Introduction to Partial Differential Equations
Exercise sheet 5

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13. Is this an applied math course?

In economics, the Black-Scholes equation is a PDE that describes the price V of an (European-style) option under some assumptions about the risk and expected return, as a function of time t and current stock price S. The equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S},$$

where r and σ are constants representing the interest rate and the stock volatility respectively. Describe the order of this equation, and whether it is elliptic, parabolic, and/or hyperbolic.

(3 points)

14. Around and around

Consider the unit circle $C = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$. In this question we will evaluate the integral

$$\int_C y \, d\sigma \qquad (*)$$

in two different ways, demonstrating that it does not depend on the choice of parametrisation.

(a) Consider a regular parameterisation Φ of a subset A and a continuous function f on A. Why (or under what conditions) is the integral unchanged by removing a point from A: *(1 bonus point)*

$$\int_A f \, d\sigma = \int_{A\setminus\{a\}} f \, d\sigma.$$

Therefore in order to compute the submanifold integration $(*)$ it is enough to use parameterisation that cover all but finitely many points of C.

(b) Consider the regular parametrisation $\Phi : (0, 2\pi) \to C$ given by $t \mapsto (\cos t, \sin t)$. Compute the integral $(*)$ using this parametrisation. *(2 points)*

(c) Consider upper and lower halves of the circle: $U_1 = \{(x, y) \in C \mid y > 0\}$ and $U_2 = \{(x, y) \in C \mid y < 0\}$. There are obvious parametrisations $\Phi_i : (-1, 1) \to U_i$ given by $\Phi_1(x) = (x, +\sqrt{1 - x^2})$ and $\Phi_2(x) = (x, -\sqrt{1 - x^2})$. Compute $(*)$ using these parametrisations. *(2 points)*

(d) Compute this integral using the divergence theorem. *(2 points)*

15. The Proof is Left as an Exercise for the Reader

Using Definitions 2.4 and 2.7, prove Lemma 2.9: The following properties hold for $a, b \in \mathbb{R}$ and $f, g \in C(A)$.

(i) Linearity: $\int_A af + bg \, d\sigma = a\int_A f \, d\sigma + b\int_A g \, d\sigma$. *(1 point)*

(ii) Order Preserving: if $f \leq g$ on A then $\int_A f \, d\sigma \leq \int_A g \, d\sigma$. *(1 point)*

(iii) Triangle Inequality: $\left|\int_A f \, d\sigma\right| \leq \int_A |f| \, d\sigma$. *(1 point)*

(iv) Transformation: If $P : \mathbb{R}^n \to \mathbb{R}^n$ is a euclidean motion (translation, reflection, rotation) and $s \in \mathbb{R}^+$ is a scaling factor then $\int_A f \, d\sigma = s^k \int_{(sP)^{-1}[A]} f \circ (sP) \, d\sigma$. *(3 points)*

16. The Black Spot

Consider the plane $\mathbb{R}^2$, a disc $B_r = \{x^2 + y^2 \leq r^2\}$ and the function $g(x, y) = \ln(x^2 + y^2)$.

(a) By calculating $\nabla g \cdot N$ for the outward pointing normal $N = \frac{1}{r}(x, y)$, show that the value of the integral

$$\int_{\partial B_r} \nabla g \cdot N \, d\sigma$$

does not depend on the radius r. *(2 points)*

(b) Can you explain this fact using the divergence theorem?
Hint: Apply the divergence theorem to the region $A_{r,R} = B_R \setminus \overline{B_r}$ for two different radii $r < R$. *(3 points)*