Introduction to Partial Differential Equations
Exercise sheet 8

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23. Subharmonic Functions

Let $\Omega \subset \mathbb{R}^n$ be an open and connected region. A continuous function $v : \overline{\Omega} \to \mathbb{R}$ is called *subharmonic* if for all $x \in \Omega$ and $r > 0$ with $B(x,r) \subset \Omega$ it lies below its spherical mean: $v(x) \leq \mathcal{S}[v](x,r)$.

(a) Let v_1, v_2 be two subharmonic functions. Show that $v = \max(v_1, v_2)$ is subharmonic. *(1 point)*

(b) Suppose that v is twice continuous differentiable. Show that v is subharmonic if and only if $-\Delta v \leq 0$ in Ω. *(3 points)*

(c) Prove that every subharmonic function obeys the *maximum principle*: If the maximum of v can be found inside Ω then v is constant. *(2 points)*

24. Liouville's Theorem

Liouville's theorem (3.10 in the script) says that if u is bounded and harmonic, then u is constant. In this question we give a geometric proof in $\mathbb{R}^2$ using ball means defined when $\overline{B(x,r)} \subset \Omega$ through

$$\mathcal{M}[v](x,r) = \frac{1}{\omega_n r^n} \int_{B(x,r)} v(y) \, \mathrm{d}y.$$

Theorem 3.5 states that if u is harmonic then it obeys $u(x) = \mathcal{M}[u](x,r)$. This beautiful proof comes from the article Nelson, 1961.

(a) Consider two points a, b in the plane which are distance $2d$ apart. Now consider two balls, both with radius $r > d$, centred on the two points respectively. Show that the area of the intersection is *(2 bonus points)*

$$\operatorname{area} B(a,r) \cap B(b,r) = 2r^2 \operatorname{acos}(dr^{-1}) - 2d\sqrt{r^2 - d^2}$$

(b) Suppose that v is a bounded function on the plane: $-C \leq v(x) \leq C$ for all x and some constant C. Show that *(2 points)*

$$\left|\mathcal{M}[v](a,r) - \mathcal{M}[v](b,r)\right| \leq \frac{2C}{\omega_2}\left(\pi - 2\operatorname{acos}(dr^{-1}) - \frac{2d}{r}\sqrt{1 - d^2 r^{-2}}\right)$$

(c) Let $u \in C^2(\mathbb{R}^2)$ be a bounded harmonic function. Complete the proof that u is constant. *(2 points)*

25. Weak Tea

As in the script, for any test function $\psi \in \mathcal{D}((0,R))$, define a test function $\tilde{\psi}_a \in \mathcal{B}(a,R)$ by

$$\tilde{\psi}_a(x) = \frac{\psi(|x-a|)}{n\omega_n |x-a|^{n-1}}.$$

(a) Describe the support of $\tilde{\psi}$ in terms of the support of ψ. *(1 point)*

(b) We have defined the spherical mean of a distribution using the formula $\mathcal{S}_a[F](\psi) = F(\tilde{\psi}_a)$ for $\psi \in \mathcal{D}((0, R))$. Compute $\mathcal{S}_a[\delta]$ for $a \neq 0$ and $a = 0$. Does it have weak mean value property? *(3 points)*

(c) Take a function $u \in C(\mathbb{R}^n)$ and fix a radius r. Consider the function $x \mapsto \mathcal{S}[u](x, r)$. Prove

$$F_{\mathcal{S}[u](x,r)}(\varphi) = F_u(x \mapsto \mathcal{S}[\varphi](x, r)).$$

(2 points)

(d) The spherical mean for distributions, as we have defined it, has the center point fixed and takes a test function on $(0, \infty)$ instead of a radius. The formula in the previous part defines a different sort of spherical mean of a distribution $\tilde{\mathcal{S}}_r[F] \in \mathcal{D}'(\mathbb{R}^n)$:

$$\tilde{\mathcal{S}}_r[F](\varphi) = F(x \mapsto \mathcal{S}[\varphi](x, r))$$

What properties would you expect of this spherical mean applied to a harmonic distribution? What are the advantages and disadvantages of this spherical mean compared to the one in the lecture script? What interesting observations can you make about this? *(Bonus points as deserved.)*

26. Back in the saddle

Suppose that $u \in C^2(\mathbb{R}^2)$ is a harmonic function with a critical point at x_0. Assume that the Hessian of u has non-zero determinant. Show that x_0 is a saddle point. Explain the connection to the maximum principle. *(4 points)*