Introduction to Partial Differential Equations
Exercise sheet 6

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17. Convoluted

The convolution of two functions $f, g : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(y) g(x-y)\, dy.$$

(a) Let $f_n(x) = 0.5n$ for $x \in [-n^{-1}, n^{-1}]$ and 0 otherwise. Show that the following bounds hold

$$\inf_{|y| \le n^{-1}} g(y) \le (g * f_n)(0) \le \sup_{|y| \le n^{-1}} g(y).$$

(2 points)

(b) Suppose now that g is continuous. Show that $(g * f_n)(0) \to g(0)$ as $n \to \infty$. *(2 points)*

(c) Show that the convolution of C_0^∞-functions on $\mathbb{R}^n$ is a bilinear, commutative, and associative operation. *(1+2+2 Points)*

18. Distributions

(a) Choose any compact set $K \subset \mathbb{R}$. Since it is bounded, there exists $R > 0$ with $K \subseteq [-R, R]$. Now choose any test function $\phi \in C_0^\infty(\mathbb{R})$ with compact support in K. Since it is continuous, $\sup_{x \in K} |\phi(x)|$ is finite. Prove the following inequality *(1 point)*

$$\left| \int_0^\infty \phi(x)\, \mathrm{d}x \right| \le 2R \sup_{x \in K} |\phi(x)|.$$

(b) Define the Heaviside function $H : \mathbb{R} \to \mathbb{R}$ by $H(x) := 1$ for $x \ge 0$ and $H(x) := 0$ for $x < 0$. Show that the distribution associated to the Heaviside function

$$F_H : C_0^\infty(\mathbb{R}) \to \mathbb{R},\ \phi \mapsto \int_0^\infty \phi(x)\, \mathrm{d}x$$

is in fact a distribution on $\mathbb{R}$ using part (a) and Definition 2.14 directly. *(1 point)*

(c) Calculate the first and second derivatives of H as a distribution. If they are regular distributions, describe the corresponding function. *(2 points)*

(d) Consider the circle $C = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$. Show that

$$G(\varphi) := \int_C \varphi\, d\sigma$$

defines a distribution in $\mathcal{D}'(\mathbb{R}^2)$. Note that the $d\sigma$ indicates this is an integration over the submanifold C. Does there exist a locally integrable function $g : \mathbb{R}^2 \to \mathbb{R}$ with

$$G(\varphi) = \int_{\mathbb{R}^2} g\, \varphi\, \mathrm{d}x$$

for all $\varphi \in C_0^\infty(\mathbb{R})$? (Hint. Use Lemma 2.15) *(2 Points + 2 Bonus Points)*

19. Delta Quadrant

(a) Prove that the support of the delta distribution δ is $\{0\}$. *(2 points)*

(b) Argue from Lemma 2.12 that δ is the limit of the standard mollifier $(\lambda_\epsilon)_{\epsilon > 0}$ as $\epsilon \downarrow 0$ as a sequence of distributions. *(1 point)*

(c) Prove for any distribution F that the convolution with δ is again F. *(2 points)*