Introduction to Partial Differential Equations
Exercise sheet 3

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7. Royale with Cheese

Recall Burgers' equation from Example 1.5 of the lecture script:

$$\dot{u} + u\partial_x u = 0,$$

for $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. In this question we will apply the method of characteristics to solve this equation for the initial condition $g(x) = 2x$.

(a) According to Theorem 1.4, there is a unique C^1 solution to this initial value problem, at least when t is small. For how long does the theorem guarantee that the solution exists uniquely? *(1 point)*

(b) Suppose that u is a solution to this equation and suppose that $(x(s), t(s))$ is a path in the domain of u. What is the s derivative of u along this path? What constraints should we place on the derivatives of x and t? *(2 points)*

(c) On an (x, t)-plane draw the characteristics. *(1 point)*

(d) Finally, derive the following solution to the initial value problem: *(2 points)*

$$u(x,t) = \frac{2x}{1+2t}.$$

Solution.

(a) The condition in the theorem depends on the bound $f''(g(x))g'(x) \geq -\alpha$. For this equation, $f''(u) = 1$ and $g'(x) = 2$, so the product is bound below easily by $\alpha = 0$. Hence the theorem says that the unique solution exists for all time.

(b) By the chain rule,

$$\frac{d}{ds}u(x(s), t(s)) = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial t}\frac{dt}{ds} = \frac{dt}{ds}\dot{u} + \frac{dx}{ds}\partial_x u.$$

If we compare this to the PDE, then we see that we should choose $t' = 1$, i.e. $t = s + t_0$, and $x' = u$. Because the initial condition is $t = 0$, we choose $t_0 = 0$.

(c) With the choice made in the previous part, the function u is constant along the characteristic. Therefore we have $x' = g(x_0) = 2x_0$, which integrates to $x(s) = 2x_0 s + x_0$. We already computed in the previous part $t(s) = s$. The characteristics are the rays $x = 2x_0 t + x_0$ for $x_0 \in \mathbb{R}$. The solution takes the value x_0 on the corresponding ray. We can see that the 'mass' (the conserved quantity) is flowing away from the origin.

(d) We have see that the characteristic has the equation

$$x = x_0 + 2x_0 t = x_0(1 + 2t),$$

and that u is constant along the characteristic. In other words, we know that for all x_0 and t

$$u(x(t), t) = u(x_0(1 + 2t), t) = g(x_0) = 2x_0.$$

We have now found the solution in implicit form. To make it explicit, rearrange the relation $x = x_0(1 + 2t)$. Then

$$u(x,t) = 2x_0 = 2\frac{x}{1+2t}.$$

8. Linear Partial Differential Equations Consider a PDE of the form $F(\nabla u(x), u(x), x) = 0$. Suppose that F is linear in the derivatives and has continuously differentiable coefficients. That is, it can be written in the form

$$F(p, z, x) = b(z, x) \cdot p + c(z, x)$$

with b and c continuously differentiable. Show that the characteristic curves $(x(s), z(s))$ for $z(s) := u(x(s))$ can be described by ODEs that are independent of $p(s) := \nabla u(x(s))$. *(4 points)*

Solution.

We try to apply the method of characteristics as we have been using it to this point. We differentiate z with respect to s using the chain rule:

$$\frac{dz}{ds} = \nabla u(x(s)) \cdot \dot{x}(s) = p(s) \cdot \dot{x}(s)$$

We see that if we choose $\dot{x} = b(z, x)$ then we continue

$$= p(s) \cdot \dot{x}(s) = p \cdot b(z, x) = -c(z, x).$$

Now we have a system of ODES, namely

$$\dot{x} = b(z, x), \quad \dot{z} = -c(z, x),$$

that only involves z and x.

Let us now give some commentary on this case compared to the general case. In particular we see here how it is the fact that the PDE has a term like $b \cdot p$ that allows us to simplify the equation for z so well. In general a first order PDE does not need to have a term $b \cdot p$ and thus the equation for $\dot{z}$ will depend on p. If the equation for $\dot{z}$ depends on p, then we also need to know how p is changing along the curve. Following Section 1.5, p changes according to

$$\frac{dp}{ds} = \left(\sum_j \partial_i \partial_j u \, \dot{x}_j \right) = \text{Hess}(u)\dot{x},$$

where $\text{Hess}(u)$ is the matrix of second derivatives of u. The problem is that this is not an ODE in x, z, p because it depends on the second derivatives of u. The trick in the general case is to look at the total derivative of F with respect to x_i:

$$\begin{aligned} 0 &= \partial_p F \cdot \partial_i p + \partial_z F \partial_i z + \partial_i F \\ &= \partial_p F \cdot \partial_i p + \partial_z F \, p_i + \partial_i F \\ 0 &= \text{Hess}(u) \nabla_p F + \partial_z F \, p + \nabla_x F. \end{aligned}$$

This equation also has a $\text{Hess}(u)$ in it, but all other terms just involve x, z, p. If we suppose that the characteristic has the property that $\dot{x} = \partial_p F$, then we can use this equation to eliminate the $\text{Hess}(u)$ from the $\dot{p}$ equation

$$\dot{p} = -\partial_z F \, p - \nabla F.$$

So $p(s)$ is described by an ODE and the assumption about $\dot{x}$ does not involve p.

Notice that our choice of characteristic $\dot{x} = \partial_p F$ also only depends on x, z, p. Finally,

$$\dot{z} = \nabla u \cdot \dot{x} = p \cdot \partial_p F$$

also only depends on x, z, p. So we have arrived at a system of ODEs in x, z, p.

The characteristics of the general case are actually the same as the ones in the simpler case, because $\dot{x} = \partial_p F = b$.

9. Solving PDEs Solve the initial value problems of the following PDEs using the method of characteristics. You may assume that g is continuously differentiable on the corresponding domain.

(a) $x_2 \partial_1 u - x_1 \partial_2 u = u$ on the domain $x_1, x_2 > 0$, with initial condition $u(x_1, 0) = g(x_1)$.

(3 points)

(b) $x_1 \partial_1 u + 3x_2 \partial_2 u + \partial_3 u = 2u$ on $x_1, x_2 \in \mathbb{R}$, $x_3 > 0$, with initial condition $u(x_1, x_2, 0) = g(x_1, x_2)$.

(3 points)

(c) $u \partial_1 u + \partial_2 u = 1$ on the domain $x_1, x_2 > 0$, with initial condition $u(x_1, x_1) = \frac{1}{2} x_1$.

(4 points)

Solution. These PDEs are all quasi-linear (discussed in class), so we may use the simplified form of the equations of the method of characteristics.

(a) This PDE is $(x_2, -x_1) \cdot p - z = 0$. The system of ODEs therefore reads in part

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1,$$

which is the well know system solved by the sinusoidal functions. From the boundary condition $(x_1, 0)$, we should choose $x_{20} = 0$. Therefore $x_2 = -x_{10} \sin s$ and $x_1 = x_{10} \cos s$ are the characteristics. Given a point (x_1, x_2) we can determine the parameters of the characteristics as

$$s = \arctan(-x_2/x_1), \quad x_{10} = \sqrt{x_1^2 + x_2^2} = |x|.$$

The ODE describing the values of u is $\dot{z} = z$, which has the solution $z(s) = e^s z(0)$. From the initial condition

$$z(0) = u(x(0)) = u(x_{10}, 0) = g(x_{10}).$$

Putting this together

$$u(x) = e^{\arctan(-x_2/x_1)} g(|x|).$$

(b) From $F = (x_1, 3x_2, 1) \cdot p - 2z$ it follows that

$$x(s) = (x_{10}e^s, x_{20}e^{3s}, x_{30} + s) = (x_{10}e^s, x_{20}e^{3s}, s),$$

where we choose our starting points for the characteristics to lie in the case of the initial conditions, which requires us to set $x_{30} = 0$. Already we can determine the appropriate parameter values for any point: $s = x_3$, $x_{10} = x_1 e^{-x_3}$, and $x_{20} = x_2 e^{-3x_3}$. The the equation for the values is $z = z(0)e^{2s}$, so

$$u(x) = e^{2s}u(x_{10}, x_{20}, 0) = e^{2x_3} g\left(x_1 e^{-x_3}, x_2 e^{-3x_3}\right).$$

(c) This PDE, $F = (z, 1) \cdot p - 1$ is a little different to the others, because of the z in the coefficients of p. This creates a linkage in the system of ODEs:

$$\dot{x}_1 = z, \quad \dot{x}_2 = 1, \quad \dot{z} = 1.$$

Fortunately, we can solve for z first this time quite easily: $z(s) = s + z(0)$. Then $x(s) = (\frac{1}{2}s^2 + sz(0) + x_{10}, s + x_{20})$. Choose $x_{20} = x_{10}$. This choice means that $x_{10} = x_2 - s$. The initial conditions give $z(0) = u(x(0)) = u(x_{10}, x_{10}) = \frac{1}{2}x_{10}$. Together this allows us to solve for s:

$$\begin{aligned} x_1 &= \frac{1}{2}s^2 + s\frac{1}{2}(x_2 - s) + x_2 - s \\ x_1 - x_2 &= \frac{1}{2}x_2 s - s \\ s &= \frac{2x_1 - 2x_2}{x_2 - 2}. \end{aligned}$$

Finally, what we are interested in is the value of the solution u on these curves, and $u(x(s)) = z(s) = s + z(0) = s + \frac{1}{2}(x_2 - s)$, ie

$$u(x) = \frac{1}{2}x_2 + \frac{1}{2}\frac{2x_1 - 2x_2}{x_2 - 2}.$$