17. Convoluted

The convolution of two functions $f, g : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(y)g(x-y)\, dy.$$

(a) Let $f_n(x) = 0.5n$ for $x \in [-n^{-1}, n^{-1}]$ and 0 otherwise. Show that the following bounds hold

$$\inf_{|y|\leq n^{-1}} g(y) \leq (g * f_n)(0) \leq \sup_{|y|\leq n^{-1}} g(y).$$

(2 points)

(b) Suppose now that g is continuous. Show that $(g * f_n)(0) \to g(0)$ as $n \to \infty$. *(2 points)*

(c) Show that the convolution of C_0^∞-functions on $\mathbb{R}^n$ is a bilinear, commutative, and associative operation. *(1+2+2 Points)*

Solution.

(a) We compute

$$\begin{aligned}(g * f_n)(0) &= \int_{-\infty}^{\infty} g(y) f_n(-y) = \int_{-n^{-1}}^{n^{-1}} g(y) \times \frac{1}{2}n \\ &\leq \frac{1}{2}n \int_{-n^{-1}}^{n^{-1}} \sup_{0\leq y\leq n^{-1}} g(y) = \frac{1}{2}n \sup_{|y|\leq n^{-1}} g(y) \times 2n^{-1} \\ &= \sup_{|y|\leq n^{-1}} g(y)\end{aligned}$$

In a similar manner, we see that $(g * f_n)(0)$ is bound below by $\inf_{|y|\leq n^{-1}} g(y)$.

(b) Clearly $\sup_{|y|<n^{-1}} g(y) \geq g(0)$. On the other hand, choose any $\epsilon > 0$. By the continuity of g, there exists $\delta > 0$ such that $|g(y) - g(0)| < \epsilon$ for all $y \in (-\delta, \delta)$. Choose N such that $N^{-1} < \delta$. That means for all $|y| < N^{-1}$ we have $|g(y) - g(0)| < \epsilon$. For all $n > N$ the interval $[-n^{-1}, n^{-1}]$ is a subset of $[-N^{-1}, N^{-1}]$. It follows that

$$\sup_{|y|<n^{-1}} g(y) \leq \sup_{|y|<N^{-1}} g(y) < \sup_{|y|<N^{-1}} (g(0) + \epsilon) = g(0) + \epsilon.$$

These two inequalities together say that $\forall \epsilon > 0\, \exists N\, \forall n > N$ it holds that

$$|\sup_{y\in I_n} g(y) - g(0)| < \epsilon.$$

This is the definition of $\sup_{|y|<n^{-1}} g(y) \to g(0)$. The same argument shows that $\inf_{|y|<n^{-1}} g(y) \to g(0)$ also. By the sandwich rule/squeeze rule, the result follows.

(c) Formal bilinearity follows from the linearity of the integral and the bilinearity of the product of functions. The smoothness and compact support of all functions involved means that the integrals always exist.

In the following we make substitution of the form $z = x - y$. The Jacobi matrix of this transformation is $-I$, so the factor is $|\det -I| = 1$. We say that these transformations are volume preserving. By dx we mean $\mathrm{d}^n x$ or $d\mu_{\mathbb{R}^n}$ or whatever your preferred notation for integration on $\mathbb{R}^n$ is.

Commutativity:

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\, dy = \int_{\mathbb{R}^n} f(x-z)g(z)\, dz = g * f(x).$$

Associativity

$$\begin{aligned} f*(g*h)(x) &= \int_{\mathbb{R}^n} f(y)(g*h)(x-y)\, dy \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)g(z)h((x-y)-z)\, dz\, dy \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)g(z)h(x-(y+z))\, dz\, dy \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(y)g(w-y)h(x-w)\, dy\, dw \\ &= \int_{\mathbb{R}^n} (f*g)(w)h(x-w)\, dw \\ &= (f*g)*h(x) \end{aligned}$$

18. Distributions

(a) Choose any compact set $K \subset \mathbb{R}$. Since it is bounded, there exists $R > 0$ with $K \subseteq [-R, R]$. Now choose any test function $\phi \in C_0^\infty(\mathbb{R})$ with compact support in K. Since it is continuous, $\sup_{x\in K} |\phi(x)|$ is finite. Prove the following inequality *(1 point)*

$$\left| \int_0^\infty \phi(x)\, \mathrm{d}x \right| \leq 2R \sup_{x\in K} |\phi(x)|.$$

(b) Define the Heaviside function $H : \mathbb{R} \to \mathbb{R}$ by $H(x) := 1$ for $x \geq 0$ and $H(x) := 0$ for $x < 0$. Show that the distribution associated to the Heaviside function

$$F_H : C_0^\infty(\mathbb{R}) \to \mathbb{R},\ \phi \mapsto \int_0^\infty \phi(x)\, \mathrm{d}x$$

is in fact a distribution on $\mathbb{R}$ using part (a) and Definition 2.14 directly. *(1 point)*

(c) Calculate the first and second derivatives of H as a distribution. If they are regular distributions, describe the corresponding function. *(2 points)*

(d) Consider the circle $C = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$. Show that

$$G(\varphi) := \int_C \varphi\, d\sigma$$

defines a distribution in $\mathcal{D}'(\mathbb{R}^2)$. Note that the $d\sigma$ indicates this is an integration over the submanifold C. Does there exist a locally integrable function $g : \mathbb{R}^2 \to \mathbb{R}$ with

$$G(\varphi) = \int_{\mathbb{R}^2} g\, \varphi\, \mathrm{d}x$$

for all $\varphi \in C_0^\infty(\mathbb{R})$? (Hint. Use Lemma 2.15) *(2 Points + 2 Bonus Points)*

Solution.

(a)

$$\begin{aligned}
\left|\int_0^\infty \phi(x)\,\mathrm{d}x\right| &\le \int_0^\infty |\phi(x)|\,\mathrm{d}x \\
&= \int_{[0,\infty]\cap K} |\phi(x)|\,\mathrm{d}x \qquad \text{since } \phi \text{ is zero outside of } K \\
&\le \int_{[0,\infty]\cap K} \sup_{x\in K} |\phi(x)|\,\mathrm{d}x \\
&= \sup_{x\in K} |\phi(x)| \int_{[0,\infty]\cap K} 1\,\mathrm{d}x \\
&\le \sup_{x\in K} |\phi(x)| \int_{[-R,R]} 1\,\mathrm{d}x \qquad \text{since } [0,\infty]\cap K \subset K \subset [-R,R] \\
&= 2R \sup_{x\in K} |\phi(x)|.
\end{aligned}$$

This is a very common idea for integrals of test functions. Because their support is compact, it has finite area. And because the functions are continuous, they obtain a maximum. These two factors then bound the integral of the test function.

(b) For the reason we just articulated, test functions are always L^1. Therefore the integral is well-defined and finite. The integral is a linear operator. So the only property we need to show is that H is continuous with respect to the semi-norms. Choose any compact set K. Then by part (a)

$$|F_H(\phi)| = \left|\int_0^\infty \phi(x)\,\mathrm{d}x\right| \le 2R \sup_{x\in K} |\phi(x)| = 2R\,\|\phi\|_{K,0}.$$

Thus the required inequality holds with $M = 1$, $\alpha_1 = 0$, and $C_1 = 2R$.

(c) We use H both for the function and the distribution F_H, as is common. In one sense, calculating the derivatives are easy, they are just $\partial_x H(\phi) = -H(\partial_x \phi)$ and $\partial_x^2 H(\phi) = -\partial_x H(\partial_x \phi) = H(\partial_x^2 \phi)$. But this does not give us an insight into their behaviour. However

$$\partial_x H(\phi) = -H(\partial_x \phi) = -\int_0^\infty \phi'\,\mathrm{d}x = -\Big[\phi\Big]_0^\infty = -\phi(\infty) + \phi(0) = \phi(0).$$

This is the delta distribution (also know as the Dirac distribution). And

$$\partial_x^2 H(\phi) = \int_0^\infty \phi''\,\mathrm{d}x = \Big[\phi'\Big]_0^\infty = -\phi'(0).$$

Neither of these distribution come from L^1_{loc} functions.

(d) G is linear in φ, so that's okay. We should check the continuity. But this is using the same general idea as (a) and (b): Choose any compact set K and test function supported in K. Then there is a ball $B(0,R)$ that contains K. Then

$$|G(\varphi)| \le \int_{C\cap B(0,R)} \sup_{x\in K} |\phi(x)|\,\mathrm{d}\sigma \le 2\pi \sup_{x\in K} |\phi(x)|.$$

The constant 2π follows since this is the maximum length of the circle C inside the ball $B(0, R)$.

There does not exist such a function g. Suppose for contradiction that it did exist, that $G(\varphi) = F_g(\varphi)$. For every point $y \notin C$ consider a small ball $B(y, r)$ that is disjoint from C. We will now apply Lemma 2.15 to this ball, $\Omega = B(y, r)$. For any test function $\varphi \in C_0^\infty(B(y, r))$ we know that it is zero on C because C and the ball are disjoint:

$$G(\varphi) = \int_C 0 \, d\sigma = 0$$

It follows from the lemma that $g = 0$ on $B(y, r)$ or more generally $g(y) = 0$ for $y \notin C$. But C is a null-set in $\mathbb{R}^2$, so we can say that $g \equiv 0$ as an L^1_{loc} function. This is a contradiction because G is not zero.

19. Delta Quadrant

(a) Prove that the support of the delta distribution δ is $\{0\}$. *(2 points)*

(b) Argue from Lemma 2.12 that δ is the limit of the standard mollifier $(\lambda_\epsilon)_{\epsilon>0}$ as $\epsilon \downarrow 0$ as a sequence of distributions. *(1 point)*

(c) Prove for any distribution F that the convolution with δ is again F. *(2 points)*

Solution.

(a) Choose any point x that isn't the origin. Then the ball $B(x, |x|)$ does not contain the origin. For any test function $\phi \in \mathcal{D}(B(x, |x|))$ we extend it by zero to a test function on $\mathbb{R}^n$. It has $\delta(\phi) = \phi(0) = 0$. By the definition of support of a distribution, this shows that $B(x, |x|)$ belongs to the complement of the support. Because this holds for all x other than the origin, the support is contained in $\{0\}$. It is either $\{0\}$ or the empty set. If it were the empty set, that would mean that δ is the zero distribution. Therefore the support is $\{0\}$.

(b) We say that $F_n \to F$ as distributions if $F_n(\phi) \to F(\phi)$ for all test functions ϕ (pointwise convergence). The distribution associated to a mollifier acts as

$$F_{\lambda_\epsilon}(\phi) = \int_{\mathbb{R}^n} \phi(x)\lambda_\epsilon(x) \, dx = \int_{\mathbb{R}^n} \phi(x)\lambda_\epsilon(-x) \, dx =: \phi_\epsilon(0),$$

using the fact that the standard mollifier is point symmetric. Lemma 2.12 says that $\phi_\epsilon(0)$ converges to $\phi(0)$. Therefore

$$F_{\lambda_\epsilon}(\phi) = \phi_\epsilon(0) \to \phi(0) = \delta(\phi).$$

(c) According to the definition $F * \delta(\phi) = F(\phi * \mathrm{P}\delta)$. In turn, the inner convolution defines a test function ψ through the formula

$$\psi(x) := (\phi * \mathrm{P}\delta)(x) = (\mathrm{P}\delta)(\mathrm{T}_x\mathrm{P}\phi) = \delta(\mathrm{P}\mathrm{T}_x\mathrm{P}\phi).$$

We should be careful when unwinding this stack of operators

$$(\mathrm{P}\mathrm{T}_x\mathrm{P}\phi)(y) = (\mathrm{T}_x\mathrm{P}\phi)(-y) = (\mathrm{P}\phi)(-y-x) = \phi(y+x).$$

Therefore $\psi(x) = \phi(0+x)$. Finally $F * \delta(\phi) = F(\psi) = F(\phi)$.