# Introduction to Partial Differential Equations
## Exercise sheet 9

Ross Ogilvie 28th October, 2024

**27. To be or not to be**

Consider the Dirichlet problem for the Laplace equation $\Delta u = 0$ on $\Omega$ with $u = g$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^n$ is an open and bounded subset and $g$ is a continuous function. We know from the weak maximum principle that there is at most one solution. In this question we see that for some domains, existence is not guaranteed.

**(a)** Consider $\Omega = B(0,1) \setminus \{0\}$, so that the boundary $\partial\Omega = \partial B(0,1) \cup \{0\}$ consists of two components. We write $g(x) = g_1(x)$ for $x \in \partial B(0,1)$ and $g(0) = g_2$. Show that there does not exist a solution for $g_1(x) = 0$ and $g_2 = 1$.

Hint. Use Lemma 3.23, even if we haven't reached it in lectures yet. *(3 points)*

**(b)** Generalise this: What are the necessary and sufficient conditions on $g$ for the Dirichlet problem to have a solution on this domain? *(3 points)*

**(c)** Generalise again: What can you say about the Dirichlet problem for bounded domains whose boundaries have isolated points? *(1 point)*

**28. Do nothing by halves**

Let $H_1^+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 > 0\}$ be the upper half-space and $H_1^0 = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = 0\}$ the dividing hyperplane. We call $R_1(x) = (-x_1, x_2, \ldots, x_n)$ reflection in the plane $H^0$.

**(a)** **A reflection principle for harmonic functions** Let $u \in C^2(\overline{H_1^+})$ be a harmonic function that vanishes on $H_1^0$. Show that the function $v : \mathbb{R}^n \to \mathbb{R}$ defined through reflection

$$v(x) = \begin{cases} u(x) & \text{for } x_1 \geq 0 \\ -u(R_1(x)) & \text{for } x_1 < 0 \end{cases}$$

is harmonic. *(3 points)*

**(b)** **Green's function for the half-space** Show that Green's function for $H_1^+$ is

$$G(x, y) = \Phi(x - y) - \Phi(R_1(x) - y).$$

*(2 points)*

**(c)** **Green's function for the half-ball** Compute the Green's function for $B^+$. *(3 points)*
Hint. Make use of both the Green's function for the ball and part (b).

**29. Teach a man to fish**

**(a)** Using the Green's function of $H_1^+$ from the previous question, derive the following formal integral representation for a solution of the Dirichlet problem $\Delta u = 0$ in $H_1^+, u|_{H_1^0} = g$

$$u(x) = \frac{2x_1}{n\omega_n} \int_{H_1^0} \frac{g(z)}{|x - z|^n} \, \mathrm{d}\sigma(z)$$

Here, 'formal' means that you do not need to prove that the integrals are finite/well-defined.

*(3 points)*

**(b)** Show that if $g$ is periodic (that is, there is some vector $L \in \mathbb{R}^{n-1}$ with $g(x+L) = g(x)$ for all $x \in \mathbb{R}^{n-1}$) then so is the solution. *(2 points)*

**(c)** Now consider the plane $n = 2$ with $g$ function with compact support. Approximate the value of $u(x)$ for large $|x|$. Feel free to modify this question as you see fit, what interesting things can you say about the growth of $u$? *(Bonus Points as deserved)*