Introduction to Partial Differential Equations
Exercise sheet 7

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20. The only constant is change

Let $\lambda_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ be the standard mollifier. Let $F \in \mathcal{D}(\Omega)$ be any distribution, not necessarily regular.

(a) For any point $a \in \Omega$, explain why $F(\lambda_\varepsilon(x-a))$ is well-defined for ε sufficiently small. *(1 point)*

(b) Expand the definitions to show $(\lambda_\varepsilon * F)(a) = F(\lambda_\varepsilon(x-a))$. *(2 points)*

(c) Suppose that F has the property that $F(\lambda_\varepsilon(x-a)) = 0$ for all a, ε (for which it is defined). Argue using Exercise 19 that $F = 0$. *(2 points)*

(d) Suppose that F has the following property: if a test function $\varphi \in \mathcal{D}(\Omega)$ has total integral zero,

$$\int_\Omega \varphi(x)\, dx = 0,$$

then $F(\varphi) = 0$. Prove that $F = F_c$ for $c \in \mathbb{R}$ the constant function. *(3 points)*
Hint. Define $c = (\lambda_r * F)(a)$.

21. Twirling towards freedom

Let $u \in C^2(\mathbb{R}^n)$ be a harmonic function. Show that the following functions are also harmonic.

(a) $v(x) = u(x+b)$ for $b \in \mathbb{R}^n$.

(b) $v(x) = u(ax)$ for $a \in \mathbb{R}$.

(c) $v(x) = u(Rx)$ for $R(x_1, \ldots, x_n) = (-x_1, x_2, \ldots, x_n)$ the reflection operator.

(d) $v(x) = u(Ax)$ for any orthogonal matrix $A \in O(\mathbb{R}^n)$.

Together these show that the Laplacian is invariant under *similarities* (Euclidean motions, reflection and rescaling). *(6 points)*

22. Harmonic Polynomials in Two Variables

(a) Let $u \in C^\infty(\mathbb{R}^n)$ be a smooth harmonic function. Prove that any derivative of u is also harmonic. *(1 point)*

(b) Choose any positive degree n. Consider the complex valued function $f_n : \mathbb{R}^2 \to \mathbb{C}$ given by $f_n(x,y) = (x+\iota y)^n$ and let $u_n(x,y)$ and $v_n(x,y)$ be its real and imaginary parts respectively. Show that u_n and v_n are harmonic. *(3 points)*

(c) A *homogeneous polynomial* of degree n in two variables is a polynomial of the form $p = \sum a_k x^k y^{n-k}$. Show that a homogeneous polynomial of degree n is harmonic if and only if it is a linear combination of u_n and v_n. *(2 points + 2 bonus points)*