Introduction to Partial Differential Equations
Exercise sheet 4

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10. Don't cross the streams Consider the PDE $\partial_x u + x\partial_y u = 0$ on the domain $y > 0$ with the boundary condition $u(x, 0) = g(x)$.

(a) Show that the boundary hyperplane $\{y = 0\}$ is non-characteristic at $(x, 0)$, except for $x = 0$. *(1 point)*

(b) What condition does the PDE impose on the boundary data g at the point $(0, 0)$? *(1 point)*

(c) Determine the characteristic curves of this PDE. *(1 point)*

(d) By considering the y-derivative of u on the boundary hyperplane, show that there is no C^1 solution with the initial data $g(x) = x$. *(2 points)*

Solution.

(a) The hyperplane is already suitable for Definition 1.6. In symbolic form, the PDE is $F(p, z, x) = p_1 + x_1 p_2 = 0$, so $\frac{\partial F}{\partial p_2} = x_1$. This is nonzero when $x_1 \neq 0$, ie every point of the hyperplane except the origin.

(b) On the boundary, we see that

$$0 = \partial_x u(x, 0) + x\partial_y u(x, 0) = \partial_x g(x) + x\partial_y u(x, 0),$$

since the x-derivative of u can be determined entirely by g. Thus at the origin we must have $g'(0) = 0$.

(c) The characteristic curves should obey $\dot{x} = 1, \dot{y} = x$, which is a closed system. First we obtain that $x = s + x_0$ and then

$$\dot{y} = s + x_0 \Rightarrow y = \frac{1}{2}s^2 + x_0 s + y_0.$$

Setting $y_0 = 0$ and eliminating s gives the family of parabolas $y = \frac{1}{2}x^2 - \frac{1}{2}x_0^2$. These foliate the plane completely without overlapping. The parabola $y = \frac{1}{2}x^2$ passes through the origin.

(d) The solution is constant along characteristic curves, so it is

$$u(x, y) = g(x_0) = x_0 = \pm\sqrt{x^2 - 2y}$$

where the sign is chosen to be the same sign as x. The y-derivative is

$$\frac{\partial u}{\partial y} = \mp\frac{1}{\sqrt{x^2 - y}}, \qquad \frac{\partial u}{\partial y}(x, 0) = -\frac{1}{x}.$$

Thus u cannot be continuously differentiable.

Notice that this initial condition does not obey the condition in part (b).

11. It's just a jump to the left

In this question we explore some other solutions to the initial value problem from Example 1.10. As we saw, for small t the method of characteristics gives a unique solution

$$u_{t<1}(x,t) = \begin{cases} 1 & \text{for } x < t \\ \frac{x-1}{t-1} & \text{for } t \leq x < 1 \\ 0 & \text{for } 1 \leq x. \end{cases}$$

(a) (Optional) Derive this solution for yourself, for extra practice.

After $t = 1$, the characteristics begin to cross and so the method cannot assign which value u should have at a point (x, t). However, we could still arbitrarily decide to choose a value of one characteristic. Consider therefore

$$v(x,t) = \begin{cases} u_{t<1} & \text{for } t < 1 \\ 1 & \text{for } x < t \\ 0 & \text{for } t \leq x \end{cases}$$

(b) Draw the corresponding characteristics diagram in the (x, t)-plane for this function. *(2 points)*

(c) Describe the graph of discontinuities $y(t)$. Compute the Rankine-Hugonoit condition for v. *(2 points)*

(d) How much mass (i.e. the integral of v over x) is being lost in the system described by v for $t > 1$? *(2 points)*

Solution.

(a) Refer to lecture script.

(b) The characteristics for $t < 1$ are described in the lecture script

$$x = \begin{cases} t + x_0 & \text{for } x_0 \leq 0 \\ t(1 - x_0) + x_0 & \text{for } 0 < x_0 \leq 1 \\ x_0 & \text{for } 1 < x_0. \end{cases}$$

However, for $t > 1$ there are two regions: $x > t$ and $x < t$. In the former the characteristics continue to be horizontal lines. In the lower region they are lines with gradient 1.

(c) The discontinuity is for $t > 1$ when the solution jumps from 0 to 1. This occurs on the line $y(t) = t$. Hence $\dot{y} = 1$. On the other hand, $f(u) = \frac{1}{2}u^2$, the value of u on the upper side of the discontinuity is $v^r(t^+, t) = 0$ and $v^l(t^-, t) = 1$. The right hand side of the Rankine-Hugonoit condition is then

$$\frac{\frac{1}{2} \cdot 0^2 - \frac{1}{2} \cdot 1^2}{0 - 1} = \frac{1}{2}.$$

This shows that v does not fulfil the condition.

(d) We know that mass is conserved away from the discontinuity. Therefore we just need to know how much is being lost across the discontinuity. We first compute the amount of mass in some interval containing the discontinuity, say $[a, b]$:

$$\int_a^b v(x,t)\,dx = \int_a^t 1\,dx + \int_t^b 0\,dx = t - a.$$

So the amount of mass in the interval is increasing by 1 unit of mass per unit of time. And then we compute how much mass is moving in and out of this interval:

$$f(u(t-1,t)) - f(u(t+1,t)) = f(1) - f(0) = \frac{1}{2}.$$

So there is a constant inflow of 0.5 units of mass per unit of time. Hence the system must be gaining 0.5, because the inflow to the interval is less than the amount the interval is gaining.

12. You're not in traffic, you are traffic

In this question we look at an equation similar to Burgers' equation that describes traffic. Let u measure the number of cars in a given distance of road, the car density. We have seen that f should be interpreted as the flux function, the number of things passing a particular point. When there are no other cars around, cars travel at the speed limit s_m. When they are bumper-to-bumper they can't move, call this density u_m.

(a) What properties do you think that f should have? Does $f(u) = s_m u \cdot (1 - u/u_m)$ have these properties? *(2 points)*

(b) Find a function f that meets your conditions, or use the f from the previous part, and write down a PDE to describe the traffic flow. *(1 point)*

(c) Find all solutions that are constant in time. *(2 points)*

(d) Consider the situation of the start of a race: to the left of the starting line, the racecars are queued up at half of the maximum density (ie $0.5u_m$). To the right of the starting line, the road is empty. Now, at time $t = 0$, the race begins. Give a discontinuous solution that obeys the Rankine-Hugonoit condition, as well as a continuous solution. *(4 points)*

Solution.

(a) The density flux of the cars should be the density of the cars multiplied by the speed they are travelling $f = us$. We already know that speed depends on the car density u, being zero for $u = u_m$ and s_m for $u = 0$. Assuming a linear relationship gives $s(u) = s_m(1 - u/u_m)$ and the f in the question. A more realistic relationship between density and speed would be non-linear, but probably still monotone and concave.

(b) Now, cars are a conserved quantity; have you ever seen a car vanish? Therefore it is reasonable to use the conservation PDE model. Differentiating f from the previous part gives

$$\dot{u} + s_m \left(1 - 2\frac{u}{u_m}\right) \partial_x u = 0.$$

(c) If a solution is constant in time, then it must be that $1 - 2u/u_m = 0$ or $\partial_x u = 0$. In either case, u must be constant. Conversely, all constant solutions solve the PDE.

(d) Choose units so that $s_m = 1$ and $u_m = 2$. The PDE is now

$$\dot{u} + (1-u)\,\partial_x u = 0.$$

The characteristics are $x = x_0 + (1 - u_0(x_0))t$, in other words

$$\begin{cases} x = x_0 & \text{for } x_0 < 0 \\ x = x_0 + t & \text{for } x_0 > 0. \end{cases}$$

Physically we can explain this as there being a region $x > t$ where the first car at the light, now driving full-speed, has not yet reached and another region $x < 0$ where the traffic constant density and so all moving at the same speed and hence the number of cars is constant. In between cars can move with greater speed and race ahead of the cars behind them, but density prevents them from moving at full speed.

This middle region is not determined by the initial conditions, so there is the possibility to have many solutions. If there was a jump between no cars and half maximum density, then the Rankine-Hugonoit condition would say it would have a slope of

$$\dot{y} = \frac{s_m \frac{1}{2} u_m (1 - \frac{1}{2} u_m/u_m) - 0(1 - 0/u_m)}{\frac{1}{2} u_m - 0} = \frac{1}{2} s_m.$$

The interpretation is that the lights turn green and all the racecars move forward at the same speed $s(\frac{1}{2}u_m) = \frac{1}{2}s_m$. You can interpret this solution as treating the racecars at the front as belonging to the half-maximum density region.

But we prefer solutions that are as regular as possible (and also racecar drivers who drive as fast as possible). By characteristics, if $u(x,t)$ is C^1 in this region then it must be constant on lines through the origin: $x = ct$ for $c \in [-1, 1]$. It must therefore be equal to some function $g(x/t) = g(c)$ with $g(0) = 1$ and $g(1) = 0$. The PDE then reduces to an ODE.

$$\begin{aligned} -\frac{x}{t^2} g' + (1-g) \cdot \frac{1}{t} g' &= 0 \\ -cg' + (1-g)g' &= 0 \\ g' \cdot (-c + 1 - g) &= 0. \end{aligned}$$

So either $g(c)$ is constant, which contradicts the endpoint conditions, or $g(c) = 1 - c$. In summary

$$u(x,t) = \begin{cases} 1 & \text{for } x < 0 \\ 1 - \frac{x}{t} & \text{for } 0 \le x \le t \\ 0 & \text{for } t < x \end{cases}$$

is a continuous solution.