Introduction to Partial Differential Equations
Exercise sheet 2

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4. **Inhomogeneous Transport Equation** First order partial differential equations share many things in common with first order ordinary differential equations (ODEs). Consider the linear inhomogeneous ODE

$$\frac{du}{dt} = f(t).$$

(a) Find a solution $u : \mathbb{R} \to \mathbb{R}$ to this equation. *(1 point)*

(b) For any initial value $c \in \mathbb{R}$, show that there is a unique solution with $u(0) = c$. *(2 points)*

We consider now the inhomogeneous transport equation

$$\partial_t u + b \cdot \nabla u = f$$

with initial value given by a function $g(x)$, namely $u(x, 0) = g(x)$. It had an explicit solution

$$u(x, t) = g(x - tb) + \int_0^t f(x + (s - t)b, s)\, ds.$$

(c) Show that the integral term itself solves the inhomogeneous transport equation. What initial value problem does it solve? *(3 points)*

(d) Prove that the solution to the initial value problem is unique. (You may assume that the solution to the homogeneous version is unique.) *(2 points)*

Solution.

(a) A solution is given by the integral $\int_0^t f(s)\, ds$.

(b) A solution to the initial value problem is

$$u(t) = c + \int_0^t f(s)\, ds.$$

This is a sum of a solution to the homogeneous equation that satisfies the initial value and a solution to the inhomogeneous equation that has an initial value of zero.

Suppose that there is another solution v. Then by linearity $u - v$ solves the homogeneous equation $\frac{du}{dt} = 0$ and therefore is a constant. But we know these solutions have the same initial value and hence $u - v = 0$ for all time.

(c) We compute the t and x derivatives of the integral term

$$\begin{aligned}
\frac{\partial}{\partial t} \int_0^t f(x + (s - t)b, s)\, ds &= f(x + (t - t)b, t) + \int_0^t (-b) \cdot \nabla f(x + (s - t)b, s)\, ds \\
&= f(x, t) - b \cdot \int_0^t \nabla f(x + (s - t)b, s)\, ds \\
\nabla \int_0^t f(x + (s - t)b, s)\, ds &= \int_0^t \nabla f(x + (s - t)b, s)\, ds.
\end{aligned}$$

The sum is equal to $f(x, t)$ as required. We find the initial value of this function by substituting $t = 0$. But then we have $\int_0^0$ which is always 0.

(d) Because this is a linear equation, the difference between two solutions of the inhomogeneous equation is a solution of the homogeneous equation with zero initial value. We have seen in lectures that the solution to the homogeneous transport equation with initial value is unique. Therefore any two solutions must be equal.

5. Method of characteristics for an Inhomogeneous PDE Use the method of characteristics to solve the following *inhomogeneous* PDE

$$x\partial_x u + y\partial_y u = 1$$

on the domain $x > 0, y \in \mathbb{R}$, with initial condition $u(1, y) = y$. Note, the function u will *not* be constant along the characteristic, but its value along the characteristic will be determined by its initial value. *(6 points)*

Solution. As before, we consider a path $(x(s), y(s))$ in the domain and compute how u changes along this path:

$$\frac{du}{ds} = \partial_x u \cdot x' + \partial_y \cdot y'.$$

If we choose a path with $x' = x$ and $y' = y$, then

$$\frac{du}{ds} = x\partial_x u + y\partial_y u = 1.$$

The characteristic is the parametric curve $x(s) = x_0 e^s$, $y(s) = y_0 e^s$. We want that the characteristic to be at the boundary $x = 1$ for $s = 0$, which means we should choose $x_0 = 1$. In non-parametric form the characteristics are the lines $y - y_0 x = 0$. These are lines which pass through the origin. The point (x_1, y_1) belongs to the characteristic with $y_0 = y_1/x_1$. Every point in the domain belongs to exactly one characteristic and every characteristic passes through the boundary condition, so there is a unique solution.

If we integrate the differential equation for u, the value of u changes along the characteristic according to $u(s) = s + u_0$ where $u_0 = u(s = 0) = u(1, y_0)$. So to find the value of u at the point (x_1, y_1), not only do we have to find which characteristic the point belongs to, we also have to find out the value of s at that point. The value of u_0 is

$$u_0 = u(s = 0) = u(1, y_0) = y_0 = y_1/x_1.$$

The value of s at (x_1, y_1) is $e^s = x_1$, which is easily seen from the equation for $x(s)$. Therefore

$$u(x_1, y_1) = s + u_0 = \ln x_1 + \frac{y_1}{x_1}.$$

Since this holds for any point (x_1, y_1), we can just write $u(x, y) = \ln x + y/x$. This is the solution. Another way to explain this is that we labelled the point (x_1, y_1) with subscript 1 to distinguish it from the parametric functions $x(s), y(s)$, but now we have finished the question we can throw away the characteristics.

Desmos: https://www.desmos.com/calculator/drlrkvbz2d

6. Duhamel's Principle

Duhamel's principle is a method to find a solution to an inhomogeneous PDE if one can solve the homogeneous PDE for any initial condition. In this exercise we give the general idea and show how it applies to the transport equation. Consider an inhomogeneous PDE on $\mathbb{R}^n \times \mathbb{R}$ of the following form

$$\partial_t u - Lu = f(x,t), \quad u(x,0) = 0,$$

where L is a linear differential operator on $\mathbb{R}^n$ with constant coefficients. The idea is to instead consider the following family of homogeneous equations

$$\partial_t u_s - Lu_s = 0, \quad u_s(x,s) = f(x,s).$$

Suppose that we can find such solutions u_s. Prove that

$$u(x,t) = \int_0^t u_s(x,t)\, ds$$

is a solution to the inhomogeneous problem. (Do not worry about convergence problems.)
Use this method to solve the inhomogeneous transport. *(2 + 4 points)*

Solution. We just compute

$$\begin{aligned}
\partial_t u(x,t) &= u_t(x,t) + \int_0^t \partial_t u_s(x,t)\, ds \\
&= f(x,t) + \int_0^t L u_s(x,t)\, ds \\
&= f(x,t) + L \int_0^t u_s(x,t)\, ds \\
&= f(x,t) + Lu(x,t).
\end{aligned}$$

Here we have interchanged the operator L and the integral without checking whether this is technically valid.

Suppose we have an inhomogeneous transport problem with zero initial condition.

$$\dot{u} + b \cdot \nabla u = f, \quad u(x,0) = 0.$$

Duhamel's principle tells us we should instead solve, for each s,

$$\dot{u}_s + b \cdot \nabla u_s = 0, \quad u_s(x,s) = f(x,s).$$

In the script we show how to solve the initial value problem when the initial value is at time 0. To solve a problem where the initial value is at time s, we need to do a change of coordinates $v_s(x,t') = u_s(x, t'+s)$. This gives us

$$\partial_{t'} v_s + b \cdot \nabla v_s = 0, \quad v_s(x,0) = f(x,s)$$

which has the solution $v_s(x,t') = f(x - bt', s)$. Thus $u_s(x,t) = f(x - b(t-s), s)$. Finally we get the solution

$$u(x,t) = \int_0^t f(x - b(t-s), s)\, ds.$$

This is the same as the solution we derived in Section 1.2.