Introduction to Partial Differential Equations
Exercise sheet 7

Ross Ogilvie 14th October, 2024

20. The only constant is change

Let $\lambda_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ be the standard mollifier. Let $F \in \mathcal{D}(\Omega)$ be any distribution, not necessarily regular.

(a) For any point $a \in \Omega$, explain why $F(\lambda_\varepsilon(x-a))$ is well-defined for ε sufficiently small. *(1 point)*

(b) Expand the definitions to show $(\lambda_\varepsilon * F)(a) = F(\lambda_\varepsilon(x-a))$. *(2 points)*

(c) Suppose that F has the property that $F(\lambda_\varepsilon(x-a)) = 0$ for all a, ε (for which it is defined). Argue using Exercise 19 that $F = 0$. *(2 points)*

(d) Suppose that F has the following property: if a test function $\varphi \in \mathcal{D}(\Omega)$ has total integral zero,

$$\int_\Omega \varphi(x)\, dx = 0,$$

then $F(\varphi) = 0$. Prove that $F = F_c$ for $c \in \mathbb{R}$ the constant function. *(3 points)*
Hint. Define $c = (\lambda_r * F)(a)$.

Solution.

(a) Ω is an open set, so any point $a \in \Omega$ has a closed neighborhood $\overline{B(a,r)} \subset \Omega$. The support of $\lambda_\varepsilon(x-a)$ is $\overline{B(a,\varepsilon)}$, which is hence contained in Ω for any $\varepsilon \leq r$.

(b) We understand from Lemma 2.16 that the convolution of a smooth function of compact support and a distribution is again smooth function. That gives $(\lambda_\varepsilon * F)(a) = F(\mathsf{T}_a \mathsf{P} \lambda_\varepsilon)$. By the definitions of the translation and point reflection operators

$$\mathsf{T}_a \mathsf{P} \lambda_\varepsilon(x) = \mathsf{P}\lambda_\varepsilon(x-a) = \lambda_\varepsilon(a-x).$$

But the standard mollifier is point reflection symmetric, so this is also $\lambda_\varepsilon(x-a)$.

(c) In light of (b), we might restate the property on F as that $\lambda_\varepsilon * F = 0$ for all ε. Therefore

$$0 = \lim_{\varepsilon \downarrow 0} \lambda_\varepsilon * F = \delta * F = F.$$

Here we have used that $\lim_{\varepsilon\downarrow 0} \lambda_\varepsilon = \delta$, Exercise 19(b), and $\delta * F = F$, Exercise 19(c).

(d) The idea is to use part (c) to prove $F - F_c = 0$. Choose any ball $\overline{B(a,r)} \subset \Omega$ and set $c = (\lambda_r * F)(a)$. This constant is independent of the choice of a, r; if a', r' is any other choice then $\varphi(x) = \lambda_{r'}(x-a') - \lambda_r(x-a)$ has

$$\int_\Omega \varphi(x)\, dx = \int_\Omega \lambda_{r'}(x-a')\, dx - \int_\Omega \lambda_r(x-a)\, dx = 1 - 1 = 0,$$

hence

$$c' - c = (\lambda_{r'} * F)(a') - (\lambda_{r'} * F)(a') = F(\lambda_{r'}(x-a') - \lambda_r(x-a)) = F(\varphi) = 0.$$

On the other hand, observe that

$$F_c(\lambda_r(x-a)) = \int_\Omega c\lambda_r(x-a)\, dx = c\int_\Omega \lambda_r(x-a)\, dx = c,$$

using that $\lambda_r(x-a)$ is the shift of a mollifier, and so has total integral 1. Putting these facts together gives

$$(F - F_c)(\lambda_\varepsilon(x-a)) = F(\lambda_\varepsilon(x-a)) - F_c(\lambda_\varepsilon(x-a)) = c - c = 0.$$

Thus we can conclude from part (c) that $F - F_c = 0$.

21. Twirling towards freedom

Let $u \in C^2(\mathbb{R}^n)$ be a harmonic function. Show that the following functions are also harmonic.

(a) $v(x) = u(x+b)$ for $b \in \mathbb{R}^n$.

(b) $v(x) = u(ax)$ for $a \in \mathbb{R}$.

(c) $v(x) = u(Rx)$ for $R(x_1, \ldots, x_n) = (-x_1, x_2, \ldots, x_n)$ the reflection operator.

(d) $v(x) = u(Ax)$ for any orthogonal matrix $A \in O(\mathbb{R}^n)$.

Together these show that the Laplacian is invariant under *similarities* (Euclidean motions, reflection and rescaling). *(6 points)*

Solution.

(a) This follows by the chain rule

$$\Delta v(x) = \sum \frac{\partial^2 u}{\partial x_i^2}(x+b) \cdot 1 = \Delta u(x+b) = 0.$$

(b) This also follows by the chain rule

$$\Delta v(x) = \sum \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}(ax) \cdot a\right) = a^2 \Delta u(ax) = 0.$$

(c) You guessed it, we apply the chain rule. Only the x_1 derivative is affected:

$$\frac{\partial^2 v}{\partial x_1^2}(x) = \frac{\partial}{\partial x_1}\left(-\frac{\partial u}{\partial x_1}(Rx)\right) = \frac{\partial^2 u}{\partial x_1^2}(Rx).$$

This shows $\Delta v(x) = \Delta u(Rx) = 0$.

(d) This is also the chain rule, with $(Ax)_i = \sum_j A_{ij} x_j$. We will write this with indices, but if you can keep everything as matrices then it is a bit shorter.

$$\begin{aligned}
\frac{\partial v}{\partial x_k}(x) &= \frac{\partial}{\partial x_k} u\Big(\sum_j A_{ij} x_j\Big) \\
&= \sum_l \frac{\partial u}{\partial x_l}\Big(\sum_j A_{ij} x_j\Big) A_{lk} \\
\frac{\partial^2 v}{\partial x_k^2}(x) &= \frac{\partial}{\partial x_k} \sum_l \frac{\partial u}{\partial x_l}\Big(\sum_j A_{ij} x_j\Big) A_{lk} \\
&= \sum_{l,m} \frac{\partial^2 u}{\partial x_m \partial x_l}\Big(\sum_j A_{ij} x_j\Big) A_{lk} A_{mk}
\end{aligned}$$

Now when we sum over k, we can group together the like derivatives and get a sum over the A multipliers. Because A is an orthogonal matrix, we have $AA^T = I$, or in other words $\delta_{lm} = \sum_k A_{lk}(A^T)_{km} = \sum_k A_{lk}A_{mk}$. This gives

$$\Delta v(x) = \sum_{l,m} \frac{\partial^2 u}{\partial x_m \partial x_l}(Ax)\left(\sum_k A_{lk}A_{mk}\right) = \sum_{l,m} \frac{\partial^2 u}{\partial x_m \partial x_l}(Ax)\delta_{lm} = \sum_l \frac{\partial^2 u}{\partial x_l^2}(Ax) = 0.$$

22. Harmonic Polynomials in Two Variables

(a) Let $u \in C^\infty(\mathbb{R}^n)$ be a smooth harmonic function. Prove that any derivative of u is also harmonic. *(1 point)*

(b) Choose any positive degree n. Consider the complex valued function $f_n : \mathbb{R}^2 \to \mathbb{C}$ given by $f_n(x, y) = (x+\iota y)^n$ and let $u_n(x, y)$ and $v_n(x, y)$ be its real and imaginary parts respectively. Show that u_n and v_n are harmonic. *(3 points)*

(c) A *homogeneous polynomial* of degree n in two variables is a polynomial of the form $p = \sum a_k x^k y^{n-k}$. Show that a homogeneous polynomial of degree n is harmonic if and only if it is a linear combination of u_n and v_n. *(2 points + 2 bonus points)*

Solution.

(a) Since the function is smooth, it is in particular thrice continuously differentiable. Thus we can interchange the order of partial derivatives

$$\Delta(\partial_i u) = \sum_k \partial_k^2 \partial_i u = \sum_k \partial_i \partial_k^2 u = \partial_i \Delta u = 0.$$

(b) There are two approaches. The simplest is to extend the Laplacian linearly to complex valued functions. All normal rules of calculus apply and we get

$$\Delta f_n = n(n-1)(x+\iota y)^{n-2} + n(n-1)(\iota^2)(x+\iota y)^{n-2} = 0.$$

But perhaps this feels undeserved. Let's instead compute more directly. By binomial expansion we have

$$u_n + \iota v_n = \sum_{k=0}^n \binom{n}{k} \iota^k x^{n-k} y^k = \sum_{0 \le 2j \le n} \binom{n}{2j}(-1)^j x^{n-2j} y^{2j} + \iota \sum_{0 \le 2j+1 \le n} \binom{n}{2j+1}(-1)^{2j} x^{n-2j-1} y^{2j+1}.$$

Differentiating gives

$$\begin{aligned}\Delta u_n &= \sum_{0 \le 2j \le n-1} (n-2j)(n-2j-1)\binom{n}{2j}(-1)^j x^{n-2j-2} y^{2j} + \sum_{1 \le 2j \le n} (2j)(2j-1)\binom{n}{2j}(-1)^j x^{n-2j} y^{2j-2} \\ &= \sum_{0 \le 2j \le n-1} \left[(n-2j)(n-2j-1)\binom{n}{2j} - (2j+2)(2j+1)\binom{n}{2j+2}\right](-1)^j x^{n-2j-2} y^{2j}.\end{aligned}$$

The result now follows from the definition of the binomial coefficients.

$$(n-2j)(n-2j-1)\binom{n}{2j} = (n-2j)(n-2j-1)\frac{n!}{(n-2j)!(2j)!} = \frac{n!}{(n-2j-2)!(2j)!}$$

$$(2j+2)(2j+1)\binom{n}{2j+2} = (2j+2)(2j+1)\frac{n!}{(n-2j-2)!(2j+2)!} = \frac{n!}{(n-2j-2)!(2j)!}$$

Likewise for v_n.

(c) Any linear combination of u_n and v_n is harmonic since Δ is a linear operator. For the converse, we prove this by induction. Note that the set of homogeneous polynomials is closed under addition and scaling. Further it is closed under differentiation:

$$\partial_x \sum_{k=0}^{n} a_k x^k y^{n-k} = \sum_{k=1}^{n} k a_k x^{k-1} y^{n-k} = \sum_{j=0}^{n-1} (j+1) a_{j+1} x^j y^{(n-1)-j}$$

For $n = 0$ and $n = 1$ the result holds because u_n, v_n span all polynomials.

Suppose now it holds up to degree n. Let $p = p_0 x^{n+1} y^0 + p_1 x^n y^1 + \dots$ be a homogeneous harmonic polynomial of degree $n+1$. Define $q = p - p_0 u_{n+1} - \frac{1}{n} p_1 v_{n+1}$. Then this does not have the terms $x^{n+1}y^0$ or $x^n y^1$. Note that $\partial_x q$ is again a homogeneous harmonic polynomial and its degree is n, so $\partial_x q = a u_n + b v_n$ for some constants a and b. But $\partial_x q$ has no term with $x^n y^0$ or $x^{n-1} y^1$, hence $a = b = 0$. This shows that q is constant with respect to x, and the only possibility is then that $q = A y^{n+1}$. But this is only harmonic for $A = 0$. We conclude therefore that $p = p_0 u_{n+1} + \frac{1}{n} p_1 v_{n+1}$.