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23. Subharmonic Functions

Let $\Omega \subset \mathbb{R}^n$ be an open and connected region. A continuous function $v : \overline{\Omega} \to \mathbb{R}$ is called *subharmonic* if for all $x \in \Omega$ and $r > 0$ with $B(x, r) \subset \Omega$ it lies below its spherical mean: $v(x) \leq \mathcal{S}[v](x, r)$.

(a) Let v_1, v_2 be two subharmonic functions. Show that $v = \max(v_1, v_2)$ is subharmonic. *(1 point)*

(b) Suppose that v is twice continuous differentiable. Show that v is subharmonic if and only if $-\Delta v \leq 0$ in Ω. *(3 points)*

(c) Prove that every subharmonic function obeys the *maximum principle*: If the maximum of v can be found inside Ω then v is constant. *(2 points)*

Solution.

(a) The inequality $v_i(x) \leq v(x)$ for all points gives

$$\frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} v_i(y)\, \mathrm{d}\sigma(y) \leq \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} v(y)\, \mathrm{d}\sigma(y)$$

and so $v_i(x) \leq \mathcal{S}[v_i](x, r) \leq \mathcal{S}[v](x, r)$. It follows $v(x) \leq \mathcal{S}[v](x, r)$.

(b) We know that

$$\frac{\partial}{\partial r}\mathcal{S}[v](x, r) = \frac{1}{n\omega_n} \int_{B(0,1)} \Delta v(x_0 + rz)\, \mathrm{d}z.$$

Suppose that $\Delta v \geq 0$. Then this shows that $\mathcal{S}(r)$ is a non-decreasing function. On the other hand $v(x) = \mathcal{S}(0)$. Thus $v(x) = \mathcal{S}(0) \leq \mathcal{S}(r)$.

For the converse, we can not say from $v(x) = \mathcal{S}(0) \leq \mathcal{S}(r)$ directly that $\mathcal{S}$ is a non-decreasing function of r. For example, $\mathcal{S}$ could initially increase, but then oscillate. But we do have the intuition that it must increase initially. We use the following argument to prove this rigorously.

Suppose that there is a point x_0 with $\Delta v(x_0) < 0$. By continuity there is a ball $\overline{B(x_0, R)}$ on which $\Delta v(x) < \frac{1}{2}\Delta v(x_0)$. But then for all $r \leq R$

$$\frac{\partial}{\partial r}\mathcal{S}[v](x_0, r) = \frac{1}{n\omega_n} \int_{B(0,1)} \Delta v(x_0 + rz)\, \mathrm{d}z \leq \frac{1}{n\omega_n} \int_{B(0,1)} \frac{1}{2}\Delta v(x_0)\, \mathrm{d}z = \frac{1}{2}\Delta v(x_0).$$

Integrating this from $r = 0$ to R gives

$$\mathcal{S}(R) \leq \frac{1}{2}\Delta v(x_0) R + \mathcal{S}(0) = \frac{1}{2}\Delta v(x_0) R + v(x_0) < v(x_0).$$

This is similar to the proof in (a), in that we use the strict estimate $\frac{1}{2}\Delta v(x_0) < 0$ in the final step. This shows that $v(x_0)$ lies above the spherical mean $\mathcal{S}(R)$ and so is not subharmonic. (Actually, it lies above all the spherical means $\mathcal{S}(r)$ for $r \leq R$.) The contrapositive statement is that if v is subharmonic then there are no points with $\Delta v < 0$.

(c) Suppose that v does indeed obtain a maximum at some $x_0 \in \Omega$, so that $v(x_0) - v(x) \geq 0$. Let $B(x_0, R) \subset \Omega$. The mean of a constant is just the same constant $\mathcal{S}[v(x_0)](y, r) = v(x_0)$. For $r < R$ we have that

$$0 \geq v(x_0) - \mathcal{S}[v](x_0, r) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x_0,r)} v(x_0) - v(x) \, \mathrm{d}\sigma(x) \geq 0.$$

It follows that $v(x_0) - v(x) = 0$ for all $x \in B(x_0, r)$ (we will show this below, if you are unsure how to prove this). It follows that $v(x) = v(x_0)$ for all points $x \in B(x_0, R)$. Because Ω is connected, this shows that $v(x) = v(x_0)$ on all of Ω. (The full version of this last step is in the lecture script.)

Let us justify the claim made above. Suppose that f is continuous and non-negative, and

$$\int_{B(0,r)} f \, d\sigma = 0.$$

We want to show that $f(x) = 0$ for all $x \in B(0, r)$. Suppose there is a point x_1 where $f(x_1) > 0$. Then by continuity there is an open ball $U = B(x_1, \varepsilon)$ such that for all $x \in U$ it holds that $f(x) \geq \frac{1}{2} f(x_1)$. Then

$$\begin{aligned}\int_{\partial B(0,r)} f(x) \, \mathrm{d}\sigma(x) &\geq \int_{\partial B(0,r)\cap U} f(x) \, \mathrm{d}\sigma(x) \geq \int_{\partial B(0,r)\cap U} \frac{1}{2} f(x_1) \, \mathrm{d}\sigma(x) \\ &\geq \frac{1}{2} f(x_1) \, \sigma(\partial B(0, r) \cap U) \\ &> 0\end{aligned}$$

because $\partial B(x_0, r) \cap U$ is an open subset of the sphere and therefore has positive measure. This is a contradiction. Therefore such x_1 cannot exist; $f(x)$ is zero on the sphere.

A note on the need for this proof. Notice it is not enough to use a neighbourhood U where $f(x) > 0$ because integrals do not preserve *strict* inequalities. Hence we use this trick with $f(x) > \frac{1}{2} f(x_1)$ so in the last step we can apply a strict inequality. This is a common trick in analysis when dealing with continuous functions and limits (remember, the integral is a limit of a sum).

Indeed, if $f < g$ are integrable functions we can't say $\int_E f < \int_E g$. For example, take E to be a null set, so that both integrals are zero. However this is the only counterexample. Try to prove this yourself, and feel free to ask for a hint/help. The proof I give above though, which only applies to continuous functions, is meant to be more intuitive and geometric. It avoids using facts about the surface measure $d\sigma$ other than $\partial B(x_0, r) \cap U$ has positive measure.

24. Liouville's Theorem

Liouville's theorem (3.10 in the script) says that if u is bounded and harmonic, then u is constant. In this question we give a geometric proof in $\mathbb{R}^2$ using ball means defined when $\overline{B(x,r)} \subset \Omega$ through

$$\mathcal{M}[v](x,r) = \frac{1}{\omega_n r^n} \int_{B(x,r)} v(y) \,\mathrm{d}y.$$

Theorem 3.5 states that if u is harmonic then it obeys $u(x) = \mathcal{M}[u](x,r)$. This beautiful proof comes from the article Nelson, 1961.

(a) Consider two points a, b in the plane which are distance $2d$ apart. Now consider two balls, both with radius $r > d$, centred on the two points respectively. Show that the area of the intersection is *(2 bonus points)*

$$\operatorname{area} B(a,r) \cap B(b,r) = 2r^2 \operatorname{acos}(dr^{-1}) - 2d\sqrt{r^2 - d^2}$$

(b) Suppose that v is a bounded function on the plane: $-C \leq v(x) \leq C$ for all x and some constant C. Show that *(2 points)*

$$\left|\mathcal{M}[v](a,r) - \mathcal{M}[v](b,r)\right| \leq \frac{2C}{\omega_2}\left(\pi - 2\operatorname{acos}(dr^{-1}) - \frac{2d}{r}\sqrt{1 - d^2 r^{-2}}\right)$$

(c) Let $u \in C^2(\mathbb{R}^2)$ be a bounded harmonic function. Complete the proof that u is constant. *(2 points)*

Solution.

(a) Let the distance from the point where the two circles intersect to the line connecting a and b be h. We have then that $r^2 = d^2 + h^2$. It has an angle of elevation from the centre of the balls of $\cos\theta = d/r$. The area of the sector centred at b with angle 2θ is $\frac{1}{2}r^2(2\theta) = r^2 \operatorname{acos}(dr^{-1})$. Subtracting off a triangle gives the area of the segment of $B(b,r)$ as

$$r^2 \operatorname{acos}(dr^{-1}) - dh.$$

Twice this is the area of $B(a,r) \cap B(b,r)$

(b) The integrals have the same integrand, so their difference reduces to integrals on certain domains:

$$\omega_2 r^2\big(\mathcal{M}[v](a,r) - \mathcal{M}[v](b,r)\big) = \int_{B(a,r)\setminus B(b,r)} v(y)\,\mathrm{d}y - \int_{B(b,r)\setminus B(a,r)} v(y)\,\mathrm{d}y.$$

With the triangle inequalities

$$\begin{aligned}
\omega_2 r^2 \left|\mathcal{M}[v](a,r) - \mathcal{M}[v](b,r)\right| &\leq \left|\int_{B(a,r)\setminus B(b,r)} v(y)\,\mathrm{d}y\right| + \left|\int_{B(b,r)\setminus B(a,r)} v(y)\,\mathrm{d}y\right| \\
&\leq \int_{B(a,r)\setminus B(b,r)} |v(y)|\,\mathrm{d}y + \int_{B(b,r)\setminus B(a,r)} |v(y)|\,\mathrm{d}y \\
&\leq C\int_{B(a,r)\setminus B(b,r)} \mathrm{d}y + C\int_{B(b,r)\setminus B(a,r)} \mathrm{d}y \\
&= C\left(2\pi r^2 - 2\operatorname{area} B(a,r) \cap B(b,r)\right)
\end{aligned}$$

(c) We know that a harmonic function is equal to its ball mean

$$|u(a) - u(b)| = \big|\mathcal{M}(u, a, r) - \mathcal{M}(u, b, r)\big| \leq \frac{2C}{\omega_2}\left(\pi - 2\mathrm{acos}(dr^{-1}) - \frac{2d}{r}\sqrt{1 - d^2 r^{-2}}\right).$$

This must hold for all $r > 1$. But as $r \to \infty$ the right hand side tends to 0. By the squeeze rule, it must be that $u(a) = u(b)$. Since this holds for all pairs of points, u is constant.

Note: As stated in the linked article, this proof holds in all dimensions. The key trick to bound from below the volume of the lens (the overlap between the two balls) by the volume of a ball of radius $r - d$. This avoids having to calculate the volume of the lens exactly.

25. Weak Tea

As in the script, for any test function $\psi \in \mathcal{D}((0, R))$, define a test function $\tilde{\psi}_a \in \mathcal{B}(a, R)$ by

$$\tilde{\psi}_a(x) = \frac{\psi(|x - a|)}{n\omega_n |x - a|^{n-1}}.$$

(a) Describe the support of $\tilde{\psi}$ in terms of the support of ψ. *(1 point)*

(b) We have defined the spherical mean of a distribution using the formula $\mathcal{S}_a[F](\psi) = F(\tilde{\psi}_a)$ for $\psi \in \mathcal{D}((0, R))$. Compute $\mathcal{S}_a[\delta]$ for $a \neq 0$ and $a = 0$. Does it have weak mean value property? *(3 points)*

(c) Take a function $u \in C(\mathbb{R}^n)$ and fix a radius r. Consider the function $x \mapsto \mathcal{S}[u](x, r)$. Prove

$$F_{\mathcal{S}[u](x,r)}(\varphi) = F_u(x \mapsto \mathcal{S}[\varphi](x, r)).$$

(2 points)

(d) The spherical mean for distributions, as we have defined it, has the center point fixed and takes a test function on $(0, \infty)$ instead of a radius. The formula in the previous part defines a different sort of spherical mean of a distribution $\tilde{\mathcal{S}}_r[F] \in \mathcal{D}'(\mathbb{R}^n)$:

$$\tilde{\mathcal{S}}_r[F](\varphi) = F(x \mapsto \mathcal{S}[\varphi](x, r))$$

What properties would you expect of this spherical mean applied to a harmonic distribution? What are the advantages and disadvantages of this spherical mean compared to the one in the lecture script? What interesting observations can you make about this? *(Bonus points as deserved.)*

Solution.

(a) The support of $\tilde{\psi}_a$ is the points x with $|x - a| \in \operatorname{supp}\psi$. Any compact subset of $\mathbb{R}$ is a finite union of closed intervals, so the support of $\tilde{\psi}_a$ is the finite union of annuli, all centered at a.

(b) By definition

$$\mathcal{S}_a[\delta](\psi) = \delta\left(\frac{\psi(|x-a|)}{n\omega_n\,|x-a|^{n-1}}\right) = \frac{\psi(|a|)}{n\omega_n\,|a|^{n-1}}.$$

We have explained in the lectures that $\tilde{\psi}_a(a) = 0$, because ψ is identically zero near 0, and so dividing by $|x-a|^{n-1}$ doesn't produce a singularity. It does not have the weak mean value property, because if the integral of ψ is zero there is no reason this value should be zero.

In the case $a = 0$, this means that $\mathcal{S}_0[\delta](\psi) = 0$ for all distributions ψ. In other words $\mathcal{S}_0[\delta] = 0$ is the zero distribution.

In the case $a \neq 0$, we might simplify this formula to

$$\mathcal{S}_a[\delta](\psi) = \frac{1}{n\omega_n\,|a|^{n-1}}\delta_{|a|}(\psi) \quad \Rightarrow \quad \mathcal{S}_a[\delta] = \frac{1}{n\omega_n\,|a|^{n-1}}\delta_{|a|}.$$

(c) We calculate, using Fubini and coordinate changes where necessary.

$$\begin{aligned}
F_{\mathcal{S}[u](x,r)}(\varphi) &= \int_{\mathbb{R}^n}\left(\frac{1}{n\omega_n r^{n-1}}\int_{\partial B(x,r)} u(y)\,d\sigma(y)\right)\varphi(x)\,dx \\
&= \int_{\mathbb{R}^n}\left(\frac{1}{n\omega_n r^{n-1}}\int_{\partial B(0,r)} u(x+z)\,d\sigma(z)\right)\varphi(x)\,dx \\
&= \frac{1}{n\omega_n r^{n-1}}\int_{\partial B(0,r)}\int_{\mathbb{R}^n} u(x+z)\varphi(x)\,dx\,d\sigma(z) \\
&= \frac{1}{n\omega_n r^{n-1}}\int_{\partial B(0,r)}\int_{\mathbb{R}^n} u(w)\varphi(w-z)\,dw\,d\sigma(z) \\
&= \int_{\mathbb{R}^n} u(w)\left(\frac{1}{n\omega_n r^{n-1}}\int_{\partial B(0,r)} \varphi(w-z)\,d\sigma(z)\right)\,dw \\
&= \int_{\mathbb{R}^n} u(w)\left(\frac{1}{n\omega_n r^{n-1}}\int_{\partial B(w,r)} \varphi(y)\,d\sigma(y)\right)\,dw \\
&= \int_{\mathbb{R}^n} u(w)\mathcal{S}[\varphi](w,r)\,dw \\
&= F_u(w \mapsto \mathcal{S}[\varphi](w,r)).
\end{aligned}$$

(d) A harmonic function has the property that $u(x) = \mathcal{S}[u](x,r)$ (the mean value property). We generalised this in the lecture script to the spherical mean of distributions, by saying that $\mathcal{S}_a[F]$ is a constant distribution for each $a \in \mathbb{R}^n$. Specifically, we generalised that the spherical mean is constant in r. In the situation of this question we could try to generalise the fact that the spherical mean with fixed radius is equal to the function. We expect that harmonic distributions have $U(\varphi) = \tilde{\mathcal{S}}_r[U](\varphi)$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

The disadvantage of this version is that we know something special about constant distributions, namely that they correspond to constant functions. It is less clear (a priori) what can be said about distributions that are unchanged by $\tilde{\mathcal{S}}$.

The advantage is that this version is easier to understand in terms of distribution theory. Recall from Exercise 17(d) the distribution G that comes from integrating on the unit circle.

We can modify this to

$$G_r(\varphi) = \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(0,r)} \varphi(y)\, d\sigma(y).$$

Observe that

$$\begin{aligned}
\varphi * \mathsf{P}G_r(x) &= \mathsf{P}G_r(\mathsf{T}_x\mathsf{P}\varphi) = G_r(\mathsf{P}\mathsf{T}_x\mathsf{P}\varphi) \\
&= \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(0,r)} \mathsf{P}\mathsf{T}_x\mathsf{P}\varphi(y)\, d\sigma(y) \\
&= \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(0,r)} \varphi(x+y)\, d\sigma(y) \\
&= \frac{1}{n\omega_n r^{n-1}} \int_{\partial B(x,r)} \varphi(y)\, d\sigma(y) \\
&= \mathcal{S}[\varphi](x,r),
\end{aligned}$$

and so

$$\tilde{\mathcal{S}}_r[F](\varphi) = F(x \mapsto \mathcal{S}[\varphi](x,r)) = F(\varphi * \mathsf{P}G_r) = G_r * F(\varphi).$$

Therefore this version of spherical mean is nothing other than the convolution of distributions. Consider what transpires if we take a limit as $r \to 0$

$$\lim_{r\to 0} G_r * F(\varphi) = \lim_{r\to 0} F(\mathcal{S}[\varphi](x,r)) = F(\mathcal{S}[\varphi](x,0)) = F(\varphi(x)).$$

In other words, the limit gives back the distribution F. This is the distribution version of Lemma 3.3.

If we wanted to prove Weyl's lemma using this approach, here are some steps we might take. Suppose that U is a harmonic distribution. We should prove $U = \tilde{\mathcal{S}}[U] = G_r * U$, as we mentioned above. The next hurdle is that G_r is not a smooth function. If it were, then we have U as a convolution with a smooth function, implying that U is a smooth function. Instead we approximate G_r by smooth functions. If λ_ε is a mollifier then $\widetilde{\mathsf{T}_\mathsf{r}\lambda_\varepsilon}$ is a suitable approximation. Note, this is the function we use to define u in Step 2 of our proof of Weyl's lemma. Finally, we know U is harmonic distribution that corresponds to the smooth function u, so by the compatibility of distribution derivatives and classical derivatives we conclude that u is harmonic.

26. Back in the saddle

Suppose that $u \in C^2(\mathbb{R}^2)$ is a harmonic function with a critical point at x_0. Assume that the Hessian of u has non-zero determinant. Show that x_0 is a saddle point. Explain the connection to the maximum principle. *(4 points)*

Solution. Let $a = \partial_x^2 u$, $b = \partial_x\partial_y u$, $c = \partial_y^2 u$ at x_0. Because u is harmonic, $a + d = 0$. The determinant of the Hessian of u at x_0 is $ad - b^2 = -a^2 - b^2 \le 0$. Since by assumption it is not zero, it is negative. Therefore x_0 is a saddle point.

The maximum principle implies that any extrema must occur on the boundary. We know from Analysis I or II that extrema can only occur on the boundary or at critical points. It follows that harmonic functions can only have critical points that are saddle points.