Introduction to Partial Differential Equations

Exercise sheet 5

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13. Is this an applied math course?

In economics, the Black-Scholes equation is a PDE that describes the price V of an (European-style) option under some assumptions about the risk and expected return, as a function of time t and current stock price S. The equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S},$$

where r and σ are constants representing the interest rate and the stock volatility respectively. Describe the order of this equation, and whether it is elliptic, parabolic, and/or hyperbolic.

(3 points)

Solution. The highest derivative in the equation is second order, so this is a second order PDE. To determine which type it is, bring all the terms to one side and order them as per the general form of a second order linear PDE

$$\left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + 0\frac{\partial^2 V}{\partial S \partial t} + 0\frac{\partial^2 V}{\partial t \partial S} + 0\frac{\partial^2 V}{\partial t^2}\right) + \left(rS\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t}\right) - rV = 0.$$

From this we can read off all the coefficient functions:

$$a = \begin{pmatrix} \frac{1}{2}\sigma^2 S^2 & 0 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} rS \\ 1 \end{pmatrix}, \quad c = -r.$$

We see that the matrix a is positive semi-definite and in fact its kernel is one dimensional. That makes it a parabolic PDE.

14. Around and around

Consider the unit circle $C = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$. In this question we will evaluate the integral

$$\int_C y \, d\sigma \qquad (*)$$

in two different ways, demonstrating that it does not depend on the choice of parametrisation.

(a) Consider a regular parameterisation Φ of a subset A and a continuous function f on A. Why (or under what conditions) is the integral unchanged by removing a point from A: *(1 bonus point)*

$$\int_A f \, d\sigma = \int_{A\setminus\{a\}} f \, d\sigma.$$

Therefore in order to compute the submanifold integration $(*)$ it is enough to use parameterisation that cover all but finitely many points of C.

(b) Consider the regular parametrisation $\Phi : (0, 2\pi) \to C$ given by $t \mapsto (\cos t, \sin t)$. Compute the integral $(*)$ using this parametrisation. *(2 points)*

(c) Consider upper and lower halves of the circle: $U_1 = \{(x, y) \in C \mid y > 0\}$ and $U_2 = \{(x, y) \in C \mid y < 0\}$. There are obvious parametrisations $\Phi_i : (-1, 1) \to U_i$ given by $\Phi_1(x) = (x, +\sqrt{1 - x^2})$ and $\Phi_2(x) = (x, -\sqrt{1 - x^2})$. Compute $(*)$ using these parametrisations.

(2 points)

(d) Compute this integral using the divergence theorem. *(2 points)*

Solution.

(a) First observe that if $\Phi : U \to A$ is a regular parameterisation of A then the restriction to $\tilde{U} = U \setminus \{\Phi^{-1}(a)\}$ is a regular parameterisation of $A \setminus \{a\}$. The question is reduced to explaining why the integral over U in Definition 2.4 is equal to the integral over $\tilde{U}$. This depends somewhat on the definition of integral that you are using. In Lebesgue integration, sets of measure zero can not contribute to the final value, and a point is measure zero in dimensions $k \geq 1$. With Darboux or Riemann integrations, these are defined initially on closed sets only. They are extended to open sets, or in this case punctured neighbourhoods by taking a limit of closed sets. Continuity of the integrand is sufficient then to ensure there is no difference.

(b) Using the previous part, we know that we can integrate with the parametrisation over $U = (0, 2\pi)$, and simply ignore the point $(1, 0) \in C$ that is not covered because that does not affect the value of the integral.

We must also calculate the distortion factor. The coordinate maps $\Phi : (0, 2\pi) \subset \mathbb{R} \to \mathbb{R}^2$, so its derivative is size 2×1, namely $(-\sin t, \cos t)^T$. The distortion factor therefore is

$$\det \begin{bmatrix} -\sin t & \cos t \end{bmatrix} \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} = \det \begin{bmatrix} 1 \end{bmatrix} = 1$$

We can now compute the integral

$$\int_0^{2\pi} \sin t \times 1 \, dt = -\cos t \Big|_0^{2\pi} = 0.$$

(c) Here we have that $\Phi_1' = (1, -x(1-x^2)^{-0.5})^T$. So

$$\begin{aligned} \int_{U_1} y \, d\sigma &= \int_{-1}^{1} \sqrt{1-x^2} \times \sqrt{1 + x^2(1-x^2)^{-1}} \, dx \\ &= \int_{-1}^{1} \sqrt{1 - x^2 + x^2} \, dx = 2. \end{aligned}$$

And likewise for U_2 we get an integral of -2. Together the value is zero, as in the previous part.

(d) To apply the divergence theorem, we recognise C as the boundary of the disc $\Omega = \{x^2 + y^2 \leq 1\}$, with the outward pointing normal $N = (x, y)$. (Because this it is the unit circle, this normal N is already unit length.) We now need to write the integrand y in the form $F \cdot N = (F_1, F_2) \cdot N = xF_1 + yF_2$. We see that $F = (0, 1)$ fits. The divergence of F is

$$\nabla \cdot F = \frac{\partial}{\partial x} 0 + \frac{\partial}{\partial y} 1 = 0.$$

Therefore

$$\int_C x \, d\sigma = \int_{\partial\Omega} F \cdot N \, d\mu = \int_\Omega \nabla \cdot F \, d\mu = \int_\Omega 0 \, d\mu = 0.$$

15. The Proof is Left as an Exercise for the Reader

Using Definitions 2.4 and 2.7, prove Lemma 2.9: The following properties hold for $a, b \in \mathbb{R}$ and $f, g \in C(A)$.

(i) Linearity: $\int_A af + bg\, d\sigma = a \int_A f\, d\sigma + b \int_A g\, d\sigma$. *(1 point)*

(ii) Order Preserving: if $f \leq g$ on A then $\int_A f\, d\sigma \leq \int_A g\, d\sigma$. *(1 point)*

(iii) Triangle Inequality: $\left|\int_A f\, d\sigma\right| \leq \int_A |f|\, d\sigma$. *(1 point)*

(iv) Transformation: If $P : \mathbb{R}^n \to \mathbb{R}^n$ is a euclidean motion (translation, reflection, rotation) and $s \in \mathbb{R}^+$ is a scaling factor then $\int_A f\, d\sigma = s^k \int_{(sP)^{-1}[A]} f \circ (sP)\, d\sigma$. *(3 points)*

Solution. First we make a general observation: it is sufficient to prove each of these properties in the case that A is a graph. In the general case, we have a cover O_i with subordinate partition of unity h_i. If we have proved linearity in the graph case, then the general case follows by rearranging the sum:

$$\begin{aligned}\int_A af + bg\, d\sigma &= \sum_i \int_{A_i \cap O_i} h_i(af + bg)\, d\sigma = \sum_i \int_{A_i \cap O_i} a(h_i f) + b(h_i g)\, d\sigma \\ &= \sum_i \left[a \int_{A_i \cap O_i} h_i f\, d\sigma + b \int_{A_i \cap O_i} h_i g\, d\sigma \right] \\ &= a \sum_i \int_{A_i \cap O_i} h_i f\, d\sigma + b \sum_i \int_{A_i \cap O_i} h_i g\, d\sigma \\ &= a \int_A f\, d\sigma + b \int_A g\, d\sigma.\end{aligned}$$

The proof of the general case from the graph case is similar for the other three properties. Therefore we assume below that A is a graph with parameterisation $\Phi(x) = (x, \lambda(x))$ for $\lambda : U \to \mathbb{R}^{n-k}$.

(i) In this case, like (ii) and (iii) we apply Definition 2.4 and 2.7 to write the submanifold integral as a normal Lebesgue integral, and then use the corresponding property.

$$\begin{aligned}\int_A af + bg\, d\sigma &= \int_U (af \circ \Phi + bg \circ \Phi)\sqrt{\det((\Phi')^T \Phi')} d\mu \\ &= \int_U af \circ \Phi \sqrt{\det((\Phi')^T \Phi')} d\mu + \int_U bg \circ \Phi \sqrt{\det((\Phi')^T \Phi')} d\mu \\ &= a \int_U f \circ \Phi \sqrt{\det((\Phi')^T \Phi')} d\mu + b \int_U g \circ \Phi \sqrt{\det((\Phi')^T \Phi')} d\mu \\ &= a \int_A f\, d\sigma + b \int_A g\, d\sigma.\end{aligned}$$

(ii) Note that the square root factor is always positive, so it's presence doesn't change the inequality:

$$\int_A f\, d\sigma = \int_U f \circ \Phi \sqrt{\det((\Phi')^T \Phi')} d\mu \leq \int_U g \circ \Phi \sqrt{\det((\Phi')^T \Phi')} d\mu = \int_A g\, d\sigma.$$

(iii) You can prove this like (ii), but we can also use the general proof that linearity and order preserving imply the triangle inequality for integrals. Let $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \max\{-f(x), 0\}$. By the definition of maximum, $f^+ \geq 0$ and $f^- \geq 0$. Further $f = f^+ - f^-$ and $|f| = f^+ + f^-$. Then

$$\left|\int_A f\, d\sigma\right| = \left|\int_A f^+ - f^-\, d\sigma\right| = \left|\int_A f^+\, d\sigma - \int_A f^-\, d\sigma\right| \leq \left|\int_A f^+\, d\sigma\right| + \left|\int_A f^-\, d\sigma\right|.$$

Until now we have only used the linearity of the integral. But since $f^\pm \geq 0$ then

$$\int_A f^\pm\, d\sigma \geq 0.$$

Therefore the absolute value signs can be dropped. We continue

$$\left|\int_A f\, d\sigma\right| \leq \int_A f^+\, d\sigma + \int_A f^-\, d\sigma = \int_A f^+ + f^-\, d\sigma = \int_A |f|\, d\sigma.$$

(iv) This is the only property that really requires us the use the parameterisation; in the previous parts the distortion factor was just a factor that was carried around.

The key insight to get started with this question is to understand that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 bijective transformation then for any parameterisation Φ of a submanifold A, we get a parameterisation $T \circ \Phi$ of $T[A]$. Let us give a concrete example. Consider scaling the unit circle $C \subset \mathbb{R}^2$ by s. This is the transformation $T(x, y) = (sx, sy)$. We know that $\Phi(t) = (\cos t, \sin t)$ is a parameterisation of C. Then $T \circ \Phi(t) = (s\cos t, s\sin t)$ is a parametrisation of the circle with radius s. Or another example, translation $T(x, y) = (x + a, y + b)$. Then $T \circ \Phi(t) = (\cos t + a, \sin t + b)$ is a parametrisation of the unit circle centered at (a, b). To put this in the form that we will use it, if Φ is a parameterisation of $T^{-1}[A]$ then $T \circ \Phi$ is a parameterisation of A.

The question is "how do these transformations affect the submanifold integral?" Let us begin with translation $P(x) = x + b$ for $b \in \mathbb{R}^n$. Then $(P \circ \Phi)(x) = \Phi(x) + b$ and clearly $(P \circ \Phi)' = \Phi'$. Hence

$$\begin{aligned}
\int_A f\, d\sigma &= \int_U f \circ (P \circ \Phi)\sqrt{\det(((P \circ \Phi)')^T (P \circ \Phi)')} d\mu \\
&= \int_U (f \circ P) \circ \Phi \sqrt{\det((\Phi')^T \Phi')} d\mu \\
&= \int_{P^{-1}[A]} f \circ P\, d\sigma.
\end{aligned}$$

The proofs for reflection and rotation are similar to one another. They are linear transformations so we can describe them by a constant square matrix A, namely $P(x) = Ax$. This means $(P \circ \Phi)(x) = A\Phi(x)$ and $(P \circ \Phi)' = A\Phi'$. Now we compute

$$\begin{aligned}
\int_A f\, d\sigma &= \int_U f \circ (P \circ \Phi)\sqrt{\det(((P \circ \Phi)')^T (P \circ \Phi)')} d\mu \\
&= \int_U (f \circ P) \circ \Phi \sqrt{\det((\Phi')^T A^T A \Phi')} d\mu.
\end{aligned}$$

What is special about rotations and reflections is that $A^T A = I$ is the identity matrix. Hence we can finish off the calculation

$$= \int_U (f \circ P) \circ \Phi \sqrt{\det((\Phi')^T I \Phi')} d\mu = \int_{P^{-1}[A]} f \circ P \, d\sigma.$$

Finally, we have a scaling transformation $T(x) = sx$. Actually, we can also write this as a matrix $T(x) = (sI)x$ with $A = sI$. So we reuse the calculation above. This time $A^T A = (sI)^T(sI) = s^2 I^T I = s^2 I$ and

$$\begin{aligned} \int_A f \, d\sigma &= \int_U (f \circ T) \circ \Phi \sqrt{\det((\Phi')^T (s^2 I) \Phi')} d\mu \\ &= \int_U (f \circ T) \circ \Phi \sqrt{s^{2k} \det((\Phi')^T \Phi')} d\mu \\ &= s^k \int_{T^{-1}[A]} (f \circ T) \, d\sigma. \end{aligned}$$

16. The Black Spot

Consider the plane $\mathbb{R}^2$, a disc $B_r = \{x^2 + y^2 \leq r^2\}$ and the function $g(x, y) = \ln(x^2 + y^2)$.

(a) By calculating $\nabla g \cdot N$ for the outward pointing normal $N = \frac{1}{r}(x, y)$, show that the value of the integral

$$\int_{\partial B_r} \nabla g \cdot N \, d\sigma$$

does not depend on the radius r. *(2 points)*

(b) Can you explain this fact using the divergence theorem?
Hint: Apply the divergence theorem to the region $A_{r,R} = B_R \setminus \overline{B_r}$ for two different radii $r < R$. *(3 points)*

Solution.

(a) The outward pointing unit normal of the ball of radius r is $N = r^{-1}(x, y)$ where $r = \sqrt{x^2 + y^2}$. The gradient $\nabla g = r^{-2}(2x, 2y)$. Together that gives

$$\nabla g \cdot N = r^{-3}(2x^2 + 2y^2) = r^{-3} 2r^2 = 2r^{-1}.$$

Then because r is constant on ∂B_r,

$$\int_{\partial B_r} \nabla g \cdot N \, d\sigma = \int_{\partial B_r} 2r^{-1} \, d\sigma = 2r^{-1} \int_{\partial B_r} 1 \, d\sigma.$$

Now, the integral above is just the 1 times the circumference of the circle, and therefore the value is $2r^{-1} \times 2\pi r = 4\pi$.

(b) Let us consider the difference of two of these integrals for different radii $r < R$.

$$\int_{\partial B_R} \nabla g \cdot N \, d\sigma - \int_{\partial B_r} \nabla g \cdot N \, d\sigma = \int_{\partial B_R} \nabla g \cdot N \, d\sigma + \int_{\partial B_r} \nabla g \cdot (-N) \, d\sigma = \int_{\partial A_{r,R}} \nabla g \cdot N \, d\sigma,$$

where $A_{r,R}$ is the annulus with inner radius r and outer radius R. Note, the boundary of the annulus consists of two disjoint circles and the outward pointing normal of the annulus on the inner boundary circle is the outward point normal of the disc B_r. This explains the sign in the above calculation.

If we apply the divergence theorem to the annulus, we get

$$\int_{\partial A_{r,R}} \nabla g \cdot N \, d\sigma = \int_{A_{r,R}} \Delta g \, dx,$$

and the Laplacian of g is zero:

$$\begin{aligned} \Delta g &= \frac{\partial}{\partial x} \frac{2x}{x^2 + y^2} + \frac{\partial}{\partial y} \frac{2y}{x^2 + y^2} \\ &= \frac{2}{x^2 + y^2} - \frac{4x^2}{(x^2 + y^2)^2} + \frac{2}{x^2 + y^2} - \frac{4y^2}{(x^2 + y^2)^2} \\ &= 0. \end{aligned}$$

Therefore the integral over the annulus is zero, and hence the difference of the integrals on the two circles is also zero.

If we are being precise, the Laplacian of g is zero on $\mathbb{R}^2 \setminus \{(0,0)\}$; at the origin it is not defined/singular. This is why we could not apply the divergence theorem directly to B_r.