^0 = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = 0\}$ the dividing hyperplane. We call $R_1(x) = (-x_1, x_2, \ldots, x_n)$ reflection in the plane H^0.

(a) A reflection principle for harmonic functions Let $u \in C^2(\overline{H_1^+})$ be a harmonic function that vanishes on H_1^0. Show that the function $v : \mathbb{R}^n \to \mathbb{R}$ defined through reflection

$$v(x) = \begin{cases} u(x) & \text{for } x_1 \geq 0 \\ -u(R_1(x)) & \text{for } x_1 < 0 \end{cases}$$

is harmonic. *(3 points)*

(b) Green's function for the half-space Show that Green's function for H_1^+ is

$$G(x, y) = \Phi(x - y) - \Phi(R_1(x) - y).$$

(2 points)

(c) Green's function for the half-ball Compute the Green's function for B^+. *(3 points)*
Hint. Make use of both the Green's function for the ball and part (b).

Solution.

(a) One could try to show directly that v is harmonic. Clearly it is when $x_1 \neq 0$, and it is possible to compute the necessary derivatives when $x_1 = 0$. However, there is a more general method using the uniqueness of the solution to the Dirichlet problem on the ball.

Fix any radius $r > 0$ and consider the ball B_r. Let $g = v|_{\partial B_r}$ be the restriction of this function to the sphere. This is continuous, in particular when $x_1 = 0$. There is a unique solution $\tilde{v}$ to the Laplace equation $\Delta \tilde{v} = 0$ with $\tilde{v}|_{\partial B_r} = g$. We see that $-\tilde{v} \circ R_1$ is also a solution to this equation, thus $\tilde{v} = -\tilde{v} \circ R$. This implies that $\tilde{v}$ vanishes on $B_r \cap H_1^0$.

Now consider $\tilde{v} - u$. This is also a harmonic function on $B_r \cap H_1^+$, and moreover it is identically zero on $\partial(B_r \cap H_1^+)$. The maximum principle says it has to be zero on all of $B_r \cap H_1^+$. Thus $\tilde{v} = u$ on $B_r \cap H_1^+$ and by reflection $\tilde{v} = v$ on B_r. By taking r larger and larger, we see that equality holds for all points of the plane.

(b) Let Φ be the fundamental solution to the Laplace equation. Let $G(x, y)$ be the Green's function for H_1^+. The required properties are (1) that for any $x \in H_1^+$ the function $G(x, y) - \Phi(x - y)$ is a harmonic function of y and (2) that for any $x \in H_1^+$ we have $\lim_{y \to H_1^0} G(x, y) = 0$. We saw for the unit ball that the Greens function was a difference of the fundamental solution and its reflection across the boundary of the ball. That way, the two cancelled on the boundary and gave the second property. So let use try

$$G(x, y) = \Phi(x - y) - \Phi(R_1(x) - y).$$

The first property is satisfied because $\Phi(R_1(x) - y)$ is only not harmonic when $y = R(x)$, and for $x \in H_1^+$ this only occurs when $y \in H_1^-$. To show the second property, note that $\Phi(z)$ is radially symmetric. Since $\|R(x) - y\| = \|x - R_1(y)\|$, for any $y \in H^0$ we have

$$G(x, y) = \Phi(x - y) - \Phi(x - R_1(y)) = \Phi(x - y) - \Phi(x - y) = 0.$$

Thus we have shown that G is the Greens function.

(c) To discuss the Greens function for the half-ball, we should introduce a symbol for inversion in the sphere, $\iota(x) := |x|^{-2}x$. We know from lectures then

$$G_B(x,y) = \Phi(x-y) - \Phi(|x|(\iota(x)-y)).$$

Following the ideas of the previous question, we guess that the Greens function for the half-ball is the reflection of this one

$$G(x,y) = \Phi(x-y) - \Phi(|x|(\iota(x)-y)) - \Phi(R(x)-y) + \Phi(|x|(\iota(R(x))-y)).$$

If both $x, y \in B^+$ then $\iota(x) - y$, $R(x) - y$, and $|x|(\iota(R(x)) - y)$ are never zero, so $G(x,y) - \Phi(x,y)$ is harmonic. The boundary of B^+ has two parts $\overline{B^0}$ and $\partial B^+ \cap H^+$. If $y \in \overline{B^0}$ then

$$\begin{aligned} G(x,y) &= \Phi(x-y) - \Phi(|x|(\iota(x)-y)) - \Phi(x-R(y)) + \Phi(|x|(R(\iota(x))-y)) \\ &= \Phi(x-y) - \Phi(|x|(\iota(x)-y)) - \Phi(x-y) + \Phi(|x|(\iota(x)-y)) = 0. \end{aligned}$$

On the other hand, if $y \in \partial B^+ \cap H^+$ is in the hemispherical part, then $\||x|(\iota(x)-y)\| = \|x-y\|$ as in the lecture notes, but also $\||x|(\iota(R(x))-y)\| = \|R(x)-y\|$, so

$$G(x,y) = [\Phi(x-y) - \Phi(|x|(\iota(x)-y))] - [\Phi(R(x)-y) - \Phi(|x|(\iota(R(x))-y))] = 0.$$

29. Teach a man to fish

(a) Using the Green's function of H_1^+ from the previous question, derive the following formal integral representation for a solution of the Dirichlet problem $\Delta u = 0$ in $H_1^+, u|_{H_1^0} = g$

$$u(x) = \frac{2x_1}{n\omega_n} \int_{H_1^0} \frac{g(z)}{|x-z|^n} \, \mathrm{d}\sigma(z)$$

Here, 'formal' means that you do not need to prove that the integrals are finite/well-defined. *(3 points)*

(b) Show that if g is periodic (that is, there is some vector $L \in \mathbb{R}^{n-1}$ with $g(x+L) = g(x)$ for all $x \in \mathbb{R}^{n-1}$) then so is the solution. *(2 points)*

(c) Now consider the plane $n = 2$ with g function with compact support. Approximate the value of $u(x)$ for large $|x|$. Feel free to modify this question as you see fit, what interesting things can you say about the growth of u? *(Bonus Points as deserved)*

Solution.

(a) Begin with Greens Representation formula

$$u(x) = -\int_{H^+} G_{H^+}(x,y)\Delta_y u(y) \, \mathrm{d}y - \int_{H^0} u(z) \nabla_z G_{H^+}(x,z) \cdot N \, \mathrm{d}\sigma(z).$$

The function u is harmonic, so the first integral vanishes. For the second term, $\nabla_z G_{H^+}(x,z) = -\nabla_z \Phi(x-z) + \nabla_z \Phi(R(x)-z)$ and we already computed the gradient of the fundamental solution in Theorem 3.2: $\nabla \Phi(y) = -\frac{1}{n\omega_n}\frac{y}{|y|^n}$. The normal is also easy to describe, it points in the negative x_1 direction: $N = (-1, 0, \ldots, 0)$. Therefore

$$\begin{aligned} u(x) &= -\int_{H^0} g(z)\frac{1}{n\omega_n}\left[\frac{x-z}{|x-z|^n} - \frac{R(x)-z}{|R(x)-z|^n}\right]\cdot(-1,0,\ldots,0)\,\mathrm{d}\sigma(z) \\ &= \frac{1}{n\omega_n}\int_{H^0} g(z)\left[\frac{x_1-z_1}{|x-z|^n} - \frac{-x_1-z_1}{|x-z|^n}\right]\,\mathrm{d}\sigma(z) \\ &= \frac{2x_1}{n\omega_n}\int_{H^0}\frac{g(z)}{|x-z|^n}\,\mathrm{d}\sigma(z). \end{aligned}$$

(b) Consider $u(x+L)$ and use the change of coordinates $y = z - L$

$$u(x+L) = \frac{2x_1}{n\omega_n}\int_{H^0}\frac{g(z)}{|x+L-z|^n}\,\mathrm{d}\sigma(z) = \frac{2x_1}{n\omega_n}\int_{H^0}\frac{g(y+L)}{|x-y|^n}\,\mathrm{d}\sigma(y) = u(x).$$

(c) We will suppose that g is positive. In general you can decompose g into two positive functions $g = g^+ - g^-$ consider two boundary value problems $v = g^+$ and $w = g^-$ on H^0. Then $u = v - w$ solves the original boundary value problem.

One possible upper bound is

$$u(x) = \frac{x_1}{\pi}\int_{\mathbb{R}}\frac{g(z)}{x_1^2+(x_2-z)^2}\,dz \le \frac{x_1}{\pi}\int_{\mathbb{R}}\frac{g(z)}{x_1^2}\,dz = \frac{1}{\pi x_1}\int_{\mathbb{R}} g(z)\,dz.$$

Here is a way to get a lower bound too. Since g has compact support, it is supported in an interval $[-R,R]$. For $|x| > R$ and $z \in [-R,R]$ we have that $|x| - R \le |x-z| \le |x| + R$. This leads to the bounds

$$\frac{x_1}{\pi(|x|+R)^2}\int_{\mathbb{R}} g(z)\,dz \le u(x) \le \frac{x_1}{\pi(|x|-R)^2}\int_{\mathbb{R}} g(z)\,dz.$$

In particular, we see that for x_1 small and $x_2 > R$ large that u grows approximately linearly in x_1, with gradient dependent on the total mass of g. On the other hand, for fixed x_1 we see that it falls off with the inverse square of x_2.